Lecture Notes on Real Analysis
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Contents

Chapter 1

Basic structures: topology, metrics, semi-norms, norms

1.1 Topological spaces

Definition 1.1.1. *Let X be a set and $\mathscr{O}$ a family of subsets of X. $\mathscr{O}$ is a topology on X whenever*

(1) $\cup_{i\in I} O_i \in \mathscr{O}$ *if every* $O_i \in \mathscr{O}$.

(2) $O_1 \cap O_2 \in \mathscr{O}$ *if* $O_1, O_2 \in \mathscr{O}$.

(3) $X, \emptyset \in \mathscr{O}$.

We shall say that $(X, \mathscr{O})$ is a topological space. The open sets are defined as the elements of $\mathscr{O}$, the closed sets are defined as the subsets of X whose complement is open: a union of open sets is open, a finite intersection of open sets is open, an intersection of closed sets is a closed set, a finite union of closed sets is a closed set. If $\mathscr{O}_1, \mathscr{O}_2$ are two topologies on X such that $\mathscr{O}_1 \subset \mathscr{O}_2$, we shall say that $\mathscr{O}_2$ is finer than $\mathscr{O}_1$.

We may notice that the third condition can be considered as a consequence of the two previous ones since a union (resp. an intersection) on an empty set of indices of subsets of X is the empty set (resp. X).

Examples of topological spaces.

· The most familiar example is certainly the real line $\mathbb{R}$ equipped with the standard topology: a subset O of $\mathbb{R}$ is open, when for all $x \in O$ there exists an open-interval $I =]a, b[$ such that $x \in I \subset O$. The property (1) above is satisfied as well as (2) since the intersection of two open-intervals is an open-interval. Note that the open-intervals are also open sets.

· Also $\mathbb{R}^n$ has the following standard topology: a subset O of $\mathbb{R}^n$ is open, when for all $x \in O$ there exists some open-intervals $I_j =]a_j, b_j[$ such that $x \in I_1 \times \cdots \times I_n \subset O$. The property (1) above is satisfied as well as (2) since the intersection of two open-intervals is an open-interval.

· Let us give some more abstract examples. The *Discrete Topology* on a set X is $\mathscr{P}(X)$, a topology on X for which all the subsets of X are open. Naturally, it is

the finest possible topology on X. The *Trivial Topology* on X is $\{\emptyset, X\}$: it is the coarsest topology on X, since all topologies on X are finer.
· The *Cofinite Topology* on a set X is $\mathscr{O} = \{\emptyset\} \cup \{\Omega \subset X, \Omega^c \text{ finite}\}$. It is obviously a topology since an intersection of finite sets is a finite set, and a finite union of finite sets is a finite set. Note that the cofinite topology on a finite set is the discrete topology.
· The *Cocountable topology* on a set X is $\mathscr{O} = \{\emptyset\} \cup \{\Omega \subset X, \Omega^c \text{countable}\}$. It is obviously a topology since an intersection of countable sets is a countable set, and a finite union of countable sets is a countable set. Note that the cocountable topology on a countable set is the discrete topology.
· On the other hand if $(\mathscr{O}_\alpha)_{\alpha \in A}$ is a family of topologies on a set X, $\cap_{\alpha \in A} \mathscr{O}_\alpha$ is also a topology on X. As a consequence, it is possible to define the coarsest (smallest) topology on a set X containing a family $\mathscr{A}$ of subsets of X: it is the intersection of the topologies which contain $\mathscr{A}$ (this makes sense since $\mathscr{A} \subset \mathscr{P}(X)$ which is a topology on X, so that the set of topologies containing $\mathscr{A}$ is not empty).
· If $(X, \le)$ is a totally ordered set, we define the *open-intervals* as the sets $]x_1, x_2[= \{x \in X, x_1 < x < x_2\}$ (here $x' < x''$ means $x' \le x''$ and $x' \ne x''$) or the sets

$$]-\infty, x[= \{y \in X, y < x\}, \quad]x, +\infty[= \{y \in X, x < y\}.$$

The set $\mathscr{I} = \{\cup_{a \in A} I_a, I_a \text{ open-interval}\} \cup \{X\}$ is a topology on X. The set $\mathscr{I}$ is obviously stable by union, contains the empty set and X. We note also that, since X is totally ordered, the intersection of two open-intervals is an open-interval:

$$]x_1, x_2[\bigcap]y_1, y_2[\;=] \max(x_1, y_1), \min(x_2, y_2)[,$$

with the convention $\max(-\infty, x) = x = \min(+\infty, x)$. As a consequence, taking the intersection of two elements of $\mathscr{I}$ leads to

$$\big(\cup_{a \in A} I_a\big) \cap \big(\cup_{b \in B} I_b\big) = \cup_{(a,b) \in A \times B} (I_a \cap I_b),$$

so that $\mathscr{I}$ is also stable by finite intersection.

We have seen in the section 1.1, that given a family $(\mathscr{O})_{\alpha \in A}$ of topologies on a set X, $\cap_{\alpha \in A} \mathscr{O}_\alpha$ is also a topology on X: that topology is of course weaker than each $\mathscr{O}_\alpha$.

Remark 1.1.2. Let us consider a set X, a family of topological spaces $(Y_j, \mathscr{O}_j)$ and a family of mappings $\varphi_j : X \longrightarrow Y_j$. If $\mathscr{O}$ is a topology on X such that all the φ_j are continuous, then for all $j \in J$, for all $\omega_j \in \mathscr{O}_j$, $\varphi_j^{-1}(\omega_j) \in \mathscr{O}$. Let us now consider the family $\mathscr{F} = \{\varphi_j^{-1}(\omega_j)\}_{j \in J, \omega_j \in \mathscr{O}_j}$ and we define $\mathscr{O}_\mathscr{F}$ as the intersection of the topologies on X which contain $\mathscr{F}$: this makes sense because the discrete topology $\mathscr{P}(X)$ contains $\mathscr{F}$ and an intersection of topologies on X is also a topology on X. Naturally, all the mappings φ_j are continuous for the topology $\mathscr{O}_\mathscr{F}$ and if $\widetilde{\mathscr{O}}$ is a topology on X such that all the mappings φ_j are continuous, then $\widetilde{\mathscr{O}} \supset \mathscr{F}$ and thus, by the very definition of $\mathscr{O}_\mathscr{F}$, we have $\mathscr{O}_\mathscr{F} \subset \widetilde{\mathscr{O}}$. The topology $\mathscr{O}_\mathscr{F}$ is thus the weakest topology on X such that all the mappings φ_j are continuous.

Definition 1.1.3. *Let $(X, \mathscr{O})$ be a topological space and A a subset of X. The interior of A is defined as*

$$\mathring{A} = \bigcup_{\Omega \text{ open } \subset A} \Omega, \quad \text{(the largest open set included in } A\text{, noted also as } \operatorname{int} A\text{).}$$

The closure of A is defined as

$$\bar{A} = \bigcap_{F \text{ closed } \supset A} F, \quad \text{(the smallest closed set containing } A\text{).}$$

The set A is said to be dense in X whenever $\bar{A} = X$. The boundary of A is (the closed set) defined as

$$\partial A = \bar{A} \backslash \operatorname{int} A = \bar{A} \cap \overline{A^c}.$$

Definition 1.1.4. *Let X be a topological space, $x \in X, V \subset X$. We say that V is a neighborhood of x if it contains an open set containing x. We shall note $\mathscr{V}_x$ the set of neighborhoods of x. We note that $\mathscr{V}_x$ is stable by extension ($V \in \mathscr{V}_x, W \supset V$ implies $W \in \mathscr{V}_x$), by finite intersection and that no element of $\mathscr{V}_x$ is empty. A subset of X is open if and only if it is a neighborhood of all its points.*

Let us prove that last assertion: an open set is a neighborhood of all its points by definition and conversely if $\Omega \subset X$ is a neighborhood of all its points, then for all $x \in \Omega$, there exists an open set $\omega_x \subset \Omega$ with $x \in \omega_x$, so that $\Omega = \cup_{x \in \Omega} \omega_x$ union of open sets, thus open.

Definition 1.1.5. *A topological space $(X, \mathscr{O})$ is said to be a Hausdorff space if, when $x_1 \neq x_2$ in X, there exist $U_j \in \mathscr{V}_{x_j}, j = 1, 2$ such that $U_1 \cap U_2 = \emptyset$.*

N.B. We shall see that most of the examples of topological spaces that we encounter in functional analysis are indeed Hausdorff spaces, as it is the case in particular for the metric spaces, whose definition is given in the next section. However, let us consider $\mathbb{N}$ (or any infinite set) equipped with the cofinite topology, for which the closed sets are the finite sets. Let U_0, U_1 be open sets containing respectively $0, 1$. Then $U_0 \cap U_1 \neq \emptyset$, otherwise $U_0^c \cup U_1^c = \mathbb{N}$, which is not possible since U_0^c and U_1^c are both finite. However, singletons $\{n\}$ are closed for the cofinite topology on $\mathbb{N}$. This is also the case in a Hausdorff space, since for $x_0 \neq x \in X$, there exists ω_x open such that $x_0 \notin \omega_x \ni x$ implying that $\{x_0\}^c = \cup_{x \neq x_0} \omega_x$ thus open. Within the various notions of separation for topological spaces, one may single out the notion of Hausdorff space, or T_2 space, as defined above, and the weaker notion of T_1 space, defined as topological spaces for which the singletons are closed. We have just proven that a T_2 space is T_1 but that the converse is not true in general.

A very general approach of topology is outlined in the appendix with the notions of *filters* and *ultrafilters*. We note also that for a subset A of a topological space X we have

$$x \in \bar{A} \Longleftrightarrow \forall V \in \mathscr{V}_x, V \cap A \neq \emptyset. \tag{1.1.1}$$

In fact the complement of $\bar{A}$ is the interior of A^c: $x \notin \bar{A}$ is thus equivalent to $A^c \in \mathscr{V}_x$, which is indeed the previous claim.

Exercise 1.1.6. Verify that, for A, B subsets of a topological space X,

$$(\bar{A})^c = \operatorname{int}(A^c), \quad (\operatorname{int} A)^c = \overline{A^c}$$
$$\overline{A \cup B} = \bar{A} \cup \bar{B}, \quad \operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B.$$

Show that the inclusion $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ holds and may be strict.

Lemma 1.1.7. *Let $(X, \mathscr{O})$ be a topological space and $U \subset X$. The following properties are equivalent:*

(i) $\overline{U} = X$.

(ii) $\forall \Omega \in \mathscr{O}, \Omega \neq \emptyset \Longrightarrow \Omega \cap U \neq \emptyset$.

Proof. If (i) is satisfied and if Ω is open, we have

$$\Omega \cap U = \emptyset \Longrightarrow U \subset \Omega^c \Longrightarrow X = \overline{U} \subset \Omega^c \Longrightarrow \Omega = \emptyset, \quad \text{proving } (ii).$$

Conversely, if (i) is violated, the open set $\Omega = \left(\overline{U}\right)^c \neq \emptyset$, but

$$\Omega \cap U = \left(\bar{U}\right)^c \cap U = \left(\operatorname{int}(U^c)\right) \cap U \subset U^c \cap U = \emptyset.$$

proving non-(ii). □

Definition 1.1.8. *Let X, Y be topological spaces, $f : X \longrightarrow Y$. Let $x_0 \in X$; the mapping f is said to be continuous at x_0 if*

$$\forall V \in \mathscr{V}_{f(x_0)}, \exists U \in \mathscr{V}_{x_0}, \quad f(U) \subset V.$$

The mapping f is said to be continuous on X if it is continuous at all points of X. That property is satisfied if and only if, for all open sets V of Y, $f^{-1}(V)$ is an open set of X. The mapping f is an homeomorphism if it is bijective and bicontinuous (f and f^{-1} are continuous).

Let us prove the property stated above. Let f be a continuous mapping, B an open set of Y and $x \in f^{-1}(B)$ ($f(x) \in B$ which is thus a neighborhood of $f(x)$). From the continuity of f, there exists a neighborhood U of x such that $f(U) \subset B$, which means $U \subset f^{-1}(B)$, and thus $f^{-1}(B)$ is a neighborhood of x, so that $f^{-1}(B)$ is open. Conversely if the inverse image by f of any open set is open, and if $x \in X$, V is a neighborhood of $f(x)$ containing an open set $B \ni f(x)$, $f^{-1}(B)$ is open and contains x; as a consequence, $f^{-1}(B)$ is a neighborhood of x and $f\big(f^{-1}(B)\big) \subset B \subset V$, qed.

Exercise 1.1.9. Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous at only one point.

Definition 1.1.10. *Let $(X, \mathscr{O})$ be a topological space and S be a subset of X. The induced topology on S is $\mathscr{O}_S = \{\Omega \cap S\}_{\Omega \in \mathscr{O}}$.*

Note that the properties of a topology are immediately satisfied and that $\mathscr{O}_S$ is the coarsest topology such that the canonical injection $\iota : S \to X$ is continuous. On the one hand, $\iota^{-1}(\Omega) = \Omega \cap S$ is open if $\Omega \in \mathscr{O}$ and ι is thus continuous; on the other hand, if $\mathscr{O}'$ is a topology on S that makes ι continuous, $\mathscr{O}_S$ must be contained in $\mathscr{O}'$, since $\iota^{-1}(\Omega) \in \mathscr{O}'$ for $\Omega \in \mathscr{O}$.

The next section introduces the class of metric spaces, a very useful class of topological spaces in functional analysis. Although the notion of metric space is enough to describe a large part of the most natural functional spaces, the reader may keep in mind that some interesting and natural examples of functional spaces are not metrizable. This is the case for instance of some of the test functions spaces used in distribution theory, such that the continuous functions with compact support from $\mathbb{R}$ to $\mathbb{R}$. Naturally, a good understanding of distribution theory does not necessarily require a great familiarity with non-metrizable spaces but one should nevertheless keep in mind that the developments of functional analysis raised various questions of general topology, which went much beyond the metrizable framework.

1.2 Metric Spaces

Definition 1.2.1. *Let X be a set*[1] *et $d : X \times X \to [0, +\infty[$. We say that d is a distance on X if for $x_j \in X$,*

(1) $d(x_1, x_2) = 0 \Longleftrightarrow x_1 = x_2$, *(separation),*

(2) $d(x_1, x_2) = d(x_2, x_1)$, *(symmetry),*

(3) $d(x_1, x_3) \le d(x_1, x_2) + d(x_2, x_3)$, *(triangle inequality).*

(X, d) is called a metric space. For $r > 0, x \in X$, we define the open-ball with center x and radius r, $B(x, r) = \{y \in X, d(y, x) < r\}$.

Definition 1.2.2. *Let (X, d) be a metric space. A subset Ω of X belongs to the topology $\mathscr{O}_d$ on X defined by the metric d if*

$$\forall x \in \Omega, \exists r > 0, B(x, r) \subset \Omega.$$

We note that $\mathscr{O}_d$ is a topology since the stability by union is obvious and the stability by finite intersection follows from the fact that $B(x, r_1) \cap B(x, r_2) = B(x, \min(r_1, r_2))$. Moreover the open-balls are open since, considering for $r_0 > 0, x_0 \in X$, $x \in B(x_0, r_0)$, we have with $\rho = r_0 - d(x, x_0)$ (which is > 0)

$$d(y, x) < \rho \Longrightarrow d(y, x_0) \le d(y, x) + d(x, x_0) < \rho + d(x, x_0) = r_0,$$

implying that $B(x, \rho) \subset B(x_0, r_0)$ and $B(x_0, r_0)$ open.

We note that in a metric space (X, d), for $x \in X, r \ge 0$, the set

$$\tilde{B}(x, r) = \{y \in X, d(y, x) \le r\}$$

[1] The reader of Bourbaki will have noticed the ineptitude of that first sentence.

is closed. In fact, if $d(y,x) > r$, we have $B(y, d(y,x) - r) \subset \big(\tilde{B}(x,r)\big)^c$: take z such that $d(z,y) < d(y,x) - r$. Then by the triangle inequality

$$d(z,x) \geq d(y,x) - d(z,y) > d(z,y) + r - d(z,y) = r \Longrightarrow z \notin \tilde{B}(x,r), \text{ qed.}$$

Since $\tilde{B}(x,r)$ is a closed set containing $B(x,r)$, the closure $\overline{B(x,r)}$ of the ball $B(x,r)$ is included in $\tilde{B}(x,r)$. However there are some examples where the inclusion $\overline{B(x,r)} \subset \tilde{B}(x,r)$ is strict. Take on a set X (with at least two elements) the *discrete metric,* defined by $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$. It is obviously a metric and since $\{x\}^c = \cup_{y \neq x} B(y, 1/2)$ is open, $\{x\}$ is closed and

$$B(x,1) = \{x\} = \overline{B(x,1)}, \quad \tilde{B}(x,1) = X.$$

A metric topology is always Hausdorff since for $x \neq y$, we have

$$B(x,r) \cap B(y,r) = \emptyset, \quad \text{with } r = d(x,y)/2 (> 0).$$

In fact let $z \in B(x,r) \cap B(y,r)$. By the triangle inequality $d(x,y) \leq d(x,z) + d(z,y) < 2r = d(x,y)$, which is impossible.

Definition 1.2.3. *Let (X,d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. The sequence $(x_n)_{n\in\mathbb{N}}$ is said to be converging with limit x whenever*

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall k \geq N_\epsilon, d(x_k, x) < \epsilon. \qquad \textit{We set } \lim_n x_n = x.$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is said to be a Cauchy sequence whenever

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall k, l \geq N_\epsilon, d(x_k, x_l) < \epsilon.$$

A converging sequence is a Cauchy sequence. If (X,d) is such that all Cauchy sequences are converging, we say that X is complete.

The notation $\lim x_n = x$ is legitimate since the separation induces the uniqueness of the limit: if x', x'' are limits of a sequence $(x_n)_{n\in\mathbb{N}}$, we have $0 \leq d(x', x'') \leq d(x', x_n) + d(x_n, x'')$ and since the numerical sequences $(d(x', x_n)), (d(x_n, x''))$ tend to 0, we find $d(x', x'') = 0$, i.e. $x' = x''$. On the other hand, if a sequence (x_n) is converging to x, we have $d(x_n, x) < \epsilon/2$ for $n \geq N_{\epsilon/2}$ and thus for $k, l \geq N_{\epsilon/2}$, we get $d(x_k, x_l) \leq d(x_k, x) + d(x, x_l) < \epsilon$, so that (x_n) is also a Cauchy sequence.

Proposition 1.2.4. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ be a mapping. Let $x_0 \in X$. The mapping f is continuous at x_0 if and only if for every sequence $(x_n)_{n\geq 1}$ converging with limit x_0, the sequence $(f(x_n))_{n\geq 1}$ is converging with limit $f(x_0)$.*

Proof. Assuming first that f is continuous at x_0, we know that for $\epsilon > 0$, there exists $r > 0$ such that $f(B(x_0, r)) \subset B(f(x_0), \epsilon)$. Let $(x_n)_{n\geq 1}$ be a sequence converging with limit x_0. For $n \geq N$, $x_n \in B(x_0, r)$ and thus $f(x_n) \in B(f(x_0), \epsilon)$, so that $\lim_n f(x_n) = f(x_0)$. Conversely, if f is not continuous at x_0, there exists a neighborhood V of $f(x_0)$ such that, for all neighborhoods U of x_0, $f(U) \not\subset V$. In other words, there exists $\epsilon_0 > 0$ such that for all integers $n \geq 1$, there exists $x_n \in X$ such that $d(x_n, x_0) < 1/n$ and $d(f(x_n), f(x_0)) \geq \epsilon_0$. As a consequence $\lim_n x_n = x_0$ and the sequence $(f(x_n))_{n\geq 1}$ is not converging to $f(x_0)$; □

Exercise 1.2.5. The set $\mathbb{Q}$ of rational numbers, equipped with the standard distance given by the absolute value $|x-y|$ is not complete. Consider for instance the sequence of rational numbers defined by

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \quad x_0 = 2$$

and prove that it is a Cauchy sequence of $\mathbb{Q}$ which is not converging in $\mathbb{Q}$.

The reader will see in the appendix that, given a metric space (X, d), it is possible to construct a complete metric space $(\widetilde{X}, \tilde{d})$ such that X is dense in $\widetilde{X}$ (i.e. $\overline{X} = \widetilde{X}$), $\tilde{d}_{|X\times X} = d$ and such that, for all complete metric space Y and all applications f uniformly continuous [2] from X in Y, there exists a unique uniformly continuous extension of f to $\widetilde{X}$. The space $(\widetilde{X}, \tilde{d})$ is complete and uniquely determined by the previous property (up to an isometry of metric spaces[3]). The space $(\widetilde{X}, \tilde{d})$ is called the completion of (X, d) and its construction is very close to the completion of $\mathbb{Q}$ to obtain $\mathbb{R}$. $\widetilde{X}$ is constructed as the quotient of the set $\mathscr{C}[X]$ of Cauchy sequences by the following equivalence relation: two Cauchy sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are equivalent means that $\lim_n d(x_n, y_n) = 0$. We define the distance $\tilde{d}$ on $\mathscr{C}[X]$ by $\tilde{d}((x_n), (y_n)) = \lim_n d(x_n, y_n)$ and we prove that it is well-defined and satisfies the above properties.

Remark 1.2.6. Note that completeness is a property of the metric and not of the topology, meaning that a complete metric space can be homeomorphic (see the definition 1.1.8) to a non-complete one. An example is given by the real line $\mathbb{R}$ with the standard distance given by $|x - y|$, which is a complete metric space, but homeomorphic to the open interval $]0, 1[$, which is not complete, since the Cauchy sequence $(1/k)_{k\geq 1}$ is not converging in $]0, 1[$.

1.3 Topological Vector Spaces

1.3.1 General definitions

Let us recall that a vector space E is an (additive) commutative group such that a scalar multiplication $\mathtt{k} \times E \ni (\lambda, x) \mapsto \lambda \cdot x \in E$ is defined ($\mathtt{k}$ is a commutative field), with the following axioms: for $x, y \in E, \lambda, \mu \in \mathtt{k}$,

$\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y,$ (here $+$ is the addition in E), a version of Thales theorem,

$(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x,$ (the first $+$ is the addition in $\mathtt{k}$, the next the addition in E),

$\lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x,$ ($\lambda\mu$ is the product in $\mathtt{k}$),

$1 \cdot x = x,$ (1 is the unit element in $\mathtt{k}$).

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[2] $\forall\epsilon > 0, \exists\alpha > 0, d_X(x_1, x_2) < \alpha \Longrightarrow d_Y(f(x_1), f(x_2)) < \epsilon$.

[3] A mapping F is an isometry between metric spaces (X_j, d_j) if F is bijective from X_1 on X_2 and if $d_2(F(x_1), F(y_1)) = d_1(x_1, y_1)$.

This implies that for all $x \in E$, $0_{\mathtt{k}} \cdot x = 0_E$ since $0_{\mathtt{k}} \cdot x + x = 0_{\mathtt{k}} \cdot x + 1_{\mathtt{k}} \cdot x = 1_{\mathtt{k}} \cdot x = x$. All the vector spaces that we shall consider in this text are vector spaces on the field $\mathbb{R}$ or $\mathbb{C}$, denoted by $\mathtt{k}$ in the sequel. These fields are equipped with their usual topology, given by the distance $|x-y|$ (absolute value in the real case, modulus in the complex case). We want to deal with *topological vector spaces*, i.e. to consider vector spaces equipped with a topology which is somehow compatible with the algebraic structure of the vector space. We define this more precisely.

Definition 1.3.1. *Let E be a vector space and $\mathscr{O}$ a topology on E such that the mappings $E \times E \ni (x, y) \mapsto x + y \in E$, $\quad \mathtt{k} \times E \ni (\alpha, x) \mapsto \alpha \cdot x \in E$ are continuous. We shall say that $(E, \mathscr{O})$ is a topological vector space (TVS for short).*

1.3.2 Vector spaces with a translation-invariant distance

On a vector space E we define a *translation-invariant* distance d, as a distance on E such that, for $x, y, z \in E$,

$$d(x + z, y + z) = d(x, y). \tag{1.3.1}$$

Lemma 1.3.2. *Let E be a vector space and d a translation-invariant distance on E. Assume that*

$$\begin{aligned} \forall \epsilon > 0, \exists r > 0, &\quad \{\lambda \in \mathtt{k}, |\lambda| < r\} \cdot B(0, r) \subset B(0, \epsilon), \\ \forall x \in E, \forall \epsilon > 0, \exists r > 0, &\quad \{\lambda \in \mathtt{k}, |\lambda| < r\} \cdot x \subset B(0, \epsilon), \\ \forall \lambda \in \mathtt{k}, \forall \epsilon > 0, \exists r > 0, &\quad \lambda \cdot B(0, r) \subset B(0, \epsilon). \end{aligned}$$

Then we define $\mathscr{V}_0 = \{V \subset E, \text{such that } \exists r > 0 \text{ with } B(0, r) \subset V\}$ and

$$\mathscr{O} = \{\Omega \subset E, \forall x \in \Omega, \exists V \in \mathscr{V}_0,\ x + V \subset \Omega.\}$$

Then $(E, \mathscr{O})$ is a TVS, $\mathscr{O}$ is the topology defined by the distance d and $\mathscr{V}_0$ is the set of neighborhoods of 0. For all $x \in E$, the set of neighborhoods of x is $x + \mathscr{V}_0 = \{x + V\}_{V \in \mathscr{V}_0}$.

Proof. Let us first remark that $\mathscr{O}$ is the topology on E defined by the metric d. Let $\Omega \subset E$ such that for all $x \in \Omega$, there exists $r > 0$ such that $B(x, r) \subset \Omega$, i.e. Ω is an open subset of E for the topology induced by the metric d. Since d is translation invariant, we have $B(x, r) = x + B(0, r)$: in fact, for $y \in E$, we have $d(y, x) = d(y - x, 0)$ so that

$$d(y, x) < r \Longleftrightarrow d(y - x, 0) < r \Longleftrightarrow y = x + z, \quad \text{with } d(z, 0) < r.$$

As a consequence, the open sets of (E, d) are indeed given by the property of the lemma. To prove the continuity of $E \times E \ni (x, y) \mapsto x + y \in E$, we consider for $r > 0$ a neighborhood $x_0 + y_0 + B(0, r)$ of $x_0 + y_0$. We note that

$$x_0 + B(0, \frac{r}{2}) + y_0 + B(0, \frac{r}{2}) \subset x_0 + y_0 + B(0, r)$$

since $d(z', 0) < r/2$ and $d(z'', 0) < r/2$ imply that

$$d(z' + z'', 0) \le d(z' + z'', z') + d(z', 0) = d(z'', 0) + d(z', 0) < r,$$

and the continuity property is proven. To prove the continuity of $\mathtt{k} \times E \ni (\lambda, x) \mapsto \lambda x \in E$, we consider for $\epsilon > 0$ a neighborhood $\lambda_0 x_0 + B(0, \epsilon)$ of $\lambda_0 x_0$. Using the hypothesis of the lemma, we know that there exists $r_1 > 0$ such that for $(\mu, z) \in \mathtt{k} \times E, |\mu| < r_1, d(z, 0) < r_1$, we have $d(\mu z, 0) < \epsilon/3$; moreover there exists $r_2 > 0$ such that $|\mu| < r_2$ implies $d(\mu x_0, 0) < \epsilon/3$ and there exists $r_3 > 0$ such that $d(z, 0) < r_3$ implies $d(\lambda_0 z, 0) < \epsilon/3$. This proves that for $(\mu, z) \in \mathtt{k} \times E$ such that $|\mu| < \min(r_1, r_2), d(z, 0) < \min(r_1, r_3)$, we have

$$(\lambda_0 + \mu)(x_0 + z) = \lambda_0 x_0 + \mu x_0 + \lambda_0 z + \mu z \in B(0, \epsilon),$$

proving the continuity property. □

1.3.3 Normed spaces

A case in which the verification of the assumptions of the lemma is very simple is given by the case of a *normed* vector space.

Definition 1.3.3. *Let E be a vector space and $N : E \to \mathbb{R}_+$. We shall say that N is a norm on E if for $x, y \in E, \alpha \in \mathtt{k}$,*

(1) $N(x) = 0 \Longleftrightarrow x = 0$, *(separation),*

(2) $N(\alpha x) = |\alpha| N(x)$, *(homogeneity),*

(3) $N(x + y) \le N(x) + N(y)$, *(triangle inequality).*

(E, N) will be called a normed vector space. We define on E the distance

$$d(x, y) = N(x - y). \tag{1.3.2}$$

Proposition 1.3.4. *Let (E, N) be a normed vector space and d the distance (1.3.2). The distance d is translation-invariant and the metric space (E, d) is a topological vector space.*

Proof. We see immediately that d is a translation-invariant distance; to check the continuity at $(0, 0)$, we note that for $(\lambda, x) \in \mathtt{k} \times E, \epsilon > 0$,

$$d(\lambda x, 0) = N(\lambda x) = |\lambda| N(x) < \epsilon$$

provided $d(x, 0) = N(x) < \epsilon$ and $|\lambda| < 1$. Checking the two other properties of the previous lemma amounts, for (λ_0, x_0) given in $\mathtt{k} \times E$, $|\mu| < r_1, d(z, 0) < r_2$, to look at

$$d(\mu x_0, 0) + d(\lambda_0 z, 0) = N(\mu x_0) + N(\lambda_0 z) = |\mu| N(x_0) + |\lambda_0| N(z) < \epsilon$$

provided $r_1 N(x_0) + |\lambda_0| r_2 < \epsilon$. □

Definition 1.3.5. *A Banach space is a complete normed vector space.*

A *Hilbert space* is a particular type of Banach space, for which the norm is derived from a dot-product, also called a scalar product. It is better here to discuss separately the real and the complex case. If E is a real vector space and $E \times E \ni (x, y) \mapsto \langle x, y\rangle \in \mathbb{R}$ is a bilinear symmetric form[4], which is positive-definite, i.e. such that $\langle x, x\rangle > 0$ for $x \neq 0$, we shall say that $(E, \langle\cdot, \cdot\rangle)$ is a real prehilbertian space. If E is a complex vector space and $E \times E \ni (x, y) \mapsto \langle x, y\rangle \in \mathbb{C}$ is a sesquilinear Hermitian form[5], which is positive-definite, i.e. such that $\langle x, x\rangle > 0$ for $x \neq 0$, we shall say that $(E, \langle\cdot, \cdot\rangle)$ is a complex prehilbertian space.

Lemma 1.3.6. *Let $(E, \langle\cdot, \cdot\rangle)$ be a complex (resp. real) prehilbertian space. Then $\|x\| = \langle x, x\rangle^{1/2}$ is a norm on E. Moreover, for $x, y \in E$ the Cauchy-Schwarz*[6] *inequality holds:*

$$|\langle x, y\rangle| \leq \|x\|\|y\|. \tag{1.3.3}$$

The equality above holds if an only if $x \wedge y = 0$, i.e. x and y are linearly dependent.

Proof. It is enough to deal with the complex case. We define for $t \in \mathbb{R}$,

$$p(t) = \langle x + ty, x + ty\rangle = \|x\|^2 + 2t \operatorname{Re}\langle x, y\rangle + t^2\|y\|^2,$$

and since p is a non-negative polynomial of degree (less than) two on $\mathbb{R}$, we get $(\operatorname{Re}\langle x, y\rangle)^2 \leq \|x\|^2\|y\|^2$. Writing now with $\theta \in \mathbb{R}$, $\langle x, y\rangle = e^{i\theta}|\langle x, y\rangle|$, we apply the previous result to get $|\langle x, y\rangle|^2 = (\operatorname{Re}\langle e^{-i\theta}x, y\rangle)^2 \leq \|x\|^2\|y\|^2$, which is (1.3.3). If $x \wedge y = 0$, we have $y = \lambda x$ or $x = \lambda y$ and the equality in (1.3.3) is obvious. If $x \wedge y \neq 0$ the polynomial p above takes only positive[7] values so that $(\operatorname{Re}\langle x, y\rangle)^2 < \|x\|^2\|y\|^2$. Using the same trick as above with $e^{i\theta}$, we get a strict inequality in (1.3.3). To prove now that $\langle x, x\rangle^{1/2}$ is a norm is easy: (1) and (2) in the definition 1.3.3 are obvious and we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2 \operatorname{Re}\langle x, y\rangle \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2. \qquad \square$$

N.B. Let $(E, \langle\cdot, \cdot\rangle)$ be a prehilbertian space. A direct consequence of the Cauchy-Schwarz inequality (1.3.3). is

$$\|x\| = \sup_{\|y\|=1} |\langle x, y\rangle|. \tag{1.3.4}$$

It is true for $x = 0$; also $\|x\|$ is greater than the rhs from (1.3.3), and conversely if $x \neq 0$, $\|x\| = \langle x, \frac{x}{\|x\|}\rangle$, which gives the result.

Definition 1.3.7. *A real (resp. complex) Hilbert space is a Banach space such that the norm is derived from a bilinear symmetric (resp. sesquilinear Hermitian) dot-product so that $\|x\| = \langle x, x\rangle^{1/2}$.*

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[4]It means that the mappings $E \ni x \mapsto \langle x, y\rangle \in \mathbb{R}$ are linear and $\langle x, y\rangle = \langle y, x\rangle$.

[5]It means that the mappings $E \ni x \mapsto \langle x, y\rangle \in \mathbb{C}$ are $\mathbb{C}$-linear and $\overline{\langle x, y\rangle} = \langle y, x\rangle$. In particular for $\lambda \in \mathbb{C}, x, y \in E$, $\langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle$.

[6] One should also associate to this inequality the name of Viktor Yakovlevich Bunyakovsky, who actually discovered it. References on the history of the Cauchy-Bunyakovsky-Schwarz inequality appear in http://www-history.mcs.st-and.ac.uk/history/Biographies/Bunyakovsky.html.

[7]In (mathematical) english, r positive means $r > 0$ and r nonnegative means $r \geq 0$.

Examples. The simplest example of a normed vector space is $\mathbb{R}^n$ with the Euclidean norm $\|x\|_2 = \big(\sum_{1\le j\le n} x_j^2\big)^{1/2}$, or any of the following norms for $p \in [1, +\infty]$,

$$\|x\|_p = \big(\sum_{1\le j\le n} |x_j|^p\big)^{1/p}, \quad \|x\|_\infty = \max_{1\le j\le n} |x_j|. \tag{1.3.5}$$

It is easy to prove that all the norms on $\mathbb{R}^n$ are *equivalent*, i.e. if N_1, N_2 are two norms on $\mathbb{R}^n$, $\exists C > 0, \quad \forall x \in \mathbb{R}^n, \quad C^{-1}N_2(x) \le N_1(x) \le CN_2(x)$. This implies in particular that the topologies on $\mathbb{R}^n$ defined by the distances associated to these norms by the formula (1.3.2) are all the same. Since $\mathbb{R}$ is complete, we see that $\mathbb{R}^n$ equipped with a norm is a Banach space.

The infinite-dimensional $\ell^p(\mathbb{N})$ are more interesting: we define for $p \in [1, +\infty]$,

$$\ell^p(\mathbb{N}) = \{(x_n)_{n\in\mathbb{N}}, \quad \sum_{n\in\mathbb{N}} |x_n|^p < +\infty\}, \quad \|(x_n)_{n\in\mathbb{N}}\|_p = \big(\sum_{n\in\mathbb{N}} |x_n|^p\big)^{1/p} \tag{1.3.6}$$

$$\ell^\infty(\mathbb{N}) = \{(x_n)_{n\in\mathbb{N}}, \quad \sup_{n\in\mathbb{N}} |x_n| < +\infty\}, \quad \|(x_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|. \tag{1.3.7}$$

It is easy to see that for $1 \le p \le q \le +\infty$, we have[8]

$$\ell^1(\mathbb{N}) \subset \ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N}) \subset \ell^\infty(\mathbb{N}), \quad \text{and for } x = (x_n)_{n\in\mathbb{N}} \in \ell^p, \quad \|x\|_q \le \|x\|_p,$$

and all these spaces are Banach spaces (we refer the reader to the website [9], chapter 3, for a proof of the triangle inequality). Moreover, for $1 \le p < q \le +\infty$, the norm $\|_q$ is *not* equivalent on $\ell^p(\mathbb{N})$ to the norm $\|_p$, i.e. there is no $C > 0$ such that for all $x \in \ell^p(\mathbb{N})$, $\|x\|_p \le C\|x\|_q$. Otherwise, we would have with any $N \ge 1$ integer, $x_n = n^{-1/q}$ for $1 \le n \le N$ and $x_n = 0$ for other values of n,

$$\left(\frac{(N+1)^{1-\frac{p}{q}} - 1}{1 - \frac{p}{q}}\right)^{1/q} \le \big(\sum_{1\le n\le N} n^{-\frac{p}{q}}\big)^{1/q} \le C\big(\sum_{1\le n\le N} n^{-1}\big)^{1/p} \le C(1 + \ln N)^{1/p}$$

which is impossible.

Given a measured space $(X, \mathcal{M}, \mu)$ where μ is a positive measure, $p \in [1, +\infty]$, the space $L^p(\mu)$ of class of measurable functions f such that $\int_X |f|^p d\mu < +\infty$ (for $p = +\infty$, $\operatorname{esssup}|f| < +\infty$) is a Banach space (see e.g. [9], chapter 3) with the norm

$$\left(\int_X |f|^p d\mu\right)^{1/p}, \quad \operatorname{esssup}|f| \text{ for } p = +\infty. \tag{1.3.8}$$

For Ω open subset of $\mathbb{R}^n$ we shall note simply $L^p(\Omega)$ that space for $X = \Omega$, $\mathcal{M}$ the Lebesgue σ-algebra and μ the Lebesgue measure. Note that one may consider the space $\mathcal{L}^p(\mu)$ of measurable functions such that $\int_X |f|^p d\mu < +\infty$, but that the separation axiom is not verified: the condition $\int_X |f|^p d\mu = 0$ will imply only that

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[8]For $q \ge p \ge 1$,

$$\|x\|_q^q = \sum_n |x_n|^q \le \sum_n |x_n|^p (\sup_n |x_n|)^{q-p} \le \|x\|_p^p \big(\sum_n |x_n|^p\big)^{\frac{q}{p}-1} = \|x\|_p^q, \quad \text{qed.}$$

$f = 0$, μ-almost everywhere, and this is why we have to consider $L^p(\mu)$, which is the quotient of $\mathcal{L}^p(\mu)$ by the equivalence relation of equality μ-a.e.

An other very important example is $C^0([0,1];\mathbb{R})$, the vector space of continuous mappings from $[0,1]$ to $\mathbb{R}$, equipped with the norm

$$\|u\| = \sup_{x\in[0,1]} |u(x)|. \tag{1.3.9}$$

It is a good exercise left to the reader to prove that $C^0([0,1];\mathbb{R})$ with that norm is a Banach space.

1.3.4 Semi-norms

Let us consider the space $C^0(]0,1[;\mathbb{R})$; what is the natural topology on that space? Obviously, one cannot take the norm (1.3.9) since it may be infinite (think of $u(x) = 1/x$). On the other hand it is quite natural to look at

$$p_k(u) = \sup_{k^{-1}\le x\le 1-k^{-1}} |u(x)|, \quad \text{for } 1 \le k \in \mathbb{N}.$$

p_k is not a norm since the separation property (3) in the definition 1.3.3 is not satisfied; however, one can use the $(p_k)_{k\ge1}$ to give a definition of a converging sequence $(u_n)_{n\in\mathbb{N}}$: that sequence converges to 0 means that for each $k \ge 1$, $\lim_n p_k(u_n) = 0$. This is the uniform convergence on the compact subsets of the open set $]0,1[$. This example may serve as a motivation to introduce a more general structure than the normed vector space, namely vector spaces for which the topology is defined by a countable family of *semi-norms.*

Definition 1.3.8. *Let E be a vector space and $p : E \longrightarrow \mathbb{R}_+$. We shall say that p is a semi-norm on E if for $x, y \in E, \alpha \in \mathtt{k}$,*

(1) $p(\alpha x) = |\alpha| p(x)$, *(homogeneity),*

(2) $p(x+y) \le p(x) + p(y)$, *(triangle inequality)* [9].

Let us consider a countable family $(p_k)_{k\ge1}$ of semi-norms on E. We shall say that the family $(p_k)_{k\ge1}$ is separating whenever $p_k(x) = 0$ for all $k \ge 1$ implies $x = 0$.

Let E be a vector space and $(p_k)_{k\ge1}$be a separating countable family of semi-norms on E. We define $d : E \times E \longrightarrow \mathbb{R}_+$ by the formula

$$d(x,y) = \sum_{k\ge1} 2^{-k} \frac{p_k(x-y)}{1+p_k(x-y)}. \tag{1.3.10}$$

Lemma 1.3.9. *Let E be a vector space and $(p_k)_{k\ge1}$be a separating countable family of semi-norms on E. The formula (1.3.10) defines a translation-invariant distance*

[9] We note that (1) implies $p(0) = 0$ but that the separation property (1) in the definition 1.3.3 is not satisfied in general.

on E and the metric space (E,d) is a TVS. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of E. Then

$$\lim_n x_n = 0 \Longleftrightarrow \forall k \geq 1, \lim_n p_k(x_n) = 0.$$

Assuming that $k \mapsto p_k(x)$ is increasing for all $x \in E$, we obtain that a basis of neighborhoods of 0_E is the family $(B_{k,l})_{k,l\geq 1}$ with

$$B_{k,l} = \{x \in E, p_k(x) < 1/l\},$$

i.e. each $B_{k,l} \in \mathscr{V}_0$, and for all $V \in \mathscr{V}_0$, there exists $k,l \geq 1$ such that $B_{k,l} \subset V$.

Proof. The formula above makes sense and is obviously translation-invariant and symmetric. The separation property (1) of the definition 1.2.1 is a consequence of the separating property of the family $(p_k)_{k\geq 1}$. To verify the triangle inequality for d, we note that the mapping $\mathbb{R}_+ \ni \theta \mapsto \frac{\theta}{1+\theta} = 1 - \frac{1}{1+\theta}$ is increasing so that, since $p_k(x-z) \leq p_k(x-y) + p_k(y-z)$,

$$d(x,z) = \sum_{k\geq 1} 2^{-k} \frac{p_k(x-z)}{1+p_k(x-z)} \leq \sum_{k\geq 1} 2^{-k} \frac{p_k(x-y)+p_k(y-z)}{1+p_k(x-y)+p_k(y-z)} \leq d(x,y)+d(y,z).$$

To check that the metric space (E,d) is a TVS, we use the lemma 1.3.2: for $\lambda \in \mathtt{k}, x \in E$, we have[10]

$$d(\lambda x, 0) = \sum_{k\geq 1} 2^{-k} \frac{|\lambda| p_k(x)}{1+|\lambda| p_k(x)} \leq \max(|\lambda|, 1) d(x,0)$$

so that $d(\lambda x, 0) < \epsilon$ provided $d(x,0) < \epsilon$ and $|\lambda| \leq 1$. Also $d(\lambda_0 x, 0) < \epsilon$ provided $d(x,0) < \epsilon/(1+|\lambda_0|)$; finally we consider $d(\lambda x_0, 0)$. We have for $N \geq 1$

$$d(\lambda x_0, 0) = \sum_{k\geq 1} 2^{-k} \frac{|\lambda| p_k(x_0)}{1+|\lambda| p_k(x_0)} \leq \sum_{1\leq k\leq N} 2^{-k} \frac{|\lambda| p_k(x_0)}{1+|\lambda| p_k(x_0)} + \sum_{k>N} 2^{-k}$$

and thus, for all $N \geq 1$, $0 \leq \limsup_{\lambda\to 0} d(\lambda x_0, 0) \leq \sum_{k>N} 2^{-k} = 2^{-N}$ which implies $\lim_{\lambda\to 0} d(\lambda x_0, 0) = 0$. The assumptions of the lemma 1.3.2 are satisfied and (E,d) is indeed a TVS. Let us now consider a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of E, converging to 0: then, for each $k \geq 1$, $0 \leq p_k(x_n)/(1+p_k(x_n)) \leq 2^k d(x_n, 0)$ so that

$$\lim_n p_k(x_n)/(1+p_k(x_n)) = 0 \Longrightarrow \lim_n p_k(x_n) = 0.$$

Conversely if for all $k \geq 1$, $\lim_n p_k(x_n) = 0$, we have for all $N \geq 1$,

$$0 \leq d(x_n, 0) \leq \sum_{1\leq k\leq N} 2^{-k} \frac{p_k(x_n)}{1+p_k(x_n)} + \sum_{k>N} 2^{-k}$$

and thus, for all $N \geq 1$, $0 \leq \limsup_n d(x_n, 0) \leq 2^{-N}$, i.e. $\lim_n d(x_n, 0) = 0$. Let us prove now the last statement of the lemma: we have, using $k \mapsto p_k$ increasing,

$$d(x,0) \leq \sum_{1\leq k\leq k_0} 2^{-k} \frac{p_k(x)}{1+p_k(x)} + 2^{-k_0} \leq \frac{p_{k_0}(x)}{1+p_{k_0}(x)} + 2^{-k_0} \leq p_{k_0}(x) + 2^{-k_0}.$$

[10] We use for $a,b \geq 0$, $\frac{ab}{1+ab} \leq \frac{b}{1+b}$ if $a \leq 1$ and $\frac{ab}{1+ab} \leq \frac{ab}{1+b}$ if $a \geq 1$.

