Nonabelian algebraic topology: Part I

Ronald Brown
Department of Mathematics
School of Informatics
University of Wales, Bangor

Rafael Sivera Villanueva
Departmento de Geometria y Topologia
Universidad de Valencia(Spain)

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This contains the Introductory material, planned Table of Contents, and Part I of the three Parts of this text, as well as the bibliography.

It is made available on the web in order to attract comments and suggestions. These should be sent to either of

r.brown@bangor.ac.uk

Rafael.Sivera@uv.es

It is planned to make the further parts available in the coming months.

Negotiations for publication in a series are in hand.

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Contents

Preface

Aims

Our aim for this text is to give a connected and we hope readable account of some work since about 1965 on extending to higher dimensions the theory and applications in algebraic topology of the fundamental group. This group, $\pi_1(X, x)$, for a space X with base point x, is determined by homotopy classes relative to the end points of paths , i.e. maps $f : [0, 1] \to X$ from the unit interval to X which map $0, 1$ to x. The fundamental group is one of the corner stones of basic algebraic topology, with many applications in topology, analysis, geometry, and group theory, which often, particularly in group theory, exploit nonabelian examples.

Our extension to higher dimensions of the fundamental group necessitates a parallel extension of some concepts of group theory, such as a free group. These two extensions, running side by side, allow for an account of some aspects of algebraic topology with more nonabelian features than those available from previous texts, and this explains our title. We are also able to give a new exposition of group cohomology, including nonabelian coefficients, based on analogies with homotopy theory.

Structure of the book

We divide our account into three parts, each with an Introduction.

In Part I we give some history of work on the fundamental group and groupoid, in particular explaining how the van Kampen theorem gives a method of computation of the fundamental group. We are then mainly concerned with the extension of nonabelian work to dimension 2, using the key concept, due to J.H.C. Whitehead in 1946, of *crossed module*. This is a morphism

$$\mu : M \to P$$

of groups together with an action of the group P on the right of the group M, written $(m, p) \mapsto m^p$, satisfying the two rules:

CM1) $\mu(m^p) = p^{-1}(\mu m)p$;

CM2) $m^{-1}nm = n^{\mu m}$,

for all $p \in P,\ m, n \in M$. Algebraic examples of crossed modules include normal subgroups M of P; P-modules; the inner automorphism crossed module $M \to \mathrm{A}ut\, M$; and many others. There is the beginnings of a combinatorial and also computational crossed module theory.

The standard geometric example of crossed module is the boundary morphism of the second relative homotopy group

$$\partial : \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

where X_1 is a subspace of the topological space X and $x \in X_1$. This relative homotopy group is defined in terms of certain homotopy classes of maps $I^2 \to X$. For this reason, and because they are a good model of 2-dimensional pointed homotopy theory, crossed modules are commonly seen as good candidates for 2-*dimensional groups*.

The remarkable fact is that we can calculate with these 2-dimensional structures and apply these calculations to topology using a 2-dimensional version of the van Kampen theorem for the fundamental group.

We give a substantial account of this 2-dimensional theory because the step from dimension 1 to dimension 2 involves a number of new ideas for which the reader's intuition needs to be developed. In particular, calculation with crossed modules requires some extensions of combinatorial group theory, for example to induced crossed modules. Finally in this Part, the proof of the van Kampen type theorem for crossed modules, involves a notion of *homotopy double groupoid*, based on composing squares with common edges. The intuition for this construction was the start of the theory of this book.

In Part II we extend the theory of crossed modules to *crossed complexes*, giving applications which include many basic results in homotopy theory, such as the relative Hurewicz theorem. This Part is intended as a kind of handbook of basic techniques in this border area between homology and homotopy theory.

However for the *proofs* of these results, particularly of the van Kampen type theorem and use of the tensor product and homotopy theory of crossed complexes, we have to introduce in Part III another algebraic structure, that of *cubical ω-groupoid with connection*. In principle, Part III can be read independently of the previous parts, referring back for some basic definitions.