Let $r_0 > 0$ be given and k_0 such that $2^{-k_0} < r_0/2$. We have

$$B(0, r_0) \supset \{x \in E, p_{k_0}(x) < 1/l_0\} = B_{k_0,l_0}, \quad 1/l_0 < r_0/2,$$

since $x \in B_{k_0,l_0}$ implies $d(x, 0) \le p_{k_0}(x) + 2^{-k_0} < \frac{r_0}{2} + \frac{r_0}{2} = r_0$. Conversely, we have

$$\frac{p_k(x)}{1 + p_k(x)} \le 2^k d(x, 0)$$

so that, for k, l given integers ≥ 1, there exists $r > 0$ such that $B(0, r) \subset B_{k,l}$; in fact $d(x, 0) < r$ implies $\frac{p_k(x)}{1+p_k(x)} < 2^k r$ and taking $r = 2^{-k-2}l^{-1}$ gives

$$p_k(x) \le \frac{1}{4l} + \frac{1}{4l} p_k(x) \Longrightarrow p_k(x)\big(1 - \frac{1}{4l}\big) \le \frac{1}{4l} \Longrightarrow p_k(x) \le \frac{1}{4l - 1} < \frac{1}{l}$$

since $3l \ge 3 > 1$. The proof of the lemma is complete. □

N.B. Note that for a TVS as above, whose topology is defined by a separating countable family of semi-norms, the closure of the open ball $B(x, r)$ is indeed the closed ball $\tilde{B}(x, r) = \{y \in E, d(y, x) \le r\}$ (see the remark after the definition 1.2.2). In fact, we have already seen that $\overline{B(x, r)} \subset \tilde{B}(x, r)$, so that it is enough to check the other inclusion. Using the translation-invariance, we may consider only $x_0 \in E$ such that $d(x_0, 0) = r_0$ with some $r_0 > 0$. Now, we have $d((1 - \epsilon)x_0, x_0) = d(-\epsilon x_0, 0)$ and thus, since the assumptions of the lemma 1.3.2 are proven true, we have $\lim_{\epsilon\to 0} d((1 - \epsilon)x_0, x_0) = 0$; on the other hand, $(1 - \epsilon)x_0 \in B(0, r_0)$ for $\epsilon > 0$ since

$$d((1 - \epsilon)x_0, 0) = \sum_{k\ge1} 2^{-k} \frac{(1 - \epsilon)p_k(x_0)}{1 + (1 - \epsilon)p_k(x_0)} < \sum_{k\ge1} 2^{-k} \frac{p_k(x_0)}{1 + p_k(x_0)} = d(x_0, 0) = r_0,$$

where the strict inequality above is due to the fact that $p_k(x_0) > 0$ for at least one $k \ge 1$ (otherwise $x_0 = 0$, which is incompatible with $d(x_0, 0) > 0$) and the mapping $\mathbb{R}_+ \ni \theta \mapsto \theta/(1 + \theta)$ is strictly increasing. As a result, x_0 is a limit of points of $B(0, r_0)$ and thus belongs to the closure of the open ball.

Definition 1.3.10. *Let E be a vector space and $(p_k)_{k\ge1}$ a separating countable family of semi-norms on E. The metric space (E, d) with d given by (1.3.10) is a TVS. We shall say that E is a Fréchet space when (E, d) is complete.*

N.B. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a vector space E, equipped with a separating countable family of semi-norms $(p_k)_{k\ge1}$, is a Cauchy sequence means that it is a Cauchy sequence for the metric d defined by (1.3.10). This is equivalent to the following properties:

$$\forall k \ge 1, \forall \epsilon > 0, \exists N_{\epsilon,k}, \forall n', n'' \ge N_{\epsilon,k}, \quad p_k(x_{n'} - x_{n''}) < \epsilon. \tag{1.3.11}$$

To prove this, we note first that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for d, since

$$\frac{p_k(x_{n'} - x_{n''})}{1 + p_k(x_{n'} - x_{n''})} \le 2^k d(x_{n'}, x_{n''})$$

we have $p_k(x_{n'} - x_{n''})\big(1 - 2^k d(x_{n'}, x_{n''})\big) \le 2^k d(x_{n'}, x_{n''})$. For a given $k \ge 1$, and $\epsilon > 0$, since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, we can find N such that, for $n', n'' \ge N$, $d(x_{n'}, x_{n''}) < \min(\epsilon 2^{-k-1}, 2^{-k-1})$: we get

$$\frac{1}{2} p_k(x_{n'} - x_{n''}) \le p_k(x_{n'} - x_{n''})\big(1 - 2^k d(x_{n'}, x_{n''})\big) \le 2^k d(x_{n'}, x_{n''}) < \frac{\epsilon}{2},$$

which is (1.3.11). Let us assume now that (1.3.11) holds. For all $k \ge 1$, we have

$$d(x_{n'}, x_{n''}) \le \sum_{1 \le l \le k} 2^{-l} \frac{p_l(x_{n'} - x_{n''})}{1 + p_l(x_{n'} - x_{n''})} + \sum_{l > k} 2^{-l},$$

so that for $\epsilon > 0$, choosing k_ϵ such that $2^{-k_\epsilon} < \epsilon/2$, using $M_\epsilon = \max_{1 \le l \le k_\epsilon} N_{\frac{\epsilon}{2}, l}$ (where the $N_{\frac{\epsilon}{2}, l}$ are defined in (1.3.11)), for $n', n'' \ge M_\epsilon$ we have

$$d(x_{n'}, x_{n''}) \le \sum_{1 \le l \le k_\epsilon} 2^{-l} \frac{\frac{\epsilon}{2}}{1 + \frac{\epsilon}{2}} + 2^{-k_\epsilon} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

proving that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

1.4 A review of the basic structures for TVS

1.4.1 Hilbert spaces

This is the richest structure: a Hilbert space is a complete normed vector space whose norm is derived from a dot-product (see the definition 1.3.7). The typical examples are $\ell^2(\mathbb{N})$ and more generally $L^2(\mu)$ (see (1.3.8)) and the dot-product is

$$\langle f, g \rangle = \int_X f \bar{g} d\mu.$$

One can prove that a *separable*[11] Hilbert space is isomorphic to $\ell^2(\mathbb{N})$.

1.4.2 Banach spaces

A Banach space is a complete normed vector space. The typical examples are $\ell^p(\mathbb{N})$ and more generally $L^p(\mu)$ (see (1.3.8)) for $1 \le p \le +\infty$) with the norm $(\int_X |f|^p d\mu)^{1/p}$. Other examples include $C^0([0,1]; \mathbb{R})$ and more generally $C^0(K; \mathbb{R}^N)$ where K is a compact topological space (see the next section) with the norm

$$\|u\| = \sup_{x \in K} |u(x)|_{\mathbb{R}^N}, \quad \text{(here } |_{\mathbb{R}^N} \text{ stands for a norm on } \mathbb{R}^N).$$

[11] i.e. containing a countable dense part.

1.4.3 Fréchet spaces

Fréchet spaces are complete metric vector spaces whose distance is given by a countable separating family of semi-norms $(p_k)_{k\ge 1}$ (see (1.3.10) and the definition 1.3.10). The most typical examples are $C^m(\Omega;\mathbb{C})$ where $m\in\mathbb{N},\Omega$ open subset of $\mathbb{R}^n$, the complex-valued C^m functions defined on Ω. Since it is possible (exercise) to write

$$\Omega=\cup_{l\in\mathbb{N}}K_l,\quad K_l \text{ compact},$$

we consider the countable family of seminorms $p_l(u)=\sup_{x\in K_l,|\alpha|\le m}|(\partial_x^\alpha u)(x)|$. Another example is $C^\infty(\Omega;\mathbb{C})$, the C^∞ complex-valued functions on Ω; the family

$$p_{l,m}(u)=\sup_{x\in K_l,|\alpha|\le m}|(\partial_x^\alpha u)(x)|$$

defines the topology. For Ω open subset of $\mathbb{C}$, one may also consider $\mathscr{H}(\Omega)$ the holomorphic functions on Ω, with the family of semi-norms $\sup_{x\in K_l}|u(x)|$. There are many other examples that we shall use later on, such as the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of smooth rapidly decreasing functions on $\mathbb{R}^n$: u is in $\mathscr{S}(\mathbb{R}^n)$ means that it is a smooth function on $\mathbb{R}^n$ such that, for all α,β, $p_{\alpha,\beta}(u)=\sup_{x\in\mathbb{R}^n}|x^\alpha(\partial_x^\beta u)(x)|<+\infty$. The $p_{\alpha,\beta}$ describe the topology on $\mathscr{S}(\mathbb{R}^n)$ (an example of such a function is $e^{-\|x\|^2}$ where $\|x\|$ is the Euclidean norm on $\mathbb{R}^n$).

1.4.4 More general structures

Some very interesting and natural topological vector spaces are not metrizable, such as $\mathscr{D}(\Omega)$, the smooth complex-valued compactly supported functions defined on Ω, open set of $\mathbb{R}^n$. They are important in the theory of distributions, but many aspects of that theory can be acquired without a deep understanding of the topology of $\mathscr{D}$, which is defined by an *uncountable* family of semi-norms.

1.5 Compactness

1.5.1 Compact topological spaces

Definition 1.5.1. *A topological space $(X,\mathscr{O})$ is said to be compact when it is a Hausdorff space (see the definition 1.1.5) and satisfy the Borel-Lebesgue property: if $(\Omega_i)_{i\in I}$ is a family of open sets such that $X=\cup_{i\in I}\Omega_i$, there exists a finite subset J of I such that $X=\cup_{i\in J}\Omega_i$.*

N.B. If A is a closed subset of a compact space X, then A is also compact. Using the definition 1.1.10 of the induced topology on A, the separation property is obvious and we may assume that $A\subset\cup_{i\in I}\Omega_i$, where each Ω_i is an open subset of X. Then we have

$$X=\cup_{i\in I}\Omega_i\cup A^c$$

and since A^c is open, the compactness of X implies that $X=\cup_{i\in J}\Omega_i\cup A^c$ with a finite subset J of I. As a consequence $A\subset\cup_{i\in J}\Omega_i$, proving its compactness.

Proposition 1.5.2. *Let X be a Hausdorff topological space.*
(1) *Let A, B be two compact disjoint subsets of X. Then there exist U, V open disjoint subsets of X such that $A \subset U$ and $B \subset V$.*
(2) *Let A be a compact subset of X. Then A is a closed subset of X.*

Proof. Since X is Hausdorff, for each $(x, y) \in A \times B$, there exists some open sets $U_x(y) \in \mathscr{V}_x, V_y(x) \in \mathscr{V}_y$ such that $U_x(y) \cap V_y(x) = \emptyset$. By the compactness of B, we have for all $x \in A$,

$$B \subset \cup_{1 \le j \le N_x} V_{y_j}(x) = W(x).$$

As a consequence, with $T(x) = \cap_{1 \le j \le N_x} U_x(y_j)$, we have $T(x) \cap W(x) = \emptyset$, $W(x)$ open containing B and the open set $T(x) \in \mathscr{V}_x$. By the compactness of A, we have

$$A \subset \cup_{1 \le k \le M} T(x_k).$$

We take then $U = \cup_{1 \le k \le M} T(x_k)$, $V = \cap_{1 \le k \le M} W(x_k)$, which are disjoint open sets containing respectively A, B, proving (1). Let A be a compact subset of X; if $a \notin A$, then A and $\{a\}$ are disjoint compact subsets and from the now proven (1), there exists an open set $V \in \mathscr{V}_a$ such that $V \cap A = \emptyset$, i.e. $V \subset A^c$, proving that A^c is open. □

Proposition 1.5.3. *Let $(K_i)_{i \in I}$ be a family of compact subsets of a Hausdorff space X such that $\cap_{i \in I} K_i = \emptyset$. Then there exists a finite subset J of I such that $\cap_{i \in J} K_i = \emptyset$.*

Proof. Note that from the property (b) of the proposition 1.5.2, the K_i are closed subsets of X. For a fixed $i_0 \in I$,

$$K_{i_0} \subset \cup_{i \ne i_0, i \in I} K_i^c \Longrightarrow K_{i_0} \subset \cup_{i \in J} K_i^c, \quad J \text{ finite subset of } I.$$

As a result, $\cap_{i \in J \cup \{i_0\}} K_i = \emptyset$. □

Theorem 1.5.4. *Let X, Y be topological spaces, with Y a Hausdorff space, and $f : X \longrightarrow Y$ be a continuous mapping. If X is compact, then $f(X)$ is compact.*

Proof. $f(X)$ is a Hausdorff space as a subset of a Hausdorff space. Let us assume that $f(X) \subset \cup_{i \in I} V_i$ where V_i are open subsets of Y. Then

$$X = \cup_{i \in I} \underbrace{f^{-1}(V_i)}_{\substack{\text{open} \\ \text{since } f \text{ continuous}}}$$

so that for some finite J, $X = \cup_{i \in J} f^{-1}(V_i)$, and thus $f(X) = \cup_{i \in J} f(f^{-1}(V_i)) \subset \cup_{i \in J} V_i$, proving the result. □

1.5.2 Compact metric spaces

Theorem 1.5.5 (Bolzano-Weierstrass). *Let X be a metric space. Then the two following properties are equivalent.*
(i) X is compact.
(ii) From any sequence of elements of X, on can extract a convergent subsequence.
This means that for a metric space the compactness is equivalent to the sequential compactness (as defined by (ii)).

Remark 1.5.6. If $(x_k)_{k\ge 1}$ is a sequence, a subsequence $(x_{\kappa(l)})_{l\ge 1}$ is defined by an increasing mapping $\kappa : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ $(\forall l, \kappa(l) < \kappa(l+1))$; in other words, a subsequence is

$$x_{\kappa_1}, x_{\kappa_2}, \dots, x_{\kappa_l}, x_{\kappa_{l+1}}, \dots \quad \text{with } \kappa_1 < \kappa_2 < \dots < \kappa_l < \kappa_{l+1} < \dots$$

Proof. Let us assume that (i) holds and let $(x_k)_{k\ge 1}$ be a sequence of elements of X. We have then that

$$X \supset F_1 = \overline{\{x_k\}_{k\ge 1}} \supset F_2 = \overline{\{x_k\}_{k\ge 2}} \supset \dots \supset F_n = \overline{\{x_k\}_{k\ge n}} \supset \dots$$

and $(F_n)_{n\ge 1}$ is a decreasing sequence of non-empty compact sets (since F_n is closed in a compact set). As a result, the set $\cap_{n\ge 1} F_n$ is closed $\subset X$ and thus compact; moreover $\cap_{n\ge 1} F_n \ne \emptyset$, otherwise

$$\cup_{n\ge 1} F_n^c = X$$

and by the compactness of X, we would have $X = \cup_{1\le n\le N} F_n^c = F_N^c$ since the F_n^c are increasing with n, which is not possible since $x_N \in F_N \ne \emptyset$. Let $y \in \cap_{n\ge 1} F_n$: for all $V \in \mathscr{V}_y$, for all $n \in \mathbb{N}$, $V \cap \{x_k\}_{k\ge n} \ne \emptyset$. This means that,

$$\forall \epsilon > 0, \quad \forall n \ge 1, \quad \exists k \ge n, \quad d(x_k, y) < \epsilon. \tag{1.5.1}$$

Let us then assume that we have found $x_{k_1}, \dots, x_{k_m}$ with $1 \le k_1 < k_2 < \dots < k_m$ such that $d(x_{k_j}, y) < 1/j$. Then using (1.5.1), we can find $k_{m+1} \ge 1 + k_m$ such that $d(x_{k_{m+1}}, y) < 1/(m+1)$. Eventually we have constructed an extracted subsequence $(x_{k_j})_{j\ge 1}$ of the sequence $(x_k)_{k\ge 1}$ with $\lim_j x_{k_j} = y \in X$, proving (ii).

Lemma 1.5.7 (Lebesgue numbers of a covering)**.** *Let $X = \cup_{i\in I} \Omega_i$ be an open covering of a metric space X satisfying (ii). Then*

$$\exists r_0 > 0, \forall x \in X, \exists i \in I, \quad \tilde{B}(x, r_0) \subset \Omega_i, \qquad \text{where } \tilde{B}(x, r_0) = \{y \in X, d(y, x) \le r_0\}.$$

Proof of the lemma. Otherwise, we would have

$$\forall k \ge 1, \exists x_k \in X, \forall i \in I, \quad \tilde{B}(x_k, 1/k) \cap \Omega_i^c \ne \emptyset. \tag{1.5.2}$$

From the property (ii), we would be able to extract a convergent subsequence $(x_{k_j})_{j\ge 1}$ from the sequence $(x_k)_{k\ge 1}$. Since $\lim_j x_{k_j} = y_0$ which belongs to some open set Ω_{i_0}, we get $B(y_0, r) \subset \Omega_{i_0}$ with some $r > 0$. As a result, $x_{k_j} \in B(y_0, r/2)$ for $j \ge j_0$ and we have

$$\tilde{B}(x_{k_j}, 1/k_j) \subset B(y_0, r)$$

since if $x \in \tilde{B}(x_{k_j}, 1/k_j)$, we get

$$d(x, y_0) \le d(x, x_{k_j}) + d(y_0, x_{k_j}) \le \frac{1}{k_j} + \frac{r}{2} < r \text{ if } j \ge j_0 \text{ and } k_j > 2/r.$$

As a consequence $\tilde{B}(x_{k_j}, 1/k_j) \subset \Omega_{i_0}$ which contradicts (1.5.2). The proof of the lemma is complete. □

Lemma 1.5.8 (Precompactness). *Let X be a metric space satisfying (ii). Then*

$$\forall r > 0, \exists N \in \mathbb{N}^*, \exists\ x_1, \dots, x_N \in X, \qquad X = \cup_{1 \le k \le N} \tilde{B}(x_k, r).$$

Proof of the lemma. Otherwise, we would have

$$\exists r_0 > 0, \forall N \ge 1, \forall x_1, \dots, x_N \in X, \qquad \cup_{1 \le k \le N} \tilde{B}(x_k, r_0) \ne X. \tag{1.5.3}$$

Let us assume that we have found $x_1, \dots, x_n \in X$ such that $d(x_i, x_j) > r_0$ when $i \ne j$ (note that given x_1, since $\tilde{B}(x_1, r_0) \ne X$, we can find x_2 such that $d(x_1, x_2) > r_0$). Using (1.5.3), we can find $x_{n+1} \notin \cup_{1 \le k \le n} \tilde{B}(x_k, r_0)$ and thus for

$$\text{for } k = 1, \dots, n, \quad d(x_k, x_{n+1}) > r_0.$$

Eventually, we can construct a sequence $(x_k)_{k \ge 1}$ such that $d(x_k, x_l) > r_0$ if $k \ne l$. Naturally, such a sequence cannot have a convergent subsequence, which contradicts the assumption (ii). The proof of the lemma is complete. □

Let us now conclude with the proof of the theorem. We assume that X is a metric space satisfying (ii) and that $X = \cup_{i \in I} \Omega_i$ where the Ω_i are some open subsets of X. From the lemma 1.5.7, we get that there exists $r_0 > 0$ such that for all $x \in X$, there exists $i_x \in I$ such that $\tilde{B}(x, r_0) \subset \Omega_{i_x}$. From the lemma 1.5.8, we obtain that for that $r_0 > 0$, there exists a finite sequence $x_1, \dots, x_{N_0}$ with

$$X = \cup_{1 \le k \le N_0} \tilde{B}(x_k, r_0) \subset \cup_{1 \le k \le N_0} \Omega_{i_{x_k}},$$

which is a finite covering sought after. The proof of the theorem is complete. □

Lemma 1.5.9. *The compact subsets of $\mathbb{R}^n$ (equipped with its standard topology) are the closed and bounded subsets.*

Proof. Let K be a compact subset of $\mathbb{R}^n$. By the proposition 1.5.2, K must be closed; moreover K is bounded, since $K \subset \cup_{k \ge 1} B(0, k)$ and by compactness $K \subset B(0, k_0)$ for some $k_0 \ge 1$. Conversely, let K be a closed bounded subset of $\mathbb{R}^n$: then K is a closed subset of $[-M, M]^n$ for some positive M. It suffices to show that $[-M, M]^n$ is a compact subset of $\mathbb{R}^n$ since we know from the N.B. after the definition 1.5.1 that a closed subset of a compact set is compact. To prove that $[-M, M]$ is a compact subset of $\mathbb{R}$ is easy: we consider a sequence $(x_k)_{k \ge 1}$ in $[-M, M]$ and we define

$$\liminf_k x_k = \sup_k (\inf_{n \ge k} x_n) \le \limsup_k x_k = \inf_k (\sup_{n \ge k} x_n).$$

We note that $k \mapsto b_k = \sup_{n \ge k} x_n$ (resp. $k \mapsto a_k = \inf_{n \ge k} x_n$) is a non-increasing[12] (resp. non-decreasing) sequence in $[-M, M]$, so that they are both converging and

$$a = \lim_k a_k = \sup_k a_k, \quad b = \lim_k b_k = \inf_k b_k.$$

[12] In mathematical english, a sequence $(a_k)_{k \in \mathbb{N}}$ is said to be non-decreasing whenever for all k, $a_k \le a_{k+1}$ and a sequence $(b_k)_{k \in \mathbb{N}}$ is said to be non-increasing whenever for all k, $b_k \ge b_{k+1}$. Saying that the sequence $(a_k)_{k \in \mathbb{N}}$ is increasing means $a_k < a_{k+1}$ for all k; saying that the sequence $(b_k)_{k \in \mathbb{N}}$ is decreasing means $b_k > b_{k+1}$ for all k.

Moreover since $a_k \le b_k$ and both sequence are converging we get the above inequality. Moreover, for all $\epsilon > 0, N \ge 1$, there exists $k \ge N$ such that

$$a - \epsilon < a_k \le a, \quad \exists n \ge k, \quad a - \epsilon < a_k \le x_n < a_k + \epsilon \le a + \epsilon$$

so that a (as well as b) is the limit of an extracted subsequence of the sequence $(x_k)_{k\ge 1}$. One can also prove that a (resp. b) is the smallest (resp. largest) limit point (i.e. limit of a subsequence) of the sequence $(x_k)_{k\ge 1}$. So the property (ii) is satisfied for the metric space $[-M, M]$ as well as for $[-M, M]^n$ by iterated extraction. The proof of the lemma is complete. □

N.B. It might be the proper time for stating a couple of **caveat**. We have seen that, for a metric space, the sequential compactness is equivalent to the compactness. However, there exist some topological spaces which are sequentially compact and not compact: this is the case for instance of the ordered $[0, \omega_1)$ where ω_1 is the first uncountable ordinal. Conversely, there are topological spaces which are compact and not sequentially compact: this is the case of $[0, 1]^{[0,1]}$, the mappings from $[0, 1]$ to $[0, 1]$, equipped with the product topology. There is a general theorem of topology, called the *Tychonoff theorem*, asserting that a product $X = \Pi_{i\in I} X_i$ of compact spaces is compact, where the topology on X is the natural product-topology, defined as the coarsest topology for which the projections $\pi_i : X \to X_i$ are continuous. Although that Tychonoff theorem is easy for I countable, it is one of the great success of the theory of filters [13] to provide a very simple proof of that result, whatever is the cardinality of I. We shall have little use in these lectures of uncountable products of topological spaces, but the reader should keep in mind that the metrizability theory and countable products are far from exhausting the variety of examples and notions of topological spaces.

Theorem 1.5.10 (Heine theorem)**.** *Let X, Y be metric spaces and $f : X \longrightarrow Y$ be a continuous mapping. If X is compact, f is uniformly continuous, i.e.*

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x', x'' \in X, d_X(x', x'') < \alpha \Longrightarrow d_Y\big(f(x'), f(x'')\big) < \epsilon.$$

Proof. Reductio ad absurdum: otherwise

$$\exists \epsilon_0 > 0, \forall k \ge 1, \exists x'_k, x''_k \in X, d_X(x'_k, x''_k) < 1/k, d_Y\big(f(x'_k), f(x''_k)\big) \ge \epsilon_0.$$

From the sequence $(x'_k)_{k\ge 1}$, we may extract a convergent subsequence $(x'_{k_1(l)})_{l\ge 1}$, and from the sequence $(x''_{k_1(l)})_{l\ge 1}$, we may extract a convergent subsequence $(x''_{k_1(k_2(m))})_{m\ge 1}$. With $\kappa = k_1 \circ k_2$, the sequences $(x'_{\kappa(m)})_{m\ge 1}, (x''_{\kappa(m)})_{m\ge 1}$ are both convergent (the first

[13] A thorough exposition of the theory of filters can be found in the first chapter of the Bourbaki volume *Topologie générale* [1] (see also the G. Choquet book, *Topology* [4]). As a historical note, the theory of filters and ultrafilters was invented by Henri Cartan in 1937 and developed systematically by the Bourbaki group later on. It is interesting to notice that, although that theory was criticized for being too abstract, the essentially equivalent notion of *nets* is used in the english literature (see e.g. the chapter 4 of the first volume *Functional Analysis* of [11]). We cannot resist quoting the magnificent book [11], in which the authors write on page 118 (notes of chapter IV) *"We find the filter theory of convergence very unintuitive and prefer the use of nets in all cases"* before adding a supplement on page 351 about ... the theory of filters, a discreet hommage to its efficiency.

one as a subsequence of a convergent sequence) with the same limit z, since with $z' = \lim_m x'_{\kappa(m)}, z'' = \lim_m x''_{\kappa(m)}$,

$$d_X(z', z'') \leq \underbrace{d_X(z', x'_{\kappa(m)})}_{\substack{\to 0 \\ m \to +\infty}} + \underbrace{d_X(x'_{\kappa(m)}, x''_{\kappa(m)})}_{\leq 1/\kappa(m)} + \underbrace{d_X(x''_{\kappa(m)}, z'')}_{\substack{\to 0 \\ m \to +\infty}}.$$

However, we have

$$0 < \epsilon_0 \leq d_Y\big(f(x'_{\kappa(m)}), f(x''_{\kappa(m)})\big) \leq d_Y\big(f(x'_{\kappa(m)}), f(z)\big) + d_Y\big(f(x''_{\kappa(m)}), f(z)\big)$$

although the continuity of f implies

$$\lim_m d_Y\big(f(x'_{\kappa(m)}), f(z)\big) = \lim_m d_Y\big(f(x''_{\kappa(m)}), f(z)\big) = 0,$$

and thus $0 < \epsilon_0 \leq 0$ which is impossible. □

1.5.3 Local compactness

Definition 1.5.11. *A topological space is said to be locally compact if it is a Hausdorff space (cf. the definition 1.1.5) such that each point has a compact neighborhood.*

Proposition 1.5.12. *In a locally compact topological space X, every point has a basis of compact neighborhoods, i.e. $\forall x \in X, \forall U \in \mathscr{V}_x, \exists L \text{compact}, L \in \mathscr{V}_x, L \subset U$. More generally, let K be a compact subset of a locally compact topological space and U an open set such that $K \subset U$. Then there exists an open set V with compact closure such that*

$$K \subset V \subset \overline{V} \subset U.$$

Proof. Since every point has a compact neighborhood, we can cover K with finitely many $(W_j)_{1\leq j\leq N}$ such that W_j is open with compact closure; the set $W = \cup_{1\leq j\leq N} W_j$ is also open with compact closure, since a finite union of open sets is open and the closure of a finite union is the union of the closures. If $U = X$, we can take $V = W$. Otherwise, for each $x \in U^c$, the proposition 1.5.2 shows that there exists V_x, V'_x open disjoint such that $K \subset V_x$, $\{x\} \subset V'_x$; as a result, $(U^c \cap \overline{W} \cap \overline{V_x})_{x\in U^c}$ is a family of compact sets with empty intersection: we have $V_x \cap V'_x = \emptyset$ and thus $x \notin \overline{V_x}$, so that

$$y \in \cap_{x\in U^c}\big(U^c \cap \overline{W} \cap \overline{V_x}\big) \Longrightarrow y \in U^c, y \in \overline{W} \text{and for all } x \in U^c,\ y \in \overline{V_x}$$
$$\Longrightarrow y \in \overline{V_y} \quad \text{which is impossible.}$$

From the proposition 1.5.3, we can find $x_1, \dots, x_N \in U^c$ such that

$$\emptyset = \cap_{1\leq j\leq N}\big(U^c \cap \overline{W} \cap \overline{V_{x_j}}\big) \Longrightarrow \cap_{1\leq j\leq N}\big(\overline{W} \cap \overline{V_{x_j}}\big) \subset U$$

We consider now the open set $V = W \cap \cap_{1\leq j\leq N} V_{x_j}$. We have by construction $K \subset V_{x_j} \cap U$ and thus $K \subset V \subset \overline{V} \subset \overline{W} \cap \cap_{1\leq j\leq N} \overline{V_{x_j}}$ which is compact $\subset U$. □

The typical examples of locally compact spaces are the open subsets of $\mathbb{R}^n$. On the other hand, the Hilbert space $\ell^2(\mathbb{N})$ is not locally compact since the sequence $(u_n)_{n\geq 0}$ with $u_n = (\delta_{k,n})_{k\geq 0} \in \ell^2(\mathbb{N})$ is made with unit vectors such that $\langle u_n, u_m\rangle_{\ell^2(\mathbb{N})} = \delta_{n,m}$; as a consequence, if $(u_n)_{n\geq 0}$ had a convergent subsequence $(v_j = u_{n_j})_{j\geq 0}$, we would have

$$\|v_i - v_j\|^2 = \|v_i\|^2 + \|v_j\|^2 - 2\langle v_i, v_j\rangle = 2 \quad \text{if } i \neq j,$$

so that the Cauchy criterion could not be satisfied. More generally, we shall see below that any locally compact TVS is finite dimensional.

Theorem 1.5.13. *Let E be a locally compact topological vector space. Then E is finite-dimensional.*

Remark 1.5.14. Note that the translation and the multiplication by a non-zero scalar are homeomorphisms of E, since they are continuous (because E is a TVS), bijective with a continuous inverse (the inverse mapping of $x \mapsto x + x_0$ is $y \mapsto y - x_0$ and the inverse mapping of $x \mapsto \lambda_0 x$ with $\mathtt{k} \ni \lambda_0 \neq 0$ is $y \mapsto \lambda_0^{-1} y$).

Proof. Let K be a compact neighborhood of 0. We have thus

$$K \subset \cup_{x\in K}\big(x + \frac{1}{2}\mathring{K}\big), \text{and } x + \frac{1}{2}\mathring{K} \text{ is open from the previous remark,}$$

so that by the compactness of K, we can find $x_1, \dots, x_N \in K$ such that

$$K \subset \cup_{1\leq j\leq N}(x_j + \frac{1}{2}\mathring{K}).$$

We define then $F = \mathrm{Vect}(x_1, \dots, x_N)$ the vector space generated by the $(x_j)_{1\leq j\leq N}$. We have

$$K \subset F + \frac{1}{2}\mathring{K} \Longrightarrow \frac{1}{2}K \subset F + \frac{1}{4}\mathring{K} \Longrightarrow K \subset F + F + \frac{1}{4}\mathring{K} \subset F + \frac{1}{4}\mathring{K},$$

and assuming $K \subset F + 2^{-n}\mathring{K}$, we obtain

$$K \subset F + \frac{1}{2}\mathring{K} \subset F + \frac{1}{2}K \subset F + \frac{1}{2}(F + 2^{-n}\mathring{K}) \subset F + 2^{-n-1}\mathring{K},$$

and finally

$$K \subset \cap_{n\geq 1}(F + 2^{-n}\mathring{K}). \tag{1.5.4}$$

Lemma 1.5.15. *Let E be a topological vector space. Then there exists $\mathscr{B} \subset \mathscr{V}_0$ such that for all $V \in \mathscr{V}_0$, there exists $B \in \mathscr{B}$ such that B is open, $B \subset V$ and for all scalar λ such that $|\lambda| \leq 1$, $\lambda B \subset B$(B is said to be balanced). Moreover B is absorbing, i.e. $E = \cup_{\lambda\in\mathtt{k}}\lambda B$. In other words $\mathscr{V}_0$ has a basis of balanced and absorbing neighborhoods of* 0.

Proof of the lemma. Let $V \in \mathscr{V}_0$ be given. From the continuity of the multiplication by a scalar, there exists U open $\in \mathscr{V}_0, r > 0$ such that $|\lambda| < r \Longrightarrow \lambda U \subset V$. We define

$$B = \cup_{0<|\lambda|<r}\lambda U, \quad (\text{for } \lambda \neq 0,\ \lambda U \text{ is homeomorphic to } U, \text{ thus is open}).$$

B is open, contains 0 (since $0 \in U$ thus $0 \in \lambda U$), $B \subset V$. Moreover B is balanced since for $|\mu| \le 1$,

$$\mu B = \cup_{0<|\lambda|<r} \mu\lambda U \subset \cup_{0<|\lambda|<r} \lambda U = B.$$

On the other hand, the continuity of the multiplication implies that, for $x_0 \in E$, there exists $r_0 > 0$ such that $\lambda x_0 \in B$ for $|\lambda| \le r_0$, so that $x_0 \in tB$ for $|t| \ge 1/r_0$; as a consequence, we have $E = \cup_{n\ge 1} 2^n B$, completing the proof of the lemma. □

For $V \in \mathscr{V}_0$, we can find $B \in \mathscr{V}_0$ open, $B \subset V$, $\in \mathscr{V}_0$, balanced with $E = \cup_{n\ge 1} 2^n B$. By the compactness of K, we get that $K \subset \cup_{1\le j\le N} 2^j B$ and since for $1 \le j \le N$,

$$2^j B = 2^N \underbrace{2^{j-N} B}_{\substack{\subset B \\ \text{since } B \text{ balanced}}} \subset 2^N B,$$

we get $K \subset 2^N B$, so that $2^{-N} K \subset B \subset V$ and using (1.5.4), we get

$$K \subset F + 2^{-N} \mathring{K} \subset F + V, \quad \text{for any } V \in \mathscr{V}_0.$$

As a result[14], we get $K \subset \bar{F}$.

Lemma 1.5.16. *Let F be a finite dimensional subspace in a Hausdorff topological vector space E. Then F is closed.*

That lemma implies our theorem: since K is a neighborhood of 0, the continuity of the multiplication implies that, for $x_0 \in E$, there exists $r_0 > 0$ such that $\lambda x_0 \in K$ for $|\lambda| \le r_0$, so that $x_0 \in tK$ for $|t| \ge 1/r_0$; as a consequence, we have

$$\lambda K \subset \lambda \bar{F} = \lambda F \subset F, \quad \text{so that} \quad E = \cup_{n\ge 1} 2^n K \subset F,$$

completing the proof of the theorem. □

Let us give the proof of the lemma. We may assume that a basis of F is $(e_1, \dots, e_m)$ and consider the injective linear mapping $\mathtt{k}^m \ni \alpha \mapsto L\alpha = \sum_{1\le j\le m} \alpha_j e_j \in E$. The continuity of the addition in E implies that, for $U \in \mathscr{V}_0$, there exists $V \in \mathscr{V}_0$ such that

$$\underbrace{V + \dots + V}_{m \text{ terms}} \subset U.$$

The continuity of the multiplication implies that, for $V \in \mathscr{V}_0, \exists r > 0$, such that if $|\lambda| < r$, then $\lambda e_j \in V$ for $1 \le j \le m$. Taking now $\max_{1\le j\le m} |\alpha_j| < r$ implies

$$\sum_{1\le j\le m} \alpha_j e_j \in U,$$

proving the continuity of the linear mapping L. Since the unit sphere S of $\mathtt{k}^m$ is compact (as closed and bounded, see the lemma 1.5.9), $L(S)$ is compact (cf. the

[14] In a TVS E, for $A \subset E$, we have $\bar{A} = \cap_{V\in\mathscr{V}_0}(A+V)$. In fact $x \in \bar{A}$ is equivalent to $\forall V \in \mathscr{V}_x, A \cap V \ne \emptyset$, which is equivalent to $\forall V \in \mathscr{V}_0, A \cap (V+x) \ne \emptyset$, which is equivalent to $\forall V \in \mathscr{V}_0$ $\exists a \in A, \exists v \in V, x + v = a$, which is equivalent to $x \in \cap_{V\in\mathscr{V}_0}(A-V) = \cap_{V\in\mathscr{V}_0}(A+V)$ since $v \mapsto -v$ is an homeomorphism: with $W \in \mathscr{V}_0$ given, we may find $V \in \mathscr{V}_0$ such that $-V \subset W$ and thus $A - V \subset A + W$ so that $\cap_{V\in\mathscr{V}_0}(A-V) \subset \cap_{W\in\mathscr{V}_0}(A+W)$ and the reverse equality as well.

theorem 1.5.4) and $0 \notin L(S)$. As a consequence, there exists a balanced open neighborhood $V \in \mathscr{V}_0$ such that $V \cap L(S) = \emptyset$ and

$$L^{-1}(V \cap F) \cap S = \emptyset \quad \text{otherwise } \emptyset \neq L\big(L^{-1}(V \cap F) \cap S\big) \subset V \cap F \cap L(S) = \emptyset.$$

The set $A = L^{-1}(V \cap F)$ contains 0 and such that for $\alpha \in A$, the segment $[0, \alpha] \subset A$: in fact, if $L\alpha \in V \cap F$, we have for $\theta \in [0, 1]$,

$$L(\theta\alpha) = \theta L(\alpha) \in V \cap F \quad \text{since } V \text{ is balanced and } F \text{ is a vector space.}$$

As a result, $A \subset B_1$, where B_1 is the open unit ball of $\mathtt{k}^m$: otherwise, it would contain a point α_0 with $\|\alpha_0\| \geq 1$ and the segment $[0, \alpha_0]$, which intersects the unit sphere S, contradicting the fact that A and S are disjoint. Let us consider now $x_0 \in \bar{F}$: there exists $t_0 > 0$ such that $x_0 \in t_0 V$ (continuity of multiplication, and $t_0 V$ is open) and

$$\forall W \in \mathscr{V}_0, \quad (W + x_0) \cap (t_0 V) \cap F \neq \emptyset \Longrightarrow x_0 \in \overline{t_0 V \cap F} = \overline{t_0 (V \cap F)},$$

but since[15]

$$t_0(V \cap F) = t_0 L(A) \subset t_0 L(B_1) = L(t_0 B_1) \subset \underbrace{L(t_0 \overline{B_1})}_{\text{compact}},$$

we have $x_0 \in \overline{t_0(V \cap F)} \subset L(t_0 \overline{B_1}) \subset F$ and $x_0 \in F$, completing the proof of the lemma.

N.B. The consequences of the theorem 1.5.13 are important in functional analysis. None of the natural spaces of functions that we shall consider, such as the Banach space $C^0([0, 1]; \mathbb{R})$, are finite-dimensional[16]. As a consequence, these spaces are not locally compact, which means in particular that, for an infinite-dimensional Banach space, the closed unit ball is not compact. This is a drastic change from the finite-dimensional geometry, and the reader has to keep in mind that the ordinary intuition that we have of the geometry in $\mathbb{R}^n$ is radically modified with infinite-dimensional spaces (by the way, infinite-dimensional means *not finite-dimensional*).

[15] Note that since $A = L^{-1}(V \cap F)$, we have $L(A) \subset V \cap F$; also if $x \in V \cap F$, we have $x = L(\alpha)$, thus with $\alpha \in A$: this implies $x \in L(A)$ and finally $L(A) = V \cap F$.

[16] The space $C^0([0, 1]; \mathbb{R})$ is not finite-dimensional, e.g. because it contains the functions $(e_n)_{n \in \mathbb{N}}$ defined by $e_n(x) = x^n$: these functions are independent since a polynomial cannot vanish identically on $[0, 1]$ unless it is the zero polynomial.

Chapter 2

Basic tools of Functional Analysis

2.1 The Baire theorem and its consequences

René Baire (1874 – 1932) is a french mathematician who made a lasting landmark contribution to functional analysis, known today as the *Baire Category Theorem.* We study in this section that theorem and the manifold consequences in the realm of functional analysis.