Background

Recall that two maps $f, g : X \to Y$ of topological spaces are called *homotopic* if there is a *homotopy* $H : f \simeq f'$, by which is meant a map $H : [0,1] \times X \to Y$, such that $H(0,x) = f(x), H(1,x) = f'(x)$ for all $x \in X$. In this way we get a set $[X,Y]$ of homotopy classes of maps $X \to Y$. Spaces X, Y are *homotopy equivalent*, written $X \simeq Y$, if there are maps $f : X \to Y, g : Y \to X$ such that the composites $fg,\ gf$ are homotopic to the respective identity maps $1_Y,\ 1_X$. Then $f : X \to Y$ is called a *homotopy equivalence*. Thus a basic problem is to decide if spaces X, Y are, or are not, homotopy equivalent.

It is not surprising that the fundamental group is a homotopy invariant since it is defined in terms of homotopy classes of maps. Thus if two connected spaces are homotopy equivalent, they have isomorphic fundamental groups.

Another corner stone of algebraic topology is the theory of homology, with its abelian homology groups $H_n(X),\ n \geqslant 0$, for a topological space X. The homology groups, like the fundamental group, are homotopy invariants. A homotopy equivalence induces isomorphisms of homology groups as well of fundamental groups. However the definition of homology, and the proof of homotopy invariance, are more subtle than those of the various homotopy groups. Also the converse is false: a map may induce isomorphisms of fundamental group and homology groups, and yet not be a homotopy equivalence.

Higher homotopy groups $\pi_n(X,x)$ were defined in 1932 for $n \geqslant 2$ and they are all abelian. One definition of these is in terms of homotopy classes of maps of an n-cube I^n to X which map the boundary ∂I^n of the n-cube to the base point x, and all homotopies are constant on ∂I^n.

A further important problem in algebraic topology, with many applications, is to calculate the set $[X,Y]$ of homotopy classes of maps in terms of information on X, Y. This can be solved completely in some cases. For instance, if X is a connected CW-complex and $\pi_i(Y,y) = 0$ for $i > 1$, there is a bijection of sets

$$[X,Y] \cong [\pi_1(X,x), \pi_1(Y,y)]$$

where the right hand set is conjugacy classes of morphisms of groups. We give an analogous result when $\dim X \leqslant n$ and $\pi_i(Y,y) = 0$ for $1 < i < n$.

We describe algebraic structures in dimensions greater than 1 which develop the nonabelian character of the fundamental group: they are in some sense 'more nonabelian than groups', and they reflect better the geometrical complications of higher dimensions than the known homology and homotopy groups. We show how these methods can be applied to determine homotopy invariants of spaces, and homotopy classification of maps, in cases which include some classical results, and allow results not available by classical methods.

The development of such higher dimensional, nonabelian, methods in algebraic topology has been a programme of the first author since about 1966. Its inspiration was work of Philip Higgins in 1963 generalising the notion of presentation of groups to presentation of groupoids. This suggested a generalisation of the fundamental group on a space with base point to the fundamental *groupoid* on a *set of base points*, thus allowing a more flexible modelling of the underlying geometry of a space. This modelling also allowed more calculations, through a generalisation to groupoids of the van Kampen theorem for the fundamental group. The success of groupoids at this level suggested a programme of using groupoids in higher dimensional homotopy theory, and in particular developing a higher dimensional version of the van Kampen theorem.

Brown and Higgins found in the 1970s that *higher homotopy groupoids* could be defined, with values in what we called ω-groupoids, using maps of squares or n-cubes rather than paths. The key idea is to use not groups but groupoids, and to replace the space with base point by a *filtered space*

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty,$$

namely a topological space X_∞ with an increasing sequence of subspaces. Then a new functor $\rho(X_*)$ was obtained using filtered maps $I^n_* \to X_*$, where I^n_r consists of all faces of the n cube of dimension $\leqslant r$. The homotopies are through filtered maps and keep the vertices of I^n fixed throughout.

An ω-groupoid is in the first instance a graded set with n groupoid structures in each dimension $n \geqslant 1$ satisfying a fairly complicated but geometrically clear set of laws. This idea yielded, after a struggle, new abstract structures underlying homotopy theory, which led to new understanding and new calculations. The use of groupoids, and of structures with algebraic operations not always defined, was essential for this work.

A pleasant surprise was that the investigation of the existence and use of higher homotopy groupoids led to links of ω-groupoids with more classical structures, particularly crossed modules and crossed complexes, on which J.H.C. Whitehead had done extensive work in the 1940s. His work gave key clues to the directions to take.

The notion of crossed complex arose from *relative homotopy theory*, in which occur groups $\pi_n(X, A, x)$, $n \geqslant 2$, and which are abelian for $n \geqslant 3$. They are defined in terms of homotopy classes of maps $I^n \to X$ which map to x the set J^{n-1} of all $(n-1)$-faces of I^n except the $(0,1)$-th face, map the remaining face to A, and all homotopies keep J^{n-1} fixed. Thus for a filtered space X_* one obtains the fundamental groupoid $\pi_1(X_1, X_0)$ and the relative homotopy groups $\pi_n(X_n, X_{n-1}, x), x \in X_0, n \geqslant 2$. The structure all these satisfy is called a crossed complex. So we obtain a functor Π from filtered spaces to crossed complexes.

The remarkable fact is that this functor Π can be calculated directly in some important cases by a Generalised van Kampen Theorem (GvKT). This theorem, like its version for the fundamental group or groupoid, is an example of a 'local-to-global' theorem. It gives a method for calculating the functor Π for some filtered spaces which are presented as a union of smaller pieces.

'Local-to-global' is a general term applied to a family of problems concerned with relating the behaviour of a large structure to the way it is built out of smaller pieces. Such problems are central in mathematics and science.

To express this idea in the cases of interest to us, we take from category theory the concept of *colimit*. This gives a general definition of a kind of gluing process, of building large structures out of smaller ones of the same type. Our aim is to build what we call 'functors' from topological data to algebraic data which 'preserve certain colimits'. This says intuitively that such functors allow a modelling by 'gluing' in algebra of the process of gluing in topology. In this way we will obtain precise and useful algebraic calculations of some homotopical information on spaces built out of smaller ones.

Many of the main aims of the book can be summarised by stating that we construct a diagram, which we call the *Main Equivalence (ME)* :

(ME)
$$\begin{tikzcd} & \text{(filtered spaces)} \arrow[dl, "\Pi"'] \arrow[dr, "\rho"] & \\ \text{(crossed complexes)} \arrow[rr, shift left, "\lambda"] & & (\omega\text{-groupoids}) \arrow[ll, shift left, "\gamma"] \end{tikzcd}$$

such that

(A) γ, λ give an equivalence of categories;

(B) $\gamma\rho$ is naturally equivalent to Π;

(C) ρ, and hence also Π, preserves certain colimits.

The final statement we call a Generalised van Kampen Theorem (GvKT); it allows for calculations of Π, and so of certain relative homotopy groups, to get started. Corollaries of these results include:

(i) the Brouwer degree theorem (the n-sphere S^n is $(n-1)$-connected and the homotopy classes of maps of S^n to itself are classified by an integer called the *degree* of the map);

(ii) the relative Hurewicz theorem, which relates relative homotopy and homology groups;

(iii) Whitehead's theorem that $\pi_n(A \cup \{e^2_\lambda\}, A, x)$ is a free crossed $\pi_1(A, x)$-module; and

(iv) computations of the second homotopy group, and even 2-type, of the mapping cone of the map $Bf : BG \to BH$ of classifying spaces induced by a morphism $f : G \to H$ of groups.

The last two corollaries deal with constructions in crossed modules, seen as crossed complexes of length 2. These are in general nonabelian, and so these two corollaries (which we give in Part I) are not easily reachable, or not obtainable, by traditional means.

The proof of the Generalised van Kampen Theorem uses only a little knowledge of homotopy, CW-complexes, and category theory, but it is quite elaborate. The facts (A), (B), (C) are crucial. It turns out that the functor ρ is convenient for formulating and proving theorems, while the functor Π is convenient for calculation and for relating to classical constructions, such as relative homotopy groups, and chain complexes.

Our proof of the higher dimensional, local-to-global GvKT relies on methods which allow the expression of the intuitions of (i) algebraic inverse to subdivision, and (ii) of commutative cube.