2.1.1 The Baire category theorem

Theorem 2.1.1 (Baire theorem)**.** *Let (X, d) be a complete metric space and $(F_n)_{n\ge 1}$ be a sequence of closed sets with empty interiors. Then the interior of $\cup_{n\ge1} F_n$ is also empty.*

N.B. The statement of that theorem is equivalent to say that, in a complete metric space, given a sequence $(U_n)_{n\ge1}$ of open dense sets the intersection $\cap_{n\ge1} U_n$ is also dense. In fact, if (U_n) is a sequence of open dense sets, the sets $F_n = U_n^c$ are closed and $\operatorname{int} F_n = \emptyset \Longleftrightarrow \emptyset = \operatorname{int}(U_n^c) = \left(\overline{U_n}\right)^c \Longleftrightarrow \overline{U_n} = X$, so that

$$\operatorname{int}(\cup_{n\ge1} F_n) = \emptyset \Longleftrightarrow \emptyset = \operatorname{int}(\cup_{n\ge1} U_n^c) = \operatorname{int}\left((\cap_{n\ge1} U_n)^c\right) = \left(\overline{(\cap_{n\ge1} U_n)}\right)^c$$
$$\Longleftrightarrow \overline{(\cap_{n\ge1} U_n)} = X.$$

Proof of the theorem. Let $(U_n)_{n\ge1}$ be a sequence of dense open sets. Let $x_0 \in X, r_0 > 0$ (we may assume that X is not empty, otherwise the theorem is trivial). Using the lemma 1.1.7 and the density of U_1, we obtain $B(x_0, r_0) \cap U_1 \ne \emptyset$ so that

$$\exists r_1 \in]0, r_0/2[, \quad B(x_0, r_0) \cap U_1 \supset B(x_1, 2r_1) \supset \tilde{B}(x_1, r_1) = \{y \in X, d(y, x_1) \le r_1\}.$$

Let us assume that we have constructed $x_0, x_1, \dots, x_n$ with $n \ge 1$ such that

$$B(x_k, r_k) \cap U_{k+1} \supset \tilde{B}(x_{k+1}, r_{k+1}), \quad 0 < r_{k+1} < r_k/2, \quad 0 \le k \le n-1.$$

Using the density of U_{n+1}, we obtain $B(x_n, r_n) \cap U_{n+1} \ne \emptyset$ and

$$\exists r_{n+1} \in]0, r_n/2[, \quad B(x_n, r_n) \cap U_{n+1} \supset B(x_{n+1}, 2r_{n+1}) \supset \tilde{B}(x_{n+1}, r_{n+1}).$$

Since $0 < r_n \le 2^{-n} r_0$ (induction), we have $\lim_n r_n = 0$ and $(x_n)_{n\ge 0}$ is a Cauchy sequence since for $k, l \ge n$,

$$B(x_k, r_k) \cup B(x_l, r_l) \subset B(x_n, r_n) \Longrightarrow d(x_k, x_l) < 2r_n.$$

Since the metric space X is assumed to be complete, the sequence $(x_n)_{n\ge 0}$ converges; let $x = \lim_n x_n$. We have for all $n \ge 0$, $\tilde{B}(x_{n+1}, r_{n+1}) \subset B(x_n, r_n)$ so that, for all $k \ge 1$, $\tilde{B}(x_{n+k}, r_{n+k}) \subset B(x_n, r_n)$ and thus

$$\sup_{k\ge 0} d(x_{n+k}, x_n) \le r_n \Longrightarrow d(x, x_n) \le r_n \Longrightarrow x \in \cap_{n\ge 1}\tilde{B}(x_n, r_n) \subset \cap_{n\ge 1} U_n$$

and $d(x, x_0) \le r_0$. As a result, for all $x_0 \in X$, all $r_0 > 0$, the set $\tilde{B}(x_0, r_0) \cap \cap_{n\ge 1} U_n \neq \emptyset$. This implies that $U = \cap_{n\ge 1} U_n$ is dense since, for $x_0 \in X$, for any neighborhood V of x_0, there exists $r_0 > 0$ such that $V \supset B(x_0, 2r_0) \supset \tilde{B}(x_0, r_0)$, and thus

$$V \cap U \supset \tilde{B}(x_0, r_0) \cap U \neq \emptyset \Longrightarrow x_0 \in \overline{U}. \qquad \square$$

Theorem 2.1.2. *Let X be a locally compact topological space (see the definition 1.5.11) and $(F_n)_{n\ge 1}$ be a sequence of closed sets with empty interiors. Then the interior of $\cup_{n\ge 1} F_n$ is also empty.*

Proof. The proof is essentially the same as for the previous theorem. Let $(U_n)_{n\ge 1}$ be a sequence of dense open sets. Let B_0 a non-empty open subset of X. Since U_1 is dense, the open set $B_0 \cap U_1$ is non-empty and thus is a neighborhood of a point. From the proposition 1.5.12, $B_0 \cap U_1$ contains a compact set with non-empty interior and thus

$$B_0 \cap U_1 \supset \overline{B_1}, \quad \overline{B_1} \text{ compact}, B_1 \text{ open} \neq \emptyset.$$

We get that $B_1 \cap U_2$ is a non-empty open set which contains a compact $\overline{B_2}$, B_2 open $\neq \emptyset$. Following the same procedure as in the previous proof, we may consider the compact set K defined by $K = \cap_{n\ge 1} \overline{B_n}$. The set K is non-empty, otherwise the proposition 1.5.3 would imply that $\emptyset = \cap_{1\le n\le N} \overline{B_n} = \overline{B_N}$ for some N, which is not possible since at each step, the set $\overline{B_N}$ is compact with non-empty interior. As a result, we have

$$\emptyset \neq K \subset \cap_{n\ge 1} U_n = U, \quad K \subset B_0,$$

and thus, for any open subset B_0 of X, the set $U \cap B_0 \neq \emptyset$, which means that $\overline{U} = X$. $\square$

Definition 2.1.3. Let X be a topological space and $A \subset X$.

· The subset A is said to be *rare* or *nowhere dense* when $\overset{\circ}{\overline{A}} = \emptyset$.
· The subset A is of *first category* when it is a countable union of rare subsets. Such a subset is also said to be *meager.*
· The subset A of X is of *second category* when it is not of first category.

A topological space X is a *Baire space* if for any sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets with empty interiors, the union $\cup_{n\in\mathbb{N}} F_n$ is also with empty interior. As shown above, X is a *Baire space* if and only if, for any sequence $(U_n)_{n\in\mathbb{N}}$ of dense open sets, the intersection $\cap_{n\in\mathbb{N}} U_n$ is also dense.

Remark 2.1.4. We have proven that a metric complete space, as well as a locally compact space are both Baire spaces.

Note that $\mathbb{Q}$ is a meager subset of $\mathbb{R}$, thus of first category in $\mathbb{R}$, i.e. "small" in the sense of category but $\mathbb{Q}$ is dense in $\mathbb{R}$.

The Cantor set is a compact space, and so is of second category in itself, but it is of first category in the interval [0, 1] with the usual topology: in fact defining for a compact interval $J = [a, b]$ the intervals $J_0 = [a, a + \frac{b-a}{3}]$ and $J_2 = [b - \frac{b-a}{3}, b]$, we define

$$\begin{aligned} &K_0 = [0, 1] = I, \\ &K_1 = [0, 1/3] \cup [2/3, 1] = I_0 \cup I_2, \quad && |K_1| = 2 3^{-1}, \\ &K_2 = I_{00} \cup I_{02} \cup I_{20} \cup I_{22}, && |K_2| = 2^2 3^{-2}, \\ &\dots \\ &K_n = \cup_{\alpha \in \{0,2\}^n} I_\alpha, && |K_n| = 2^n 3^{-n}, \end{aligned}$$

and the Cantor set is $\mathcal{C} = \cap_{n\ge1} K_n$ so that $\operatorname{int} \mathcal{C} = \emptyset$ since $|\mathcal{C}| \le \inf_n 2^n 3^{-n} = 0$. As a consequence $\mathcal{C}$ is rare in $[0, 1]$.

Here is an example of a set of second category in $\mathbb{R}$, i.e. "large" in the sense of category, but with Lebesgue measure 0 (small in the sense of the Lebesgue measure). We define for $\mathbb{Q} = \{x_n\}_{n\ge1}$,

$$A = \cap_{m\ge1} U_m, \quad U_m = \cup_{n\ge1}]x_n - 2^{-n-m}, x_n + 2^{-n-m}[.$$

The Lebesgue measure $|A|$ is such that

$$|A| \le \inf_{m\ge1} \sum_{n\ge1} 2^{1-n-m} = \inf_{m\ge1} 2^{-m+1} = 0.$$

If A were meager,we would have a sequence (A_k) of subsets of $\mathbb{R}$ with $\operatorname{int}(\overline{A_k}) = \emptyset$, so that

$$\mathbb{R} = A \cup A^c = \cup_k A_k \cup A^c = \cup_k \overline{A_k} \cup A^c = \cup_k \overline{A_k} \cup \cup_m U_m^c.$$

We note that $\operatorname{int}(U_m^c) = \emptyset$ since $\overline{U_m} \supset \overline{\mathbb{Q}} = \mathbb{R}$. We would have written $\mathbb{R}$ as a countable union of closed sets with empty interiors: this is not possible from the Baire theorem.

To convince the reader that the notions of size given respectively by the Lebesgue measure and by the category are unrelated, we can also give an example of a set of first category, "small" in the sense of category, but with full Lebesgue measure in $[0, 1]$. Let us assume that for any integer $k \ge 1$, we can construct a compact subset $\mathscr{C}_k$ of $[0, 1]$ such that

$$\operatorname{int}(\mathscr{C}_k) = \emptyset, \quad |\mathscr{C}_k| \ge \frac{k-1}{k}.$$

We define then $A = \cup_{k\ge1} \mathscr{C}_k$ and we have $|A| \ge \sup_{k\ge1} |\mathscr{C}_k| \ge \sup_{k\ge1} (1 - \frac{1}{k}) = 1$. Moreover, A is obviously of first category as a countable union of compact sets with empty interior. The remaining question: how construct such a $\mathscr{C}_k$? We can modify the construction of the Cantor set $\mathscr{C}$ above as follows. Let $k \geq 1$ be given and $\epsilon_0 = 1/k$. We define

$$\begin{aligned}
&K_0 = [0,1] = I, \\
&K_1 = I_0 \cup I_2,\ I_0 = [0, \frac{1}{2} - \frac{\epsilon_0}{4}],\ I_2 = [\frac{1}{2} + \frac{\epsilon_0}{4}, 1], \quad |K_1^c| = \frac{\epsilon_0}{2}, \\
&K_2 = I_{00} \cup I_{02} \cup I_{20} \cup I_{22}, \qquad |K_2^c| = 2^2 \frac{\epsilon_0}{2^4} + \frac{\epsilon_0}{2}, \\
&\dots \\
&K_n = \cup_{\alpha \in \{0,2\}^n} I_\alpha, \qquad |K_n^c| = 2^n \frac{\epsilon_0}{2^{2n}} + \cdots + 2^l \frac{\epsilon_0}{2^{2l}} + \cdots + \frac{\epsilon_0}{2},
\end{aligned}$$

so that $\mathscr{C}_k = \cap_{n \geq 1} K_n$ is compact, and by the Beppo Levi theorem, we have $|\mathscr{C}_k| = \lim_n |K_n| = \lim_n \big(1 - \epsilon_0(1 - 2^{-n})\big) = 1 - \epsilon_0 = 1 - 1/k$. Moreover, we have $\operatorname{int}(\mathscr{C}_k) = \emptyset$ since no non-empty open interval can be included in $\mathscr{C}_k$: if

$$\forall n \geq 1, \quad]x_0 - r_0, x_0 + r_0[\subset K_n = \cup_{\alpha \in \{0,2\}^n} I_\alpha,$$

then, since the I_α are disjoint intervals,

$$\forall n \geq 1, \exists \alpha_n \in \{0,2\}^n, \quad]x_0 - r_0, x_0 + r_0[\subset I_{\alpha_n}.$$

However the common length l_n of I_α is such that $2^n l_n \leq 1$ so that $\lim_n l_n = 0$.

2.1.2 The Banach-Steinhaus theorem

Let us begin with some elementary facts about linear mappings between Banach spaces.

Proposition 2.1.5. *Let E, F be normed vector spaces and $\mathcal{L}(E, F)$ be the vector space of continuous linear mappings from E into F. A linear mapping L from E to F belongs to $\mathcal{L}(E, F)$ if and only if*

$$\exists C > 0, \forall u \in E, \quad \|Lu\|_F \leq C\|u\|_E. \tag{2.1.1}$$

On the vector space $\mathcal{L}(E, F)$, we define the norm

$$\|L\| = \sup_{u \in E, \|u\|_E = 1} \|Lu\|_F. \tag{2.1.2}$$

If F is a Banach space, the vector space $\mathcal{L}(E, F)$ equipped with that norm is a Banach space.

Proof. $\mathcal{L}(E, F)$ is obviously a vector space. Moreover, if $L \in \mathcal{L}(E, F)$, the set $L^{-1}(B_F(0_F, 1))$ is open, contains 0_E and thus contains $\overline{B_E}(0_E, r_0)$ with some $r_0 > 0$. As a consequence, for $u \in E, u \neq 0$, we have

$$\|L\big(r_0 \frac{u}{\|u\|_E}\big)\|_F \leq 1, \quad \text{i.e. } \|Lu\|_F \leq r_0^{-1}\|u\|_E \quad \text{(also true for } u = 0\text{)}.$$

Conversely, if L is a linear mapping between E and F satisfying (2.1.1), then, for $\rho > 0$, we have $L^{-1}\big(B_F(0,\rho)\big) \supset B_E(0,\rho C^{-1})$, since $L(0_E) = 0_F$ and

$$0 < \|u\|_E < \rho C^{-1} \Longrightarrow \|Lu\|_F \le C\|u\|_E < \rho.$$

A a result, L is continuous at 0 and since it is a linear mapping, it is continuous everywhere: to check the continuity at u_0, we note that $u \mapsto Lu$ is the composition $u \mapsto u - u_0 \mapsto L(u-u_0) \mapsto Lu$, where the first and last mappings are translations, which are homeomorphisms. The formula (2.1.2) is well-defined on $\mathcal{L}(E,F)$, is obviously homogeneous and separated. Let $L_1, L_2 \in \mathcal{L}(E,F)$: for $u \in E$, we have

$$\|(L_1 + L_2)u\|_E \le \|L_1 u\|_E + \|L_2 u\|_E \le (\|L_1\| + \|L_2\|)\|u\|_E$$

and the triangle inequality follows. Assuming that F is a Banach space, we consider a Cauchy sequence $(L_k)_{k\ge1}$ in $\mathcal{L}(E,F)$. For each $u \in E$, the sequence $(L_k u)_{k\ge1}$ is a Cauchy sequence in the Banach space F since $\|L_k u - L_l u\|_F \le \|L_k - L_l\|\|u\|_E$, so that we can define

$$Lu = \lim_k L_k u.$$

We note also that the numerical sequence $(\|L_k\|)_{k\ge1}$ is a Cauchy sequence since, by the triangle inequality[1] we get $|\|L_k\| - \|L_l\|| \le \|L_k - L_l\|$, and thus $(\|L_k\|)_{k\ge1}$ is bounded. The mapping L is obviously linear and satisfies, for $u \in E$,

$$\|Lu\|_F \le \|Lu - L_k u\|_F + \|L_k u\|_F \le \|Lu - L_k u\|_F + \|u\|_E \sup_{k\ge1} \|L_k\|$$

and thus $\|Lu\|_F \le \|u\|_E \sup_{k\ge1} \|L_k\|$, so that $L \in \mathcal{L}(E,F)$. We check now, for $u \in E$

$$\|(L_k - L)u\|_E = \lim_l \|(L_k - L_l)u\|_E \le \|u\|_E \limsup_l \|L_k - L_l\| = \epsilon(k)\|u\|_E, \ \lim_k \epsilon(k) = 0.$$

As a consequence, $\|L - L_k\| = \sup_{\|u\|_E=1} \|(L_k - L)u\|_E \le \epsilon(k)$ and the sequence $(L_k)_{k\ge1}$ converges to L in the normed space $\mathcal{L}(E,F)$. □

Theorem 2.1.6 (Banach-Steinhaus Theorem, Principle of Uniform Boundedness)**.** *Let E be a Banach space, F be a normed vector space and $(L_j)_{j\in J}$ be a family of $\mathcal{L}(E,F)$ which is "weakly bounded", i.e. satisfies*

$$\forall u \in E, \quad \sup_{j\in J} \|L_j u\|_F < +\infty. \tag{2.1.3}$$

Then the family $(L_j)_{j\in J}$ is "strongly bounded", i.e. satisfies

$$\sup_{j\in J} \|L_j\|_{\mathcal{L}(E,F)} < +\infty. \tag{2.1.4}$$

Proof. We consider for $n \in \mathbb{N}^*$, the set $F_n = \{u \in E, \sup_{j\in J} \|L_j u\|_F \le n\}$. From the assumption of the theorem, we have $E = \cup_{n\in\mathbb{N}} F_n$. Moreover each F_n is closed: let $(u_k)_{k\ge1}$ be a sequence of elements of F_n converging with limit u. For all $j \in J$,

[1] $\|L_k\| \le \|L_k - L_l\| + \|L_l\| \Longrightarrow \|L_k\| - \|L_l\| \le \|L_k - L_l\| \Longrightarrow |\|L_k\| - \|L_l\|| \le \|L_k - L_l\|$.

L_j is continuous and thus $\lim_k L_j u_k = L_j u$. By the continuity of the norm[2], we get $\lim_k \|L_j u_k\|_F = \|L_j u\|_F$, and since $\|L_j u_k\| \le n$, we get $\|L_j u\| \le n$ for all $j \in J$ and $u \in F_n$. Note also that F_n is symmetric and convex: for $u_0, u_1 \in F_n$, $u_\theta = (1-\theta)u_0 + \theta u_1$, $\theta \in [0,1]$, we have $u_\theta \in F_n$ since

$$\|L_j u_\theta\| \le (1-\theta)\|L_j u_0\| + \theta\|L_j u_1\| \le (1-\theta)n + \theta n = n.$$

Applying the Baire theorem to the Banach space E, we see that there must exist some $n_0 \in \mathbb{N}^*$ such that $\operatorname{int}(F_{n_0}) \ne \emptyset$. In other words, F_{n_0} should contain an interior point u_0, and since F_{n_0} is symmetric, $-u_0$ is also an interior point, as well as the whole segment $[-u_0, u_0]$ by convexity. As a consequence, 0 is an interior point of F_{n_0}. This implies that there exists $\rho_0 > 0$ such that $\overline{B}(0, \rho_0) \subset F_{n_0}$, i.e.

$$\|u\| \le \rho_0 \Longrightarrow \sup_{j \in J} \|L_j u\| \le n_0, \quad \text{so that } \forall j \in J, \forall u \ne 0, \quad \|L_j \rho_0 \frac{u}{\|u\|}\| \le n_0,$$

implying that $\forall j \in J, \|L_j u\| \le \frac{n_0}{\rho_0}\|u\|$ and thus $\forall j \in J, \|L_j\| \le \frac{n_0}{\rho_0}$ which is (2.1.4). The proof of the theorem is complete. □

We can prove the same theorem in a much more general context than the framework of Banach spaces. We shall limit ourselves to the case of Fréchet spaces, which are complete metric spaces whose topology is defined by a countable separating family of semi-norms $(p_k)_{k \ge 1}$ (see the definition 1.3.10). There is no loss of generality to assume that the sequence p_k is non-decreasing , i.e. for all $u \in E, k \ge 1, p_k(u) \le p_{k+1}(u)$, since we may replace the semi-norm p_k by the semi-norm $\sum_{1 \le j \le k} p_j$.

Let us recall that for E, F topological vector space, $\mathcal{L}(E, F)$ is the vector space of continuous linear mappings from E into F; when $E = F$, we shall write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$. When the topology on E and F is given by a countable separating family of (non-decreasing) semi-norms $(p_k)_{k \ge 1}$ on E and $(q_l)_{l \ge 1}$ on F, the continuity of a linear mapping L from E to F is equivalent to

$$\forall l \ge 1, \exists k \ge 1, \exists C > 0, \forall u \in E \quad q_l(Lu) \le C p_k(u). \tag{2.1.5}$$

In fact, since L is linear, its continuity is equivalent to the continuity at 0. If L is continuous at 0, $l \ge 1$, the set $L^{-1}(\{v \in F, q_l(v) < 1\})$ is open, contains 0_E and thus contains $\{u \in E, p_k(u) \le r_0\}$ with some $k \ge 1, r_0 > 0$. As a consequence, for $u \in E, \epsilon > 0$, we have

$$q_l\Big(L\big(r_0 \frac{u}{p_k(u) + \epsilon}\big)\Big) \le 1, \quad \text{i.e. } \forall \epsilon > 0, \forall u \in E,\ q_l(Lu) \le r_0^{-1}\big(p_k(u) + \epsilon\big),$$

which gives (2.1.5), $q_l(Lu) \le r_0^{-1} p_k(u)$. Conversely, if L is a linear mapping between E and F satisfying (2.1.5), then, for $l \ge 1, \rho > 0$, we have

$$L^{-1}\big(\{v \in F, q_l(v) \le \rho\}\big) \supset \{u \in E, p_k(u) \le \rho C^{-1}\}$$

since $p_k(u) \le \rho C^{-1} \Longrightarrow q_l(Lu) \le C p_k(u) \le \rho$. As a result, L is continuous at 0.

[2]Let E be a normed space. The mapping $E \ni x \mapsto \|x\| \in \mathbb{R}_+$ is Lipschitz continuous: we have already seen in the previous footnote that the triangle inequality implies $|\|x_1\| - \|x_2\|| \le \|x_1 - x_2\|$.

A subset B of a topological vector space E is said to be *bounded* if

$$\forall U \in \mathscr{V}_0, \exists s > 0, \forall t \geq s, \quad B \subset tU. \tag{2.1.6}$$

When the topology of E is given by a countable separating family of semi-norms $(p_k)_{k\geq 1}$, it follows from the lemma 1.3.9 that a subset B is bounded when

$$\forall k \geq 1, \quad \sup_{u\in B} p_k(u) < +\infty. \tag{2.1.7}$$

A family $\mathcal{F} \subset \mathcal{L}(E, F)$ is equicontinuous when

$$\forall l \geq 1, \exists k \geq 1, \exists C > 0, \forall L \in \mathcal{F}, \forall u \in E, \quad q_l(Lu) \leq Cp_k(u). \tag{2.1.8}$$

Theorem 2.1.7 (Principle of Uniform Boundedness)**.** *Let E, F be topological vector spaces whose topology is given by a countable separating (non-decreasing) family of semi-norms, and assume that E is a Fréchet space. Let $(L_j)_{j\in J}$ be a family of $\mathcal{L}(E, F)$ which is "weakly bounded", i.e. satisfies*

$$\forall u \in E, \quad \{L_j u\}_{j\in J} \text{ is bounded in } F. \tag{2.1.9}$$

Then the family $\{L_j\}_{j\in J}$ is "strongly bounded", i.e. satisfies

$$\forall B \text{ bounded of } E, \quad \cup_{j\in J} L_j(B) \text{ is bounded in } F \tag{2.1.10}$$

and the family $(L_j)_{j\in J}$ is equicontinuous.

Proof. Let B_0 be a bounded subset of E, $l_0 \geq 1$. Since, for all $u \in E$, $\{L_j u\}_{j\in J}$ is bounded in F, we have

$$\forall u \in E, \ \sup_{j\in J} q_{l_0}(L_j u) < +\infty.$$

As a consequence, we have $E = \cup_{n\geq 1} F_n$, $F_n = \{u \in E, \sup_{j\in J} q_{l_0}(L_j u) \leq n\}$. Moreover each F_n is closed: let $(u_k)_{k\geq 1}$ be a sequence of elements of F_n converging with limit u. For all $j \in J$, L_j is continuous and thus $\lim_k L_j u_k = L_j u$. By the continuity of the semi-norm[3] we get $\lim_k q_{l_0}(L_j u_k) = q_{l_0}(L_j u)$, and since $q_{l_0}(L_j u_k) \leq n$, we get $q_{l_0}(L_j u) \leq n$ for all $j \in J$ and $u \in F_n$. Note also that F_n is symmetric and convex: for $u_0, u_1 \in F_n$, $u_\theta = (1-\theta)u_0 + \theta u_1$, $\theta \in [0, 1]$, we have $u_\theta \in F_n$ since

$$q_{l_0}(L_j u_\theta) \leq (1-\theta) q_{l_0}(L_j u_0) + \theta q_{l_0}(L_j u_1) \leq (1-\theta)n + \theta n = n.$$

Applying the Baire theorem to the complete metric space E, we see that there must exist some $n_0 \in \mathbb{N}^*$ such that $\operatorname{int}(F_{n_0}) \neq \emptyset$. In other words, F_{n_0} should contain an interior point u_0, and since F_{n_0} is symmetric, $-u_0$ is also an interior point, as well as the whole segment $[-u_0, u_0]$ by convexity. As a consequence, 0 is an interior point of F_{n_0}. This implies that there exists $\rho_0 > 0, k_0 \geq 1$ such that

$$p_{k_0}(u) \leq \rho_0 \Longrightarrow \sup_{j\in J} q_{l_0}(L_j u) \leq n_0,$$

[3] The triangle inequality for a semi-norm q implies $|q(v_1) - q(v_2)| \leq q(v_1 - v_2)$.

so that

$$\forall j \in J, \forall u \in E, \forall \epsilon > 0, \quad q_{l_0}\Big(L_j \rho_0 \big(\frac{u}{p_{k_0}(u) + \epsilon}\big)\Big) \le n_0,$$

implying that

$$\forall j \in J, \forall u \in E, \forall \epsilon > 0, \quad q_{l_0}(L_j u) \le \frac{n_0}{\rho_0}\big(p_{k_0}(u) + \epsilon\big)$$

and thus $\forall j \in J, \forall u \in E, \quad q_{l_0}(L_j u) \le n_0 p_{k_0}(u)/\rho_0$, which is the equicontinuity (2.1.8). Since the bounded set B_0 satisfies $\sup_{u \in B_0} p_{k_0}(u) = M_0 < +\infty$, we have

$$B_0 \subset \{u \in E, \sup_{j \in J} q_{l_0}(L_j u) \le M_0 n_0/\rho_0\}$$

which implies that

$$\cup_{j \in J} L_j(B_0) \subset \{v, q_{l_0}(v) \le M_1\}, \quad \text{with } M_1 = M_0 n_0/\rho_0,$$

so that $\sup_{v \in \cup_{j \in J} L_j(B_0)} q_{l_0}(v) < +\infty$ and $\cup_{j \in J} L_j(B_0)$ is indeed bounded. The proof of the theorem is complete. □

Corollary 2.1.8. *Let E be a Fréchet space, F be a topological vector space whose topology is defined by a countable separating family of semi-norms and $(L_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{L}(E, F)$ such that, for all $u \in E$, the sequence $(L_n u)_{n \in \mathbb{N}}$ converges in F. Then defining L on E by $Lu = \lim_n L_n u$, we obtain that $L \in \mathcal{L}(E, F)$.*

Proof. The mapping L is obviously linear, and the previous theorem implies that the sequence $(L_n)_{n \in \mathbb{N}}$ is equicontinuous, i.e.

$$\forall l \ge 1, \exists k \ge 1, \exists C > 0, \forall n \in \mathbb{N}, \forall u \in E, \quad q_l(L_n u) \le C p_k(u).$$

The continuity of q_l and the convergence of $(L_n u)_{n \in \mathbb{N}}$ imply that

$$\forall l \ge 1, \exists k \ge 1, \exists C > 0, \forall u \in E, \quad q_l(Lu) \le C p_k(u),$$

which is the continuity of L. □

N.B. The previous theorem can be proven for more general spaces than the Fréchet spaces, and in particular the local convexity does not play any rôle in the proof; nevertheless it is important that the topological vector space E is a Baire space and F is a Hausdorff TVS. On the other hand, the previous corollary will be very useful for distribution theory; at this moment one may simply point out that the type of convergence of the L_n is indeed very weak ("simple" convergence), and the fact that the continuity is not lost in the limiting process is an important consequence of the Baire theorem, proving that a simple limit of continuous linear mappings (say between Fréchet spaces) is still continuous.

Remark 2.1.9. Another point is concerned with the notion of *boundedness* in a topological vector space, as given by the definition (2.1.6). That notion is pretty obvious in a Banach space but more subtle, even in a Fréchet space, where it is given by (2.1.7). In particular, in a locally convex Hausdorff space E (such as a Fréchet space), no neighborhood of 0 is bounded unless E is normable. It is not difficult to see that, in a general TVS, compact subsets are bounded.

A possible collision – and confusion – of terminology exists around the word boundedness: that word can be used with a different meaning in a metric space (not necessarily a vector space). In a metric space (X, d) a subset A is said to be d-bounded if its diameter is finite, i.e. if $\sup_{x,y\in A} d(x, y) < +\infty$. Now if (X, d) is a metric topological vector space, the notions of boundedness and d-boundedness may differ. Consider for instance the real line $\mathbb{R}$ with the metric (which is translation invariant) $d(x, y) = \frac{|x-y|}{1+|x-y|}$: $\mathbb{R}$ itself is d-bounded and not bounded.

Anyhow, when we deal with a topological vector space E, we shall stick with the TVS definition of boundedness, as given by (2.1.6), even if E is a metric TVS.

2.1.3 The open mapping theorem

Theorem 2.1.10. *Let E, F be Banach spaces and let A be a bijective mapping belonging to $\mathcal{L}(E, F)$. Then A is an isomorphism, i.e.*

$$\exists \beta, \gamma > 0, \quad \forall u \in E, \quad \beta \|u\|_E \le \|Au\|_F \le \gamma \|u\|_E. \tag{2.1.11}$$

Proof. First of all, we note that since $A \in \mathcal{L}(E, F)$, the second inequality in (2.1.11) is satisfied. Moreover, as A is bijective, the inverse mapping A^{-1} must be shown to be continuous, i.e. $\|A^{-1}v\|_E \le C\|v\|_F$ which is equivalent[4] to the first inequality in (2.1.11). To prove this, we first define for $N \in \mathbb{N}^*$ the set

$$\Phi_N = \overline{A\big(B_E(0, N)\big)}.$$

We note that each Φ_N is closed (and also symmetric and convex) and that $E = \cup_{N\in\mathbb{N}^*}\Phi_N$ since A is onto. Using the Baire theorem, we find $N_0 \ge 1$ such that Φ_{N_0} contains an interior point, and since Φ_{N_0} is symmetric and convex, 0 is indeed an interior point so that

$$\exists R_0 > 0, \quad \overline{B_F(0, R_0)} \subset \overline{A\big(B_E(0, N_0)\big)}.$$

Defining $A_0 = N_0 R_0^{-1} A$, which is a bijective mapping of $\mathcal{L}(E, F)$, we have also $\overline{B_F(0, 1)} \subset \overline{A_0\big(B_E(0, 1)\big)}$. Let v_0 be in the closed unit ball of F: from the previous inclusion, we can find $u_0 \in \overline{B_E(0, 1)}$ such that

$$\|v_0 - A_0u_0\|_F \le 1/2, \quad \text{so that } \exists u_1 \in \overline{B_E(0, 1)} \text{ with } \|2(v_0 - A_0u_0) - A_0u_1\| \le 1/2$$

which means that we have found $u_0, u_1 \in \overline{B_E(0, 1)}$ with $\|v_0 - A_0u_0 - A_0 2^{-1}u_1\| \le 2^{-2}$. Inductively, if we assume that we can find $u_0, u_1, \dots, u_n \in \overline{B_E(0, 1)}$ such that

[4] A mapping in $\mathcal{L}(E, F)$ may satisfy (2.1.11) without being an isomorphism: the shift operator S, defined on $\ell^2(\mathbb{N})$ by

$$S((u_n)_{n\ge0}) = (v_n)_{n\ge0}, \quad v_0 = 0, \quad v_n = u_{n-1}, \text{ for } n \ge 1,$$

is bounded from $\ell^2(\mathbb{N})$ into itself and even isometric since $\|Su\| = \|u\|$, but is not onto since the sequence $(\delta_{0,n})_{n\ge0}$ is not in its range.

$\|v_0 - \sum_{0\le k\le n} A_0 2^{-k} u_k\| \le 2^{-n-1}$, it is also possible to find $u_{n+1} \in \overline{B_E(0,1)}$ such that

$$\|2^{n+1}\big(v_0 - \sum_{0\le k\le n} A_0 2^{-k} u_k\big) - u_{n+1}\| \le 2^{-1}.$$

Eventually, we can construct a sequence $(u_n)_{n\ge 0}$ in $\overline{B_E(0,1)}$ so that

$$\|v_0 - \sum_{0\le k\le n} A_0 2^{-k} u_k\| \le 2^{-n-1}. \tag{2.1.12}$$

The sequence $(U_n = \sum_{0\le k\le n} 2^{-k} u_k)_{n\ge 0}$ is a Cauchy sequence since $\|2^{-k} u_k\| \le 2^{-k}$ and thus converge. As a consequence, with $U = \lim_n U_n$, $v_n = A_0 U_n$, the continuity of A_0 and (2.1.12) give

$$v_0 = \lim_n v_n = \lim_n A_0 U_n = A_0 U \Longrightarrow v_0 = A_0 U, \|U\|_E \le 2$$

which proves that $\overline{B_F(0,1)} \subset A_0\big(\overline{B_E(0,2)}\big)$. Let us now consider $u \ne 0$ in E; then $A_0 u \ne 0$ (A_0 is bijective and thus one-to-one) and, from the previous inclusion, there exists $u' \in E$ such that

$$\overline{B_F(0,1)} \ni \frac{A_0 u}{\|A_0 u\|} = A_0 u', \quad \|u'\| \le 2 \Longrightarrow \frac{u}{\|A_0 u\|} = u' \Longrightarrow \|u\| \le 2\|A_0 u\|,$$

implying (2.1.11) with $\beta = \frac{R_0}{2N_0}$ (note that the inequality (2.1.11) is trivially satisfied for $u = 0$). The proof of the theorem is complete. □

Remark 2.1.11. Let E be a vector space, N be a subspace and $p : E \longrightarrow E/N$ the canonical mapping $p(u) = u + N$. The space E/N is a vector space with the addition $p(u_1) + p(u_2) = p(u_1 + u_2)$ and the multiplication by a scalar $\lambda p(u) = p(\lambda u)$, which are well-defined operations. If E is a Banach space and N is a closed subspace, then the quotient space E/N is a Banach space with norm

$$\|p(u)\|_{E/N} = \inf_{w\in N} \|u + w\|_E.$$

The homogeneity and triangle inequality are easy to verify and the separation follows from the fact that if $\lim_k (u + w_k) = 0, w_k \in N$ then $\lim_k w_k = -u$, so that $-u$ and thus u belong to the closure of N, which is N, ensuring $p(u) = 0_{E/N} = N$. Moreover, E/N is a Banach space: if $(u_k)_{k\ge 1}$ is a sequence of E such that

$$\sum_{k\ge 1} \|p(u_k)\|_{E/N} < +\infty,$$

we can find a sequence $(w_k)_{k\ge 1}$ in N with $\|u_k + w_k\|_E \le \|p(u_k)\|_{E/N} + 2^{-k}$, so that $\sum_{k\ge 1}(u_k + w_k)$ is a converging series in the Banach space E. As a result, there exists $v \in E$ so that

$$0 = \lim_n \|\sum_{1\le k\le n} (u_k + w_k) - v\|_E \Longrightarrow 0 = \lim_n \|\sum_{1\le k\le n} p(u_k) - p(v)\|_{E/N},$$

proving the completeness. Moreover, the mapping p is open, i.e. sends open sets onto open sets: since it is linear, it is enough to verify that $p(B_E(0,1))$ is a neighborhood of 0. But we have

$$B_{E/N}(0,1) = p(B_E(0,1))$$

since if $\|p(u)\|_{E/N} < 1$, there exists $w \in N$ such that $\|u + w\|_E < 1$ and $p(u) = p(u + w) \in p(B_E(0,1))$. Conversely, if $\|u\|_E < 1$, then $\|p(u)\|_{E/F} \le \|u\|_E < 1$.

Corollary 2.1.12.
(1) *Let E, F be Banach spaces and $A \in \mathcal{L}(E,F)$. If A is onto, then A is open.*
(2) *Let E be a vector space and let N_1, N_2 be two norms such that $(E, N_j)(j = 1,2)$ are Banach space and such that $N_1 \lesssim N_2$, i.e. $\exists C > 0, \forall u \in E, \quad N_1(u) \le C N_2(u)$. Then $N_2 \lesssim N_1$ so that*

$$\exists C > 0, \forall u \in E, \quad C^{-1} N_2(u) \le N_1(u) \le C N_2(u),$$

i.e. the two norms are equivalent.

Proof. For E, F Banach spaces and A onto $\in \mathcal{L}(E,F)$, the continuity of A implies that $\ker A$ is a closed subspace of E and, denoting by $p : E \longrightarrow E/\ker A$ the canonical mapping, the quotient mapping

$$\widetilde{A} : E/\ker A \longrightarrow F, \quad \widetilde{A}(p(u)) = Au$$

is well-defined and bijective. From the previous theorem, $\widetilde{A}$ is an isomorphism and thus is an open mapping, as well as the canonical mapping p (from the previous remark) so that $A = \widetilde{A} \circ p$ is also open, providing the first point. The second property is due to the fact that the assumption expresses that the identity map is bijective linear continuous from (E, N_2) onto (E, N_1) and thus is an isomorphism; as a result it satisfies with a positive β (and for all $u \in E$)

$$\beta N_2(u) \le N_1(\operatorname{Id} u) = N_1(u)$$

which is the sought result. □

Remark 2.1.13. That corollary can be extended far beyond the Banach space framework. In particular for complete metric TVS with translation-invariant distances (e.g. Fréchet spaces), if $A \in \mathcal{L}(E,F)$ is onto, then it is open; if A is bijective, it is an isomorphism. Moreover if E is a vector space and $\mathscr{T}_j$, $j = 1,2$ are topologies on E such that $(E, \mathscr{T}_j)$ are TVS given by complete metrics with translation-invariant distances so that $\mathscr{T}_1 \subset \mathscr{T}_2$, then $\mathscr{T}_1 = \mathscr{T}_2$.

2.1.4 The closed graph theorem

Theorem 2.1.14. *Let E, F be Banach spaces and $A : E \longrightarrow F$ be a linear map. The following properties are equivalent.*
(i) A is continuous.
(ii) The graph of A, $\Gamma_A = \{(u, Au)\}_{u \in E}$ is closed in $E \times F$.

Proof. Note that Γ_A is a vector subspace of $E \times F$ which is the range of the linear mapping L_A given by $E \ni u \mapsto (u, Au) \in E \times F$. If A is continuous, then $\|Au\|_F \le C\|u\|_E$ and the mapping L_A is also continuous since

$$\|L_A u\|_{E\times F} = \|u\|_E + \|Au\|_F \le (C+1)\|u\|_E.$$

Consequently, the range of L_A is closed, since if both $(u_k)_{k\ge 1}$ and $(Au_k)_{k\ge 1}$ are converging respectively to u, v, then, by continuity of A, $v = \lim_k Au_k = Au$. Let us show now the reverse implication, assuming that Γ_A is closed. We note that Γ_A is a Banach space, as a closed subspace of the Banach space $E \times F$ and that

$$\pi_1 : \Gamma_A \longrightarrow E, \quad \pi_1((u, Au)) = u,$$

is linear bijective and continuous since $\|\pi_1((u, Au))\|_E = \|u\|_E$: applying the open mapping theorem, we find that π_1 is an isomorphism, implying that π_1^{-1} is continuous. As a consequence, considering $\pi_2 : \Gamma_A \longrightarrow F, \quad \pi_2((u, Au)) = Au$, which is a linear continuous mapping ($\|\pi_2((u, Au))\|_F = \|Au\|_F \le \|(u, Au)\|_{E\times F}$), we have

$$A = \pi_2 \circ \pi_1^{-1} \Longrightarrow A \text{ continuous.} \qquad \square$$

2.2 The Hahn-Banach theorem

2.2.1 The Hahn-Banach theorem, Zorn's lemma

Let E be a topological vector space; we define the topological dual E^* of E as $\mathcal{L}(E, \mathtt{k})$, the vector space of linear continuous forms on E. Of course when E is finite-dimensional, the topological dual E^* is equal to the algebraic dual E', which is defined as the vector space of linear forms on E, i.e. linear mappings from E to $\mathtt{k}$. However, when E is infinite-dimensional (i.e. not finite-dimensional), we shall see that E^* is much smaller than E'. We shall devote most of our attention to describing the properties of E^*, so when we speak about the dual space of a topological vector space E, it will always mean the topological dual; if we want to deal with E', we shall speak explicitely of the algebraic dual of E. As far as notations are concerned, for a vector space E and $\xi \in E', x \in E$, we shall write $\xi \cdot x$ instead of $\xi(x)$.

Theorem 2.2.1 (The Hahn-Banach theorem)**.** *Let E be a vector space, M be a subspace of E, p be a semi-norm on E (see the definition 1.3.8), and ξ be a linear form on M such that*

$$\forall x \in M, \quad |\xi \cdot x| \le p(x). \tag{2.2.1}$$

Then there exists $\widetilde{\xi} \in E'$, such that $\widetilde{\xi}_{|M} = \xi$ and $\quad \forall x \in E,\ |\widetilde{\xi} \cdot x| \le p(x)$.

Proof. (1) *We start with the real case, i.e.* $\mathtt{k} = \mathbb{R}$. We may assume that $M \ne E$ (otherwise there is nothing to prove). Considering $x_1 \in E\backslash M$, we define

$$M_1 = M \oplus \mathbb{R}x_1, \quad (\text{note that } M \cap \mathbb{R}x_1 = \{0\})\ .$$

On M_1, we want to define a linear form ξ_1 extending ξ and still satisfying (2.2.1). Let us first remark that, if such a ξ_1 exists, we would have

$$\forall y \in M, \forall t \in \mathbb{R}, \quad \xi_1 \cdot (y \oplus t x_1) = \xi \cdot y + t\xi_1 \cdot x_1,$$

so that we have only to find a proper $\xi_1 \cdot x_1$. On the other hand, we note that for $x, y \in M$

$$\xi \cdot x - p(x - x_1) = \xi \cdot (x + y) - \xi \cdot y - p(x - x_1) \leq p(x + y) - \xi \cdot y - p(x - x_1) \\ \leq -\xi \cdot y + p(x_1 + y).$$

As a consequence, we have

$$a = \sup_{x \in M} \big(\xi \cdot x - p(x - x_1)\big) \leq \inf_{y \in M} \big(p(x_1 + y) - \xi \cdot y\big) = b.$$

Let us choose $\sigma = \xi_1 \cdot x_1 \in [a, b]$; then for $y \in M, t \in \mathbb{R}$, we define $\xi_1 \in M_1'$ by

$$\xi_1 \cdot (y + tx_1) = \xi \cdot y + t\sigma,$$

so that $\quad \xi \cdot y + \xi \cdot (-y) - p(-y - x_1) \leq \xi \cdot y + \sigma \leq \xi \cdot y + p(x_1 + y) - \xi \cdot y \quad$ and

$$-p(-x_1 - y) \leq \xi_1 \cdot (y + x_1) = \xi \cdot y + \sigma \leq p(x_1 + y) \Longrightarrow |\xi_1 \cdot (y + x_1)| \leq p(x_1 + y).$$

Now if $t \in \mathbb{R}^*$, we have for $y \in M$

$$|\xi_1 \cdot (y + tx_1)| = |t||\xi_1 \cdot (t^{-1}y + x_1)| \leq |t| p(t^{-1}y + x_1) = p(y + tx_1)$$

and since $\xi_1 \cdot y = \xi \cdot y$, we get that $\xi_1 \in M_1'$ and $\forall z \in M_1, |\xi_1 \cdot z| \leq p(z)$.

We shall resort now to a very abstract argument involving the so-called Zorn's lemma.

Lemma 2.2.2 (Zorn's lemma)**.** *Let $(X, \leq)$ be a non-empty inductive ordered set: the relation $\leq$ is an order relation*[5] *on the set X such that if Y is a totally ordered subset of X (i.e. $\forall y', y'' \in Y, y' \leq y''$ or $y'' \leq y'$), there exists $x \in X$ which is an upper bound for Y (i.e. $\forall y \in Y, y \leq x$). Then there exists a maximal element in X, i.e.*

$$\exists x_+ \in X, \forall x \in X, \quad x_+ \leq x \Longrightarrow x_+ = x.$$

Remark 2.2.3. We shall not prove that lemma, a hard piece of mathematics which can be shown to be equivalent to the axiom of choice as well as to the Zermelo's theorem: *The Axiom of Choice* says that if $(X_i)_{i \in I}$ is a family of sets such that for all $i \in I, X_i \neq \emptyset$, then the Cartesian product $\prod_{i \in I} X_i$ is not empty as well[6]. On the other hand, *Zermelo's Theorem* states that, on any set X, one can define an order relation $\leq$ which makes $(X, \leq)$ a well-ordered set, i.e. such that any non-empty subset of X has a smallest element: $\forall Y \subset X, Y \neq \emptyset, \exists y_0 \in Y, \forall y \in Y, \quad y_0 \leq y)$.

[5] $\forall x, y, z \in X, \quad x \leq x, \quad \big[x \leq y, y \leq x \Longrightarrow x = y\big], \quad \big[x \leq y, y \leq z \Longrightarrow x \leq z\big].$

[6] The Cartesian product $\prod_{i \in I} X_i$ is defined as the set of mappings x from I to $\cup_{i \in I} X_i$ such that, for all $i \in I$, $x(i) \in X_i$. A particular case of interest occurs when $\forall i \in I, X_i = X$; then we note $\prod_{i \in I} X_i = X^I$ which is the set of mappings from I to X. A more academic remark is concerned with the case when $I = \emptyset$: in that case, $\prod_{i \in \emptyset} X_i$ is not empty since it has a single element which is the mapping whose graph is the empty set. In fact the real point of the axiom of choice is concerned with the cases where I is infinite and in particular non-countable.

N.B. Obviously the set $\mathbb{N}$ of the natural integers with the usual order is indeed well-ordered, and this is the basis for the familiar *induction* reasoning; considering a sequence $(\mathscr{P}_n)_{n\in\mathbb{N}}$ of statements such that $\mathscr{P}_0$ is true and $\forall n \in \mathbb{N}, \mathscr{P}_n \Longrightarrow \mathscr{P}_{n+1}$ we define

$$S = \{n \in \mathbb{N}, \mathscr{P}_n \text{ is not true}\}.$$

If S is not empty, then it has a smallest element s_0 and necessarily $s_0 > 0$ since $\mathscr{P}_0$ is true; as a consequence $s_0 - 1 \in S^c$, so that $\mathscr{P}_{s_0-1}$ is true, implying that $\mathscr{P}_{s_0}$ is true, contradicting $s_0 \in S$. As a result, S should be empty and $\mathscr{P}_n$ is true for all $n \in \mathbb{N}$. In some sense, Zorn's lemma, or the principle of transfinite induction could be used in a similar way to handle a non-countable family of statements satisfying properties analogous to those of the countable family mentioned above. Of course, it is not difficult to equip a countable set X with an order relation which makes it a well-ordered set: it suffices to use the bijection with a subset of $\mathbb{N}$. However, the set $\mathbb{Q}$ of rational numbers (which is countable), with the standard order is *not* a well-ordered set; consider for instance $T = \{x \in \mathbb{Q}_+, x^2 \ge 2\}$, a set which is bounded from below without a smallest element (exercise). This means that to construct an order relation on $\mathbb{Q}$ which makes it a well-ordered set, one has to use a different order than the classical one and, for instance, one may use an explicit bijection between $\mathbb{Q}$ and $\mathbb{N}$ (exercise). The real difficulties begin when you want to construct an order relation on $\mathbb{R}$ which makes it a well-ordered set; naturally, one cannot use the standard order, e.g. since $]0, 1]$ does not have a smallest element, although it has the greatest lower bound 0. So the construction of that order relation has no relationship with the standard order on the real line and is in fact a result of set theory, dealing with order relations on $\mathscr{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$.

Let us now go back to the proof of our theorem. We consider the set

$$\mathcal{X} = \big\{(N,\eta), N \text{ vector subspace of } E,\ N \supset M,\ \eta \in N', \eta_{|_M} = \xi,\ \forall x \in N, |\eta \cdot x| \le p(x)\big\}$$

with the order relation $(N_1, \eta_1) \le (N_2, \eta_2)$ meaning $N_1 \subset N_2, \quad \eta_{2|_{N_1}} = \eta_1$. It is a matter of routine left to the reader to check that it is an order relation. Let us now consider a totally ordered subset $\mathcal{Y} = \{(N_i, \eta_i)\}_{i\in I}$ of $\mathcal{X}$. We define

$$N = \cup_{i\in I} N_i.$$

We have $N \supset M$ and N is a vector subspace of E since for $x, y \in N$, there exists $i, j \in I$ such that $x \in N_i, y \in N_j$ and the total order property implies that $N_i \subset N_j$ or $N_j \subset N_i$ in such a way that for $\lambda, \mu \in \mathbb{R}$, $\lambda x + \mu y \in N_i \cup N_j \subset N$. We note now that, if $x \in N_i \cap N_j$, since $N_i \subset N_j, \eta_{j|_{N_i}} = \eta_i$ or $N_j \subset N_i, \eta_{i|_{N_j}} = \eta_j$ we have

$$\eta_i \cdot x = \eta_j \cdot x$$

and we may define η on N so that $\eta \cdot x = \eta_i \cdot x$, if $x \in N_i$. We verify that $\eta \in N'$, since for $x, y \in N$, there exists $i \in I$ so that $x, y \in N_i$ and the linearity of η follows from the linearity of η_i. Moreover the very definition of η ensures that $\eta_{|_M} = \xi$ and

$$\forall x \in N, \exists i \in I, x \in N_i, \quad \eta \cdot x = \eta_i \cdot x \Longrightarrow |\eta \cdot x| = |\eta_i \cdot x| \le p(x),$$

so that (N, η) belongs to $\mathcal{X}$ and is indeed an upper bound for $\mathcal{Y}$, proving that $(\mathcal{X}, \leq)$ is an inductive set. Since $\mathcal{X}$ contains (M, ξ), it is non-empty and thus, applying Zorn's lemma, it has a maximal element (N, η). If N were different from E, the construction of the beginning of the proof would provide an element $(N_1, \eta_1) \in \mathcal{X}$ strictly larger than (N, η), a situation which is not compatible with the status of maximal element of (N, η). Finally we have proven that $N = E$, which gives the result of the theorem in the real case.

(2) *We tackle now the complex case* $\mathtt{k} = \mathbb{C}$. We define for $x \in M$,

$$u \cdot x = \frac{1}{2}(\xi \cdot x + \overline{\xi \cdot x})$$

which is an $\mathbb{R}$-linear mapping from M (which can be seen also as a real vector space) to $\mathbb{R}$. We have for all $x \in M$, $|u \cdot x| \leq p(x)$. Applying the now proven result for the real case, we can find an extension v of u, $\mathbb{R}$-linear from E to $\mathbb{R}$ such that $\forall x \in E, |v \cdot x| \leq p(x)$. Let us now define for $x \in E$,

$$\eta \cdot x = v \cdot x - i\big(v \cdot (ix)\big).$$

The mapping η is $\mathbb{C}$-linear since if $z = a + ib, a, b \in \mathbb{R}, x, y \in E$,

$$\begin{aligned}
\eta \cdot (zx) &= v \cdot (zx) - i\big(v \cdot (izx)\big) \\
&= v \cdot (ax + ibx) - i\big(v \cdot (iax - bx)\big) \\
&= a(v \cdot x) + b(v \cdot (ix)) - ia(v \cdot (ix)) + ib(v \cdot x) \\
&= (a + ib)(v \cdot x) - i(a + ib)(v \cdot (ix)) \\
&= z\big(\eta \cdot x\big),
\end{aligned}$$

and moreover $\eta \cdot (x + y) = \eta \cdot x + \eta \cdot y$ by $\mathbb{R}$ linearity. We also have $\eta_{|M} = \xi$ since for $x \in M$, ix also belongs to M and since v extends u and ξ is $\mathbb{C}$-linear, we get

$$\begin{aligned}
\eta \cdot x = v \cdot x - i\big(v \cdot (ix)\big) &= u \cdot x - i\big(u \cdot (ix)\big) = \operatorname{Re}(\xi \cdot x) - i \operatorname{Re}(\xi \cdot (ix)) \\
&= \operatorname{Re}(\xi \cdot x) - i \operatorname{Re}(i(\xi \cdot x)) = \operatorname{Re}(\xi \cdot x) + i \operatorname{Im}(\xi \cdot x) = \xi \cdot x.
\end{aligned}$$

Last, we check $|\eta \cdot x|$ for $x \in E$. We have $|\eta \cdot x| = e^{i\theta_x}(\eta \cdot x)$ so that, since v is real-valued,

$$|\eta \cdot x| = \eta \cdot (e^{i\theta_x} x) = \underbrace{v \cdot (e^{i\theta_x} x)}_{\in \mathbb{R}} - i \underbrace{\big(v \cdot (ie^{i\theta_x} x)\big)}_{\in \mathbb{R}}$$

and since $|\eta \cdot x| \in \mathbb{R}$ we get $v \cdot (ie^{i\theta_x} x) = 0$ and $|\eta \cdot x| = v \cdot (e^{i\theta_x} x) \leq p\big(e^{i\theta_x} x\big) = p(x)$, completing the proof of the theorem. □

2.2.2 Corollary on the topological dual

Theorem 2.2.4. *Let E be a Fréchet space (see the definition 1.3.10), M be a closed subspace of E and $x_0 \in E$. Then the following properties are equivalent:*

(i) $x_0 \notin M$,

(ii) $\exists \xi \in E^*, \quad \xi \cdot x_0 = 1, \quad \ker \xi \supset M$.

Proof. The property (ii) implies trivially (i), since it gives $x_0 \in (\ker \xi)^c \subset M^c$. Let us prove the converse. Let $x_0 \in E \backslash M$ and $(p_k)_{k \geq 1}$ be a countable family of semi-norms describing the topology on E. Since M is closed, there exists $U_0 \in \mathscr{V}_0$ such that $(x_0 + U_0) \cap M = \emptyset$ and

$$\exists k_0 \geq 1, \exists R_0 > 0, \quad \text{such that} \quad U_0 \supset \{x \in E, p_{k_0}(x) < R_0\}.$$

We consider $M_1 = M \oplus \mathtt{k} x_0$ and $\xi \in M_1'$ defined by $\xi \cdot (x \oplus t x_0) = t$. We have for $t \in \mathtt{k}^*, x \in M$, $p_{k_0}\big(-\frac{x}{t} - x_0\big) \geq R_0$, otherwise $p_{k_0}\big(-\frac{x}{t} - x_0\big) < R_0$ and $-\frac{x}{t} - x_0 \in U_0$, implying $-\frac{x}{t} = x_0 - \frac{x}{t} - x_0 \notin M$. As a consequence, for $t \in \mathtt{k}^*, x \in M$,

$$|\xi \cdot (x \oplus t x_0)| = |t| \leq \frac{|t|}{R_0} p_{k_0}\big(-\frac{x}{t} - x_0\big) = \frac{1}{R_0} p_{k_0}\big(x + t x_0\big) \quad \text{and } \xi \cdot x = 0,$$

so that, for $y \in M_1$, $|\xi \cdot y| \leq R_0^{-1} p_{k_0}(y)$. Using the Hahn-Banach theorem 2.2.1, we can find an extension $\tilde\xi$ of ξ to the whole E such that, for $y \in E$, $|\tilde\xi \cdot y| \leq R_0^{-1} p_{k_0}(y)$. This implies that $\tilde\xi \in E^*$ and since $\tilde\xi \cdot x_0 = \xi \cdot x_0 = 1$ as well as $\tilde\xi_{|M_1} = \xi, \xi_{|M} = 0$, the linear form $\tilde\xi$ satisfies (ii). □

2.3 Examples of Topological Vector Spaces

We have already seen a couple of examples of TVS in the section 1.4. Here we examine in more details some various examples of Fréchet spaces.

2.3.1 $C^0(\Omega)$, Ω open subset of $\mathbb{R}^n$.

We consider an open subset Ω of $\mathbb{R}^n$ and we start with the proof of the following lemma.

Lemma 2.3.1. *Let Ω be an open subset of $\mathbb{R}^n$. There exists a sequence $(K_j)_{j \geq 1}$ of compact subsets of Ω such that*

$$\Omega = \cup_{j \geq 1} K_j, \quad K_j \subset \operatorname{int}(K_{j+1}). \tag{2.3.1}$$

If K is a compact subset of Ω, there exists $j \in \mathbb{N}^*$ such that $K \subset K_j$.

Proof. We define first for $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$d(x, A) = \inf_{a \in A} |x - a|, \quad \text{where } |\cdot| \text{ is a norm on } \mathbb{R}^n. \tag{2.3.2}$$

The function $x \mapsto d(x, A)$ is Lipschitz continuous since

$$|x_1 - a| \leq |x_1 - x_2| + |x_2 - a| \Longrightarrow \forall a \in A,\ d(x_1, A) \leq |x_1 - x_2| + |x_2 - a|,$$

so that $d(x_1, A) \leq |x_1 - x_2| + d(x_2, A)$ and finally

$$|d(x_1, A) - d(x_2, A)| \leq |x_1 - x_2|. \tag{2.3.3}$$

We have also that

$$d(x, A) = 0 \Longleftrightarrow x \in \bar A, \tag{2.3.4}$$

since the former is equivalent to the existence of a sequence $(a_l)_{l\geq 1}$ with $a_l \in A$ and $|a - a_l| \leq 1/l$. Given an open set Ω of $\mathbb{R}^n$, we define for $j \geq 1$,

$$K_j = \{x \in \mathbb{R}^n, d(x, \Omega^c) \geq 1/j, |x| \leq j\}.$$

We note from the continuity of $d(\cdot, \Omega^c)$ and of the norm that K_j is a closed subset of $\mathbb{R}^n$; moreover it is also bounded and thus is a compact subset of $\mathbb{R}^n$, and in fact of Ω since $d(x, \Omega^c) > 0$ implies $x \notin \overline{\Omega^c} = \Omega^c$ (Ω is open). We have also for $j \geq 1$ that

$$K_j \subset \{x \in \mathbb{R}^n, d(x, \Omega^c) > \frac{1}{j+1}, |x| < j+1\} \text{ which is open} \subset K_{j+1},$$

so that $K_j \subset \operatorname{int} K_{j+1}$. Finally, taking $x \in \Omega$, we have $d(x, \Omega^c) > 0$ (Ω^c is closed) and thus

$$j \geq \max\Big(\frac{1}{d(x, \Omega^c)}, E(|x|) + 1\Big) \Longrightarrow x \in K_j,$$

proving $\Omega = \cup_{j\geq 1} K_j$ and the lemma, since the very last statement of this lemma follows from $K \subset \Omega = \cup_{j\geq 1} \operatorname{int} K_{j+1}$, which implies the result by the Borel-Lebesgue property and the fact that the sequence (K_j) is non-decreasing. □

We can define now a countable separating family of semi-norms $(p_j)_{j\geq 1}$ on $C^0(\Omega)$, the vector space of (complex-valued) continuous functions defined on Ω with

$$p_j(u) = \sup_{x \in K_j} |u(x)|, \quad \text{(which makes sense since } u(K_j) \text{ is a compact subset of } \mathbb{C}).$$

Note that this family is non-decreasing, obviously made with semi-norms, and separating from the fact that $\Omega = \cup_{j\geq 1} K_j$. Let us prove that it is a complete space; we consider a Cauchy sequence $(u_l)_{l\geq 1}$, i.e. a sequence satisfying

$$\forall j \geq 1, \forall \epsilon > 0, \exists N_{\epsilon,j}, \forall l', l'' \geq N_{\epsilon,j}, \quad p_j(u_{l'} - u_{l''}) < \epsilon.$$

As a consequence, for each $j \geq 1$, the sequence $(u_{l|K_j})_{l\geq 1}$ converges uniformly to a continuous function v_j on K_j. Since $K_j \subset K_{j+1}$, we have $v_{j+1|K_j} = v_j$ and we can define v unambiguously on Ω by $v_{|K_j} = v_j$. That function v is continuous on Ω since for $j \geq 1$, $v_{|\operatorname{int} K_{j+1}} = v_{j+1|\operatorname{int} K_{j+1}}$ which is continuous and $\Omega \supset \cup_{j\geq 1} \operatorname{int} K_{j+1} \supset \cup_{j\geq 1} K_j = \Omega$. We find easily as well that

$$\forall j \geq 1, \quad \lim_l p_j(u_l - v) = 0,$$

so that the sequence $(u_l)_{l\geq 1}$ converges in $C^0(\Omega)$. The reader will check in the exercises that the vector space $C^0(\Omega)$ with that topology is not normable.

2.3.2 $C^m(\Omega)$, Ω open subset of $\mathbb{R}^n$, $m \in \mathbb{N}$.

With the family of compact sets K_j as above, we consider the family of semi-norms

$$p_j(u) = \sup_{x \in K_j, |\alpha| \leq m} |(\partial_x^\alpha u)(x)|, \tag{2.3.5}$$

where we have used the multi-index notation with

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| = \alpha_1 + \dots + \alpha_n. \tag{2.3.6}$$

For a multi-index α we define $\alpha! = \alpha_1! \dots \alpha_n!$ and for $\xi \in \mathbb{R}^n$, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ so that we have, for $\xi, \eta \in \mathbb{R}^n$, using Taylor's formula[7],

$$\frac{(\xi+\eta)^\alpha}{\alpha!} = \sum_{\substack{\beta,\gamma \in \mathbb{N}^n \\ \beta+\gamma=\alpha}} \frac{\xi^\beta}{\beta!} \frac{\eta^\gamma}{\gamma!}. \tag{2.3.7}$$

We get a Fréchet space following essentially the same arguments as for the previous example. The only point to verify is the following, that we formulate in one dimension for simplicity, leaving to the reader to filling the details in higher dimension. Take a sequence $(u_l)_{l\ge1}$ of C^1 functions on Ω, open interval of $\mathbb{R}$, which converges in $C^0(\Omega)$ as well as $(u_l')_{l\ge1}$. We define

$$v_0 = \lim_l u_l, \quad v_1 = \lim_l u_l'.$$

Then $v_0 \in C^1(\Omega)$ and $v_0' = v_1$ (exercise).

2.3.3 $C^\infty(\Omega)$, Ω open subset of $\mathbb{R}^n$.

With the family of compact sets K_j as above, we consider the countable family of semi-norms

$$p_{j,m}(u) = \sup_{x \in K_j, |\alpha| \le m} |(\partial_x^\alpha u)(x)|. \tag{2.3.8}$$

We get a Fréchet space following essentially the same arguments as for the previous example.

2.3.4 The space of holomorphic functions $\mathcal{H}(\Omega)$, Ω open subset of $\mathbb{C}$.

This is a Fréchet space with the family of semi-norms

$$p_j(u) = \sup_{z \in K_j} |u(z)|,$$

where the compact sets K_j are as in the lemma 2.3.1. It is a remarkable fact, due to the Cauchy formula, that whenever a sequence $(u_l)_{l\ge1}$ of holomorphic functions in Ω is converging uniformly on the compact subsets of Ω, the limit (say in $C^0(\Omega)$) is also holomorphic (exercise).

[7] In fact, $\frac{(\xi+\eta)^\alpha}{\alpha!} = \sum_\gamma \frac{\eta^\gamma}{\gamma!} \partial_\eta^\gamma \frac{(\xi+\eta)^\alpha}{\alpha!}{}_{|\eta=0} = \sum_{\gamma, \gamma\le\alpha} \frac{\eta^\gamma}{\gamma!} \frac{\xi^{\alpha-\gamma}}{(\alpha-\gamma)!}$, which is the sought formula (the inequality $\gamma \le \alpha$ means $\forall j, \gamma_j \le \alpha_j$).

2.3.5 The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing functions.

We define, using the multi-index notation for $\alpha \in \mathbb{N}^n$, $x \in \mathbb{R}^n, x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$,

$$\mathscr{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n), \forall \alpha, \beta \in \mathbb{N}^n, x^\alpha \partial_x^\beta u \in L^\infty(\mathbb{R}^n)\}, \tag{2.3.9}$$

$$\text{for } u \in \mathscr{S}(\mathbb{R}^n),\ p_k(u) = \sup_{\substack{\max(|\alpha|,|\beta|)\le k\\ x\in\mathbb{R}^n}} |x^\alpha (\partial_x^\beta u)(x)|, \tag{2.3.10}$$

is a family of semi-norms which makes $\mathscr{S}(\mathbb{R}^n)$ a Fréchet space[8]. A good example of such functions is given by $u(x) = e^{-\|x\|^2}$ where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$, and more generally by

$$v_{A,p}(x) = p(x)e^{-\langle Ax, x\rangle},$$

where A is a $n \times n$ complex-valued symmetric matrix so that $\operatorname{Re} A \gg 0$ and p is a polynomial. The Schwartz space plays an important rôle in Fourier analysis and we shall return to its study in chapter 3.

2.4 Ascoli's theorem

2.4.1 An example and the statement

Let us start with a simple example: we consider a sequence $(u_k)_{k\ge1}$ of continuous functions from $[0,1]$ to $\mathbb{R}$ and we assume that it is simply converging, i.e. for all $x \in [0,1]$, the sequence $(u_k(x))_{k\ge1}$ converges. Defining $u(x) = \lim_k u_k(x)$, we know that in general u need not to be continuous (exercise). We are looking for a simple and tractable condition ensuring that u is continuous: a good way to get this is to obtain the uniform convergence, i.e. the convergence in the Banach space $C^0([0,1])$. We shall assume some *equicontinuity* property for the sequence (u_k). To simplify matters in this presentation, let us assume that

$$\exists L > 0,\ \forall k \ge 1, \forall x_1, x_2 \in [0,1], \quad |u_k(x_1) - u_k(x_2)| \le L|x_1 - x_2|.$$

Then we have for $x, t \in [0,1]$,

$$\begin{aligned}|u_k(x) - u_l(x)| &\le |u_k(x) - u_k(t)| + |u_k(t) - u_l(t)| + |u_l(t) - u_l(x)|\\ &\le 2L|x-t| + |u_k(t) - u(t)| + |u(t) - u_l(t)|,\end{aligned}$$

so that if $t \in D \subset [0,1]$, $\|u_k - u_l\| \le 2L \sup_{x\in[0,1]} d(x,D) + \sup_{t\in D} |u_k(t) - u(t)| + \sup_{t\in D} |u_l(t) - u(t)|$. We choose now $\epsilon > 0$ and we take D_ϵ as a finite subset of $[0,1]$, such that

$$2L \sup_{x\in[0,1]} d(x, D_\epsilon) \le \epsilon/3 \quad \text{(it is enough to consider } D_\epsilon = \frac{\epsilon}{3L}\mathbb{N} \cap [0,1]).$$

[8] Laurent Schwartz (1915-2002) is the french mathematician (http://www-history.mcs.st-and.ac.uk/history/Biographies/Schwartz.html) who introduced the space $\mathscr{S}(\mathbb{R}^n)$ as the *Spherical Functions*, since they can be viewed as C^∞ functions on the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ which are flat (vanishing as well as all their derivatives) at the "north pole" ($\mathbb{S}^n$ is a compactification of $\mathbb{R}^n$ and the stereographic projection maps $\mathbb{S}^n \backslash \{\text{NPole}\}$ onto $\mathbb{R}^n$). L. Schwartz is not related to Herman Amandus Schwarz, a german mathematician co-credited with Cauchy for the Cauchy-Schwarz inequality (1.3.3).

Since D_ϵ is finite, for $k \ge N_\epsilon$, $\sup_{t\in D} |u_k(t) - u(t)| \le \epsilon/3$ and $\|u_k - u_l\| \le \epsilon$ for $k, l \ge N_\epsilon$, proving that $(u_k)_{k\ge1}$ is a Cauchy sequence in the Banach space $C^0([0,1])$ and the sought result. The following theorem is providing a generalization of the previous discussion, for which we point out that, for this result, the key property of $[0,1]$ is to be a metric compact space, for the target $\mathbb{R}$ to be a complete metric space and of course that the equicontinuity should hold.

Theorem 2.4.1. *Let X be a compact metric space and Y be a complete metric space. Let $(u_\alpha)_{\alpha\in A}$ be a family of continuous mappings from X to Y such that*

(1) *$(u_\alpha)_{\alpha\in A}$ is pointwise relatively compact,*

(2) *$(u_\alpha)_{\alpha\in A}$ is equicontinuous.*

Then $(u_\alpha)_{\alpha\in A}$ is strongly relatively compact.

Remark 2.4.2. We need first to clarify the meaning of the assumptions: the family $(u_\alpha)_{\alpha\in A}$ is pointwise relatively compact means that, for each $x \in X$, the set $\{u_\alpha(x)\}_{\alpha\in A}$ has a compact closure in Y. The equicontinuity of the family $(u_\alpha)_{\alpha\in A}$ means

$$\forall\epsilon > 0, \exists\delta > 0, \forall\alpha \in A, \forall x', x'' \in X, d(x', x'') < \delta \Longrightarrow d\big(u_\alpha(x'), u_\alpha(x'')\big) < \epsilon \quad (2.4.1)$$

where the first d is the distance on X and the second one the distance on Y. The strong relative compactness of the family $(u_\alpha)_{\alpha\in A}$ means that, given a sequence $(u_{\alpha_j})_{j\in\mathbb{N}}$, there exists a subsequence $(u_{\alpha_{j_k}})_{k\in\mathbb{N}}$ converging uniformly to a continuous function u:

$$\forall\epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall k, k \ge N_\epsilon \Longrightarrow \sup_{x\in X} d\big(u_{\alpha_{j_k}}(x), u(x)\big) < \epsilon.$$

We may notice that, thanks to the compactness of X, the equicontinuity property (2.4.1) is a consequence of the weaker

$$\forall x_0 \in X, \forall\epsilon > 0, \exists\delta > 0, \forall\alpha \in A, \forall x \in X, d(x, x_0) < \alpha \Longrightarrow d\big(u_\alpha(x), u_\alpha(x_0)\big) < \epsilon. \quad (2.4.2)$$

This can be proven by *reductio ad absurdum*: assuming that (2.4.2) holds and that (2.4.1) is violated, we obtain

$$\exists\epsilon_0 > 0, \forall k \in \mathbb{N}^*, \exists\alpha_k \in A, \exists x'_k, x''_k \in X, \text{ with } d(x'_k, x''_k) < 1/k$$
$$\text{and } d\big(u_{\alpha_k}(x'_k), u_{\alpha_k}(x''_k)\big) \ge \epsilon_0.$$

Since X is compact, one may extract subsequences of the sequences $(x'_k), (x''_k)$ and assume that they are both convergent in X with the same limit x_0. As a consequence, we get a contradiction since

$$0 < \epsilon_0 \le \limsup_k d\big(u_{\alpha_k}(x'_k), u_{\alpha_k}(x''_k)\big)$$
$$\le \limsup_k d\big(u_{\alpha_k}(x'_k), u_{\alpha_k}(x_0)\big) + \limsup_k d\big(u_{\alpha_k}(x_0), u_{\alpha_k}(x''_k)\big) = 0,$$

where the last equality follows from (2.4.2). Note that we have followed the reasoning of the proof of the Heine theorem 1.5.10.

Remark 2.4.3. A more more elegant statement can be proven, involving the introduction of a topology on the set $C(X;Y)$: for X, Y metric spaces with Y complete and X compact, we define (see the proposition 2.4.6) a distance D on $C(X;Y)$ by

$$D(u,v) = \sup_{x\in X} d_Y\big(u(x), v(x)\big). \tag{2.4.3}$$

The metric space $C(X;Y)$ is then complete[9]. We consider now a subset F of $C(X;Y)$ and we assume that (1) is satisfied, i.e. $\forall x \in X, \{u(x)\}_{u\in F}$ has a compact closure in Y. Assuming as well the equicontinuity (2), the Ascoli theorem says that the closure of F is compact in the metric space $C(X;Y)$. More general statements hold as well, and in particular, it is enough to assume that X is compact (and not necessarily compact metrizable).

Remark 2.4.4. We note also that a compact metric space is *separable*, i.e. contains a countable dense subset: we have from the compactness of the metric space X, $k \in \mathbb{N}^*$,

$$X = \cup_{x\in X} B(x, 1/k) = \cup_{1\le j\le N_k} B(x_{j,k}, 1/k)$$

and the countable set $\{x_{j,k}\}_{\substack{1\le k\\ 1\le j\le N_k}}$ is dense in X.

We begin with a key lemma on the diagonal process.

2.4.2 The diagonal process

Lemma 2.4.5. *Let $(a_{ij})_{i,j\in\mathbb{N}^*}$ be an infinite matrix of elements of a metric space A. We assume that each line is relatively compact, i.e. for all $i \in \mathbb{N}^*$, the set $\{a_{i,j}\}_{j\ge1}$ is relatively compact. Then, there exists a strictly increasing mapping ν from $\mathbb{N}^*$ into itself such that, for all $i \in \mathbb{N}^*$, the sequence $\big(a_{i,\nu(k)}\big)_{k\in\mathbb{N}^*}$ converges.*

Proof of the lemma. The reader will find the definition of a subsequence in 1.5.6.

· We can extract a converging subsequence $(a_{1,n_1(k)})_{k\ge1}$ from the first line $(a_{1,j})_{j\ge1}$,

· We can extract a converging subsequence $(a_{2,n_1(n_2(k))})_{k\ge1}$ from $(a_{2,n_1(k)})_{j\ge1}$.

· We can extract a converging subsequence $(a_{3,n_1(n_2(n_3(k)))})_{k\ge1}$ from $(a_{3,n_1(n_2(k))})_{j\ge1}$.

... For all $i \ge 1$, we can extract a converging subsequence

$$\big(a_{i,(n_1\circ\cdots\circ n_i)(k)}\big)_{k\ge1}.$$

Note that the mappings n_l are strictly increasing from $\mathbb{N}^*$ into itself and thus satisfy $\forall k \ge 1, n_l(k) \ge k$ (true for $k = 1$ and $n_l(k+1) > n_l(k) \ge k$ gives $n_l(k+1) \ge k+1$). We define

$$b_{i,k} = a_{i,\nu(k)}, \quad \text{with} \quad \nu(k) = (n_1 \circ \cdots \circ n_k)(k).$$

The mapping ν sends $\mathbb{N}^*$ into itself and is strictly increasing:

$$\nu(k+1) = (n_1\circ\cdots\circ n_{k+1})(k+1) \overbrace{\ge}^{\text{since } n_{k+1}(k+1)\ge k+1} (n_1\circ\cdots\circ n_k)(k+1) \underbrace{>}_{n_1\circ\cdots\circ n_k\nearrow\text{strict}} (n_1\circ\cdots\circ n_k)(k) = \nu(k).$$

[9] In particular, if Y is a Banach space (and X a compact space), $C(X;Y)$ is a vector space which is a Banach space with the norm $\|u\|_{C(X;Y)} = \sup_{x\in X}\|u(x)\|_Y$.

Moreover, the sequence $(b_{i,k})_{k,k>i}$ is a subsequence of the converging sequence

$$\big(a_{i,(n_1\circ\cdots\circ n_i)(k)}\big)_{k\geq 1}$$

since for $k > i \geq 1$, $\nu(k) = (n_1 \circ \cdots \circ n_i)\big((n_{i+1} \circ \cdots \circ n_k)(k)\big)$ and

$$\mu_i(k+1) = (n_{i+1}\circ\cdots\circ n_{k+1})(k+1) \geq (n_{i+1}\circ\cdots\circ n_k)(k+1) > (n_{i+1}\circ\cdots\circ n_k)(k) = \mu_i(k).$$

As a result, the sequence $(a_{i,\nu(k)})_{k\geq 1}$ is converging, which proves the lemma. □

Proof of the Ascoli theorem. Using the lemma 2.4.4, we consider $X_0 = \{x_i\}_{i\geq 1}$ a countable dense part of X. We consider a sequence $(u_{\alpha_j})_{j\in\mathbb{N}}$ and we note $v_j = u_{\alpha_j}$. Considering the infinite matrix $\big(v_j(x_i)\big)_{i,j\geq 1}$, thanks to the assumption (1), we see that for all $i \geq 1$ the line $\{v_j(x_i)\}_{j\geq 1}$ is relatively compact and, using the lemma 2.4.5, we find $\nu : \mathbb{N}^* \to \mathbb{N}^*$ strictly increasing such that $\big(v_{\nu(k)}(x_i)\big)_{k\geq 1}$ converges for all $i \geq 1$. Let $\epsilon > 0$ be given and $\alpha > 0$ such that (2.4.1) holds. We have $X = \cup_{i\geq 1} B(x_i, \alpha)$ and by the compactness of X, we can find M such that

$$X = \cup_{1\leq i\leq M} B(x_i, \alpha).$$

Since the sequences $\big(v_{\nu(k)}(x_i)\big)_{k\geq 1}$ are convergent,

$$\exists N_\epsilon, \forall i \in \{1, \dots, M\}, \forall k, l \geq N_\epsilon, \quad d\big(v_{\nu(k)}(x_i), v_{\nu(l)}(x_i)\big) < \epsilon. \tag{2.4.4}$$

Let $x \in X$: there exists $i \in \{1, \dots, M\}$ such that $x \in B(x_i, \alpha)$, and from the assumption (2), we have

$$\forall j \geq 1 \quad d\big(v_j(x), v_j(x_i)\big) < \epsilon. \tag{2.4.5}$$

As a consequence, we get for $k, l \geq N_\epsilon$ and all $x \in X$,

$$\begin{aligned} &d\big(v_{\nu(k)}(x), v_{\nu(l)}(x)\big) \\ &\quad\leq \underbrace{\big(v_{\nu(k)}(x), v_{\nu(k)}(x_i)\big)}_{<\epsilon \text{ from } (2.4.5)} + \underbrace{d\big(v_{\nu(k)}(x_i), v_{\nu(l)}(x_i)\big)}_{<\epsilon \text{ from } (2.4.4)} + \underbrace{d\big(v_{\nu(l)}(x_i), v_{\nu(l)}(x)\big)}_{<\epsilon \text{ from } (2.4.5)} < 3\epsilon, \end{aligned}$$

so that the uniform Cauchy criterion is satisfied for the sequence $(v_{\nu(k)})_{k\geq 1}$. The result of the theorem will be a consequence of the following

Proposition 2.4.6. *Let X, Y be metric spaces with Y complete and X compact and $\mathcal{C}(X, Y)$ be the set of continuous mappings from X to Y. The following formula defines a distance on $\mathcal{C}(X, Y)$,*

$$D(u, v) = \sup_{x\in X} d_Y\big(u(x), v(x)\big), \tag{2.4.6}$$

and makes it a complete metric space.

Proof. Let $u, v \in \mathcal{C}(X, Y)$: then $D(u, v) < +\infty$, otherwise we would be able to find a sequence $x_k \in X$ such that $d_Y\big(u(x_k), v(x_k)\big) \geq k$. From the compactness of X, we can extract a convergent subsequence $(x_{k_l})_{l \geq 1}$ with limit a. Then we have

$$\begin{aligned} k_l &\leq d_Y\big(u(x_{k_l}), v(x_{k_l})\big) \\ &\leq \underbrace{d_Y\big(u(x_{k_l}), u(a)\big)}_{\substack{\longrightarrow 0 \\ l \to +\infty}} + d_Y\big(u(a), v(a)\big) + \underbrace{d_Y\big(v(x_{k_l}), v(a)\big)}_{\substack{\longrightarrow 0 \\ l \to +\infty}} \end{aligned} \tag{2.4.7}$$

which is impossible since $\lim_l k_l = +\infty$. Moreover the separation and symmetry properties of D are obviously satisfied; for $u, v, w \in \mathcal{C}(X, Y)$, we have

$$d_Y\big(u(x), w(x)\big) \leq d_Y\big(u(x), v(x)\big) + d_Y\big(v(x), w(x)\big)$$

which implies readily that D satisfies the triangle inequality. Let us now consider a Cauchy sequence $(u_k)_{k \geq 1}$ in the metric space $\mathcal{C}(X, Y)$; since Y is complete, for all $x \in X$, the sequence $(u_k(x))_{k \geq 1}$ converges and we define $u(x) = \lim_k u_k(x)$. We have, since (u_k) is a Cauchy sequence in $\mathcal{C}(X, Y)$,

$$d\big(u_k(x), u(x)\big) = \lim_l d\big(u_k(x), u_l(x)\big) \leq \limsup_l D(u_k, u_l) = \tau(k), \quad \lim_k \tau(k) = 0,$$

and thus $\lim_k \Big(\sup_{x \in X} d\big(u_k(x), u(x)\big)\Big) = 0$. Let us prove that the function u is continuous: otherwise, we could find $x_0 \in X$, a sequence $(x_j)_{j \geq 1}$ with $\lim_j x_j = x_0$ and $\epsilon_0 > 0$ such that $d\big(u(x_j), u(x_0)\big) \geq \epsilon_0$. We would have for $j, k \geq 1$,

$$\begin{aligned} 0 < \epsilon_0 &\leq d\big(u(x_j), u_k(x_j)\big) + d\big(u_k(x_j), u_k(x_0)\big) + d\big(u_k(x_0), u(x_0)\big) \\ &\leq 2\tau(k) + d\big(u_k(x_j), u_k(x_0)\big), \end{aligned}$$

and thus, since u_k is continuous for all k,

$$0 < \epsilon_0 \leq 2\tau(k) + \limsup_j d\big(u_k(x_j), u_k(x_0)\big) = 2\tau(k) \Longrightarrow 0 < \epsilon_0 \leq \lim_k 2\tau(k) = 0,$$

which is impossible. We have proven that $u \in \mathcal{C}(X, Y)$ and $\lim_k D(u_k, u) = 0$, completing the proof of the proposition. □

Going back to the proof of the theorem, we see from the previous proposition and the fact that the sequence $(v_{\nu(k)})$ satisfies the uniform Cauchy criterion that it converges in $\mathcal{C}(X, Y)$, which proves the theorem 2.4.1. □

2.5 Duality in Banach spaces

2.5.1 Definitions

For E, F Banach spaces, we have defined the Banach space $\mathcal{L}(E, F)$ in the proposition 2.1.5 with the norm (2.1.2). We recall that the topological dual of E is the Banach space $E^* = \mathcal{L}(E, \mathtt{k})$ of continuous linear forms. When $\xi \in E^*, x \in E$, we shall write $\xi \cdot x$ instead of $\xi(x)$.

Theorem 2.5.1. *Let E be a Banach space and E^* its topological dual. Then*

$$\forall x \in E, \quad \|x\|_E = \sup_{\|\xi\|_{E^*}=1} |\xi \cdot x|.$$

Proof. From the proposition 2.1.5, we have $\|\xi\|_{E^*} = \sup_{x\in E, \|x\|_E=1} |\xi \cdot x|$. Let $0 \neq x_0 \in E$. Applying the Hahn-Banach theorem 2.2.1 with $M = \mathtt{k}x_0$, $p(x) = \|x\|_E$, defining on M the linear form η by $\eta \cdot \lambda x_0 = \lambda \|x_0\|_E$, we have $|\eta \cdot \lambda x_0| \le \|\lambda x_0\| = p(\lambda x_0)$ and we find a linear form ξ_0 defined on E such that

$$|\xi_0 \cdot x_0| = \|x_0\|_E, \quad \forall x \in E, \ |\xi_0 \cdot x| \le \|x\|_E.$$

As a consequence, $\xi_0 \in E^*$ with $\|\xi_0\| = 1$. Finally we have proven

$$\|x_0\|_E = |\xi_0 \cdot x_0| \le \sup_{\|\xi\|_{E^*}=1} |\xi \cdot x_0| \le \|x_0\|_E.$$ □

2.5.2 Weak convergence on E

Using the very general notion introduced in the remark 1.1.2, we can define the weak topology on a Banach space as follows.

Definition 2.5.2. *Let E be a Banach space. The weak topology $\sigma(E, E^*)$ on E is the weakest topology such that for all $\xi \in E^*$ the mappings $E \ni x \mapsto \langle \xi, x\rangle_{E^*,E} \in \mathtt{k}$ are continuous.*

Remark 2.5.3. Let E be a Banach space. For each $\xi \in E^*$, we define the semi-norm p_ξ on E by $p_\xi(x) = |\langle \xi, x\rangle_{E^*,E}|$; the properties of the definition 1.3.8 are obviously satisfied. Moreover the family $(p_\xi)_{\xi\in E^*}$ is separating from the theorem 2.5.1. The neighborhoods of 0 for the weak topology on E, say $\mathscr{V}_0$, have the following basis: taking Ξ a finite subset of E^* and $r > 0$, we define

$$W_{\Xi,r} = \{x \in E, \forall \xi \in \Xi, p_\xi(x) < r\}. \tag{2.5.1}$$

Note that the $W_{\Xi,r}$ are convex and symmetric. Every neighborhood of 0 for the weak topology contains a $W_{\Xi,r}$ which is also a neighborhood of 0 for that topology. The neighborhoods $\mathscr{V}_x$ of a point x are defined as $\mathscr{V}_x = \{x + V\}_{V\in\mathscr{V}_0}$; E equipped with that topology is a TVS. Note that the separating property of the family $(p_\xi)_{\xi\in E^*}$ is implying that the weak topology is separated (i.e. Hausdorff, see the definition 1.1.5): in fact $\{0\}$ is closed for the weak topology, since for $x_0 \neq 0$, from the theorem 2.5.1, there exists $\xi_0 \in E^*$ such that $\langle \xi_0, x_0\rangle = 1$, so that

$$0 \notin x_0 + \{x \in E, p_{\xi_0}(x) < 1\} : \quad \text{otherwise, } 1 = \langle \xi_0, x_0\rangle = \langle \xi_0, \underbrace{x_0 + x}_{=0}\rangle - \langle \xi_0, x\rangle < 1.$$

Moreover, to check that the addition is continuous, we take $x_1, x_2 \in E$, W_{Ξ_0,r_0} as above a neighborhood of zero (Ξ_0 finite and $r_0 > 0$), and we try to find $W_{\Xi_j,r_j}, j = 1, 2$ such that

$$x_1 + W_{\Xi_1,r_1} + x_2 + W_{\Xi_2,r_2} \subset x_1 + x_2 + W_{\Xi_0,r_0}.$$

It is enough to take $W_{\Xi_j,r_j}=W_{\Xi_0,r_0/2}$. Checking the continuity of the multiplication by a scalar is similar: given $\lambda_0\in \mathtt{k}, x_0\in E$, W_{Ξ_0,r_0} as above, we want to find W_{Ξ_1,r_1} and $t_1>0$ such that

$$\forall t\in\mathbb{R},\ |t|\le t_1,\quad (\lambda_0+\theta t)(x_0+W_{\Xi_1,r_1})\subset \lambda_0 x_0+W_{\Xi_0,r_0}.$$

It is enough to require $t_1 W_{\Xi_1,r_1}\cup\lambda_0 W_{\Xi_1,r_1}\subset W_{\Xi_0,r_0/3},\quad t_1x_0\in W_{\Xi_0,r_0/3}$; this is satisfied for $\Xi_1=\Xi_0,\quad |\lambda_0|r_1<r_0/3,\quad t_1r_1<r_0/3$.

Remark 2.5.4. Let E be a Banach space; the weak topology $\sigma(E,E^*)$ on E is weaker than the norm-topology on E (also called the strong topology): this is obvious from the very definition of the weak topology since all the mappings $x\mapsto\langle\xi,x\rangle$ are continuous for the norm-topology since $p_\xi(x)=|\langle\xi,x\rangle|\le\|\xi\|_{E^*}\|x\|_E$.

Let E be a Banach space and $x\in E$; a sequence $(x_n)_{n\in\mathbb{N}}$ in E is weakly converging to x means that

$$\forall\xi\in E^*,\quad \lim_n\langle\xi,x_n\rangle_{E^*,E}=\langle\xi,x\rangle_{E^*,E}.\quad\text{We write}\quad x_n\rightharpoonup x,\tag{2.5.2}$$

or to avoid confusion between the arrows $\rightharpoonup$ and $\to$, we may write $x_n \xrightarrow[\sigma(E,E^*)]{} x$.

Proposition 2.5.5. *Let E be a Banach space and $(x_n)_{n\in\mathbb{N}}$ be a weakly converging sequence with limit x in E. Then $\|x_n\|_E$ is bounded and $\|x\|_E\le\liminf_n\|x_n\|_E$. If $(\xi_n)_{n\in\mathbb{N}}$ is a strongly converging sequence in E^* with limit ξ, then $\lim_n\langle\xi_n,x_n\rangle_{E^*,E}=\langle\xi,x\rangle_{E^*,E}$.*

Proof. We consider the sequence of linear forms on E^* given by $E^*\ni\xi\mapsto\langle\xi,x_n\rangle$. Since for all $\xi\in E^*$, the numerical sequence $\langle\xi,x_n\rangle$ is converging, we may apply the corollary 2.1.8 of the Banach-Steinhaus theorem to get that $E^*\ni\xi\mapsto\langle\xi,x\rangle$ is continuous on E^*, i.e

$$\exists C>0,\forall\xi\in E^*,\quad |\langle\xi,x\rangle|\le C\|\xi\|_{E^*}.$$

Using the theorem 2.5.1, this implies $\|x\|_E\le C$. The Banach-Steinhaus theorem 2.1.6 implies also that the norms of the linear forms $E^*\ni\xi\mapsto\langle\xi,x_n\rangle$ make a bounded sequence, and since that norm is $\|x_n\|_E$, we get that sequence $(\|x_n\|_E)$ is bounded. We have for $\xi\in E^*$ with $\|\xi\|_{E^*}=1$, using again the theorem 2.5.1,

$$|\langle\xi,x\rangle|=\lim_n|\langle\xi,x_n\rangle|\le\liminf_n\|x_n\|_E\Longrightarrow\|x\|_E\le\liminf_n\|x_n\|_E.$$

Moreover, we have

$$|\langle\xi_n,x_n\rangle-\langle\xi,x\rangle|\le|\langle\xi_n-\xi,x_n\rangle|+|\langle\xi,x_n-x\rangle|\le\underbrace{\|\xi_n-\xi\|_{E^*}}_{\to0}\sup_n\|x_n\|_E+\underbrace{|\langle\xi,x_n-x\rangle|}_{\to0},$$

which implies $\lim_n\langle\xi_n,x_n\rangle=\langle\xi,x\rangle$. $\square$

Remark 2.5.6. When the Banach space E is infinite-dimensional, the weak topology $\sigma(E, E^*)$ is strictly weaker than the strong topology given by the norm of E. Let us prove that the unit sphere of E, $S = \{x \in E, \|x\|_E = 1\}$ is not closed in the weak topology $\sigma(E, E^*)$ if E is not finite-dimensional. Let us consider $x_0 \in E$ with $\|x_0\|_E < 1$; let W_{Ξ_0,r_0} be a neighborhood of zero for the weak topology as in (2.5.1). We claim that

$$(x_0 + W_{\Xi_0,r_0}) \cap S \neq \emptyset. \tag{2.5.3}$$

This will imply that x_0 belongs to the closure of S for the $\sigma(E, E^*)$ topology. To prove (2.5.3), we consider the finite subset $\Xi_0 = \{\xi_j\}_{1\le j\le N}$ of E^*; each $\ker \xi_j$ is a closed hyperplane, and since E is infinite-dimensional, $\cap_{1\le j\le N} \ker \xi_j$ is not reduced to $\{0\}$ (otherwise the mapping $E \ni x \mapsto L(x) = (\langle \xi_j, x\rangle)_{1\le j\le N} \in \mathbb{R}^N$ would be injective and L would be an isomorphism from E onto $L(E)$, implying E finite-dimensional). Taking now a non-zero $x_1 \in \cap_{1\le j\le N} \ker \xi_j$, we see that with the continuous function f on $\mathbb{R}$ given by $f(\theta) = \|x_0 + \theta x_1\|$

$$f(\mathbb{R}_+) \supset [\|x_0\|, +\infty[\Longrightarrow \exists \theta \in \mathbb{R}, x_0 + \theta x_1 \in S.$$

This proves (2.5.3) since $x_0 + \theta x_1 \in x_0 + W_{\Xi_0,r_0}$ because $\langle \xi_j, x_1\rangle = 0$ for all $j \in \{1, \dots, N\}$.