For (i), cubical ideas are essential, since there is an easy notion of subdividing a cube by hyperplanes parallel to the faces, and is is not so hard to envisage an algebraic structure on cubical sets which will model reversing this process. These ideas are clearly related to local-to-global problems. The idea of (ii) is more subtle than that of (i); it needs the full relation between the above algebraic categories to express it; and it is required in order to show that a morphism on $\rho(X_*)$ is well defined, i.e. is independent of the choices apparently made in its construction.

Thus the full structure of the diagram is necessary for our proofs, and not only that of the Generalised van Kampen Theorem (GvKT). We also use this equivalence of algebraic categories to formulate the further important properties of homotopy and of tensor product of crossed complexes, and to show how these model homotopies and tensor products of filtered spaces.

These methods do not *replace* traditional homological methods, partly because of the restrictive conditions on the colimits to which the GvKT applies. However, when it does apply, it can give direct and precise homotopical information not available by other means. Further, these methods have opened out new directions in algebraic topology and related areas.

Another theme in the book, containing results on crossed complexes and the functor Π, can be shown in the following diagram of categories and functors, which we call the Main Diagram (MD), in which Top is the category of topological spaces and continuous maps, and a notion of homotopy is assumed developed for filtered spaces and crossed complexes:

(MD)
$$\begin{tikzcd} (\text{filtered spaces}) \arrow[rr, shift left, "\Pi"] \arrow[dr, "U"'] & & (\text{crossed complexes}) \arrow[ll, shift left, "\mathsf{B}"] \arrow[dl, "B"] \\ & \mathsf{Top} & \end{tikzcd}$$

satisfying the following properties:

(i) Π preserves homotopies.

(ii) There is a natural equivalence $\Pi\mathsf{B} \simeq 1$. This shows that the topology and the algebra are well related.

(iii) U is a 'forgetful' functor and $B = U\mathsf{B}$. This gives a so-called *classifying space* BC of a crossed complex C.

(iv) If X is a CW-complex with skeletal filtration X_*, and C is a crossed complex, there is a natural bijection
$$[X, BC] \cong [\Pi X_*, C],$$
where the right hand side denotes homotopy classes of morphisms in the category of crossed complexes.

This last result allows for some explicit computation of homotopy classes of maps of spaces even in cases where the fundamental groups are involved, as for example for maps of surfaces to the projective plane.

A central aspect of these homotopy classification applications is a notion of tensor product $A \otimes B$ for crossed complexes A, B and of homotopy defined as morphism $\mathcal{I} \otimes B \to C$, where $\mathcal{I}$ is the 'unit interval' groupoid or crossed complex, with two objects $0, 1$ and only one arrow $\iota : 0 \to 1$. The definition of this tensor product seems formidable. However it relies on an equivalent definition for ω-groupoids, which follows geometrically from the fact that $I^m \times I^n \cong I^{m+n}$. The transfer from ω-groupoids to crossed complexes uses the equivalence of these structures. However, for the applications of crossed complexes it is sufficient to take the definition in this context on trust, and this is what we do in Part II, with the proofs involving ω-groupoids left to Part III. The proof of the above homotopy classification result also requires results from the theory of simplicial sets, and these we have to assume in Part II.

As explained earlier, Part I is devoted to the Main Equivalence and applications in dimension 2, that is to the theory and application of crossed modules.

Prerequisites:

Large parts of this book can be read by a graduate student acquainted with general topology, the fundamental group, notions of homotopy, and some basic methods of category theory. Many of these areas, including the concept of groupoid and its uses, are covered in Brown's Topology text [30], which is in the process of being prepared for web publication.

Some aspects of category theory perhaps less familiar to a graduate student are summarised in an Appendix, particularly the notion of adjoint functor, and the preservation of colimits by a left adjoint functor. This is a basic tool of algebraic computation for those algebraic structures which are built up in several levels, since it can often show that a colimit of such a structure can be built up level by level.

Some knowledge of homology theory could be useful at a few points.

For the notion of classifying space of a crossed module or crossed complex we will need results from the theory of realisations of simplicial or cubical sets. The results needed are summarised in an Appendix.

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