Examples of weak convergence

We consider the space $L^p(\mathbb{R})$ for some $p \in [1, +\infty[$ (we shall see in the section 2.5.6 that, for $p \in [1, +\infty[$, the dual space of L^p is canonically identified with $L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$). We want to provide some examples of a sequence $(u_k)_{k\in\mathbb{N}}$ of $L^p(\mathbb{R})$ weakly converging to 0, but not strongly converging to 0. Here we assume $1 < p < +\infty$.

A first phenomenon is *strong oscillations*: take $u_k(x) = e^{ikx}\mathbf{1}_{[0,1]}(x)$: the L^p norm of u_k is constant equal to 1 but for $v \in L^{p'}$, the sequence $\langle u_k, v\rangle = \int u_k(x)\bar{v}(x)dx$ has limit zero (a consequence of the Riemann-Lebesgue lemma).

The sequence $(u_k)_{k\in\mathbb{N}}$ may also *concentrate at a point*: take $u_k(x) = k^{1/p}u_1(kx)$, where u_1 has norm 1 in L^p. Here also the L^p-norm of u_k is constant equal to 1. However for $v \in L^{p'}$, $\langle u_k, v\rangle = \int u_k(x)\bar{v}(x)dx = \int u_1(t)\bar{v}(t/k)dt k^{-\frac{1}{p'}}$, with $p, p' \in]1, +\infty[$. With $\varphi, \psi \in C^0_c(\mathbb{R})$ we have

$$\begin{aligned}|\langle u_k, v\rangle| &\le |\langle u_k, v - \varphi\rangle| + |\langle u_k - \psi_k, \varphi\rangle| + |\langle \psi_k, \varphi\rangle| \\ &\le \|u_1\|_{L^p}\|v - \varphi\|_{L^{p'}} + \|u_1 - \psi\|_{L^p}\|\varphi\|_{L^{p'}} + |\langle \psi_k, \varphi\rangle|,\end{aligned}$$

which implies $\limsup_k |\langle u_k, v\rangle| \le \|u_1\|_{L^p}\|v - \varphi\|_{L^{p'}} + \|u_1 - \psi\|_{L^p}\|\varphi\|_{L^{p'}}$, and this gives the weak convergence to 0 since p, p' are both in $]1, +\infty[$.

The sequence $(u_k)_{k\in\mathbb{N}}$ may also *escape to infinity*: take $u_k(x) = u_0(k + x)$, where u_0 has norm 1 in L^p. Reasoning as above, we need only to check $\int \psi(x + k)\varphi(x)dx$, for $\varphi, \psi \in C^0_c(\mathbb{R})$: that quantity is 0 for k large enough.

2.5.3 Weak-$*$ convergence on E^*

Definition 2.5.7. *Let E be a Banach space and E^* its topological dual. The weak-$*$ topology on E^*, denoted by $\sigma(E^*, E)$, is the weakest topology such that the mappings*

$E^* \ni \xi \mapsto \xi \cdot x \in \mathtt{k}$ are continuous for all $x \in E$. A sequence $(\xi_k)_{k\in\mathbb{N}}$ of E^* is weally-$*$ converging means that $\forall x \in E$, the sequence $(\xi_k \cdot x)_{k\in\mathbb{N}}$ converges.

Proposition 2.5.8. *Let E be a Banach space and $(\xi_n)_{n\in\mathbb{N}}$ be a weakly-$*$ converging sequence with limit ξ in E^*. Then $\|\xi_n\|_{E^*}$ is bounded and $\|\xi\|_{E^*} \le \liminf_n \|\xi_n\|_{E^*}$. Let $(x_n)_{n\in\mathbb{N}}$ be a strongly converging sequence in E with limit x. Then we have*

$$\lim_n \langle \xi_n, x_n\rangle_{E^*,E} = \langle \xi, x\rangle_{E^*,E}.$$

Proof. We have for $x \in E$ with $\|x\|_E = 1$,

$$|\langle \xi, x\rangle| = \lim_n |\langle \xi_n, x\rangle| \le \liminf_n \|\xi_n\|_{E^*} \Longrightarrow \|\xi\|_{E^*} \le \liminf_n \|\xi_n\|_E.$$

From the Banach-Steinhaus theorem 2.1.6, the sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded in the normed space E^* and we define $\sup_n \|\xi_n\|_{E^*} = M < \infty$. We have then

$$|\langle \xi_n, x_n\rangle - \langle \xi, x\rangle| \le |\langle \xi_n, x_n - x\rangle| + |\langle \xi_n - \xi, x\rangle| \le M\|x_n - x\|_E + |\langle \xi_n - \xi, x\rangle|,$$

and since $\lim_n \|x_n - x\|_E = 0 = \lim_n \langle \xi_n - \xi, x\rangle$, we obtain the result. □

Theorem 2.5.9. *Let E be a separable Banach space. The closed unit ball of E^* equipped with the weak-$*$ topology is (compact and) sequentially compact.*

Proof. Let $(\xi_j)_{j\in\mathbb{N}}$ be a sequence of E^* with $\sup_{j\in\mathbb{N}} \|\xi_j\|_{E^*} \le 1$. Let $\{x_i\}_{i\in\mathbb{N}}$ be a countable dense part of E. For each $i \in \mathbb{N}$, we define $y_i : E^* \longrightarrow \mathtt{k}$ by $y_i(\xi) = \xi \cdot x_i$. Let us now consider the matrix with entries $(\xi_j \cdot x_i)_{i,j\in\mathbb{N}}$. For all $i \in \mathbb{N}$, we have

$$\sup_{j\in\mathbb{N}} |\xi_j \cdot x_i| \le \|x_i\|_E$$

so that we can apply the diagonal process given by the lemma 2.4.5 and find ν strictly increasing from $\mathbb{N}$ to $\mathbb{N}$ such that $\forall i \in \mathbb{N}$, the sequence $(\xi_{\nu(k)} \cdot x_i)_{k\in\mathbb{N}}$ is converging. As a consequence, for $x \in E$,

$$\begin{aligned}
&|\xi_{\nu(k)} \cdot x - \xi_{\nu(l)} \cdot x| \\
&\qquad \le |\xi_{\nu(k)} \cdot x - \xi_{\nu(k)} \cdot x_i| + |\xi_{\nu(k)} \cdot x_i - \xi_{\nu(l)} \cdot x_i| + |\xi_{\nu(l)} \cdot x_i - \xi_{\nu(l)} \cdot x| \\
&\qquad\qquad \le 2\|x - x_i\|_E + |\xi_{\nu(k)} \cdot x_i - \xi_{\nu(l)} \cdot x_i|.
\end{aligned}$$

Let $\epsilon > 0$ be given and $x \in E$. Let $i \in \mathbb{N}$ such that $\|x - x_i\|_E < \epsilon/4$; since the sequence $(\xi_{\nu(k)} \cdot x_i)_{k\in\mathbb{N}}$ is converging, for $k, l \ge N_\epsilon$, $|\xi_{\nu(k)} \cdot x_i - \xi_{\nu(l)} \cdot x_i| < \epsilon/2$ and thus for $k, l \ge N_\epsilon$, $|\xi_{\nu(k)} \cdot x - \xi_{\nu(l)} \cdot x| < \epsilon$, proving the weak convergence of the sequence $(\xi_{\nu(k)})_{k\in\mathbb{N}}$. □

Remark 2.5.10. Let E be a Banach space and E^* its topological dual. For $x \in E, \xi \in E^*$, we define $p_x(\xi) = |\xi \cdot x|$. For each $x \in E$, p_x is (trivially) a semi-norm on E^*. The family $(p_x)_{x\in E}$ is a separating[10] (uncountable) family of semi-norms on E^*. We shall say that U is a neighborhood of 0 in the weak-$*$ topology if it contains a finite intersection of sets

$$V_{p_x,r} = \{\xi \in E^*, p_x(\xi) < r\}, \quad x \in E, r > 0.$$

The family of semi-norms $(p_x)_{x\in E}$ describes the weak-$*$ topology on E^*, also denoted by $\sigma(E^*, E)$.

[10]If for some $\xi \in E^*$, we have $\forall x \in E, p_x(\xi) = 0$, it means $\forall x \in E, \xi \cdot x = 0$, i.e. $\xi = 0_{E^*}$.

Remark 2.5.11. Let E be a Banach space and E^* its topological dual. It is also possible to define on E the weak topology, denoted by $\sigma(E, E^*)$, given by the family of semi-norms $(p_\xi)_{\xi \in E^*}$ such that $p_\xi(x) = |\xi \cdot x|$. That family is separating since for $x \in E$, $\xi \cdot x = 0$ for all $\xi \in E^*$ implies $x = 0$, thanks to the Theorem 2.5.1. We shall say that U is a neighborhood of 0 in the weak topology if it contains a finite intersection of sets

$$V_{p_\xi, r} = \{x \in E, p_\xi(x) < r\}, \quad \xi \in E^*, r > 0.$$

The family of semi-norms $(p_\xi)_{\xi \in E^*}$ describes the weak topology on E.

Remark 2.5.12. Given a Banach space E and its topological dual E^*, we can define on E^* several weak topologies: the weak-$*$ topology $\sigma(E^*, E)$ described above, but also the weak topology on E^*, $\sigma(E^*, E^{**})$, where E^{**} is the *bidual* of E, i.e. the topological dual of the Banach space E^*. Note that the weak topology on E^* is stronger than the weak-$*$ topology, since $E \subset E^{**}$ as shown below.

2.5.4 Reflexivity

Proposition 2.5.13. *Let E be a Banach space. The bidual of E is defined as the (topological) dual of E^*. The mapping $E \ni x \mapsto j(x) \in E^{**}$ defined by*

$$\langle j(x), \xi \rangle_{E^{**}, E^*} = \langle \xi, x \rangle_{E^*, E}$$

is linear isometric and is an isomorphism on its image $j(E)$ which is a closed subspace of E^{**}. A Banach space is said to be reflexive when j is bijective (this implies in particular that E^{**} and E are isometrically isomorphic).

Proof. For $x \in E$, we have

$$\|j(x)\|_{E^{**}} = \sup_{\|\xi\|_{E^*}=1} |\langle j(x), \xi \rangle_{E^{**}, E^*}| = \sup_{\|\xi\|_{E^*}=1} |\langle \xi, x \rangle_{E^*, E}| \underbrace{=}_{\text{thm 2.5.1}} \|x\|_E, \tag{2.5.4}$$

and thus j is isometric and obviously linear. The image $j(E)$ is closed: whenever a sequence $(j(x_k))_{k \geq 1}$ converges, it is also a Cauchy sequence as well as $(x_k)_{k \geq 1}$ since $\|x_k - x_l\|_E \leq \|j(x_k - x_l)\|_{E^{**}} = \|j(x_k) - j(x_l)\|_{E^{**}}$. As a result, the sequence $(x_k)_{k \geq 1}$ converges to some limit $x \in E$, and the continuity of j (consequence of the isometry property) ensures $\lim_k j(x_k) = j(x)$, proving that $j(E)$ is closed, and thus a Banach space for the norm of E^{**}. The mapping $j : E \longrightarrow j(E)$ is an isometric isomorphism of Banach spaces. □

Remark 2.5.14. Let E be a Banach space; then the bidual of E^* is equal to the dual of E^{**}, so that $\big(E^*\big)^{**} = \big((E^{**})\big)^*$, that we shall note simply E^{***}: we have by definition

$$\big(E^*\big)^{**} = \Big(\big(E^*\big)^*\Big)^*, \quad \text{as well as } \big((E^{**})\big)^* = \Big(\big(E^*\big)^*\Big)^*.$$

Theorem 2.5.15 (Banach-Alaoglu)**.** *Let E be a Banach space. The closed unit ball $\mathcal{B}$ of E^* is compact for the weak-$*$ topology.*

Proof. For each $x \in E$, the mapping $E^* \ni \xi \mapsto \xi \cdot x \in \mathbb{C}$ is continuous in the weak-$*$ topology; since $|\xi \cdot x| \le \|\xi\|_{E^*}\|x\|_E$ we see that

$$\mathcal{B} \subset \prod_{x \in E}(\|x\|_E D_1), \quad D_1 = \{z \in \mathbb{C}, |z| \le 1\},$$

and the product topology on $\prod_{x\in E}(\|x\|_E D_1)$ induces the weak-$*$ topology on $\mathcal{B}$. We shall use the following theorem.

Theorem 2.5.16 (Tychonoff)**.** *Let $(X_i)_{i\in I}$ be a family of compact spaces. Then the product $\prod_{i\in I} X_i$ equipped with the product topology is a compact space.*

The set $\mathcal{B}$ is thus a closed subset of a compact set and is thus compact. □

Proposition 2.5.17. *Let E be a Banach space and B its closed unit ball. The following properties are equivalent.*

(i) E is reflexive,

(ii) E^* is reflexive,

(iii) B is weakly compact, i.e. compact for the $\sigma(E, E^*)$ topology.

Proof. Let us assume that (i) is satisfied. Then the mapping j defined by the proposition 2.5.13 is an isometric isomorphism from E to E^{**} and the weak-$*$ topology on E is well-defined as the topology $\sigma(E = E^{**}, E^*)$, which is simply the weak topology on E. The Banach-Alaoglu theorem implies that the unit ball of $E^{**} = E$, which is thus B, is weak-$*$ compact, i.e. is weakly compact, proving (iii).

Lemma 2.5.18. *Let E be a Banach space, B its closed unit ball and j be defined by the proposition 2.5.13. Then j is an homeomorphism of the topological space $\big(E, \sigma(E, E^*)\big)$ onto a dense subspace of the topological space $\big(E^{**}, \sigma(E^{**}, E^*)\big)$. The set $j(B)$ is dense for the $\sigma(E^{**}, E^*)$ topology in the closed unit ball of E^{**}.*

Proof of the lemma. The mapping $j : E \to j(E) \subset E^{**}$ is bijective and continuous whenever E is equipped with the weak topology $\sigma(E, E^*)$ and E^{**} with the weak-$*$ topology $\sigma(E^{**}, E^*)$: we consider a semi-norm q_ξ on E^{**}, $\xi \in E^*$, defined by

$$q_\xi(X) = |\langle X, \xi\rangle_{E^{**},E^*}|.$$

We evaluate for $x \in E$, $q_\xi(j(x)) = |\langle j(x), \xi\rangle_{E^{**},E^*}| = |\langle \xi, x\rangle_{E^*,E}| = p_\xi(x)$, where p_ξ is a semi-norm on E (for the weak topology). The previous equality proves that j is an homeomorphism from E to $j(E)$. A consequence of the isometry property of j given in the proposition 2.5.13 is that $j(B)$ is included in the closed unit ball B_{**} of E^{**}. Let $\widetilde{B}$ be the closure for $\sigma(E^{**}, E^*)$ of $j(B)$. First of all, B_{**} is $\sigma(E^{**}, E^*)$ compact from the Banach-Alaoglu theorem and thus is $\sigma(E^{**}, E^*)$ closed, so that $\widetilde{B} \subset B_{**}$. If there is some $X_0 \in B_{**} \backslash \widetilde{B}$, the `Hahn-Banach theorem implies` that there exists $\xi_0 \in E^*, \alpha \in \mathbb{R}, \epsilon > 0$ with

$$\forall x \in B, \quad \mathrm{Re}\langle \xi_0, x\rangle < \alpha < \alpha + \epsilon < \mathrm{Re}\langle X_0, \xi_0\rangle.$$

Since $0 \in B$, this implies $\alpha > 0$. We may thus multiply the previous inequality by $1/\alpha$ and find $\xi_1 \in E^*, \epsilon_1 > 0$ such that

$$\forall x \in B, \quad \mathrm{Re}\langle \xi_1, x\rangle < 1 < 1 + \epsilon_1 < \mathrm{Re}\langle X_0, \xi_1\rangle.$$

Using that B is stable by multiplication by $z \in \mathbb{C}$ with $|z| = 1$, we get $\|\xi_1\|_{E^*} \le 1$, implying that $1 + \epsilon_1 < \mathrm{Re}\langle X_0, \xi_1\rangle \le \|X_0\|_{E^{**}} \le 1$ which is impossible. The proof of the lemma is complete. □

Going back to the proof of the proposition, we assume that (iii) holds. Then, using the previous lemma, we see that j is continuous from

$$(E, \sigma(E, E^*)) \text{ in } (E^{**}, \sigma(E^{**}, E^*))$$

and B is compact for the $(E, \sigma(E, E^*))$ topology, we infer that $j(B)$ is compact. But the same lemma gives that $j(B)$ is dense for the $\sigma(E^{**}, E^*)$ topology in the closed unit ball of E^{**}, so $j(B)$ is closed and equal to the closed unit ball of E^{**}, implying that j is onto and (i).

We know now that (i) is equivalent to (iii), so that (ii) is equivalent to the compactness of the closed unit ball B_* of E^* in the topology $\sigma(E^*, E^{**})$. The Banach-Alaoglu theorem shows that B_* is compact for $\sigma(E^*, E)$ and if (i) holds, that topology is $\sigma(E^*, E^{**})$, so that (i) implies (ii).

Finally we assume that (ii) holds, i.e. E^* is reflexive. Let us first consider the norm-closed subspace $j(E)$ of E^{**}. The space E^{**} is reflexive since $E^* = E^{***}$ by (ii) and thus $E^{**} = E^{****}$. As a consequence, the unit ball of E^{**} is compact for the topology $\sigma(E^{**}, E^{***}) = \sigma(E^{**}, E^*)$ and thus the unit ball of the norm-closed subspace $j(E)$ is compact for the $\sigma(j(E), E^*) = \sigma(j(E), (j(E))^*)$ topology, which proves that $j(E)$ and thus E is reflexive. The proof of the proposition is complete. □

2.5.5 Examples

The Banach spaces c_0, ℓ^p

These are spaces of sequences of complex numbers $(x_k)_{k\ge1}$. We have

$$c_0 = \{(x_k)_{k\ge1},\ \lim_k x_k = 0\}, \quad \|(x_k)_{k\ge1}\| = \sup_{k\ge1} |x_k|, \tag{2.5.5}$$

$$\text{for } p \ge 1, \quad \ell^p = \{(x_k)_{k\ge1},\ \sum_{k\ge1} |x_k|^p < +\infty\}, \quad \|(x_k)_{k\ge1}\| = \big(\sum_{k\ge1} |x_k|^p\big)^{1/p}, \tag{2.5.6}$$

$$\ell^\infty = \{(x_k)_{k\ge1},\ \sup_{k\ge1} |x_k| < +\infty\}, \quad \|(x_k)_{k\ge1}\| = \sup_{k\ge1} |x_k|. \tag{2.5.7}$$

We leave to the reader as an exercise to check that these spaces are Banach spaces (see e.g. the *Théorème 3.2.5* in [9]) and ℓ^2 is a Hilbert space. Note also that the space c_0 is a closed subspace of ℓ^∞ (exercise). The spaces c_0, ℓ^p, for $1 \le p < +\infty$ are separable since the finite sequences of complex numbers with rational real and imaginary part are dense (exercise). The space ℓ^∞ is *not* separable (see e.g. the *Exercice 5.2* in *"Quatre-vingt exercices corrigés"* on the page [9]).

Duality results

Let us prove that $c_0^* = \ell^1$. We consider the mapping

$$\begin{array}{ccc} c_0 \times \ell^1 & \longrightarrow & \mathbb{C} \\ (x,y) & \mapsto & \sum_{k\ge 1} x_k \overline{y_k} := (x,y) \end{array} \quad \text{and we have } |(x,y)| \le \|x\|_{c_0}\|y\|_{\ell^1}. \tag{2.5.8}$$

As a consequence, we have a mapping $\ell^1 \ni y \mapsto j(y) \in c_0^*$ with $j(y)\cdot x = (x,y)$. The mapping j is linear, sends ℓ^1 into c_0^* (from (2.5.8)) and that inequality proves as well that j is continuous: $\|j(y)\|_{c_0^*} \le \|y\|_{\ell^1}$. On the other hand, for a given y in ℓ^1, $N \in \mathbb{N}^*$, choosing $x_k = y_k/|y_k|$ when $y_k \neq 0$ and $k \le N$, $x_k = 0$ otherwise, we have $x = (x_k)_{k\ge 1} \in c_0$, $\|x\|_{c_0} \le 1$,

$$\|j(y)\|_{c_0^*} = \sup_{\|x\|_{c_0}\le 1} |(x,y)| \ge \sum_{1\le k\le N} |y_k|, \quad \text{for all } N \ge 1,$$

so that $\|j(y)\|_{c_0^*} = \|y\|_{\ell^1}$. As a result $j(\ell^1)$ is a closed subspace of c_0^* which is isomorphic to ℓ^1. We need to prove that j is onto. Let us take $\xi \in c_0^*$; we define for $j \ge 1$, $e_j = (\delta_{j,k})_{k\ge 1}$ $(\in c_0)$. We choose some real numbers θ_j so that $e^{i\theta_j}\xi\cdot e_j = |\xi\cdot e_j|$ and we consider $x = (e^{i\theta_1},\dots,e^{i\theta_n},0,0,0\dots) \in c_0$, $\|x\|_{c_0} = 1$, so that

$$\xi\cdot x = \sum_{1\le j\le n} e^{i\theta_j}\xi\cdot e_j = \sum_{1\le j\le n} |\xi\cdot e_j|.$$

As a result, we have for all $n \ge 1$, $\sum_{1\le j\le n} |\xi\cdot e_j| \le \|\xi\|_{c_0^*}\|x\|_{c_0} = \|\xi\|_{c_0^*}$, proving that $y = (\xi\cdot e_j)_{j\ge 1} \in \ell^1$. Now, we have for $x = (x_j)_{j\ge 1} \in c_0$, by the continuity of ξ,

$$\xi\cdot x = \lim_{n\to+\infty} \sum_{1\le j\le n} x_j(\xi\cdot e_j) = (x,(\xi\cdot e_j)_{j\ge 1}) = (x,y),$$

proving that $\xi = j(y)$ for some $y \in \ell^1$ and the sought surjectivity.

We leave to the reader the proof that $(\ell^1)^* = \ell^\infty$, which is somewhat analogous to the previous one.

Let us now prove that $(\ell^\infty)^*$, which is the bidual of ℓ^1, is (much) larger than ℓ^1. The space c_0 is a closed proper subspace of ℓ^∞, and the corollary of the Hahn-Banach theorem 2.2.4 allows us to construct $\xi_0 \in (\ell^\infty)^*$ such that

$$\xi_{0|c_0} = 0, \quad \xi_0\cdot x_0 = 1, \quad x_0 = (1,1,1,\dots) \in \ell^\infty\backslash c_0. \tag{2.5.9}$$

As a consequence, the mapping $j : \ell^1 \longrightarrow (\ell^1)^{**} = (\ell^\infty)^*$, defined in the proposition 2.5.13, is not onto since there is no $y \in \ell^1$ such that $j(y) = \xi_0$: otherwise, we would have for $x \in \ell^\infty$,

$$\langle \xi_0, x\rangle_{(\ell^\infty)^*,\ell^\infty} = \langle j(y), x\rangle_{(\ell^1)^{**},(\ell^1)^*} = \langle x, y\rangle_{(\ell^1)^*,\ell^1} = \sum_{j\ge 1} \overline{x_j} y_j,$$

and since $\langle \xi_0, e_j\rangle_{(\ell^\infty)^*,\ell^\infty} = 0$, that would imply $y_j = 0$ for all $j \ge 1$, and $\xi_0 = 0$, contradicting (2.5.9). The next proposition is summarizing the situation.

Proposition 2.5.19. *We consider the spaces* c_0, ℓ^p *defined above. When* $1 < p < +\infty$ *we define* $p' \in]1, +\infty[$ *by the identity* $\frac{1}{p} + \frac{1}{p'} = 1$*. Then we have*

$$(\ell^1)^* = \ell^\infty, (\ell^1)^{**} \neq \ell^1, \quad \ell^1 \text{ is not reflexive,} \tag{2.5.10}$$

$$1 < p < \infty, \quad (\ell^p)^* = \ell^{p'}, (\ell^p)^{**} = \ell^p, \quad \ell^p \text{ is reflexive } {\scriptstyle (1<p<\infty)}, \tag{2.5.11}$$

$$\ell^\infty \text{ is not reflexive,} \tag{2.5.12}$$

$$c_0^* = \ell^1, c_0^{**} = (\ell^1)^* = \ell^\infty \neq c_0, \quad c_0 \text{ is not reflexive.} \tag{2.5.13}$$

Proof. The first and the fourth line are proven above, the second line will be proven in the next section in a more general setting, the third line is a consequence of the proposition 2.5.17, since ℓ^1 is not reflexive. □

2.5.6 The dual of $L^p(X, \mathcal{M}, \mu)$, $1 \leq p < +\infty$.

Let $(X, \mathcal{M}, \mu)$ be a measured space and μ a positive measure. We consider the Banach spaces $L^p(X, \mathcal{M}, \mu)$ and we want to determine their dual spaces whenever $1 \leq p < +\infty$ and the measure μ is σ-finite. The definitions and first properties of these spaces can be found for instance in the section 3.2 of the third chapter, *Espaces de fonctions intégrables*, on the page [9]. When $X = \mathbb{R}^n$ and μ is the Lebesgue measure, we shall simply write $L^p(\mathbb{R}^n)$ to denote that space. For $1 \leq p < +\infty$, we shall note p' the conjucate index such that

$$\frac{1}{p} + \frac{1}{p'} = 1$$

($p' = p/(p-1)$ if $1 < p < +\infty$ and $p' = +\infty$ if $p = 1$).

Theorem 2.5.20. *Let* $(X, \mathcal{M}, \mu)$ *be a measured space,* μ *a* σ*-finite positive measure and* $1 \leq p < +\infty$*. We shall note* $L^p(X, \mathcal{M}, \mu) = L^p(\mu)$*. Let* $\xi \in (L^p(\mu))^*$*. Then there exists a unique* $g \in L^{p'}(\mu)$ *such that*

$$\forall f \in L^p(\mu), \quad \langle \xi, f \rangle = \int_X f g d\mu, \quad \|\xi\|_{(L^p(\mu))^*} = \|g\|_{L^{p'}(\mu)},$$

so that, for $1 \leq p < +\infty$, $\quad (L^p(\mu))^* = L^{p'}(\mu)$.

N.B. We may consider the sesquilinear mapping

$$\begin{array}{rccc} \Phi : L^p(\mu) \times L^{p'}(\mu) & \longrightarrow & \mathbb{C} \\ (f, g) & \mapsto & \int_X f\bar{g} d\mu. \end{array}$$

which is well-defined, thanks to the Hölder inequality, $|\Phi(f, g)| \leq \|f\|_{L^p} \|g\|_{L^{p'}}$ (see e.g. the *Théorème 3.1.5* in the third chapter of [9]). Let us check that the mapping $L^{p'}(\mu) \ni g \mapsto \Phi_g \in (L^p(\mu))^*$ given by $\Phi_g(f) = \Phi(f, g)$ is isometric, i.e.

$$\|\Phi_g\|_{(L^p)^*} = \sup_{\|f\|_{L^p}=1} \left| \int_X f\bar{g} d\mu \right| = \|g\|_{L^{p'}}. \tag{2.5.14}$$

In fact the inequality $\|\Phi_g\|_{(L^p)^*} \le \|g\|_{L^{p'}}$ follows from the Hölder inequality and for a given $0 \ne g \in L^{p'}$ and $1 < p < +\infty$ we have, with

$$f = \frac{g}{|g|}|g|^{p'/p}\mathbf{1}_{g\ne 0}\|g\|_{L^{p'}}^{-p'/p}, \quad \|f\|_{L^p}^p = \int_X |g|^{p'}d\mu\|g\|_{L^{p'}}^{-p'} = 1,$$

the equality $\int_X f\bar{g}d\mu = \int_X |g|^{1+\frac{p'}{p}}d\mu\|g\|_{L^{p'}}^{-p'/p} = \|g\|_{L^{p'}}^{-\frac{p'}{p}+p'} = \|g\|_{L^{p'}}$. The same type of argument works for $p = 1$: here $p' = +\infty$ and for $0 \ne g \in L^\infty$ we choose $\epsilon > 0$ such that $\mu\big(\{|g| \ge \|g\|_{L^\infty} - \epsilon\}\big) > 0$ and we set

$$f = \frac{g}{|g|}\frac{\mathbf{1}\big(|g| \ge \|g\|_{L^\infty} - \epsilon\big)}{\mu\big(\{|g| \ge \|g\|_{L^\infty} - \epsilon\}\big)}, \quad \text{so that } \|f\|_{L^1} = 1,$$

and

$$\Phi_g(f) = \int_X |g|\frac{\mathbf{1}\big(|g| \ge \|g\|_{L^\infty} - \epsilon\big)}{\mu\underbrace{\big(\{|g| \ge \|g\|_{L^\infty} - \epsilon\}\big)}_{G_\epsilon}}d\mu = \frac{1}{\mu(G_\epsilon)}\int_{\|g\|_{L^\infty}-\epsilon\le|g|\le\|g\|_{L^\infty}} |g|d\mu$$

$$\ge \frac{1}{\mu(G_\epsilon)}(\|g\|_{L^\infty} - \epsilon)\mu(G_\epsilon) = \|g\|_{L^\infty} - \epsilon.$$

As a result $\|\Phi_g(f)\|_{(L^1)^*} = \|g\|_{L^\infty}$. As a result the mapping

$$\psi : L^{p'}(\mu) \longrightarrow (L^p(\mu))^*, \quad \psi(g) = \Phi_g$$

is injective and isometric and thus has a closed image isomorphic to $L^{p'}(\mu)$. The main difficulty of the above theorem is the proof that ψ is indeed *onto* when $1 \le p < +\infty$. We have already seen some examples (see (2.5.10)) showing that for $p = \infty$, the dual space of L^∞, i.e. the bidual of L^1 is much larger than L^1 and that the mapping ψ is not onto in general in that case[11].

Proof of the theorem. Let then $1 \le p < \infty$ and $\xi \in (L^p(\mu))^*$. *We assume first that* $\mu(X) < \infty$. For $E \in \mathcal{M}$, we define

$$\lambda(E) = \xi(\mathbf{1}_E). \tag{2.5.15}$$

If A, B are measurable and disjoint, we have $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$, which implies that λ is finitely additive. Let us consider $E = \cup_{j\in\mathbb{N}}E_j$ with $E_j \cap E_k = \emptyset$ if $j \ne k$ (all $E_j \in \mathcal{M}$). With $A_k = \cup_{j\le k}E_k$, we have

$$\|\mathbf{1}_E - \mathbf{1}_{A_k}\|_{L^p}^p = \int_{E\backslash A_k} d\mu = \mu(E\backslash A_k).$$

By the Lebesgue dominated convergence theorem, we know that $\lim_k \mu(E\backslash A_k) = 0$, and since ξ is continuous on L^p, we get that $\lim_k \lambda(A_k) = \lambda(E)$, i.e.

$$\lambda(E) = \sum_{k\in\mathbb{N}} \lambda(E_j),$$

[11] It is true however that ψ is an isometric one-to-one mapping, even for $p = \infty$: for $g \in L^1$, we have $\Phi_g(\frac{g}{|g|}\mathbf{1}_{\{g\ne 0\}}) = \|g\|_{L^1}$.

so that λ is a complex measure. Moreover if $\mu(E) = 0$, we have $\mathbf{1}_E = 0$ μ-a.e. and $\mathbf{1}_E = 0$ in L^p implying $\lambda(E) = 0$. As a result we have $\lambda \ll \mu$. We may apply the Radon-Nikodym theorem: there exists $g \in L^1(\mu)$ such that

$$\xi(\mathbf{1}_E) = \lambda(E) = \int_E g d\mu = \int_X g \mathbf{1}_E d\mu.$$

Thus, by the linearity of ξ, for any simple function f (finite linear combination of characteristic functions of measurable sets) we get

$$\xi(f) = \int_X f g d\mu, \qquad \text{which is true as well for } f \in L^\infty(\mu), \tag{2.5.16}$$

since a function in $L^\infty(\mu)$ is a uniform limit of simple functions. If $p = 1$, for all $E \in \mathcal{M}$, we have

$$\left|\int_X \mathbf{1}_E g d\mu\right| = |\xi(\mathbf{1}_E)| \le \|\xi\|_{(L^1)^*} \|\mathbf{1}_E\|_{L^1} = \mu(E) \|\xi\|_{(L^1)^*},$$

and thus $|g(x)| \le \|\xi\|_{(L^1)^*}$ μ-a.e., implying

$$\|g\|_{L^\infty(\mu)} \le \|\xi\|_{(L^1)^*}. \tag{2.5.17}$$

If $1 < p < \infty$, we consider a measurable function α such that $\alpha g = |g|$, and we define

$$f_n = \mathbf{1}_{E_n} |g|^{p'-1} \alpha, \quad E_n = \{|g| \le n\}.$$

We have $|\alpha| = 1$ on the set $\{g \ne 0\}$ and $p(p'-1) = p'$ so that

$$|f|_n^p = \mathbf{1}_{E_n} |g|^{p'}, \quad |f_n| \le n^{p'},$$

and applying (2.5.16) to the L^∞ function f_n, we get

$$\xi(f_n) = \int_X \mathbf{1}_{E_n} |g|^{p'-1} \alpha g d\mu = \int_{E_n} |g|^{p'} d\mu$$

and $\left|\int_{E_n} |g|^{p'} d\mu\right| \le \|\xi\|_{(L^p)^*} \|f_n\|_{L^p} = \|\xi\|_{(L^p)^*} \left(\int_{E_n} |g|^{p'} d\mu\right)^{1/p}$ and this implies

$$\left|\int_{E_n} |g|^{p'} d\mu\right|^{1-\frac{1}{p}=\frac{1}{p'}} \le \|\xi\|_{(L^p)^*}.$$

The Beppo-Levi theorem then implies that $\|g\|_{L^{p'}} \le \|\xi\|_{(L^p)^*}$. Since ξ and $f \mapsto \int f g d\mu$ coincide (and are continuous) on $L^\infty(\mu)$, which is dense in $L^p(\mu)$, they coincide on $L^p(\mu)$ and $\|\xi\|_{(L^p)^*} = \|g\|_{L^{p'}}$. The proof is complete in the case $\mu(X) < \infty$. *Let us now assume that $\mu(X) = +\infty$.*

Lemma 2.5.21. *There exists $w \in L^1(\mu)$ such that $\forall x \in X, 0 < w(x) < 1$.*

Proof. Since μ is σ-finite, we have $X = \cup_{n\ge 1} E_n, E_n \in \mathcal{M}, \mu(E_n) < \infty$. We define

$$w_n(x) = \frac{\mathbf{1}_{E_n}(x)}{2^n(1+\mu(E_n))}, \quad w(x) = \sum_{n\ge 1} w_n(x). \tag{2.5.18}$$

Since $X = \cup_{n\ge 1} E_n$, we have always $w(x) > 0$ and

$$w(x) \le \sum_{n\ge 1} 2^{-n}(1+\mu(E_n))^{-1} < \sum_{n\ge 1} 2^{-n} = 1, \int_X w d\mu = \sum_{n\ge 1} \frac{\mu(E_n)}{2^n(1+\mu(E_n))} < \infty. \square$$

We consider now the finite measure $d\nu = w d\mu$ ($\nu(X) = \int_X w d\mu < \infty$) and the linear isometries

$$\left.\begin{matrix} L^p(\nu) & \longrightarrow & L^p(\mu) \\ F & \mapsto & F w^{1/p} \end{matrix}\right\}, \quad \left\{\begin{matrix} L^p(\mu) & \longrightarrow & L^p(\nu) \\ f & \mapsto & f w^{-1/p} \end{matrix}\right., \tag{2.5.19}$$

noting that we have

$$\|F\|^p_{L^p(\nu)} = \int_X |F|^p w d\mu = \|F w^{1/p}\|^p_{L^p(\mu)}, \ \|f\|^p_{L^p(\mu)} = \int_X |f|^p w^{-1} d\nu = \|f w^{-1/p}\|^p_{L^p(\nu)}.$$

As a consequence, if $\xi \in (L^p(\mu))^*$ we can define $\eta \in (L^p(\nu))^*$ by

$$\forall F \in L^p(\nu), \ \langle \eta, F\rangle_{(L^p(\nu))^*, L^p(\nu)} = \langle \xi, w^{1/p} F\rangle_{(L^p(\mu))^*, L^p(\mu)}, \text{ and } \|\eta\|_{(L^p(\nu))^*} = \|\xi\|_{(L^p(\mu))^*}.$$

We can use the proven result on finite measures to find $G \in L^{p'}(\nu)$ such that $\|G\|_{L^{p'}(\nu)} = \|\eta\|_{(L^p(\nu))^*}$ with $\langle \eta, F\rangle_{(L^p(\nu))^*, L^p(\nu)} = \int_X F G d\nu$ so that

$$\langle \xi, f\rangle_{(L^p(\mu))^*, L^p(\mu)} = \int_X f w^{-1/p} G w d\mu = \int_X f g d\mu, \quad g = G w^{1-\frac{1}{p}},$$

and, if $p' < \infty$, $\|\xi\|^p_{(L^p(\mu))^*} = \|G\|^p_{L^{p'}(\nu)} = \int_X |G|^{p'} w d\mu = \int_X (|G| w^{1-\frac{1}{p}})^{p'} d\mu = \|g\|^p_{L^{p'}(\mu)}$. If $p = 1, p' = \infty$, we have $g = G$ and $\|\xi\|_{(L^1(\mu))^*} = \|G\|_{L^\infty(\nu)} = \|g\|_{L^\infty(\nu)}$. The proof of the theorem is complete. $\square$

2.5.7 Transposition

Definition 2.5.22. *Let E, F be Banach spaces and $A \in \mathcal{L}(E, F)$. The transposed of A is the mapping ${}^tA$ of $\mathcal{L}(F^*, E^*)$ defined by*

$$\forall \eta \in F^*, \forall x \in E, \quad \langle {}^tA\eta, x\rangle_{E^*, E} = \langle \eta, Ax\rangle_{F^*, F}.$$

We have

$$\|A\|_{\mathcal{L}(E,F)} = \|{}^tA\|_{\mathcal{L}(F^*, E^*)}. \tag{2.5.20}$$

We note that ${}^tA$ is obviously a linear mapping and that ${}^tA\eta \in E^*$ for $\eta \in F^*$ since $\sup_{\|x\|_E = 1} |\langle {}^tA\eta, x\rangle_{E^*, E}| = \sup_{\|x\|_E = 1} |\langle \eta, Ax\rangle_{E^*, E}| \le \|\eta\|_{F^*} \|A\|_{\mathcal{L}(E,F)}$. On the other hand, the theorem 2.5.1 implies that

$$\begin{aligned} \|A\|_{\mathcal{L}(E,F)} &= \sup_{\|x\|_E = 1} \|Ax\|_F = \sup_{\|x\|_E = 1, \|\eta\|_{F^*} = 1} |\langle \eta, Ax\rangle_{F^*, F}| \\ &= \sup_{\|x\|_E = 1, \|\eta\|_{F^*} = 1} |\langle {}^tA\eta, x\rangle_{E^*, E}| = \sup_{\|\eta\|_{F^*} = 1} \|{}^tA\eta\|_{E^*} = \|{}^tA\|_{\mathcal{L}(F^*, E^*)}. \end{aligned}$$

2.6 Appendix

2.6.1 Filters

Definition 2.6.1. *Let X be a set. A set $\mathscr{F} \subset \mathscr{P}(X)$ is said to be a filter on X if*

(1) $F \in \mathscr{F}, X \supset V \supset F \Longrightarrow V \in \mathscr{F}$,

(2) $F_1, F_2 \in \mathscr{F} \Longrightarrow F_1 \cap F_2 \in \mathscr{F}$,

(3) $\emptyset \notin \mathscr{F}$.

Of course (2) is equivalent to the fact that a finite intersection of elements of $\mathscr{F}$ is still an element of $\mathscr{F}$. Let us give a couple of examples.

Let $(X, \mathscr{O})$ be a non-empty topological space, $x \in X$, and $\mathscr{V}_x$ the set of neighborhoods of x. Since $V \in \mathscr{V}_x$ is equivalent to $\exists \Omega \in \mathscr{O}, x \in \Omega \subset V$, the properties (1), (3) are obviously satisfied as well as (2) from (2) in the definition 1.1.1.

Let X be a set and $\emptyset \neq A \subset X$. We define $\mathscr{F}$ as the set of subsets of X containing A: it is obviously a filter on X.

Let X be an infinite set and $\mathscr{F}$ be the set of subsets of X with a finite complement: (3) is satisfied since X is infinite, (1) is obvious as well as (2) since a finite union of finite sets is finite.

Definition 2.6.2. *Let X be a set and $\mathscr{F}_1, \mathscr{F}_2$ be filters on X. We shall say that $\mathscr{F}_2$ is finer than $\mathscr{F}_1$ whenever $\mathscr{F}_1 \subset \mathscr{F}_2$.*

Remark 2.6.3. Let $(\mathscr{F}_j)_{j \in J}$ be a non-empty family of filters on a set X; then $\cap_{j \in J} \mathscr{F}_j$ is (obviously) a filter on X.

Proposition 2.6.4. *Let X be a set and $\mathscr{E}$ be a family of subsets of X such that for any finite family $\{E_j\}_{1 \leq j \leq N} \subset \mathscr{E}$, $\cap_{1 \leq j \leq N} E_j \neq \emptyset$ (this property is called the non-empty-finite-intersection-property). Then there exists a unique filter $\widetilde{\mathscr{E}}$ on X such that $\widetilde{\mathscr{E}} \supset \mathscr{E}$ and if $\mathscr{F}$ is a filter on X containing $\mathscr{E}$, one has $\mathscr{F} \supset \widetilde{\mathscr{E}}$. The filter $\widetilde{\mathscr{E}}$ is called the filter generated by $\mathscr{E}$ and is the intersection of the filters containing $\mathscr{E}$.*

Proof. We define $\widetilde{\mathscr{E}} = \{Y \subset X, \exists E_1, \dots, E_N \in \mathscr{E}, \quad Y \supset \cap_{1 \leq j \leq N} E_j\}$. It is a filter on X since the non-empty-finite-intersection-property ensures that (3) is satisfied and (1), (2) are obvious. If $\mathscr{F}$ is a filter as in the proposition, it must contain all the subsets $\cap_{1 \leq j \leq N} E_j$ whenever $E_j \in \mathscr{E}$ and thus $\widetilde{\mathscr{E}}$. The last statement and the uniqueness follow from the remark 2.6.3. □

Chapter 3

Introduction to the Theory of Distributions

3.1 Test Functions and Distributions

3.1.1 Smooth compactly supported functions

Let Ω be an open subset of $\mathbb{R}^n$; we define $C_c^\infty(\Omega)$ as the vector space of complex-valued compactly supported functions defined on Ω. Even in the case $n = 1$ and $\Omega = \mathbb{R}$, it is not completely obvious that this space is not reduced to $\{0\}$. We leave to the reader as an exercise to check that the function

$$\rho_0(t) = \begin{cases} e^{-t^{-1}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases} \tag{3.1.1}$$

is a C^∞ function on $\mathbb{R}$. Starting with ρ_0, we may define a function ρ on $\mathbb{R}^n$ by

$$\rho(x) = \rho_0(1 - \|x\|^2) \tag{3.1.2}$$

and we see right away that $\rho \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \rho = \bar{B}(0,1)$. Here we have defined the support of ρ as the closure of the set $\{x \in \mathbb{R}^n, \rho(x) \neq 0\}$. Although that definition is fine when we deal with a continuous function, it will produce strange results if we want to define the support of a function in $L^1(\mathbb{R})$: for instance the characteristic function of $\mathbb{Q}$ is 0 a.e. and thus 0 as a function of $L^1(\mathbb{R})$, nevertheless the above set is $\mathbb{R}$. It is better to use the following definition, say for a function in $u \in L^1_{\text{loc}}(\Omega)$, Ω open subset of $\mathbb{R}^n$:

$$\operatorname{supp} u = \{x \in \Omega, \nexists U \text{open} \in \mathscr{V}_x, u_{|U} = 0\}, \quad (\operatorname{supp} u)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x, u_{|U} = 0\}. \tag{3.1.3}$$

The above definition makes sense for an L^1_{loc} function with $u_{|U} = 0$ meaning $u = 0$ a.e. in U. The smooth compactly supported functions are very useful as mollifiers, as shown by the next proposition.

Proposition 3.1.1. *Let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For $\epsilon > 0$, we define $\phi_\epsilon(x) = \epsilon^{-n}\phi(x\epsilon^{-1})$. Then, if $f \in C_c^m(\mathbb{R}^n)$, $\lim_{\epsilon\to 0_+} \phi_\epsilon * f = f$ (convergence in $C_c^m(\mathbb{R}^n)$) and if $f \in L^p(\mathbb{R}^n)$ with $1 \le p < +\infty$, $\lim_{\epsilon\to 0_+} \phi_\epsilon * f = f$ (convergence in $L^p(\mathbb{R}^n)$). In both cases the function $\phi_\epsilon * f$ is C^∞.*

Proof. We write

$$(\phi_\epsilon * f)(x) - f(x) = \int \phi_\epsilon(x-y)f(y)dy - f(x) = \int \phi(y)\big(f(x-\epsilon y) - f(x)\big)dy,$$

so that, if $\operatorname{supp}\phi \subset \bar{B}(0, R_0)$,

$$|(\phi_\epsilon * f)(x) - f(x)| \le \int |\phi(y)|dy \sup_{|x_1 - x_2| \le \epsilon R_0} |f(x_1) - f(x_2)|.$$

The function f is continuous and compactly supported, so is uniformly continuous on $\mathbb{R}^n$ (an easy consequence of the Heine theorem 1.5.10), thus

$$\lim_{\epsilon \to 0_+} \big(\sup_{x \in \mathbb{R}^n} |(\phi_\epsilon * f)(x) - f(x)|\big) = 0,$$

yielding the uniform convergence of $\phi_\epsilon * f$ towards f. If f is C_c^m, a simple differentiation under the integral sign (see e.g. the *Théorème 3.3.2.* in [9]) gives as well the uniform convergence of the derivatives, up to order m. The smoothness of $\phi_\epsilon * f$ for $\epsilon > 0$ is due to the same theorem when $f \in C_c^m(\mathbb{R}^n)$, since we have $(\phi_\epsilon * f)(x) = \int \phi_\epsilon(x-y)f(y)dy$.

Remark 3.1.2. *We have not defined a topology on the vector space $C_c^m(\mathbb{R}^n)$, but at the moment it will be enough for us to say that a sequence $(u_k)_{k\in\mathbb{N}}$ of functions in $C_c^m(\mathbb{R}^n)$ is converging if it converges in $C^m(\mathbb{R}^n)$ and if there exists a compact set K such that, for all $k \in \mathbb{N}$, $\operatorname{supp} u_k \subset K$.*

We note in particular that these conditions are satisfied by the "sequences" $(\phi_\epsilon * f)_{\epsilon > 0}$ since for $\epsilon \le 1$, $\operatorname{supp}(\phi_\epsilon * f) \subset \operatorname{supp} f + \operatorname{supp}\phi_\epsilon \subset \operatorname{supp} f + \operatorname{supp}\phi$.

Let us now take $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$. With $\psi \in C_c^0(\mathbb{R}^n)$, we have

$$f * \phi_\epsilon - f = (f - \psi) * \phi_\epsilon + \psi * \phi_\epsilon - \psi + \psi - f,$$

so that

$$\begin{aligned}\|f * \phi_\epsilon - f\|_{L^p(\mathbb{R}^n)} &\le (1 + \|\phi\|_{L^1})\|f - \psi\|_{L^p(\mathbb{R}^n)} + \|\psi * \phi_\epsilon - \psi\|_{L^p(\mathbb{R}^n)} \\ &\le (1 + \|\phi\|_{L^1})\|f - \psi\|_{L^p(\mathbb{R}^n)} + \underbrace{|\operatorname{supp}\phi + \epsilon|}_{\text{Lebesgue measure}}{}^{1/p}\|\psi * \phi_\epsilon - \psi\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Since $\psi \in C_c^\infty(\mathbb{R}^n)$, the previous convergence argument implies the inequality

$$\limsup_{\epsilon \to 0_+} \|f * \phi_\epsilon - f\|_{L^p(\mathbb{R}^n)} \le (1 + \|\phi\|_{L^1})\|f - \psi\|_{L^p(\mathbb{R}^n)}, \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^n).$$

The density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ (see e.g. the *Théorème 3.4.1* in [9]) yields the result. For $\epsilon > 0, R > 0$, all the functions

$$\psi_{R,\epsilon}(y) = \sup_{|x| \le R} |(\partial_x^\alpha \phi_\epsilon)(x-y)f(y)|$$

belong to $L^1(\mathbb{R}^n_y)$ since

$$\int \psi_{R,\epsilon}(y)dy \le \|f\|_{L^p(\mathbb{R}^n)}\Big(\int \sup_{|x| \le R} |(\partial_x^\alpha \phi_\epsilon)(x-y)|^{p'}dy\Big)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

and $\operatorname{supp}\phi \subset \bar{B}_{R_0}$ gives that $|x - y| \le \epsilon R_0, |x| \le R$ imply $|y| \le \epsilon R_0 + R$, and the finiteness of the integral above, proving the smoothness of $\phi_\epsilon * f$ for $\epsilon > 0$. □

N.B. The result of the proposition does not extend to the case $p = \infty$, since the uniform convergence of the continuous function $f * \phi_\epsilon$ would imply the continuity of the limit.

It will be also useful to use the compactly supported functions to construct some partitions of unity and, to begin with, to find C_c^∞ functions identically equal to 1 near a compact set.

Lemma 3.1.3. *Let Ω be an open subset of $\mathbb{R}^n$ and K be a compact subset of Ω. Then there exists a function $\varphi \in C_c^\infty(\Omega; [0,1])$ such that $\varphi = 1$ on a neighborhood of K.*

Proof. We claim that there exists $\epsilon_0 > 0$ such that $K + \epsilon_0 B_1 \subset \Omega$, ($B_1$ is the open unit ball). First we note that

$$d(K, \Omega^c) = \inf_{x \in K, y \in \Omega^c} |x - y| > 0, \tag{3.1.4}$$

otherwise, we could find sequences $(x_k)_{k\ge1}$ in K, $(y_k)_{k\ge1}$ in Ω^c such that $\lim_k |x_k - y_k| = 0$, and since K is compact, we may suppose that (x_k) converges with limit $x \in K$, implying $\Omega^c \ni \lim_k y_k = x$, which is impossible since $K \subset \Omega$. As a result, we have with $\epsilon_0 = d(K, \Omega^c)$

$$K + \epsilon_0 B_1 \subset \Omega,$$

otherwise, we could find $|t| < 1, x \in K$ such that $x + \epsilon_0 t = y \in \Omega^c$, implying $|x - y| < \epsilon_0 = d(K, \Omega^c)$, which is impossible. With the function ρ defined in 3.1.2, we define with $0 < \epsilon \le \frac{\epsilon_1}{2} < \frac{\epsilon_0}{4}$,

$$\varphi(x) = \int \mathbf{1}_{K+\epsilon_1 \bar{B}_1}(y) \rho\big((x-y)\epsilon^{-1}\big)\epsilon^{-n} dy \Big(\int \rho(t)dt\Big)^{-1}.$$

The function φ is C^∞ and such that

$$\operatorname{supp}\varphi \subset K + \epsilon_1 \bar{B}_1 + \epsilon \bar{B}_1 \subset K + \frac{3}{2}\epsilon_1 \bar{B}_1 \subset \underbrace{K + \frac{3}{4}\epsilon_0 \bar{B}_1}_{\text{compact}} \subset K + \epsilon_0 B_1 \subset \Omega.$$

Moreover $\varphi = 1$ on $K + \frac{\epsilon_1}{2}\bar{B}_1$ (which is a neighborhood of K), since if $x \in K + \frac{\epsilon_1}{2}\bar{B}_1$, we have, for y satisfying $|x - y| \le \epsilon$, that $y \in K + \frac{\epsilon_1}{2}\bar{B}_1 + \epsilon \bar{B}_1 \subset K + \epsilon_1 \bar{B}_1$. As a result, with $\tilde{\rho} = \rho\big(\int \rho(t)dt\big)^{-1}$, for $x \in K + \frac{\epsilon_1}{2}\bar{B}_1$, we have

$$1 = \int \tilde{\rho}((x-y)\epsilon^{-1})\epsilon^{-n} dy = \int \tilde{\rho}((x-y)\epsilon^{-1})\epsilon^{-n} \mathbf{1}_{K+\epsilon_1 \bar{B}_1}(y) dy = \varphi(x).$$

We note also that, since $\tilde{\rho} \ge 0$ with integral 1, $\mathbf{1}_L(y) \in [0,1]$, we have, for all $x \in \mathbb{R}^n$, $0 \le \varphi(x) \le 1$. The proof of the lemma is complete. □

3.1.2 Distributions

Definition 3.1.4. *Let Ω be an open set of $\mathbb{R}^n$ and let $T : C_c^\infty(\Omega) \longrightarrow \mathbb{C}$ be a linear form with the following continuity property,*

$$\forall K \, compact \subset \Omega, \exists C_K > 0, \exists N_K \in \mathbb{N}, \forall \varphi \in C_K^\infty(\Omega), \ |\langle T, \varphi\rangle| \le C_K \sup_{\substack{|\alpha|\le N_K \\ x\in\mathbb{R}^n}} |(\partial_x^\alpha \varphi)(x)|, \tag{3.1.5}$$

where $C_K^\infty(\Omega) = \{\varphi \in C_c^\infty(\Omega), \operatorname{supp}\varphi \subset K\}$.

N.B. We shall use also the notation $\mathscr{D}(\Omega)$ for the space of test functions $C_c^\infty(\Omega)$ and $\mathscr{D}'(\Omega)$ for the space of distributions on Ω. We have not introduced a topology on $\mathscr{D}(\Omega)$ but we have defined a notion of converging sequence with the remark 3.1.2. It would have been certainly more elegant to start with the display of the natural topological structure on $\mathscr{D}(\Omega)$, at the (heavy) cost of having to deal with a non-metrizable locally convex topology defined by an uncountable family of semi-norms. The study of inductive limits of increasing sequences of Fréchet spaces is outlined in the appendix **??**. Anyhow, one should think of $\mathscr{D}'(\Omega)$ as the topological dual of $\mathscr{D}(\Omega)$, a view supported by the next lemmas and remarks.

Remark 3.1.5. With $\mathscr{D}_K(\Omega) = C_K^\infty(\Omega)$, we have, using the sequence of compact sets $(K_j)_{j\ge1}$ of the lemma 2.3.1

$$\mathscr{D}(\Omega) = \cup_{j\ge1} \mathscr{D}_{K_j}(\Omega)$$

and it is not difficult to see that each $\mathscr{D}_{K_j}(\Omega)$ is a Fréchet space with the natural countable family of semi-norms given by $p_{K_j,m}(u) = \sup_{\substack{|\alpha|\le m \\ x\in K_j}} |(\partial_x^\alpha u)(x)|$. If we want to use the countable family $p_{K_j,m}$, we end-up with the topology on the Fréchet space $C^\infty(\Omega)$ as described in the subsection 2.3.3; the actual topology on $\mathscr{D}(\Omega)$ is finer and it is important to understand that, with ρ defined in (3.1.2) (say with $n = 1$), the sequence $(u_k)_{k\in\mathbb{N}}$, given by

$$u_k(x) = \rho(x - k)$$

does converge to 0 in the Fréchet space $C^\infty(\mathbb{R})$ but is *not* convergent in $C_c^\infty(\mathbb{R})$, since the second condition of the remark 3.1.2 is not satisfied: there is no compact subset K of $\mathbb{R}$ such that $\forall k \in \mathbb{N}, \operatorname{supp} u_k \subset K$.

Remark 3.1.6. Note that a linear form T on $C_c^\infty(\Omega)$ is a distribution if and only if, for all compact subsets K of Ω, its restriction to the Fréchet space $\mathscr{D}_K(\Omega)$ is continuous.

A L^1_{loc} function is a distribution: for Ω open subset of $\mathbb{R}^n$, for $f \in L^1_{\text{loc}}(\Omega)$, we define for $\varphi \in \mathscr{D}(\Omega)$

$$\langle T, \varphi\rangle = \int f(x)\varphi(x)dx \Longrightarrow |\langle T, \varphi\rangle| \le \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\operatorname{supp}\varphi} |f(x)|dx, \tag{3.1.6}$$

so that (3.1.5) is satisfied with $C_K = \int_K |f(x)|dx, N_K = 0$. Moreover the canonical mapping from $L^1_{\text{loc}}(\Omega)$ into $\mathscr{D}'(\Omega)$ is injective, as shown by the next lemma.

Lemma 3.1.7. *Let Ω be an open subset of $\mathbb{R}^n$, $f \in L^1_{loc}(\Omega)$ such that, for all $\varphi \in \mathscr{D}(\Omega)$, $\int f(x)\varphi(x)dx = 0$. Then we have $f = 0$.*

Proof. Let K be a compact subset of Ω and $\chi \in \mathscr{D}(\Omega)$ equal to 1 on a neighborhood of K as in the lemma 3.1.3. With ϕ as in the proposition 3.1.1, we get that $\lim_{\epsilon\rightarrow 0_+} \phi_\epsilon * (\chi f) = \chi f$ in $L^1(\mathbb{R}^n)$. We have

$$\big(\phi_\epsilon * (\chi f)\big)(x) = \int f(y) \underbrace{\chi(y)\phi\big((x-y)\epsilon^{-1}\big)\epsilon^{-n}}_{=\varphi_x(y)} dy, \quad \operatorname{supp}\varphi_x \subset K, \varphi_x \in \mathscr{D}(\Omega),$$

and from the assumption of the lemma, we obtain $\big(\phi_\epsilon * (\chi f)\big)(x) = 0$ for all x, implying $\chi f = 0$ from the convergence result; the conclusion follows. □

We note that it makes sense to restrict a distribution $T \in \mathscr{D}'(\Omega)$ to an open subset $U \subset \Omega$: just define

$$\langle T_{|U}, \varphi\rangle_{\mathscr{D}'(U),\mathscr{D}(U)} = \langle T, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}, \tag{3.1.7}$$

and $T_{|U}$ is obviously a distribution on U. With this in mind, we can define the support of a distribution exactly as in (3.1.8).

Definition 3.1.8. *Let Ω be an open subset of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. We define the support of T as*

$$\operatorname{supp} T = \{x \in \Omega, \forall U \textit{ open} \in \mathscr{V}_x,\ T_{|U} \neq 0\}. \tag{3.1.8}$$

We define the C^∞ singular support of T as

$$\operatorname{singsupp} T = \{x \in \Omega, \forall U \textit{ open} \in \mathscr{V}_x,\ T_{|U} \notin C^\infty(U)\}. \tag{3.1.9}$$

Note that the support and the singular support are closed subset of Ω since their complements in Ω are open: we have

$$(\operatorname{supp} T)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x,\ T_{|U} = 0\}, \tag{3.1.10}$$

$$(\operatorname{singsupp} T)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x,\ T_{|U} \in C^\infty(U)\}. \tag{3.1.11}$$

A simple consequence of that definition is that, for $T \in \mathscr{D}'(\Omega)$, $\varphi \in \mathscr{D}(\Omega)$,

$$\operatorname{supp}\varphi \subset (\operatorname{supp} T)^c \Longrightarrow \langle T, \varphi\rangle = 0. \tag{3.1.12}$$

3.1.3 First examples of distributions

The Dirac mass

We define for $\varphi \in C^0_c(\mathbb{R}^n)$, $\langle \delta_0, \varphi\rangle = \varphi(0)$; the property (3.1.5) is satisfied with $C_K = 1, N_K = 0$. We have $\operatorname{supp}\delta_0 = \{0\}$. From this, the Dirac mass cannot be an L^1_{loc} function, otherwise, since it is 0 a.e., it would be 0. Let ϕ, ϵ as in the proposition 3.1.1: then we have from that proposition

$$\lim_{\epsilon\rightarrow 0_+} \int \phi_\epsilon(x)\varphi(x)dx = \varphi(0),$$

so that the Dirac mass appears as the weak limit of $\epsilon^{-n}\phi(x\epsilon^{-1})$.

The simple layer

We consider in $\mathbb{R}^n$ the hypersurface $\Sigma = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, x_n = f(x')\}$, where $f \in C^1(\mathbb{R}^{n-1})$. We define for $\varphi \in C_c^0(\mathbb{R}^n)$,

$$\langle \delta_\Sigma, \varphi \rangle = \int_{\mathbb{R}^{n-1}} \varphi\big(x', f(x')\big)\big(1 + |\nabla f(x')|^2\big)^{1/2} dx'.$$

The property (3.1.5) is satisfied with $C_K = area(\Sigma \cap K), N_K = 0, \operatorname{supp} \delta_\Sigma = \Sigma$, and since Σ has Lebesgue measure 0 in $\mathbb{R}^n$, the simple layer potential cannot be an L^1_{loc} function.

The principal value of $1/x$

We define for $\varphi \in C_c^1(\mathbb{R})$,

$$\langle \text{pv}\,\frac{1}{x}, \varphi \rangle = \lim_{\epsilon \to 0_+} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x} dx. \tag{3.1.13}$$

Let us check that this limit exists. We have for parity reasons,

$$\begin{aligned}\int_{|x|\geq\epsilon} \frac{\varphi(x)}{x} dx &= \int_\epsilon^{+\infty} \big(\varphi(x) - \varphi(-x)\big)\frac{dx}{x} \\ &= \big[\ln x\big(\varphi(x) - \varphi(-x)\big)\big]_{x=\epsilon}^{x=+\infty} - \int_\epsilon^{+\infty} \big(\varphi'(x) + \varphi'(-x)\big) \ln x dx\end{aligned}$$

and thus, using that $\lim_{\epsilon \to 0_+} \epsilon \ln \epsilon = 0$, $\ln|x| \in L^1_{\text{loc}}(\mathbb{R})$, we get

$$\langle \text{pv}\,\frac{1}{x}, \varphi \rangle = -\int_0^{+\infty} \big(\varphi'(x) + \varphi'(-x)\big) \ln x dx = -\int_{\mathbb{R}} \varphi'(x)(\ln|x|) dx,$$

yielding $|\langle \text{pv}\,\frac{1}{x}, \varphi \rangle| \leq \int_{\operatorname{supp} \varphi'} |\ln|x|| dx \|\varphi'\|_{L^\infty}$.

3.1.4 Continuity properties

Definition 3.1.9. *Let Ω be an open subset of $\mathbb{R}^n$ and let $(\varphi_j)_{j\geq 1}$ be a sequence of functions in $C_c^\infty(\Omega)$. We shall say that $\lim_j \varphi_j = 0$ in $C_c^\infty(\Omega)$ when the two following conditions are satisfied:*
(1) *there exists a compact set $K \subset \Omega$, such that $\forall j \geq 1, \operatorname{supp} \varphi_j \subset K$,*
(2) *$\lim_j \varphi_j = 0$ in the Fréchet space $C_K^\infty(\Omega)$, i.e. $\forall \alpha \in \mathbb{N}^n$, $\lim_j \big(\sup_{x \in K} |(\partial_x^\alpha \varphi_j)(x)|\big) = 0$.*

Proposition 3.1.10. *Let Ω be an open subset of $\mathbb{R}^n$ and T be a linear form defined on $C_c^\infty(\Omega)$. The linear form T is a distribution on Ω if and only if it is sequentially continuous.*

Proof. Assuming $|\langle T, \varphi \rangle| \leq C_K \max_{|\alpha| \leq N_K} \|\partial_x^\alpha \varphi\|_{L^\infty}$ for all $\varphi \in C_K^\infty(\Omega)$ and all K compact $\subset \Omega$ implies readily the sequential continuity. Conversely, if T does not satisfy (3.1.5), we have

$$\exists K_0 \text{compact} \subset \Omega, \forall k \geq 1, \forall N \in \mathbb{N}, \exists \varphi_{k,N} \in C_{K_0}^\infty(\Omega), |\langle T, \varphi_{k,N} \rangle| > k \max_{|\alpha| \leq N} \|\partial_x^\alpha \varphi_{k,N}\|_{L^\infty}.$$

From the strict inequality, we infer that the function $\varphi_{k,N}$ is not identically 0, and we may define

$$\psi_k = \frac{\varphi_{k,k}}{k \max_{|\alpha|\le k} \|\partial_x^\alpha \varphi_{k,k}\|_{L^\infty}}, \quad \text{so that } |\langle T, \psi_k\rangle| > 1.$$

But the sequence $(\psi_k)_{k\ge 1}$ converges to 0 since $\operatorname{supp}\psi_k \subset K_0$ and for $|\beta| \le k$, $\|\partial_x^\beta \psi_k\|_{L^\infty} \le 1/k$, implying for each multi-index β that $\lim_k \|\partial_x^\beta \psi_k\|_{L^\infty} = 0$. The sequential continuity is violated since $|\langle T, \psi_k\rangle| > 1$ and the converse is proven. □

Definition 3.1.11. *Let Ω be an open subset of $\mathbb{R}^n$, $T \in \mathscr{D}'(\Omega)$ and $N \in \mathbb{N}$. The distribution T will be said of finite order N if*

$$\exists N \in \mathbb{N}, \forall K \textit{compact} \subset \Omega, \exists C_K > 0, \forall \varphi \in C_K^\infty(\Omega), |\langle T, \varphi\rangle| \le C_K \sup_{\substack{|\alpha|\le N \\ x\in\mathbb{R}^n}} |(\partial_x^\alpha \varphi)(x)|. \tag{3.1.14}$$

The vector space of distributions of order N on Ω will be denoted by ${\mathscr{D}'}^N(\Omega)$. The vector space ${\mathscr{D}'}^0(\Omega)$ is called the space of Radon measures on Ω.

Proposition 3.1.12. *Let Ω be an open subset of $\mathbb{R}^n$ and $m \in \mathbb{N}$. The vector space ${\mathscr{D}'}^m(\Omega)$ is equal to the sequentially continuous[1] linear forms on $C_c^m(\Omega)$: if $T \in {\mathscr{D}'}^m(\Omega)$, it can be extended to a sequentially continuous linear form on $C_c^m(\Omega)$. If T is a sequentially continuous linear form on $C_c^m(\Omega)$, then $T \in {\mathscr{D}'}^m(\Omega)$.*

Proof. Let us first consider $T \in {\mathscr{D}'}^m(\Omega)$, $\varphi \in C_c^m(\Omega)$. Applying the proposition 3.1.1, we find a sequence $(\varphi_k)_{k\ge1}$ in $C_c^\infty(\Omega)$, converging in $C_c^m(\Omega)$ with limit φ. Since we may assume that all the functions φ_k and φ are supported in a fixed compact subset K of Ω, we have, according to the estimate (3.1.14),

$$|\langle T, \varphi_k - \varphi_l\rangle| \le C \max_{|\alpha|\le m} \|\partial_x^\alpha(\varphi_k - \varphi_l)\|_{L^\infty} = Cp(\varphi_k - \varphi_l),$$

where p is the norm in the Banach space $C_K^m(\Omega)$. Since the sequence $(\varphi_k)_{k\ge1}$ converges in $C_K^m(\Omega)$, we get that the sequence $(\langle T, \varphi_k\rangle)_{k\ge1}$ is a Cauchy sequence in $\mathbb{C}$, thus converges; moreover, if for some compact subset L of Ω, $(\psi_k)_{k\ge1}$ is another sequence of $C_L^m(\Omega)$ converging to φ, we have

$$|\langle T, \psi_k - \varphi_k\rangle| \le C' \max_{|\alpha|\le m} \|\partial_x^\alpha(\varphi_k - \psi_k)\|_{L^\infty} = C'p(\varphi_k - \psi_k) \le C'p(\varphi_k - \varphi) + C'p(\varphi - \psi_k)$$

and $\lim_k \langle T, \psi_k - \varphi_k\rangle = 0$ so that, we can extend the linear form to $C_c^m(\Omega)$ by defining $\langle T, \varphi\rangle = \lim_k \langle T, \varphi_k\rangle$. We get also immediately that (3.1.14) holds with $N = m$ and $C_K^\infty(\Omega)$ replaced by $C_K^m(\Omega)$, so that T is obviously sequentially continuous.

Let us now consider a sequentially continuous linear form T on $C_c^m(\Omega)$; reproducing the proof of the proposition 3.1.10, we get that the estimate (3.1.14) holds with $N = m$, proving that $T \in {\mathscr{D}'}^m(\Omega)$. The proof of the proposition is complete. □

Remark 3.1.13. We have already proven directly that functions in $L^1_{\text{loc}}(\Omega)$(see (3.1.6)), the Dirac mass and a simple layer (see the section 3.1.3) are distributions of order 0. It is an exercise left to the reader to prove that the distribution $\operatorname{pv}\frac{1}{x}$ defined in (3.1.13) is of order 1 and not of order 0.

[1] The convergence of a sequence in $C_c^m(\Omega)$ is analogous to the convergence given in the definition 3.1.9, except that (2) is required in the Banach space $C_K^m(\Omega)$, i.e. $|\alpha| \le m$.

3.1.5 Partitions of unity and localization

Theorem 3.1.14 (Partition of unity)**.** *Let Ω be an open subset of $\mathbb{R}^n$, K a compact subset of Ω and $\Omega_1, \dots, \Omega_m$ open subsets of Ω such that $K \subset \Omega_1 \cup \dots \cup \Omega_m$. Then for $1 \le j \le m$, there exists $\psi_j \in C_c^\infty(\Omega_j; [0,1])$ and V open such that*

$$\Omega \supset V \supset K, \ \forall x \in V, \sum_{1 \le j \le m} \psi_j(x) = 1,$$

and for all $x \in \Omega, \sum_{1 \le j \le m} \psi_j(x) \in [0,1]$.

Proof. The case $m = 1$ of the theorem is proven in the lemma 3.1.3. We consider now $m > 1$ and we note that, since $x \in K$ implies $x \in$ one of the Ω_j,

$$K \subset \cup_{x \in K} B(x, r_x), \quad \bar{B}(x, r_x) \subset \text{one of the } \Omega_j, \quad r_x > 0.$$

From the compactness of K, we get that $K \subset \cup_{1 \le l \le N} B(x_l, r_{x_l})$ and we may assume that

$$\begin{aligned}
&\bar{B}(x_l, r_{x_l}) \subset \Omega_1, \quad \text{for } 1 \le l \le N_1, \\
&\bar{B}(x_l, r_{x_l}) \subset \Omega_2, \quad \text{for } N_1 < l \le N_2, \\
&\dots\dots\dots\dots\dots \\
&\bar{B}(x_l, r_{x_l}) \subset \Omega_m, \quad \text{for } N_{m-1} < l \le N_m = N.
\end{aligned}$$

We define then the compact sets

$$K_1 = \cup_{1 \le l \le N_1} \bar{B}(x_l, r_{x_l}), \quad \dots \quad , K_m = \cup_{N_{m-1} < l \le N_m} \bar{B}(x_l, r_{x_l}),$$

and we have $K \subset \cup_{1 \le j \le m} K_j$, and for each j, $K_j \subset \Omega_j$. Using the lemma 3.1.3, we find $\varphi_j \in C_c^\infty(\Omega_j; [0,1])$ such that $\varphi_j = 1$ on a neighborhood $V_j (\subset \Omega_j)$ of K_j. We define then

$$\begin{aligned}
&\psi_1 = \varphi_1, \\
&\psi_2 = \varphi_2(1 - \varphi_1), \\
&\dots\dots \\
&\psi_j = \varphi_j(1 - \varphi_1) \dots (1 - \varphi_{j-1}),
\end{aligned}$$

so that $\psi_j \in C_c^\infty(\Omega_j; [0,1])$ and we have

$$\sum_{1 \le j \le m} \psi_j = \sum_{1 \le j \le m} \varphi_j \Big(\prod_{1 \le k < j} (1 - \varphi_k) \Big) = 1 - \prod_{1 \le k \le m} (1 - \varphi_k), \tag{3.1.15}$$

since the formula (second equality above) is true for $m = 1$ and inductively,

$$\begin{aligned}
\sum_{1 \le j \le m+1} \varphi_j \Big(\prod_{1 \le k < j} (1 - \varphi_k) \Big) &= 1 - \prod_{1 \le k \le m} (1 - \varphi_k) + \varphi_{m+1} \prod_{1 \le k \le m} (1 - \varphi_k) \\
&= 1 - (1 - \varphi_{m+1}) \prod_{1 \le k \le m} (1 - \varphi_k) = 1 - \prod_{1 \le k \le m+1} (1 - \varphi_k).
\end{aligned}$$

We have thus for $x \in \cup_{1 \le j \le m} V_j$ (which is a neighborhood of K in Ω), using (3.1.15) and $\varphi_j = 1$ on V_j, $\sum_{1 \le j \le m} \psi_j(x) = 1$. On the other hand, (3.1.15) and φ_j valued in $[0,1]$ show that $\sum_{1 \le j \le m} \psi_j(x) \in [0,1]$ for all x. The proof is complete. □

Theorem 3.1.15. *Let Ω be an open set of $\mathbb{R}^n$ and $(\Omega_j)_{j\in J}$ be an open covering of Ω: each Ω_j is open and $\cup_{j\in J}\Omega_j = \Omega$. Let us assume that for each $j \in J$, we are given $T_j \in \mathscr{D}'(\Omega_j)$ in such a way that*

$$T_{j|\Omega_j\cap\Omega_k} = T_{k|\Omega_j\cap\Omega_k}. \tag{3.1.16}$$

Then there exists a unique $T \in \mathscr{D}'(\Omega)$ such that for all $j \in J$, $T_{|\Omega_j} = T_j$.

Proof. Uniqueness: if T, S are such distributions, we get that $(T - S)_{|\Omega_j} = 0$, so that for all $j \in J$, $\Omega_j \subset (\operatorname{supp}(T - S))^c$ and thus $\Omega = \cup_{j\in J}\Omega_j \subset (\operatorname{supp}(T - S))^c$, i.e. $T - S = 0$.
Existence: let $\varphi \in \mathscr{D}(\Omega)$ and let us consider the compact set $K = \operatorname{supp}\varphi$. We have $K \subset \cup_{j\in M}\Omega_j$ with M a finite subset of J. Using the theorem on partitions of unity, we find some function $\psi_j \in C_c^\infty(\Omega_j)$ for $j \in M$ such that $\sum_{j\in M}\psi_j = 1$ on a neighborhood of K. As a consequence, we have $\varphi = \sum_{j\in M}\psi_j\varphi$ and we define

$$\langle T, \varphi\rangle = \sum_{j\in M}\langle T_j, \psi_j\varphi\rangle.$$

The required estimates (3.1.5) are easily checked, but the linearity and the independence with respect to the decomposition deserve some attention. Assume that we have $\varphi = \sum_{k\in N}\phi_k\varphi$, where N is a finite subset of J and $\phi_k \in C_c^\infty(\Omega_k)$: we have

$$\sum_{k\in N}\langle T_k, \phi_k\varphi\rangle = \sum_{j\in M,k\in N}\langle T_k, \phi_k\psi_j\varphi\rangle \underbrace{=}_{\text{from (3.1.16)}} \sum_{j\in M,k\in N}\langle T_j, \phi_k\psi_j\varphi\rangle = \sum_{j\in M}\langle T_j, \psi_j\varphi\rangle,$$

proving that T is defined independently of the decomposition. The linearity follows at once. The proof is complete. □

3.1.6 Weak convergence of distributions

We have not defined a topology on the space of test functions $\mathscr{D}(\Omega)$, although we gave the definition of convergence of a sequence (see the definition 3.1.9); we shall need also a simple notion of weak-dual convergence of a sequence of distributions, which is the $\sigma(\mathscr{D}', \mathscr{D})$ convergence.

Definition 3.1.16. *Let Ω be an open set of $\mathbb{R}^n$, $(T_j)_{j\geq 1}$ be a sequence of $\mathscr{D}'(\Omega)$ and $T \in \mathscr{D}'(\Omega)$. We shall say that $\lim_j T_j = T$ in the weak-dual topology if*

$$\forall\varphi \in \mathscr{D}(\Omega), \quad \lim_j\langle T_j, \varphi\rangle = \langle T, \varphi\rangle. \tag{3.1.17}$$

Remark 3.1.17. We have already seen (see the section 3.1.3) that for $\rho \in C_c^\infty(\mathbb{R}^n)$, $\epsilon > 0$, $\rho_\epsilon(x) = \epsilon^{-n}\rho(x\epsilon^{-1})$, $\lim_{\epsilon\to 0_+}\rho_\epsilon = \delta_0\int\rho(t)dt$. Moreover, on $\mathscr{D}'(\mathbb{R})$, we have with $T_\lambda(x) = e^{i\lambda x}$, $\lim_{\lambda\to+\infty}T_\lambda = 0$ since for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\int_{\mathbb{R}} e^{i\lambda x}\varphi(x)dx = (i\lambda)^{-1}\int_{\mathbb{R}}\frac{d}{dx}(e^{i\lambda x})\varphi(x)dx = -(i\lambda)^{-1}\int_{\mathbb{R}} e^{i\lambda x}\varphi'(x)dx.$$

Theorem 3.1.18. *Let Ω be an open set of $\mathbb{R}^n$, $(T_j)_{j\ge 1}$ be a sequence of $\mathscr{D}'(\Omega)$ such that, for all $\varphi \in \mathscr{D}(\Omega)$, the (numerical) sequence $(\langle T_j, \varphi\rangle)_{j\ge 1}$ converges. Defining the linear form T on $\mathscr{D}(\Omega)$, by $\langle T, \varphi\rangle = \lim_j \langle T_j, \varphi\rangle$, we obtain that T belongs to $\mathscr{D}'(\Omega)$.*

Proof. This is an important consequence of the Banach-Steinhaus theorem 2.1.8; let us consider a compact subset K of Ω. Then defining $T_{j,K}$ as the restriction of T_j to the Fréchet space $\mathscr{D}_K(\Omega)$, we see that the assumptions of the corollary 2.1.8 are satisfied since $T_{j,K}$ belongs to the topological dual of $\mathscr{D}_K(\Omega)$, according to the remark 3.1.6. As a consequence the restriction of T to $\mathscr{D}_K(\Omega)$ belongs to the topological dual of $\mathscr{D}_K(\Omega)$ and from the same remark 3.1.6, it gives that $T \in \mathscr{D}'(\Omega)$. □

N.B. The reader may note that we have used $E = \mathscr{D}(\Omega) = \cup_{j\in\mathbb{N}} \mathscr{D}_{K_j}(\Omega) = \cup_j E_j$, and that our definition of the topological dual of E as linear forms T on E such that, for all j, $T_{|E_j} \in$ the topological dual of the Fréchet space E_j. This structure allows us to use the Banach-Steinhaus theorem, although we have not defined a topology on E; this observation is a good introduction to the more abstract setting of LF spaces, the so-called inductive limits of Fréchet spaces.

3.2 Differentiation of distributions, multiplication by C^∞ functions

3.2.1 Differentiation

Definition 3.2.1. *Let Ω be an open set of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. We define the distributions $\partial_{x_j} T$ and for a multi-index $\alpha \in \mathbb{N}^n$ (see (2.3.6)), $\partial_x^\alpha T$ by*

$$\langle \partial_{x_j} T, \varphi\rangle = -\langle T, \partial_{x_j}\varphi\rangle, \quad \langle \partial_x^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, \partial_x^\alpha \varphi\rangle. \tag{3.2.1}$$

We note that $\partial_x^\alpha T$ is indeed a distribution on Ω, since the mappings $\varphi \mapsto \partial_x^\alpha \varphi$ are continuous on each Fréchet space $\mathscr{D}_K(\Omega)$.

Remark 3.2.2. If $\lim_j T_j = T$ in the weak-dual topology of $\mathscr{D}'(\Omega)$, then, for all multi-indices α, $\lim_j \partial_x^\alpha T_j = \partial_x^\alpha T$ (in the weak-dual topology): we have, for each $\varphi \in \mathscr{D}(\Omega)$,

$$\langle \partial_x^\alpha T_j, \varphi\rangle = (-1)^{|\alpha|}\langle T_j, \partial_x^\alpha \varphi\rangle \underset{j\to+\infty}{\longrightarrow} (-1)^{|\alpha|}\langle T, \partial_x^\alpha \varphi\rangle = \langle \partial_x^\alpha T, \varphi\rangle.$$

Remark 3.2.3. If $u \in C^1(\Omega)$, its derivative $\partial_{x_j} u$ as a distribution coincides with the distribution defined by the continuous function $\partial u/\partial x_j$: for $\varphi \in \mathscr{D}(\Omega)$,

$$\langle \partial_{x_j} u, \varphi\rangle = -\langle u, \partial_{x_j}\varphi\rangle = -\int u(x)\frac{\partial \varphi}{\partial x_j}(x)dx = \int \frac{\partial u}{\partial x_j}(x)\varphi(x)dx = \langle \frac{\partial u}{\partial x_j}, \varphi\rangle.$$

Also, if $u, v \in C^0(\Omega)$ are such that $\partial_{x_1} u = v$ in $\mathscr{D}'(\Omega)$, then the function u admits v as a partial derivative with respect to x_1. To prove this, we may assume that u, v are both compactly supported in Ω: in fact it is enough to prove that for $\chi \in C_c^\infty(\Omega)$

identically equal to 1 near a point x_0, the function χu (compactly supported) has a partial derivative with respect to x_1 which is $\chi v + u\partial_{x_1}\chi$ (compactly supported) and we know that in $\mathscr{D}'(\Omega)$ we have

$$\langle \partial_{x_1}(\chi u), \varphi\rangle = -\langle u, \chi\partial_{x_1}\varphi\rangle = -\langle u, \partial_{x_1}(\chi\varphi)\rangle + \langle u, \varphi\partial_{x_1}\chi\rangle = \langle \partial_{x_1}u, \chi\varphi\rangle + \langle u\partial_{x_1}\chi, \varphi\rangle$$

which implies a particular case of Leibniz' formula $\partial_{x_1}(\chi u) = \chi\partial_{x_1}u + u\partial_{x_1}\chi = \chi v + u\partial_{x_1}\chi$. Assuming then that u, v are compactly supported, we have from the proposition 3.1.1, $u = \lim_\epsilon(u * \phi_\epsilon)$ in $C^0_c(\Omega)$ and the functions $u * \phi_\epsilon \in C^\infty_c(\Omega)$. Also we have, with the ordinary differentiation,

$$(\partial_{x_1}(u*\phi_\epsilon))(x) = \int u(y)(\partial_{x_1}\phi_\epsilon)(x-y)dy = \langle u(\cdot), -\partial_{y_1}\big(\phi_\epsilon(x-\cdot)\big)\rangle = \int v(y)\phi_\epsilon(x-y)dy,$$

and $\lim_\epsilon(v * \phi_\epsilon) = v$ in $C^0_c(\Omega)$. As a result the sequences $(u * \phi_\epsilon), (\partial_{x_1}(u * \phi_\epsilon))$ are both uniformly converging sequences of (compactly supported) continuous functions with respective limits u, v, and this implies that the continuous function u has v as a partial derivative with respect to x_1.

3.2.2 Examples

Defining the Heaviside function H as $\mathbf{1}_{\mathbb{R}_+}$, we get

$$H' = \delta_0 \tag{3.2.2}$$

since for $\varphi \in \mathscr{D}(\mathbb{R})$, we have $\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^{+\infty} \varphi'(t)dt = \varphi(0)$. Still in one dimension, we have

$$\langle \delta_0^{(k)}, \varphi\rangle = (-1)^k\varphi^{(k)}(0), \tag{3.2.3}$$

since it is true for $k = 0$ and inductively $\langle \delta_0^{(k+1)}, \varphi\rangle = -\langle \delta_0^{(k)}, \varphi'\rangle = -(-1)^k\varphi'^{(k)}(0) = (-1)^{k+1}\varphi^{(k+1)}(0)$. Looking at the definition (3.1.13), we see that we have proven

$$\operatorname{pv}\Big(\frac{1}{x}\Big) = \frac{d}{dx}(\ln|x|), \qquad \text{(distribution derivative).} \tag{3.2.4}$$

Let f be a finitely-piecewise C^1 function defined on $\mathbb{R}$: it means that there is an increasing finite sequence of real numbers $(a_n)_{1\le n\le N}$, so that f is C^1 on all closed intervals $[a_n, a_{n+1}]$ for $1 \le n < N$ and on $]-\infty, a_1]$ and $[a_N, +\infty[$. In particular, the function f has a left-limit $f(a_n^-)$ and a right-limit $f(a_n^+)$ which may be different. Let us compute the distribution derivative of f; for $\varphi \in \mathscr{D}(\mathbb{R})$, since f is locally integrable, we have, setting $a_0 = -\infty, a_{N+1} = +\infty$,

$$\begin{aligned}
\langle f', \varphi\rangle = -\langle f, \varphi'\rangle &= -\int_{\mathbb{R}} f(x)\varphi'(x)dx = -\sum_{0\le n\le N}\int_{a_n}^{a_{n+1}} f(x)\varphi'(x)dx \\
&= \sum_{0\le n\le N}\int_{a_n}^{a_{n+1}} \frac{df}{dx}(x)\varphi(x)dx + \sum_{0\le n\le N}\big(f(a_n^+)\varphi(a_n) - f(a_{n+1}^-)\varphi(a_{n+1})\big) \\
&= \int \varphi(x)\Big(\sum_{0\le n\le N}\frac{df}{dx}(x)\mathbf{1}_{[a_n,a_{n+1}]}(x)\Big) + \sum_{1\le n\le N} f(a_n^+)\varphi(a_n) - \sum_{1\le n\le N} f(a_n^-)\varphi(a_n),
\end{aligned}$$

so that we have obtained the so-called *formula of jumps*

$$f' = \sum_{0\le n\le N} \frac{df}{dx}\mathbf{1}_{[a_n,a_{n+1}]} + \sum_{1\le n\le N} \big(f(a_n^+) - f(a_n^-)\big)\delta_{a_n}, \tag{3.2.5}$$

where δ_{a_n} is the Dirac mass at a_n, defined by $\langle \delta_{a_n}, \varphi\rangle = \varphi(a_n)$.

We consider now the following determination of the logarithm given for $z \in \mathbb{C}\backslash\mathbb{R}_-$ by

$$\operatorname{Log} z = \oint_{[1,z]} \frac{d\xi}{\xi}, \tag{3.2.6}$$

which makes sense since $\mathbb{C}\backslash\mathbb{R}_-$ is star-shaped with respect to 1, i.e. the segment $[1, z] \subset \mathbb{C}\backslash\mathbb{R}_-$ for $z \in \mathbb{C}\backslash\mathbb{R}_-$. Since the function Log coincides with ln on $\mathbb{R}_+^*$ and is holomorphic on $\mathbb{C}\backslash\mathbb{R}_-$, we get by analytic continuation that

$$e^{\operatorname{Log} z} = z, \quad \text{for } z \in \mathbb{C}\backslash\mathbb{R}_-. \tag{3.2.7}$$

Also by analytic continuation, we have for $|\operatorname{Im} z| < \pi, \quad \operatorname{Log}(e^z) = z$. We want now to study the distributions on $\mathbb{R}$,

$$u_y(x) = \operatorname{Log}(x + iy), \quad \text{where } y \ne 0 \text{ is a real parameter.}$$

We leave as an exercise for the reader to prove that

$$\lim_{y\to 0_\pm} \operatorname{Log}(x + iy) = \ln|x| \pm i\pi\big(1 - H(x)\big), \tag{3.2.8}$$

where the limits are taken in the sense of the definition 3.1.16; also the reader can check

$$\frac{1}{x \pm i0} = \operatorname{pv}\big(\frac{1}{x}\big) \mp i\pi\delta_0, \tag{3.2.9}$$

where we have defined

$$\langle \frac{1}{x \pm i0}, \varphi\rangle = \lim_{\epsilon\to 0_+} \int \frac{\varphi(x)}{x \pm i\epsilon} dx \tag{3.2.10}$$

(part of the exercise is to prove that these limits exist for $\varphi \in \mathscr{D}(\mathbb{R})$). We conclude that section of examples with a more general lemma on a simple ODE.

Lemma 3.2.4. *Let I be an open interval of $\mathbb{R}$. The solutions in $\mathscr{D}'(I)$ of $u' = 0$ are the constants. The solutions in $\mathscr{D}'(I)$ of $u' = f$ make a one-dimensional affine subspace of $\mathscr{D}'(I)$.*

Proof. We assume first that $f = 0$; if u is a constant, then it is of course a solution. Conversely, let us assume that $u \in \mathscr{D}'(I)$ satisfies $u' = 0$. Let $\chi_0 \in C_c^\infty(I)$ such that $\int_{\mathbb{R}} \chi_0(x)dx = 1$; then we have for any $\varphi \in C_c^\infty(I)$, with $J(\varphi) = \int_{\mathbb{R}} \varphi(x)dx$, $\psi(x) = \int_{-\infty}^x \big(\varphi(t) - J(\varphi)\chi_0(t)\big)dt$, noting that ψ belongs[2] to $C_c^\infty(I)$,

$$\langle u, \varphi - J(\varphi)\chi_0\rangle = \langle u, \psi'\rangle = -\langle u', \psi\rangle = 0,$$

[2]The function ψ is obviously smooth and if φ, χ_0 are both supported in $\{a \le x \le b\}, a, b \in I$, so is ψ, thanks to the condition $\int \chi_0 = 1$.

which gives $\langle u, \varphi\rangle = J(\varphi)\langle u, \chi_0\rangle$, i.e. $u = \langle u, \chi_0\rangle$ proving that u is indeed a constant. We have proven that the solutions $u \in \mathscr{D}'(I)$ of $u' = 0$ are simply the constants. If $f \in \mathscr{D}'(I)$, we need only to construct a solution v_0 of $v_0' = f$ and then use the previous result to obtain that the set of solutions of $u' = f$ is $v_0 + \mathbb{R}$. Let us construct such a solution v_0. For $\varphi \in \mathscr{D}(I)$, we define with the same ψ as above,

$$\langle v_0, \varphi\rangle = -\langle f, \psi\rangle. \tag{3.2.11}$$

It is a distribution since for $\operatorname{supp}\varphi$ compact $\subset I$, we define (the compact set) $K_1 = \operatorname{supp}\varphi \cup \operatorname{supp}\chi_0$, and we have

$$|\langle v_0, \varphi\rangle| = |\langle f, \psi\rangle| \le C_{K_1} \max_{0\le j\le N_{K_1}} \|\psi^{(j)}\|_{L^\infty} \le C \max_{0\le j\le (N_{K_1}-1)_+} \|\varphi^{(j)}\|_{L^\infty}.$$

Moreover the formula (3.2.11) implies the sought result

$$\langle v_0', \varphi\rangle = -\langle v_0, \varphi'\rangle = \langle f, \psi_{\varphi'}\rangle = \langle f, \varphi\rangle,$$

since $\psi_{\varphi'}(x) = \int_{-\infty}^x \big(\varphi'(t) - J(\varphi')\chi_0(t)\big)dt = \varphi(x)$ because $J(\varphi') = 0$. The proof of the lemma is complete. □

3.2.3 Product by smooth functions

We define now the product of a C^∞ (resp. C^N) function by a distribution (resp. of order N).

Definition 3.2.5. *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. For $f \in C^\infty(\Omega)$, we define the product $f \cdot u$ as the distribution defined by*

$$\langle f \cdot u, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle u, f\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}. \tag{3.2.12}$$

If u is of order N and $f \in C^N(\Omega)$, we define the product $f \cdot u$ as the distribution of order N defined by

$$\langle f \cdot u, \varphi\rangle_{\mathscr{D}'^N(\Omega),C_c^N(\Omega)} = \langle u, f\varphi\rangle_{\mathscr{D}'^N(\Omega),C_c^N(\Omega)}. \tag{3.2.13}$$

Remark 3.2.6. Since the multiplication by a $C^\infty(\Omega)$ (resp. $C^N(\Omega)$) function is a continuous linear operator from $C_c^\infty(\Omega)$ (resp. $C_c^N(\Omega)$) into itself, we get that the above formulas actually define the products as distributions on Ω with the right order (see the proposition 3.1.12). Also the product defined in the second part coincides with the first definition whenever $f \in C_c^\infty(\Omega)$ and if $u \in L^1_{\text{loc}}(\Omega)$, $f \in C^0(\Omega)$, the usual product fu coincides with the $f \cdot u$ defined here, thanks to the lemma 3.1.7.

The next theorem is providing an extension to the classical Leibniz' formula for the derivatives of a product.

Theorem 3.2.7. *Let Ω be an open set of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}^n$ be a multi-index (see (2.3.6)). Then we have*

$$\frac{\partial_x^\alpha(fu)}{\alpha!} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \frac{\partial_x^\beta(f)}{\beta!}\frac{\partial_x^\gamma(u)}{\gamma!}. \tag{3.2.14}$$

Proof. We get immediately by induction on $|\alpha|$ the formula

$$\frac{\partial_x^\alpha(fu)}{\alpha!} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma}\frac{\partial_x^\beta(f)}{\beta!}\frac{\partial_x^\gamma(u)}{\gamma!}, \quad \text{with } \sigma_{\beta,\gamma}\in\mathbb{R}_+.$$

To find the $\sigma_{\beta,\gamma}$, we choose $f(x)=e^{x\cdot\xi}, u(x)=e^{x\cdot\eta}$, with $\xi,\eta\in\mathbb{R}^n$. We find then for all $\xi,\eta\in\mathbb{R}^n$, the identity

$$\frac{(\xi+\eta)^\alpha}{\alpha!} = \frac{\partial_x^\alpha(e^{x\cdot(\xi+\eta)})}{\alpha!}_{|x=0} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma}\frac{\partial_x^\beta(e^{x\cdot\xi})}{\beta!}\frac{\partial_x^\gamma(e^{x\cdot\eta})}{\gamma!}_{|x=0} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma}\frac{\xi^\beta}{\beta!}\frac{\eta^\gamma}{\gamma!},$$

and the formula (2.3.7) shows that for β,γ such that $\beta+\gamma=\alpha$

$$\sigma_{\beta,\gamma} = \partial_\xi^\beta\partial_\eta^\gamma\big(\frac{(\xi+\eta)^\alpha}{\alpha!}\big)_{|\xi=\eta=0} = 1,$$

completing the proof of the theorem. □

Examples. Let f be a continuous function on $\mathbb{R}$ and δ_0 be the Dirac mass at 0. The product $f\cdot\delta_0$ is equal to $f(0)\delta_0$: since δ_0 is a distribution of order 0, we can multiply it by a continuous function and if $\varphi\in C_c^0(\mathbb{R})$, we have

$$\langle f\cdot\delta_0,\varphi\rangle = \langle\delta_0, f\varphi\rangle = f(0)\varphi(0) = \langle f(0)\delta_0,\varphi\rangle \Longrightarrow f\cdot\delta_0 = f(0)\delta_0. \tag{3.2.15}$$

On the other hand if $f\in C^1(\mathbb{R})$ we have

$$f\cdot\delta_0' = f(0)\delta_0' - f'(0)\delta_0, \tag{3.2.16}$$

since the Leibniz' formula (3.2.14) gives $f(0)\delta_0' = (f\cdot\delta_0)' = f'\cdot\delta_0 + f\cdot\delta_0' = f'(0)\delta_0 + f\cdot\delta_0'$. In particular $x\delta_0' = -\delta_0$.

3.2.4 Division of distribution on $\mathbb{R}$ by x^m

We want now to address the question of division of a function (or a distribution) by a polynomial; a typical example is the division of 1 by the linear function x expressed by the identity

$$x\,\mathrm{pv}(1/x) = 1 \tag{3.2.17}$$

which is an immediate consequence of (3.1.13). We note also from the previous examples that, for any constant c, we have $x\big(\mathrm{pv}(1/x)+c\delta_0\big)=1$. The next theorem shows that $T=\mathrm{pv}(1/x)+c\delta_0$ are the only distributions solutions of the equation $xT=1$.

Theorem 3.2.8. *Let $m\geq 1$ be an integer.*
(1) If $u\in\mathscr{D}'(\mathbb{R})$ is such that $x^m u=0$, then $u=\sum_{0\leq j<m} c_j\delta_0^{(j)}$.
(2) Let $v\in\mathscr{D}'(\mathbb{R})$; there exists $u\in\mathscr{D}'(\mathbb{R})$ such that $v=x^m u$.

Proof. Let us first prove (1). For $\varphi, \chi_0 \in C_c^\infty(\mathbb{R})$ with $\chi_0 = 1$ near 0, we have

$$\varphi(x) = \underbrace{\sum_{0 \le j < m} \frac{\varphi^{(j)}(0)}{j!} x^j}_{p_{\varphi,m}(x)} + \underbrace{\int_0^1 \frac{(1-t)^{m-1}}{(m-1)!} \varphi^{(m)}(tx) dt\, x^m}_{\psi_{m,\varphi}(x)}, \quad \psi_{m,\varphi} \in C^\infty(\mathbb{R}),$$

and thus, since $x^m u = 0$,

$$\begin{aligned}\langle u, \varphi \rangle = \overbrace{\langle x^m u, x^{-m}(1-\chi_0)\varphi \rangle}^{=0} + \langle u, \chi_0 \varphi \rangle = \langle u, \chi_0 p_{m,\varphi} \rangle + \overbrace{\langle x^m u, \chi_0 \psi_{\varphi,m} \rangle}^{=0} \\ = \sum_{0 \le j < m} \frac{\varphi^{(j)}(0)}{j!} \langle u, \chi_0 \rangle = \sum_{0 \le j < m} \langle c_j \delta_0^{(j)}, \varphi \rangle,\end{aligned}$$

which the sought result. To obtain (2), for $\varphi \in C_c^\infty(\mathbb{R})$, and a given $v_0 \in \mathscr{D}'(\mathbb{R})$, we define, using the above notations,

$$\langle u, \varphi \rangle = \langle v_0, \chi_0 \psi_{m,\varphi} \rangle + \langle v_0, x^{-m}(1-\chi_0)\varphi \rangle.$$

This defines obviously a distribution on $\mathbb{R}$ and $\langle x^m u, \varphi \rangle = \langle u, x^m \varphi \rangle$; for the function $\phi(x) = x^m \varphi(x)$, we have $p_{\phi,m} = 0, x^m \psi_{m,\phi}(x) = x^m \varphi(x)$, so that the smooth functions $\psi_{m,\phi} = \varphi$,

$$\langle x^m u, \varphi \rangle = \langle v_0, \chi_0 \varphi \rangle + \langle v_0, x^{-m}(1-\chi_0) x^m \varphi \rangle = \langle v_0, \varphi \rangle. \qquad \square$$

3.3 Distributions with compact support

3.3.1 Identification with $\mathscr{E}'$

Let Ω be an open subset of $\mathbb{R}^n$. We have already seen that the space $C^\infty(\Omega)$ (also denoted by $\mathscr{E}(\Omega)$) is a Fréchet space. Denoting by $\mathscr{E}'(\Omega)$ the topological dual of $\mathscr{E}(\Omega)$, we can consider $T \in \mathscr{E}'(\Omega)$ as a distribution $\tilde{T}$ on Ω by defining

$$\langle \tilde{T}, \varphi \rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle T, \varphi \rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} \quad \text{(this makes sense since } \mathscr{D}(\Omega) \subset \mathscr{E}(\Omega)).$$

The linearity is obvious and the continuity of T as a linear form on the Fréchet space $\mathscr{E}(\Omega)$ implies that there exists $C > 0, N \in \mathbb{N}$, K compact subset of Ω such that

$$\forall \varphi \in \mathscr{E}(\Omega), \quad |\langle T, \varphi \rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)}| \le C \sup_{|\alpha| \le N,\ x \in K} |(\partial_x^\alpha \varphi)(x)|.$$

This estimates also proves that $\tilde{T}$ belongs to $\mathscr{D}'(\Omega)$; moreover, it has compact support in the sense of the definition (3.1.8): we have $\langle \tilde{T}, \varphi \rangle = 0$ for $\varphi \in C_c^\infty(\Omega)$, $\operatorname{supp} \varphi \subset K^c$, so that $\tilde{T}_{|K^c} = 0$ and thus $\operatorname{supp} \tilde{T} \subset K$. The next theorem proves that we can identify the space $\mathscr{E}'(\Omega)$ with the distributions on Ω with compact support, denoted by $\mathscr{D}'_{\text{comp}}(\Omega)$.

Theorem 3.3.1. *Let Ω be an open subset of $\mathbb{R}^n$. The mapping $\iota : \mathscr{E}'(\Omega) \to \mathscr{D}'_{comp}(\Omega)$, defined as above by $\iota(T) = \tilde{T}$ is bijective.*

Proof. The mapping ι is linear and if $\iota(T) = 0$, we know that T vanishes on all functions of $\mathscr{D}(\Omega)$.

Lemma 3.3.2. *Let Ω be an open subset of $\mathbb{R}^n$. The space $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$.*

Proof of the lemma. We consider a sequence $(K_j)_{j\ge 1}$ of compact subsets of Ω such that the lemma 2.3.1 is satisfied. For each $j \ge 1$, we may use the lemma 3.1.3 to construct a function $\chi_j \in \mathscr{D}(\Omega)$ with $\chi_j = 1$ near K_j. For a given $\varphi \in \mathscr{E}(\Omega)$, the sequence $(\varphi\chi_j)_{j\ge 1}$ of functions in $\mathscr{D}(\Omega)$ converges in $\mathscr{E}(\Omega)$ to φ, thanks to the last property of the lemma 2.3.1, proving the lemma. □

Since T is continuous on $\mathscr{E}(\Omega)$, $\langle T, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \lim_j \langle T, \varphi\chi_j\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)}, = 0$ since T vanishes on $\mathscr{D}(\Omega)$. Let us consider now $T \in \mathscr{D}'_{\text{comp}}(\Omega)$ with $\operatorname{supp} T = L$ (compact subset of Ω). Using the lemma 3.1.3, we consider $\chi_0 \in \mathscr{D}(\Omega)$ such that $\chi_0 = 1$ on a neighborhood of L. For $\varphi \in \mathscr{E}(\Omega)$, we define $S \in \mathscr{E}'(\Omega)$ by

$$\langle S, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \langle T, \chi_0\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} \quad \text{(note that } |\langle S, \varphi\rangle| \le C \sup_{|\alpha|\le N,\ x\in \operatorname{supp}\chi_0} |\partial_x^\alpha \varphi|),$$

We have $\iota(S) = T$ because

$$\langle \iota(S), \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle S, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \langle T, \chi_0\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle \chi_0 T, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)},$$

and since for $\varphi \in \mathscr{D}(\Omega)$, the function $(1-\chi_0)\varphi$ vanishes on an open neighborhood V of L implying

$$\operatorname{supp}\big((1-\chi_0)\varphi\big) \subset V^c \subset L^c \Longrightarrow \langle T, (1-\chi_0)\varphi\rangle = 0,$$

so that $\iota(S) = \chi_0 T = \chi_0 T + \underbrace{(1-\chi_0)T}_{=0} = T$. The proof of the theorem is complete.

□

Remark 3.3.3. We can then identify $\mathscr{D}'_{\text{comp}}(\Omega)$ with $\mathscr{E}'(\Omega)$, and we may note that for $T \in \mathscr{D}'_{\text{comp}}(\Omega)$ with $\operatorname{supp} T = L$, T is of finite order N, and for all neighborhoods K of L, there exists $C > 0$ such that, for all $\varphi \in \mathscr{E}(\Omega)$,

$$|\langle T, \varphi\rangle| \le C \sup_{|\alpha|\le N,\ x\in K} |(\partial_x^\alpha \varphi)(x)|. \tag{3.3.1}$$

In general, it is not possible to take $K = L$ in the above estimate.

3.3.2 Distributions with support at a point

The next theorem characterizes the distributions supported in $\{0\}$.

Theorem 3.3.4. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$ and let $u \in \mathscr{D}'(\Omega)$ such that $\operatorname{supp} u = \{x_0\}$. Then $u = \sum_{|\alpha|\le N} c_\alpha \delta_{x_0}^{(\alpha)}$, where the c_α are some constants.*

Proof. Let $\varphi \in C^\infty(\Omega)$; we have for $x \in V_0 \subset$ open neighborhood of x_0 (included in Ω), N_0 the order of u,

$$\varphi(x) = \sum_{|\alpha| \le N_0} \frac{(\partial_x^\alpha \varphi)(x_0)}{\alpha!}(x-x_0)^\alpha + \underbrace{\int_0^1 \frac{(1-\theta)^{N_0}}{N_0!} \varphi^{(N_0+1)}(x_0 + \theta(x-x_0)) d\theta}_{\psi(x), \quad \psi \in C^\infty(V_0)} (x-x_0)^{N_0+1},$$

and thus for $\chi \in C_c^\infty(V_0), \chi = 1$ near x_0,

$$\langle u, \varphi \rangle = \langle u, \chi_0 \varphi \rangle = \sum_{|\alpha| \le N_0} \frac{(\partial_x^\alpha \varphi)(x_0)}{\alpha!} \langle u, \chi_0(x)(x-x_0)^\alpha \rangle + \langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1} \rangle. \tag{3.3.2}$$

We have also

$$|\langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1} \rangle| \le C_0 \sup_{|\alpha| \le N_0} |\partial_x^\alpha \big(\chi_0(x)\psi(x)(x-x_0)^{N_0+1}\big)|. \tag{3.3.3}$$

We can take $\chi_0(x) = \rho(\frac{x-x_0}{\epsilon})$, where $\rho \in C_c^\infty(\mathbb{R}^n)$ is supported in the unit ball B_1, $\rho = 1$ in $\frac{1}{2}B_1$ and $\epsilon > 0$. We have then

$$\begin{aligned}\chi_0(x)\psi(x)(x-x_0)^{N_0+1} &= \epsilon^{N_0+1} \rho(\frac{x-x_0}{\epsilon}) \psi\big(x_0 + \epsilon \frac{(x-x_0)}{\epsilon}\big) \frac{(x-x_0)^{N_0+1}}{\epsilon^{N_0+1}} \\ &= \epsilon^{N_0+1} \rho_1(\frac{x-x_0}{\epsilon})\end{aligned}$$

with $\rho_1(t) = \rho(t)\psi(x_0 + \epsilon t) t^{N_0+1}$, so that $\rho_1 \in C_c^\infty(\mathbb{R}^n)$ is supported in the unit ball B_1 has all its derivatives bounded independently of ϵ. From (3.3.3), we get for all $\epsilon > 0$,

$$|\langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1} \rangle| \le C_0 \sup_{|\alpha| \le N_0} \epsilon^{N_0+1-|\alpha|} |(\partial_t^\alpha \rho_1)(\frac{x-x_0}{\epsilon})| \le C_1 \epsilon,$$

which implies that the left-hand-side of (3.3.3) is zero. The result of the theorem follows from (3.4.15). □

3.4 Tensor products

Let X be an open subset of $\mathbb{R}^m$, Y be an open subset of $\mathbb{R}^n$ and $f \in C_c^\infty(X), g \in C_c^\infty(Y)$. The tensor product $f \otimes g$ is defined by $(f \otimes g)(x,y) = f(x)g(y)$ and belongs to $C_c^\infty(X \times Y)$. Now if $T \in \mathscr{D}'(X), S \in \mathscr{D}'(Y)$, we want to define a distribution $T \otimes S \in \mathscr{D}'(X \times Y)$ such that

$$\langle T \otimes S, f \otimes g \rangle = \langle T, f \rangle \langle S, g \rangle.$$

This triggers several questions: is such a construction possible? Is the definition above sufficient to determine unambiguously the distribution $T \otimes S$? We shall answer positively to these questions, but we first address a related question of derivation of an "integral" depending on a parameter.

3.4.1 Differentiation of a duality product

Theorem 3.4.1. *Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, U an open subset of $\mathbb{R}^m$ and $\phi \in C^\infty(\Omega \times U)$ such that*

$$\forall t \in U, \exists V_t \in \mathscr{V}_t, \exists K_t \text{ compact subset of } \Omega, \quad \forall s \in V_t, \quad \operatorname{supp} \phi(\cdot, s) \subset K_t. \tag{3.4.1}$$

Then the function f defined on U by $f(t) = \langle u, \phi(\cdot, t)\rangle$ makes sense and belongs to $C^\infty(U)$. Moreover we have for all $\alpha \in \mathbb{N}^m$, $(\partial_t^\alpha f)(t) = \langle u, (\partial_t^\alpha \phi)(\cdot, t)\rangle$.

Proof. The function f makes sense since for all $t \in U$, the function $\phi(\cdot, t)$ belongs to $C_c^\infty(\Omega)$. Let $t_0 \in U$ and B_0 be a closed ball with center t_0 and positive radius r_0 included in V_{t_0} given by (3.4.1). For $|h| \le r_0$, we have

$$f(t_0 + h) - f(t_0) = \langle u, \underbrace{\phi(\cdot, t_0 + h) - \phi(\cdot, t_0)}_{\text{supported in } K_{t_0},}\rangle$$

and using Taylor's formula with integral remainder, we get

$$f(t_0 + h) - f(t_0) = \langle u, (\partial_t \phi)(\cdot, t_0)\rangle h + \underbrace{\langle u, \int_0^1 (1 - \theta) \overbrace{\partial_s^2 \phi(\cdot, t_0 + \theta h)}^{\text{support in } K_{t_0}} d\theta\rangle h^2}_{r(t_0,h)}.$$

We have, since $K_{t_0} \times B_0$ is a compact subset of $\Omega \times U$,

$$|r(t_0, h)| \le |h|^2 C_0 \sup_{x \in K_{t_0}, |\alpha| \le N_0} \int_0^1 (1 - \theta) |(\partial_x^\alpha \partial_s^2 \phi) \underbrace{(x, t_0 + \theta_0 h)}_{\in K_{t_0} \times B_0} | d\theta \le C_1 |h|^2,$$

proving the differentiability of f on U along with $df(t) = \langle u, \partial_t \phi(\cdot, t)\rangle$. Inductively, we get that f is smooth and the result of the theorem. □

Corollary 3.4.2. *Let X, Y be open subsets of $\mathbb{R}^n, \mathbb{R}^m$, $\phi \in C^\infty(X \times Y)$ and $u \in \mathscr{D}'(X)$.*
(1) If ϕ is compactly supported in $X \times Y$, the function ψ defined by $\psi(y) = \langle u, \phi(\cdot, y)\rangle$ belongs to $C_c^\infty(Y)$.
(2) If $u \in \mathscr{E}'(X)$, the function ψ defined by $\psi(y) = \langle u, \phi(\cdot, y)\rangle$ belongs to $C^\infty(Y)$.

Proof. To prove (1), we need only to verify (3.4.1): we have indeed for all $y \in Y$

$$\operatorname{supp} \phi(\cdot, y) \subset proj_X(\operatorname{supp} \phi) \quad \text{which is a compact subset of } X,$$

which implies that $\psi \in C^\infty(Y)$; moreover the function $\phi(\cdot, y) = 0$ on the open subset of Y, $\big(proj_Y(\operatorname{supp} \phi)\big)^c$, and thus $\operatorname{supp} \psi \subset proj_Y(\operatorname{supp} \phi)$ which is a compact subset of Y. To obtain (2), we consider $\chi \in C_c^\infty(X)$ equal to 1 near the compact support of u. We have then $u = \chi u$ and consequently,

$$\langle u, \phi(\cdot, y)\rangle = \langle u, \phi(\cdot, y)\chi(\cdot)\rangle.$$

The function $\Phi(x, y) = \phi(x, y)\chi(x)$ is smooth on $X \times Y$ and $\operatorname{supp} \Phi(\cdot, y) \subset \operatorname{supp} \chi$ so that we can apply the theorem 3.4.1 whose assumptions are satisfied. □

3.4.2 Pull-back by the affine group

Let us now recall the definition of the affine group of $\mathbb{R}^n$: it is the group of mappings from $\mathbb{R}^n$ into itself of the form $x \mapsto Ax + t = \theta_{A,t}(x)$ where $A \in Gl(n, \mathbb{R})(n \times n$ invertible matrices) and $t \in \mathbb{R}^n$. When A is the identity, $\Theta_{\mathrm{Id},t}$ is simply the translation of vector t; we have also $\theta_{A,t}^{-1} = \Theta_{A^{-1},-A^{-1}t}$. If u belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Theta_{A,t}$ is in the affine group of $\mathbb{R}^n$, we can define the *pull-back* of u by the map Θ by the identity

$$\Theta^*_{A,t}u = u \circ \Theta_{A,t}, \quad \text{so that } (\Theta^*_{A,t}u)(x) = u(Ax+t). \tag{3.4.2}$$

As a result for $\varphi \in C^0_c(\mathbb{R}^n)$, we find

$$\langle \Theta^*_{A,t}u, \varphi\rangle = \int_{\mathbb{R}^n} u(Ax+t)\varphi(x)dx = \int_{\mathbb{R}^n} u(y)\varphi(A^{-1}y - A^{-1}t)|\det A|^{-1}dy. \tag{3.4.3}$$

We want to use that formula to define the pull-back of a distribution on $\mathbb{R}^n$ by an affine transformation.

Definition 3.4.3. *Let $A \in Gl(n,\mathbb{R}), t \in \mathbb{R}^n$, $\Theta_{A,t}$ the affine transformation defined above and let $u \in \mathscr{D}'(\mathbb{R}^n)$. We define the distribution $\Theta^*_{A,t}u$ by the identity*

$$\langle \Theta^*_{A,t}u, \varphi\rangle = \langle u, \varphi \circ \Theta^{-1}_{A,t}\rangle |\det A|^{-1}. \tag{3.4.4}$$

Remark 3.4.4. (1) Note that this defines a distribution on $\mathbb{R}^n$, since the mapping $\varphi \mapsto \varphi \circ \Theta^{-1}_{A,t}$ is an isomorphism of $\mathscr{D}(\mathbb{R}^n)$. Moreover, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the previous definition ensures that $\Theta^*_{A,t}u = u \circ \Theta_{A,t}$, thanks to the lemma 3.1.7.
(2) The mapping $u \mapsto \Theta^*_{A,t}u$ is sequentially continuous from $\mathscr{D}'(\mathbb{R}^n)$ into itself.
(3) A distribution u on $\mathbb{R}^n$ is even (resp. odd) if $\Theta^*_{-\mathrm{Id},0}u = u$ (resp. $-u$). Using the notation

$$\check{u} = \Theta^*_{-\mathrm{Id},0}u \quad (\text{for a function } u,\ \check{u}(x) = u(-x)), \tag{3.4.5}$$

u is even means $\check{u} = u$, odd means $\check{u} = -u$.

3.4.3 Homogeneous distributions

Definition 3.4.5. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. The distribution u is said to be homogeneous with degree λ if for all $t > 0$, $u(t\cdot) = t^\lambda u(\cdot)$ (here $u(t\cdot) = \theta^*_{t\,\mathrm{Id},0}u$).*

Proposition 3.4.6. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. The distribution u is homogeneous of degree λ if and only if the Euler equation is satisfied, namely*

$$\sum_{1\le j\le n} x_j \partial_{x_j} u = \lambda u. \tag{3.4.6}$$

Proof. A distribution u on $\mathbb{R}^n$ is homogeneous of degree λ means:

$$\forall \varphi \in C^\infty_c(\mathbb{R}^n), \forall t > 0, \qquad \langle u(y), \varphi(y/t)t^{-n}\rangle = t^\lambda \langle u(x), \varphi(x)\rangle,$$

which is equivalent to $\forall \varphi \in C^\infty_c(\mathbb{R}^n), \forall s > 0, \langle u(y), \varphi(sy)s^{n+\lambda}\rangle = \langle u(x), \varphi(x)\rangle$, also equivalent to

$$\forall \varphi \in C^\infty_c(\mathbb{R}^n), \qquad \frac{d}{ds}\big(\langle u(y), \varphi(sy)s^{n+\lambda}\rangle\big) = 0 \quad \text{on } s > 0. \tag{3.4.7}$$

Note that the differentiability property is due to the theorem 3.4.1 and that

$$\langle u(y), \varphi(sy)s^{n+\lambda}\rangle = \langle u(x), \varphi(x)\rangle \quad \text{at } s = 1.$$

As a consequence, applying the theorem 3.4.1, we get that the homogeneity of degree λ of u is equivalent to

$$\forall s > 0, \quad \langle u(y), s^{n+\lambda-1}\big((n+\lambda)\varphi(sy) + \sum_{1\le j\le n} (\partial_j\varphi)(sy)sy_j\big)\rangle = 0,$$

also equivalent to $0 = \langle u(y), (n+\lambda+\sum_{1\le j\le n} y_j\partial_j)\big(\varphi(sy)\big)\rangle$ and by the definition of the differentiation of a distribution, it is equivalent to $(n+\lambda)u - \sum_{1\le j\le n} \partial_j(y_j u) = 0$, which is (3.4.6) by the Leibniz rule (3.2.14). □

Remark 3.4.7. (1) The Dirac mass at 0 in $\mathbb{R}^n$ is homogeneous of degree $-n$: we have for $t > 0$

$$\langle\delta_0(tx), \varphi(x)\rangle = \langle\delta_0(y), \varphi(y/t)t^{-n}\rangle = t^{-n}\varphi(0) = t^{-n}\langle\delta_0, \varphi\rangle.$$

(2) If T is an homogeneous distribution of degree λ, then $\partial_x^\alpha T$ is also homogeneous with degree $\lambda - |\alpha|$: taking the derivative of the Euler equation (3.4.6), we get

$$\partial_{x_k} u + \sum_{1\le j\le k} x_j\partial_{x_j}\partial_{x_k} u - \lambda\partial_{x_k} u = 0,$$

proving that $\partial_{x_k} u$ is homogeneous of degree $\lambda - 1$ and the result by iteration.
(3) It follows immediately from the definition (3.1.13) that the distribution $\mathrm{pv}(\frac{1}{x})$ is homogeneous of degree -1. The same is true for the distributions $\frac{1}{x\pm i0}$ as it is clear from (3.2.9)and (3.2.10).
(4) For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -1$ we define the $L^1_{\mathrm{loc}}(\mathbb{R})$ functions

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x \le 0., \end{cases} \qquad \chi_+^\lambda = \frac{x_+^\lambda}{\Gamma(\lambda+1)}. \tag{3.4.8}$$

The distributions χ_+^λ and x_+^λ are homogeneous of degree λ and by an analytic continuation argument, we can prove that χ_+^λ may be defined for any $\lambda \in \mathbb{C}$, is an homogeneous distribution of degree λ and satisfies

$$\chi_+^\lambda = (\frac{d}{dx})^k(\chi_+^{\lambda+k}), \quad \chi_+^{-k} = \delta_0^{(k-1)}, \quad k \in \mathbb{N}^*.$$

Lemma 3.4.8. *Let $(u_j)_{1\le j\le m}$ be non-zero homogeneous distributions on $\mathbb{R}^n$ with distinct degrees $(\lambda_j)_{1\le j\le m}$ ($j \ne k$ implies $\lambda_j \ne \lambda_k$). Then they are independent in the complex vector space $\mathscr{D}'(\mathbb{R}^n)$.*

Proof. We assume that $m \ge 2$ and that there exists some complex numbers $(c_j)_{1\le j\le m}$ such that $\sum_{1\le j\le m} c_j u_j = 0$. Then applying the operator $\mathcal{E} = \sum_{1\le j\le m} x_j\partial_{x_j}$, we get for all $k \in \mathbb{N}$,

$$0 = \sum_{1\le j\le m} c_j\mathcal{E}^k(u_j) = \sum_{1\le j\le m} c_j\lambda_j^k u_j.$$

We consider now the Vandermonde matrix $m \times m$

$$V_m = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \dots & & & \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \dots & \lambda_m^{m-1} \end{pmatrix}, \quad \det V_m = \prod_{1 \le j < k \le m} (\lambda_k - \lambda_j) \neq 0.$$

We note that for $\varphi \in C_c^\infty(\mathbb{R}^n)$, and $X \in \mathbb{C}^m$ given by

$$X = \begin{pmatrix} c_1 \langle u_1, \varphi \rangle \\ c_2 \langle u_2, \varphi \rangle \\ \dots\dots\dots \\ c_m \langle u_m, \varphi \rangle \end{pmatrix},$$

we have $V_m X = 0$, so that $X = 0$, i.e. $\forall j, \forall \varphi \in C_c^\infty(\mathbb{R}^n), c_j \langle u_j, \varphi \rangle = 0$, i.e. $c_j u_j = 0$ and since u_j is not the zero distribution, we get the sought conclusion $c_j = 0$ for all j. □

3.4.4 Tensor products of distributions

We begin with a lemma.

Lemma 3.4.9. *Let $\phi \in C_c^\infty(]0,1[^n)$; one can find a sequence of functions in*

$$\mathrm{Vect}(\otimes^n C_c^\infty(]0,1[) \quad \textit{(the vector space generated by the tensor products)}$$

converging to ϕ in $C_c^\infty(]0,1[^n)$ in the sense of the definition 3.1.9.

Proof. We define for $k \in \mathbb{Z}^n$, $\hat{\phi}(k) = \int e^{-2i\pi x \cdot k} \phi(x) dx$, and we note that, with $\Delta = \sum_{1 \le j \le n} \partial_{x_j}^2$, $m \in \mathbb{N}$,

$$\hat{\phi}(k) = (1 + |k|^2)^{-m} \int (1 - \frac{1}{4\pi^2} \Delta)^m \big(e^{-2i\pi x \cdot k}\big) \phi(x) dx$$
$$= (1 + |k|^2)^{-m} \int e^{-2i\pi x \cdot k} \big((1 - \frac{1}{4\pi^2} \Delta)^m \phi\big)(x) dx$$

so that

$$|\hat{\phi}(k)| \le (1 + |k|^2)^{-m} C_m \max_{|\alpha| \le 2m} \|\partial_x^\alpha \phi\|_{L^\infty}. \tag{3.4.9}$$

As a result the series $\Phi(x) = \sum_{k \in \mathbb{Z}^n} \hat{\phi}(k) e^{2i\pi x \cdot k}$ converges and is a smooth function, periodic with periods $\mathbb{Z}^n$: we need only to check that $\sum_{k \in \mathbb{Z}^n} (1 + |k|)^{-n-1} < +\infty$.[3] Moreover,

$$\text{for } x \in [0,1]^n, \qquad \Phi(x) = \phi(x). \tag{3.4.10}$$

[3] In fact, with $Q_k = k + (0,1)^n$ we have, replacing the Euclidean norm $|k|$ by the (equivalent) sup-norm $\|k\| = \max_{1 \le j \le k} |k_j|$, we have for $x \in Q_k$, $k_j < x_j < k_j + 1$ and thus

$$\|x\| = \max |x_j| \le 1 + \|k\| \Longrightarrow 1 + \|x\| \le 2 + \|k\|$$

and $\sum_{k \in \mathbb{Z}^n} (2 + \|k\|)^{-n-1} \le \int \sum_{k \in \mathbb{Z}^n} \mathbf{1}_{Q_k}(x) (1 + \|x\|)^{-n-1} dx = \int (1 + \|x\|)^{-n-1} dx < +\infty$.

We verify this first for $n = 1$. We have in that case

$$\Phi(x) = \lim_{N\to+\infty} \int \sum_{|k|\le N} e^{2i\pi k(x-y)}\phi(y)dy,$$

and since $\sum_{|k|\le N} e^{2i\pi kt} = 1 + 2\operatorname{Re} \sum_{1\le k\le N} e^{2i\pi kt} = 1 + 2\operatorname{Re}\big(e^{2i\pi t}\frac{e^{2i\pi Nt}-1}{e^{2i\pi t}-1}\big)$

$$= 1 + 2\operatorname{Re}\big(e^{i\pi(N+1)t}\frac{\sin(\pi Nt)}{\sin(\pi t)}\big) = \frac{\sin(\pi t(2N+1))}{\sin(\pi t)},$$

we get that, since $\phi \in C_c^\infty(]0,1[)$, and for $x \in]0,1[$,

$$\begin{aligned}\Phi(x) &= \lim_{N\to+\infty} \int \frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\phi(y)dy\\ &= \lim_{N\to+\infty} \Big(\int_0^1 \frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\big(\phi(y)-\phi(x)\big)dy + \phi(x)\int_0^1 \sum_{|k|\le N} e^{2i\pi k(x-y)}dy\Big)\\ &= \phi(x),\end{aligned}$$

because with $\psi \in C^\infty(\mathbb{R}^2)$, $\theta(s) = \frac{s}{\sin \pi s}$ (which is in $C^\infty(\mathbb{R}\backslash\pi\mathbb{Z}^*)$ and in particular on $]-1,+1[$), we have

$$\int_0^1 \frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\big(\phi(y)-\phi(x)\big)dy$$

$$= \int_0^1 \sin\big(\pi(x-y)(2N+1)\big)\overbrace{\psi(x,y)\theta(x-y)}^{\substack{\text{smooth of } y \text{ on } [0,1]\\ \text{since } x\in]0,1[}}dy \underset{N\to+\infty}{\longrightarrow} 0,$$

since with $\omega \in C^\infty([0,1])$, we have $\int_0^1 \sin\big(\pi(x-y)(2N+1)\big)\omega(y)dy =$

$$\Big[\frac{\cos\big(\pi(x-y)(2N+1)\big)}{\pi(2N+1)}\omega(y)\Big]_{y=0}^{y=1} - \int_0^1 \frac{\cos\big(\pi(x-y)(2N+1)\big)}{\pi(2N+1)}\omega'(y)dy.$$

We have proven (3.4.10) for $n = 1$ and $x \in]0,1[$. Since Φ, ϕ are both smooth on $[0,1]$ the equality holds as well for $x \in \{0,1\}$.

N.B. We could have used the Riemann-Lebesgue lemma (see e.g. the lemma 3.4.4 in [9]), but we have preferred a simple self-contained argument with an integration by parts since there was no shortage of regularity for the function ω.

To handle the case $n \ge 2$, we use an induction and in $n+1$ dimensions, we have for $\phi \in C_c^\infty(]0,1[^{n+1})$,

$$\forall x \in [0,1]^n, \quad \Phi(x, x_{n+1}) = \sum_{k\in\mathbb{Z}^n} \int_{(0,1)^n} e^{2i\pi(x-y)\cdot k}\phi(y, x_{n+1})dy = \phi(x, x_{n+1}),$$

and thus $\forall x \in [0,1]^n, \forall x_{n+1} \in [0,1], \Phi(x, x_{n+1}) =$

$$\sum_{k\in\mathbb{Z}^n} \int_{(0,1)^n} e^{2i\pi(x-y)\cdot k} \Big(\sum_{k_{n+1}\in\mathbb{Z}} \int_0^1 e^{2i\pi(x_{n+1}-y_{n+1})k_{n+1}} \phi(y, y_{n+1}) dy_{n+1} \Big) dy = \phi(x, x_{n+1}),$$

which is (3.4.10) since the series are uniformly converging. Since $\operatorname{supp}\phi \subset]0,1[^n$, there exists $\epsilon_0 > 0$ such that[4] $\operatorname{supp}\phi \subset [\epsilon_0, 1-\epsilon_0]^n$, and with $\chi \in C_c^\infty(]0,1[)$ equal to 1 on $[\epsilon_0, 1-\epsilon_0]$, we have

$$\chi(x_1)\dots\chi(x_n)\phi(x) = \phi(x) = \sum_{k\in\mathbb{Z}^n} e^{2i\pi x\cdot k} \hat{\phi}(k)\chi(x_1)\dots\chi(x_n). \tag{3.4.11}$$

The series is uniformly converging as well as all its derivatives, thanks to the fast decay of $\hat{\phi}(k)$ expressed by (3.4.9), and the functions

$$\sum_{|k|\le N} e^{2i\pi x_1 k_1} \dots e^{2i\pi x_n k_n} \hat{\phi}(k)\chi(x_1)\dots\chi(x_n)$$

belong to $\operatorname{Vect}(\otimes^n C_c^\infty(]0,1[)$ with fixed compact support in $]0,1[^n$. The proof of the lemma is complete. □

As a consequence, we get the following result.

Proposition 3.4.10. *Let X be an open subset of $\mathbb{R}^m$, Y be an open subset of $\mathbb{R}^n$. $\operatorname{Vect} C_c^\infty(X) \otimes C_c^\infty(Y)$ is dense in $C_c^\infty(X \times Y)$.*

Proof. Let K be a compact subset of $X \times Y$. For each point $(x,y) \in K$, we can find some open bounded intervals $I_1, \dots, I_m, J_1, \dots, J_n$ of $\mathbb{R}$ such that

$$(x,y) \in Q = I_1 \times \dots \times I_m \times J_1 \times \dots \times J_n \subset X \times Y.$$

As a result, we can cover K with a finite number of open "cubes" $(Q_l)_{1\le l\le N}$ included in $X \times Y$. Using a partition of unity given by the theorem 3.1.14, we can find $\psi_l \in C_c^\infty(Q_l)$ such that $\sum_{1\le l\le N} \psi_l(x) = 1$ for $x \in V$ open such that $K \subset V \subset X \times Y$. For $\varphi \in C_c^\infty(X \times Y)$, $\operatorname{supp}\varphi = K$ compact subset of $X \times Y$, we have

$$\varphi = \sum_{1\le l\le N} \varphi\psi_l, \quad \varphi\psi_l \in C_c^\infty(Q_l).$$

We can then apply the lemma 3.4.9 for each $\varphi\psi_l$ (rescaling the cube Q_l to $]0,1[^n$) to obtain the conclusion of the proposition. □

Theorem 3.4.11. *Let X be an open subset of $\mathbb{R}^m$, Y be an open subset of $\mathbb{R}^n$, and $u \in \mathscr{D}'(X), v \in \mathscr{D}'(Y)$. Then there exists a unique $w \in \mathscr{D}'(X \times Y)$ such that, $\forall \phi \in \mathscr{D}(X), \forall \psi \in \mathscr{D}(Y)$,*

$$\langle w, \phi \otimes \psi \rangle_{\mathscr{D}'(X\times Y), \mathscr{D}(X\times Y)} = \langle u, \phi \rangle_{\mathscr{D}'(X), \mathscr{D}(X)} \langle v, \psi \rangle_{\mathscr{D}'(Y), \mathscr{D}(Y)}, \tag{3.4.12}$$

where $(\phi \otimes \psi)(x,y) = \phi(x)\psi(y)$. We shall denote w by $u \otimes v$ and call it the tensor product of u and v.

[4] In fact, each projection $K_j = proj_j(\operatorname{supp}\phi)$ is a compact subset of $]0,1[$, thus $0 < \inf_{t\in K_j} t \le \sup_{t\in K_j} t < 1$.

Proof. The uniqueness follows from the proposition 3.4.10. To find such a w, we define for $\Phi \in C_c^\infty(X \times Y)$, with obvious notations,

$$\langle w, \Phi\rangle = \big\langle v(y), \langle u(x), \Phi(x,y)\rangle\big\rangle. \tag{3.4.13}$$

As a matter of fact, thanks to the corollary 3.4.2 (1), the function $Y \ni y \mapsto \langle u(\cdot), \Phi(\cdot, y)\rangle$ belongs to $C_c^\infty(Y)$ so that (3.4.13) makes sense. Using the theorem 3.4.1, we obtain $\partial_y^\alpha \langle u(\cdot), \Phi(\cdot, y)\rangle = \langle u(\cdot), \partial_y^\alpha \Phi(\cdot, y)\rangle$. If $K = \operatorname{supp} \Phi$ (compact subset of $X \times Y$), both projections $proj_X K, proj_Y K$ are compact so that

$$|\langle u(\cdot), \partial_y^\alpha \Phi(\cdot, y)\rangle| \le C_1 \sup_{|\beta| \le N_1,\ x \in proj_X K} |(\partial_x^\beta \partial_y^\alpha \Phi)(x,y)|$$

and thus

$$\begin{aligned} |\big\langle v(y), \langle u(x), \Phi(x,y)\rangle\big\rangle| &\le C_2 \sup_{\substack{|\alpha| \le N_2 \\ y \in proj_Y K}} |\partial_y^\alpha \langle u(\cdot), \Phi(\cdot, y)\rangle| \\ &\le C_1 C_2 \sup_{\substack{|\beta| \le N_1, |\alpha| \le N_2 \\ (x,y) \in K}} |(\partial_x^\beta \partial_y^\alpha \Phi)(x,y)|, \end{aligned}$$

implying that w is indeed a distribution on $X \times Y$. Since the formula (3.4.12) follows from (3.4.13), this concludes the proof of the theorem. □

Remark 3.4.12. (1) The uniqueness ensures that $w = u \otimes v$ is also defined by

$$\langle w, \Phi\rangle = \big\langle u(x), \langle v(y), \Phi(x,y)\rangle\big\rangle, \tag{3.4.14}$$

a formula for which (3.4.12) also holds.
(2) If $u \in L^1_{\mathrm{loc}}(X), v \in L^1_{\mathrm{loc}}(Y)$, then $u \otimes v$ belongs to $L^1_{\mathrm{loc}}(X \times Y)$ and is defined by $u(x)v(y)$, thanks to the lemma 3.1.7 and to the proposition 3.4.10.
(3) For $u \in \mathscr{D}'(X), v \in \mathscr{D}'(Y)$, we have

$$\operatorname{supp}(u \otimes v) = \operatorname{supp} u \times \operatorname{supp} v. \tag{3.4.15}$$

In fact, if $\Phi \in C_c^\infty(X \times Y)$ with $\operatorname{supp} \Phi \subset X \times (\operatorname{supp} v)^c$ or with $\operatorname{supp} \Phi \subset (\operatorname{supp} u)^c \times Y$, it follows from (3.4.14) or (3.4.13) that $\langle u \otimes v, \Phi\rangle = 0$; this holds as well when

$$\operatorname{supp} \Phi \subset (\operatorname{supp} u \times \operatorname{supp} v)^c = \big((\operatorname{supp} u)^c \times Y\big) \cup \big(X \times (\operatorname{supp} v)^c\big),$$

since $\operatorname{supp} \Phi \subset \Omega_1 \cup \Omega_2$ with Ω_j open subset of $X \times Y$ and, thanks to the theorem 3.1.14, the compactly supported $\Phi = \Phi_1 + \Phi_2$, with $\operatorname{supp} \Phi_j \subset \Omega_j$ (it is also a direct consequence of the theorem 3.1.15 since $(u \otimes v)_{|\Omega_j} = 0$). We have proven that $\operatorname{supp}(u \otimes v) \subset \operatorname{supp} u \times \operatorname{supp} v$. Conversely, if $x_0 \in \operatorname{supp} u, y_0 \in \operatorname{supp} v$, and U, V are respective open neighborhoods of x_0, y_0 in X, Y, we can find $\phi_0 \in C_c^\infty(U), \psi_0 \in C_c^\infty(V)$ such that $\langle u, \phi_0\rangle \ne 0$ and $\langle v, \psi_0\rangle \ne 0$. As a result $\phi_0 \otimes \psi_0 \in C_c^\infty(U \times V)$ and $\langle u \otimes v, \phi_0 \otimes \psi_0\rangle = \langle u, \phi_0\rangle\langle v, \psi_0\rangle \ne 0$, so that $(u \otimes v)_{|U \times V}$ is not zero, proving that $(x_0, y_0) \in \operatorname{supp}(u \otimes v)$ and the sought result.
(4) With the notations of the previous theorem, we have obviously from the expression (3.4.13) and the theorem 3.4.1 that $\partial_x^\alpha \partial_y^\beta (u \otimes v) = (\partial_x^\alpha u) \otimes (\partial_y^\beta v)$.

Proposition 3.4.13. *Let $n \in \mathbb{N}^*$, U be an open subset of $\mathbb{R}^{n-1}$, I an interval of $\mathbb{R}$. Let $u \in \mathscr{D}'(U \times I)$ such that $\partial_{x_n} u = 0$. Then, there exists $v \in \mathscr{D}'(U)$ such that $u = v \otimes 1$. In other words, the differential equation $\partial_{x_n} u = 0$ has the only solutions $u(x', x_n) = v(x')$.*

Proof. From the remark 3.4.12 (3) above, the tensor products $v(x') \otimes 1$ are indeed solutions of $\partial_{x_n} u = 0$. Conversely the proposition is proven for $n = 1$ by the lemma 3.2.4. Let us assume $n \geq 2$; we consider $\chi_0 \in C_c^\infty(I)$ such that $\int \chi_0(t)dt = 1$ and we define $v \in \mathscr{D}'(U)$ by the identity

$$\langle v, \varphi \rangle_{\mathscr{D}'(U),\mathscr{D}(U)} = \langle u, \varphi \otimes \chi_0 \rangle_{\mathscr{D}'(U\times I),\mathscr{D}(U\times I)}.$$

For $\varphi \in \mathscr{D}(U), \psi \in \mathscr{D}(I)$, we have with $J(\psi) = \int \psi(t)dt$,

$$\langle v \otimes 1, \varphi \otimes \psi \rangle = \langle u, \varphi \otimes \chi_0 \rangle J(\psi).$$

From the proof of the lemma 3.2.4, we see that $\psi - \chi_0 J(\psi) = \theta'$ with $\theta \in C_c^\infty(I)$, and we get $\langle u, \varphi \otimes \big(\chi_0 J(\psi) - \psi\big) \rangle = \langle u, \partial_{x_n}(\varphi \otimes \theta) \rangle = 0$ so that $\langle v \otimes 1, \varphi \otimes \psi \rangle = \langle u, \varphi \otimes \psi \rangle$, which is the sought result. □

3.5 Convolution

We want to define the convolution of two distributions on $\mathbb{R}^n$, provided one of them has compact support. Assuming first that $u \in L^1_{\text{comp}}(\mathbb{R}^n), v \in L^1_{\text{loc}}(\mathbb{R}^n), \phi \in C_c^\infty(\mathbb{R}^n)$ the integral

$$\iint u(x-y)v(y)\phi(x)dxdy = \iint u(x)v(y)\phi(x+y)dxdy, \tag{3.5.1}$$

makes sense since x and $x + y$ are moving in a compact set in the last integral (and so is y). This formula allows us to define

$$(u * v)(x) = \int u(x-y)v(y)dy = \int u(y)v(x-y)dy$$

and can naturally be extended to $u, v \in L^1(\mathbb{R}^n)$ so that $\|u*v\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)}\|v\|_{L^1(\mathbb{R}^n)}$, making $L^1(\mathbb{R}^n)$ a Banach algebra (without unit). The inequality of Young (see e.g. the Théorème 6.2.1 in [9]) is a non-trivial extension of that inequality. Anyhow, at the moment, we want to use the formula (3.5.1) for our general definition.

3.5.1 Convolution $\mathscr{E}'(\mathbb{R}^n) * \mathscr{D}'(\mathbb{R}^n)$

Definition 3.5.1. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We define the convolution $u * v$ by the following bracket of duality*

$$\langle u * v, \phi \rangle_{\mathscr{D}'(\mathbb{R}^n),\mathscr{D}(\mathbb{R}^n)} = \big\langle u(x), \langle v(y), \phi(x+y) \rangle \big\rangle = \big\langle v(y), \langle u(x), \phi(x+y) \rangle \big\rangle. \tag{3.5.2}$$

We note that the theorem 3.4.1 shows that the function $\mathbb{R}^n \ni x \mapsto \langle v(y), \phi(x+y)\rangle$ is C^∞ and thus that the first definition makes sense from the corollary 3.4.2 (2). To check the second equality above, we note that with $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the support of u, we have $\chi u = u$ and thus from the remark 3.4.12(1) and the formula (3.4.13),

$$\big\langle u(x), \langle v(y), \phi(x+y)\rangle\big\rangle = \big\langle u(x), \langle v(y), \chi(x)\phi(x+y)\rangle\big\rangle = \langle u(x)\otimes v(y), \chi(x)\phi(x+y)\rangle,$$

which is also equal to $\big\langle v(y), \langle u(x), \chi(x)\phi(x+y)\rangle\big\rangle = \big\langle v(y), \langle u(x), \phi(x+y)\rangle\big\rangle$. This proves as well that $u * v$ is a distribution on $\mathbb{R}^n$ since the mapping $C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \Phi \in C_c^\infty(\mathbb{R}^{2n})$, with $\Phi(x,y) = \phi(x+y)\chi(x)$ is continuous.

Remark 3.5.2. We note that whatever is $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the support of u, we have for $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$,

$$\langle u * v, \phi\rangle = \langle u(x)\otimes v(y), \chi(x)\phi(x+y)\rangle. \tag{3.5.3}$$

Proposition 3.5.3. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We have*

$$\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v. \tag{3.5.4}$$

Proof. Note first that $\operatorname{supp} u + \operatorname{supp} v$ is a closed subset of $\mathbb{R}^n$ as the sum of a compact set and a closed set (exercise). Now if $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\phi \subset (\operatorname{supp} u + \operatorname{supp} v)^c$, then

$$\operatorname{supp}\big((x,y)\mapsto \phi(x+y)\big) \subset (\operatorname{supp} u \times \operatorname{supp} v)^c. \tag{3.5.5}$$

In fact, if $(x_0, y_0) \in \operatorname{supp} u \times \operatorname{supp} v$, then $x_0 + y_0 \in \operatorname{supp} u + \operatorname{supp} v \subset (\operatorname{supp}\phi)^c$, the latter being open so that there exists U open in $\mathscr{V}_0$ with $\phi(x_0 + U + y_0 + U) = 0$. As a consequence, the open set $(x_0 + U) \times (y_0 + U) \subset \big(\operatorname{supp}((x,y)\mapsto \phi(x+y))\big)^c$ and this implies $(x_0, y_0) \in \big(\operatorname{supp}((x,y)\mapsto \phi(x+y))\big)^c$ and proves (3.5.5), so that (3.5.3), (3.4.15) give the conclusion of the proposition. ☐

Remark 3.5.4. For u, v both in $\mathscr{E}'(\mathbb{R}^n)$, the formula (3.5.2) ensures that $u*v = v*u$.

3.5.2 Regularization

Proposition 3.5.5. *Let $u \in \mathscr{D}'(\mathbb{R}^n), \rho \in C_c^\infty(\mathbb{R}^n)$. Then $\rho * u$ belongs to $C^\infty(\mathbb{R}^n)$.*

Proof. We have from the definitions, with $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near $\operatorname{supp}\rho$, $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle \rho * u, \phi\rangle = \langle \rho(x)\otimes u(y), \chi(x)\phi(x+y)\rangle = \langle u(y), \langle \rho(x), \chi(x)\phi(x+y)\rangle\rangle, \tag{3.5.6}$$

and we note that $\langle \rho(x), \chi(x)\phi(x+y)\rangle = \int \rho(x)\phi(x+y)dx = \int \rho(x-y)\phi(x)dx$. As a result, we have

$$\langle \rho * u, \phi\rangle = \langle u(y), \int \underbrace{\rho(x-y)\phi(x)}_{\in C_c^\infty(\mathbb{R}^{2n})} dx\rangle = \int \phi(x)\langle u(y), \rho(x-y)\rangle dx$$

where the last equality is due to the theorem 3.4.1[5] which gives also that $\psi(x) = \langle u(y), \rho(x-y)\rangle$ is C^∞; we have proven $\rho * u = \psi$ and the result. We note also the formula following from (3.5.6)

$$\langle \rho * u, \phi\rangle = \langle u, \check\rho * \phi\rangle. \tag{3.5.7}$$

□

Lemma 3.5.6. *Let Ω be an open subset of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. There exists a sequence $(\psi_j)_{j\ge 1}$ in $\mathscr{D}(\Omega)$ such that $\lim_j \psi_j = T$ in the weak-dual topology sense of the definition 3.1.16.*

Proof. We consider first a sequence $(K_j)_{j\ge1}$ of compact subsets of Ω as in the lemma 2.3.1 and a sequence $(\chi_j)_{j\ge1}$ such that $\chi_j \in C_c^\infty(\operatorname{int} K_{j+1}), \chi_j = 1$ near K_j (see the lemma 3.1.3). In the weak-dual topology sense, we have $\lim_j \chi_j T = T$: let $\varphi \in \mathscr{D}(\Omega)$, $K = \operatorname{supp}\varphi$. From the lemma 2.3.1, there exists j such that $\operatorname{supp}\varphi \subset K_j$ and thus $\varphi\chi_j = \varphi$, implying $\langle T\chi_j, \varphi\rangle = \langle T, \chi_j\varphi\rangle = \langle T, \varphi\rangle$. We can also consider the compactly supported distribution $\chi_j T$ and see it as a distribution on $\mathbb{R}^n$. We take now a function $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\int \rho(x)dx = 1$. According to the first example in the section 3.1.3, we define ρ_ϵ (it tends to the Dirac mass at 0 in the weak-dual topology when $\epsilon \to 0_+$). For $\varphi \in \mathscr{D}(\Omega)$, using (3.5.7), we have

$$\langle \rho_\epsilon * (\chi_j T), \varphi\rangle = \langle \chi_j T, \check\rho_\epsilon * \varphi\rangle. \tag{3.5.8}$$

Considering now a decreasing sequence of positive numbers (ϵ_j) with limit 0 such that

$$\operatorname{supp}\chi_j + \epsilon_j \operatorname{supp}\rho \subset \operatorname{int}(K_{j+1}) \subset \Omega,$$

and we define $T_j = \rho_{\epsilon_j} * \chi_j T$. We have from the proposition 3.5.3 that $\operatorname{supp} T_j$ is compact included in Ω and also that $T_j \in C^\infty$ (proposition 3.5.5). Going back to (3.5.8), for a fixed φ, we can find j such that $\operatorname{supp}\varphi \subset K_{j-1}$ for $j \ge j_0$, implying that

$$\operatorname{supp}(\check\rho_{\epsilon_j} * \varphi) \subset K_{j-1} + \epsilon_j \operatorname{supp}\rho \subset \operatorname{supp}\chi_{j-1} + \epsilon_{j-1}\operatorname{supp}\rho \subset K_j,$$

implying that $\chi_j(\check\rho_{\epsilon_j} * \varphi) = \check\rho_{\epsilon_j} * \varphi$ and $\langle \rho_{\epsilon_j} * (\chi_j T), \varphi\rangle = \langle T, \check\rho_{\epsilon_j} * \varphi\rangle$. The result follows from the proposition 3.1.1 (implying $\lim_j(\check\rho_{\epsilon_j} * \varphi) = \varphi$ in $C_c^\infty(\Omega)$) and the (sequential) continuity of the distribution T. □

Proposition 3.5.7. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We have*

$$\operatorname{singsupp}(u * v) \subset \operatorname{singsupp} u + \operatorname{singsupp} v. \tag{3.5.9}$$

Proof. We can choose $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the singsupp u, $\psi \in C^\infty$ equal to 1 near the singular support of v. We have from the proposition 3.5.5

$$u * v = (\chi u) * v + \underbrace{\overbrace{((1-\chi)u)}^{\in C_c^\infty(\mathbb{R}^n)} * v}_{\in C^\infty(\mathbb{R}^n)} \equiv (\chi u) * (\psi v) + \underbrace{\overbrace{(\chi u)}^{\in \mathscr{E}'(\mathbb{R}^n)} * \overbrace{((1-\psi)v)}^{\in C^\infty(\mathbb{R}^n)}}_{\in C^\infty(\mathbb{R}^n)} \mod C^\infty(\mathbb{R}^n)$$

and thus we get for all $\epsilon > 0$

$$\operatorname{singsupp}(u * v) \subset \operatorname{supp}\psi + \operatorname{supp}\psi \subset \operatorname{singsupp} u + \epsilon\bar B_1 + \operatorname{singsupp} v + \epsilon \bar B_1,$$

which gives the result. □

[5]For $\Phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in \mathscr{D}'(\mathbb{R}^n)$, $\langle 1 \otimes u, \Phi\rangle = \langle u(y), \int \Phi(x,y)dx\rangle = \int \langle u(y), \Phi(x,y)\rangle dx$.

3.5.3 Convolution with a proper support condition

Looking at the formula (3.5.1), we see that we can extend it easily for $L^1_{\text{loc}}(\mathbb{R}^n)$ functions u, v so that the mapping

$$\operatorname{supp} u \times \operatorname{supp} v \ni (x, y) \mapsto x + y = \sigma(x, y) \in \mathbb{R}^n \tag{3.5.10}$$

is *proper*, i.e. such that $\sigma^{-1}(K)$ is compact for K compact subset of $\mathbb{R}^n$. In fact if $u, v \in L^1_{\text{loc}}(\mathbb{R}^n)$ are such that the map σ of (3.5.10) is proper, the function $u * v$ defined by

$$(u * v)(x) = \int u(x-y)v(y)dy$$

is also $L^1_{\text{loc}}(\mathbb{R}^n)$, since for K compact subset of $\mathbb{R}^n$, we have

$$\begin{aligned}\iint |u(x-y)||v(y)|\mathbf{1}_K(x)dydx &= \iint |u(x)||v(y)|\mathbf{1}_K(x+y)dxdy \\ &= \iint_{\sigma^{-1}(K)} |u(x)||v(y)|dxdy < \infty, \quad \text{since } \sigma^{-1}(K) \text{ is compact in } \mathbb{R}^{2n}.\end{aligned}$$

We can extend as well the convolution product of distributions u, v, provided σ in (3.5.10) is proper. Before doing so, we prove a simple lemma.

Lemma 3.5.8. *Let $F_1, \dots, F_m$ be closed subsets of $\mathbb{R}^n$ such that the mapping $\sigma : F_1 \times \cdots \times F_m \to \mathbb{R}^n$, defined by $\sigma(x_1, \dots, x_m) = x_1 + \cdots + x_m$ is proper. Defining for $\epsilon > 0$, $F_{j,\epsilon} = \{x \in \mathbb{R}^n, |x - F_j| \le \epsilon\}$, the mapping $\sigma_\epsilon : F_{1,\epsilon} \times \cdots \times F_{m,\epsilon} \to \mathbb{R}^n$, defined by $\sigma_\epsilon(x_1, \dots, x_m) = x_1 + \cdots + x_m$ is also proper.*

Proof. We note first that $F_{j,\epsilon} = F_j + \epsilon \bar{B}_1$ ($\bar{B}_1$ is the closed unit ball of $\mathbb{R}^n$) is closed as the sum of a compact and a closed set. Let K be compact subset of $\mathbb{R}^n$; if $(x_1, \dots, x_m) \in \sigma_\epsilon^{-1}(K)$, then there exists $y_j \in F_j, t_j \in \mathbb{R}^n, |t_j| \le \epsilon$, such that $x_j = y_j + t_j$, $\sum_{1 \le j \le m}(y_j + t_j) \in K$ and thus $\sum_{1\le j \le m} y_j \in K + m\epsilon \bar{B}_1$, so that $(y_j)_{1 \le j \le m} \in \sigma^{-1}(K + m\epsilon\bar{B}_1)$, a compact subset of $\prod F_j$. As a consequence, $(x_j)_{1\le j\le m} \in \sigma^{-1}(K + m\epsilon\bar{B}_1) + \epsilon \bar{B}_{1,nm}$ ($\bar{B}_{1,nm}$ is the closed unit ball of $\mathbb{R}^{nm}$), which is compact. As a result, $\sigma_\epsilon^{-1}(K)$ is compact as a closed subset of $\prod F_{j,\epsilon}$ (σ_ϵ is continuous) included in a compact set. □

Definition 3.5.9. *Let $u_1, \dots, u_m \in \mathscr{D}'(\mathbb{R}^n)$ such that the mapping σ*

$$\prod_{1\le j\le m} \operatorname{supp} u_j \ni (x_j)_{1\le j\le m} \mapsto \sum_{1\le j\le m} x_j \in \mathbb{R}^n \quad \textit{is proper.} \tag{3.5.11}$$

For $\epsilon > 0$, we take $\chi_{j,\epsilon} \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \chi_{j,\epsilon} \subset \operatorname{supp} u_j + \epsilon \bar{B}_1$ and $\operatorname{supp} \chi_{j,\epsilon}$ is 1 on a neighborhood of $\operatorname{supp} u_j$. We define then

$$\langle u_1 * \cdots * u_m, \phi\rangle_{\mathscr{D}'(\mathbb{R}^n), \mathscr{D}(\mathbb{R}^n)} = \langle u_1 \otimes \cdots \otimes u_m, \tilde{\phi}\rangle_{\mathscr{D}'(\mathbb{R}^{nm}), \mathscr{D}(\mathbb{R}^{nm})} \tag{3.5.12}$$

with $\tilde{\phi}(x_1, \dots, x_m) = \prod_{1\le j\le m} \chi_{j,\epsilon}(x_j)\phi(\sum_{1\le j\le m} x_j)$: we note that $\tilde{\phi}$ is in $\mathscr{D}(\mathbb{R}^{nm})$ since

$$\operatorname{supp} \tilde{\phi} \subset \{(x_j)_{1\le j\le m} \in \prod_{1\le j\le m} \operatorname{supp} \chi_{j,\epsilon} \textit{ with } \sigma((x_j)) \in \operatorname{supp} \phi\}$$

which is compact from the previous lemma and (3.5.11).

It is also easy to prove that this definition does not depend on the choices of the functions $\chi_{j,\epsilon}$ having the properties listed above and that this definition coincides with the definition of convolution in the previous section. In particular, we can prove the associativity of the convolution using the identity (3.5.12), *provided the condition* (3.5.11) *is satisfied.* As a counterexample we can take $u_1 = 1, u_2 = \delta_0', u_3 = H$ and we have since $1 * \delta_0' = 0, \delta_0' * H = \delta_0$,

$$(u_1 * u_2) * u_3 = 0, \quad u_1 * (u_2 * u_3) = 1 * \delta_0 = 1.$$

Naturally the hypothesis (3.5.11) is violated here since the mapping σ defined on $\mathbb{R} \times \{0\} \times \mathbb{R}_+$ is not proper: $\sigma^{-1}(\{0\}) \supset \{(-N, 0, N)\}_{N \in \mathbb{N}}$. The assumption (3.5.11) is satisfied for $m = 2$ if $\operatorname{supp} u_1$ is compact and also for distributions on $\mathbb{R}$ with support in $\mathbb{R}_+$. We get also that

$$\forall u \in \mathscr{D}'(\mathbb{R}^n), \quad u * \delta = u, \quad \text{since } \langle u(x_1) \otimes \delta(x_2), \phi(x_1 + x_2)\rangle = \langle u, \phi\rangle. \tag{3.5.13}$$

and for $u, v \in \mathscr{D}'(\mathbb{R}^n)$ such that (3.5.11) holds

$$\partial_x^\alpha (u * v) = (\partial_x^\alpha u) * v = u * (\partial_x^\alpha v), \tag{3.5.14}$$

since $\langle \partial_x^\alpha (u * v), \phi\rangle = (-1)^{|\alpha|}\langle u * v, \partial_x^\alpha \phi\rangle = (-1)^{|\alpha|}\langle u(x) \otimes v(y), (\partial^\alpha \phi)(x + y)\rangle = \langle (\partial_x^\alpha u)(x) \otimes v(y), \phi(x + y)\rangle$ and putting inside the brackets the cut-off functions χ_ϵ does not change the outcome of the computation.

3.6 Some fundamental solutions

3.6.1 Definitions

Definition 3.6.1. *We consider a constant coefficients differential operator*

$$P = P(D) = \sum_{|\alpha| \le m} a_\alpha D_x^\alpha, \quad \textit{where } a_\alpha \in \mathbb{C}, D_x^\alpha = \frac{1}{(2i\pi)^{|\alpha|}} \partial_x^\alpha. \tag{3.6.1}$$

A distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ is called a fundamental solution of P when $PE = \delta_0$.

We note that if $f \in \mathscr{E}'(\mathbb{R}^n)$ and E is a fundamental solution of P, we have from (3.5.14), (3.5.13),

$$P(E * f) = PE * f = \delta_0 * f = f,$$

which allows to find a solution of the Partial Differential Equation (PDE for short) $P(D)u = f$, at least when f is a compactly supported distribution.

Examples. We have on the real line already proven (see (3.2.2)) that $\frac{dH}{dt} = \delta_0$, so that the Heaviside function is a fundamental solution of d/dt (note that from the lemma 3.2.4, the other fundamental solutions are $C + H(t)$). This also implies that

$$\partial_{x_1}\big(H(x_1) \otimes \delta_0(x_2) \otimes \cdots \otimes \delta_0(x_n)\big) = \delta_0(x), \quad \text{(the Dirac mass at 0 in } \mathbb{R}^n).$$

Let $N \in \mathbb{N}$. With x_+^λ defined in (3.4.8), we get, since $\partial_{x_1}^{N+1}(x_{1,+}^{N+1}) = H(x_1)(N+1)!$, that

$$(\partial_{x_1} \dots \partial_{x_n})^{N+2}\Big(\prod_{1 \le j \le n}\Big(\frac{x_{j,+}^{N+1}}{(N+1)!}\Big)\Big) = \delta_0(x).$$

The last example has the following interesting consequence.

Proposition 3.6.2. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and Ω a bounded open set. Then $u_{|\Omega}$ is a derivative of finite order of a continuous function.*

Proof. We consider for $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on Ω the distribution $\chi u \in \mathscr{E}'(\mathbb{R}^n)$ whose restriction to Ω coincides with $u_{|\Omega}$. The distribution χu has finite order N (see the remark 3.3.3). We have with $E(x) = \prod_{1\le j\le n} \frac{x_{j,+}^{N+1}}{(N+1)!}$

$$\chi u = \chi u * \delta_0 = (\partial_{x_1} \dots \partial_{x_n})^{N+2}\big(\chi u * E\big). \tag{3.6.2}$$

Since the function E is C^N with Nth derivatives (Lipschitz) continuous, we may consider the function ψ defined by

$$\psi(x) = \langle \chi(y)u(y), E(x-y)\rangle.$$

Since χu is compactly supported with order N, we have with K compact subset of $\mathbb{R}^n$,

$$|\psi(x+h) - \psi(x)| \le C \sup_{|\alpha|\le N, y\in K} |\partial_y^\alpha\big(E(x+h-y) - E(x-y)\big)|.$$

Since the function E is C^N with Nth derivatives Lipschitz continuous, we find that ψ is Lipschitz continuous. We have from the definitions, with $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle E * \chi u, \phi\rangle = \langle E(x) \otimes (\chi u)(y), \phi(x+y)\rangle = \langle (\chi u)(y), \langle E(x), \phi(x+y)\rangle\rangle,$$

and we note that $\langle E(x), \phi(x+y)\rangle = \int E(x-y)\phi(x)dx$. As a result, we have

$$\langle E * \chi u, \phi\rangle = \langle u(y), \int \underbrace{\chi(y)E(x-y)\phi(x)}_{\in C_c^N(\mathbb{R}^{2n})} dx\rangle = \int \phi(x)\langle (\chi u)(y), E(x-y)\rangle dx$$

where the last equality is due to the theorem 3.4.1[6] and gives also that $\psi = \chi u * E$. The result follows from the continuity of ψ and (3.6.2). □

3.6.2 The Laplace and Cauchy-Riemann equations

We define the Laplace operator Δ in $\mathbb{R}^n$ as

$$\Delta = \sum_{1\le j\le n} \partial_{x_j}^2. \tag{3.6.3}$$

In one dimension, we have from (3.2.2) that $\frac{d^2}{dt^2}(t_+) = \delta_0$ and for $n \ge 2$ the following result describes the fundamental solutions of the Laplace operator. In $\mathbb{R}^2_{x,y}$, we define the operator $\bar\partial$ (a.k.a. the Cauchy-Riemann operator) by

$$\bar\partial = \frac{1}{2}(\partial_x + i\partial_y). \tag{3.6.4}$$

[6]For $\Phi \in C_c^N(\mathbb{R}^n \times \mathbb{R}^n)$, $v \in \mathscr{D}'(\mathbb{R}^n)$, $\mathrm{order}(v) \le N$ $\langle 1 \otimes v, \Phi\rangle = \langle v(y), \int \Phi(x,y)dx\rangle = \int \langle v(y), \Phi(x,y)\rangle dx$.

Theorem 3.6.3. *We have $\Delta E = \delta_0$ with $\|\cdot\|$ standing for the Euclidean norm,*

$$E(x) = \frac{1}{2\pi}\ln\|x\|, \quad \text{for } n = 2, \tag{3.6.5}$$

$$E(x) = \|x\|^{2-n}\frac{1}{(2-n)|S^{n-1}|}, \qquad \text{for } n \ge 3, \text{ with } |S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{3.6.6}$$

$$\bar\partial\big(\frac{1}{\pi z}\big) = \delta_0, \quad \text{with } z = x+iy \text{ (equality in } \mathscr{D}'(\mathbb{R}^2_{x,y})). \tag{3.6.7}$$

Proof. We start with $n \ge 3$, noting that the function $\|x\|^{2-n}$ is L^1_{loc} and homogeneous with degree $2-n$, so that $\Delta\|x\|^{2-n}$ is homogeneous with degree $-n$ (see the remark 3.4.7 (2)). Moreover, the function $\|x\|^{2-n} = f(r^2), r^2 = \|x\|^2, f(t) = t_+^{1-\frac{n}{2}}$ is smooth outside 0 and we can compute there

$$\Delta(f(r^2)) = \sum_j \partial_j(f'(r^2)2x_j) = \sum_j f''(r^2)4x_j^2 + 2nf'(r^2) = 4r^2f''(r^2) + 2nf'(r^2),$$

so that with $t = r^2$,

$$\Delta(f(r^2)) = 4t(1-\frac{n}{2})(-\frac{n}{2})t^{-\frac{n}{2}-1} + 2n(1-\frac{n}{2})t^{-\frac{n}{2}} = t^{-\frac{n}{2}}(1-\frac{n}{2})(-2n+2n) = 0.$$

As a result, $\Delta\|x\|^{2-n}$ is homogeneous with degree $-n$ and supported in $\{0\}$. From the theorem 3.3.4, we obtain that

$$\underbrace{\Delta\|x\|^{2-n} = c\delta_0}_{\substack{\text{homogeneous}\\ \text{degree } -n}} + \sum_{1\le j\le m}\underbrace{\sum_{|\alpha|=j} c_{j,\alpha}\delta_0^{(\alpha)}}_{\substack{\text{homogeneous}\\ \text{degree } -n-j}}.$$

The lemma 3.4.8 implies that for $1 \le j \le m$, $0 = \sum_{|\alpha|=j} c_{j,\alpha}\delta_0^{(\alpha)}$ and $\Delta\|x\|^{2-n} = c\delta_0$. It remains to determine the constant c. We calculate, using the previous formulas for the computation of $\Delta(f(r^2))$, here with $f(t) = e^{-\pi t}$,

$$\begin{aligned}
c = \langle \Delta\|x\|^{2-n}, e^{-\pi\|x\|^2}\rangle &= \int \|x\|^{2-n}e^{-\pi\|x\|^2}\big(4\|x\|^2\pi^2 - 2n\pi\big)dx \\
&= |S^{n-1}|\int_0^{+\infty} r^{2-n+n-1}e^{-\pi r^2}(4\pi^2r^2 - 2n\pi)dr \\
&= |S^{n-1}|\big(\frac{1}{2\pi}[e^{-\pi r^2}(4\pi^2r^2-2n\pi)]^0_{+\infty} + \frac{1}{2\pi}\int_0^{+\infty} e^{-\pi r^2}8\pi^2 r dr\big) \\
&= |S^{n-1}|(-n+2),
\end{aligned}$$

giving (3.6.6). For the convenience of the reader, we calculate explicitely $|S^{n-1}|$. We have indeed

$$\begin{aligned}
1 = \int_{\mathbb{R}^n} e^{-\pi\|x\|^2}dx &= |S^{n-1}|\int_0^{+\infty} r^{n-1}e^{-\pi r^2}dr \\
\underbrace{=}_{r=t^{1/2}\pi^{-1/2}} &|S^{n-1}|\pi^{(1-n)/2}\int_0^{+\infty} t^{\frac{n-1}{2}}e^{-t}\frac{1}{2}t^{-1/2}dt\pi^{-1/2} = |S^{n-1}|\pi^{-n/2}2^{-1}\Gamma(n/2).
\end{aligned}$$

Turning now our attention to the Cauchy-Riemann equation, we see that $1/z$ is also $L^1_{\text{loc}}(\mathbb{R}^2)$, homogeneous of degree -1, and satisfies $\bar\partial(z^{-1}) = 0$ on the complement of $\{0\}$, so that the same reasoning as above shows that

$$\bar\partial(\pi^{-1}z^{-1}) = c\delta_0.$$

To check the value of c, we write $c = \langle\bar\partial(\pi^{-1}z^{-1}), e^{-\pi z\bar z}\rangle = \int_{\mathbb{R}^2} e^{-\pi z\bar z}\pi^{-1}z^{-1}\pi z dxdy = 1$, which gives (3.6.7). We are left with the Laplace equation in two dimensions and we note that with $\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$, $\frac{\partial}{\partial\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$, we have in two dimensions

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial\bar z} = 4\frac{\partial}{\partial\bar z}\frac{\partial}{\partial z}. \tag{3.6.8}$$

Solving the equation $4\frac{\partial E}{\partial z} = \frac{1}{\pi z}$ leads us to try $E = \frac{1}{2\pi}\ln|z|$ and we check directly[7] that $\frac{\partial}{\partial z}\big(\ln(z\bar z)\big) = z^{-1}$

$$\Delta(\frac{1}{2\pi}\ln|z|) = \pi^{-1}2^{-2}4\frac{\partial}{\partial\bar z}\frac{\partial}{\partial z}\big(\ln(z\bar z)\big) = \pi^{-1}\frac{\partial}{\partial\bar z}\big(z^{-1}\big) = \delta_0. \qquad \square$$

3.6.3 Hypoellipticity

Definition 3.6.4. *Let P be a linear operator of type (3.6.1). We shall say that P is hypoelliptic when for all open subsets Ω of $\mathbb{R}^n$ and all $u \in \mathscr{D}'(\Omega)$, we have*

$$\operatorname{singsupp} u = \operatorname{singsupp} Pu. \tag{3.6.9}$$

It is obvious that $\operatorname{singsupp} Pu \subset \operatorname{singsupp} u$, so the hypoellipticity means that $\operatorname{singsupp} u \subset \operatorname{singsupp} Pu$, which is a very interesting piece of information since we can then determine the singularities of our (unknown) solution u, which are located at the same place as the singularities of the source f, which is known when we try to solve the equation $Pu = f$.

Theorem 3.6.5. *Let P be a linear operator of type (3.6.1) such that P has a fundamental solution E satisfying*

$$\operatorname{singsupp} E = \{0\}. \tag{3.6.10}$$

Then P is hypoelliptic. In particular the Laplace and the Cauchy-Riemann operators are hypoelliptic.

N.B. The condition (3.6.10) appears as an iff condition for the hypoellipticity of the operator P since it is also a consequence of the hypoellipticity property.

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[7]Noting that $\ln(x^2+y^2)$ and its first derivatives are $L^1_{\text{loc}}(\mathbb{R}^2)$, we have for $\varphi \in C_c^\infty(\mathbb{R}^2)$, $\langle\frac{\partial}{\partial z}\big(\ln|z|^2\big), \varphi\rangle =$

$$\frac{1}{2}\iint_{\mathbb{R}^2}(-\partial_x\varphi + i\partial_y\varphi)\ln(x^2+y^2)dxdy = \iint\varphi(x,y)(xr^{-2} - iyr^{-2})dxdy = \iint(x-iy)^{-1}\varphi(x,y)dxdy.$$

Proof. Assume that (3.6.10) holds, let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. We consider $f = Pu \in \mathscr{D}'(\Omega)$, $x_0 \notin \operatorname{singsupp} f$, $\chi_0 \in C_c^\infty(\Omega)$, $\chi_0 = 1$ near x_0. We have from the proposition 3.5.5 that

$$\chi u = \chi u * PE = (P\chi u) * E = ([P,\chi]u) * E + \underbrace{\overbrace{(\chi f)}^{\in C_c^\infty(\mathbb{R}^n)} * E}_{\in C^\infty(\mathbb{R}^n)}$$

and thus, using the the proposition 3.5.7 for singular supports, we get

$$\operatorname{singsupp}(\chi u) \subset \operatorname{singsupp}([P,\chi]u) + \operatorname{singsupp} E = \operatorname{singsupp}([P,\chi]u) \subset \operatorname{supp}(u\nabla\chi),$$

and since χ is identically 1 near x_0, we get that $x_0 \notin \operatorname{supp}(u\nabla\chi)$, implying $x_0 \notin \operatorname{singsupp}(\chi u)$, proving that $x_0 \notin \operatorname{singsupp} u$ and the result. □

3.7 Appendix

3.7.1 The Gamma function

The gamma function Γ is a meromorphic function on $\mathbb{C}$ given for $\operatorname{Re} z > 0$ by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt. \tag{3.7.1}$$

For $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$; another interesting value is $\Gamma(1/2) = \sqrt{\pi}$. The functional equation

$$\Gamma(z+1) = z\Gamma(z) \tag{3.7.2}$$

is easy to prove for $\operatorname{Re} z > 0$ and can be used to extend the Γ function into a meromorphic function with simple poles at $-\mathbb{N}$ and $\operatorname{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}$. For instance, for $-1 < \operatorname{Re} z \le 0$ with $z \ne 0$ we define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad \text{where we can use (3.7.1) to define } \Gamma(z+1).$$

More generally for $k \in \mathbb{N}$, $-1 - k < \operatorname{Re} z \le -k$, $z \ne -k$, we can define

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)\dots(z+k)}.$$

There are manifold references on the Gamma function. One of the most comprehensive is certainly the chapter VII of the Bourbaki volume *Fonctions de variable réelle* [2].

Chapter 4

Introduction to Fourier Analysis

4.1 Fourier Transform of tempered distributions

4.1.1 The Fourier transformation on $\mathscr{S}(\mathbb{R}^n)$

Let $n \geq 1$ be an integer. The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is defined in the section 2.3.5, is a Fréchet space, as the space of C^∞ functions u from $\mathbb{R}^n$ to $\mathbb{C}$ such that, for all multi-indices[1] $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| < +\infty.$$

A simple example of such a function is $e^{-|x|^2}$, ($|x|$ is the Euclidean norm of x) and more generally if A is a symmetric positive definite $n \times n$ matrix the function

$$v_A(x) = e^{-\pi \langle Ax, x\rangle}$$

belongs to the Schwartz class.

Definition 4.1.1. *For $u \in \mathscr{S}(\mathbb{R}^n)$, we define its Fourier transform $\hat{u}$ as*

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x \cdot \xi} u(x) dx. \tag{4.1.1}$$

Lemma 4.1.2. *The Fourier transform sends continuously $\mathscr{S}(\mathbb{R}^n)$ into itself.*

Proof. Just notice that $\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) = \int e^{-2i\pi x\xi} \partial_x^\alpha (x^\beta u)(x) dx (2i\pi)^{|\beta| - |\alpha|} (-1)^{|\beta|}$. □

Lemma 4.1.3. *For a symmetric positive definite $n \times n$ matrix A, we have*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi \langle A^{-1}\xi, \xi\rangle}. \tag{4.1.2}$$

[1] Here we use the multi-index notation: for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we define

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{1 \leq j \leq n} \alpha_j.$$

Proof. In fact, diagonalizing the symmetric matrix A, it is enough to prove the one-dimensional version of (4.1.2), i.e. to check

$$\int e^{-2i\pi x\xi}e^{-\pi x^2}dx = \int e^{-\pi(x+i\xi)^2}dx e^{-\pi\xi^2} = e^{-\pi\xi^2},$$

where the second equality is obtained by taking the ξ-derivative of $\int e^{-\pi(x+i\xi)^2}dx$: we have indeed

$$\frac{d}{d\xi}\Big(\int e^{-\pi(x+i\xi)^2}dx\Big) = \int e^{-\pi(x+i\xi)^2}(-2i\pi)(x+i\xi)dx = -i\int \frac{d}{dx}\big(e^{-\pi(x+i\xi)^2}\big)dx = 0.$$

For $a>0$, we obtain $\int_{\mathbb{R}} e^{-2i\pi x\xi}e^{-\pi a x^2}dx = a^{-1/2}e^{-\pi a^{-1}\xi^2}$, which is the sought result in one dimension. If $n\ge 2$, and A is a positive definite symmetric matrix, there exists an orthogonal $n\times n$ matrix P (i.e. ${}^tPP = \mathrm{Id}$) such that

$$D = {}^tPAP, \quad D = \mathrm{diag}(\lambda_1,\dots,\lambda_n), \text{ all } \lambda_j>0.$$

As a consequence, we have, since $|\det P| = 1$,

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi}e^{-\pi\langle Ax,x\rangle}dx &= \int_{\mathbb{R}^n} e^{-2i\pi(Py)\cdot\xi}e^{-\pi\langle APy,Py\rangle}dy = \int_{\mathbb{R}^n} e^{-2i\pi y\cdot({}^tP\xi)}e^{-\pi\langle Dy,y\rangle}dy\\ (\text{with } \eta = {}^tP\xi) \quad &= \prod_{1\le j\le n}\int_{\mathbb{R}} e^{-2i\pi y_j\eta_j}e^{-\pi\lambda_j y_j^2}dy_j = \prod_{1\le j\le n}\lambda_j^{-1/2}e^{-\pi\lambda_j^{-1}\eta_j^2}\\ = (\det A)^{-1/2}e^{-\pi\langle D^{-1}\eta,\eta\rangle} &= (\det A)^{-1/2}e^{-\pi\langle {}^tPA^{-1}P\ {}^tP\xi,{}^tP\xi\rangle} = (\det A)^{-1/2}e^{-\pi\langle A^{-1}\xi,\xi\rangle}.\end{aligned}$$ □

Proposition 4.1.4. *The Fourier transformation is an isomorphism of the Schwartz class and for $u\in\mathscr{S}(\mathbb{R}^n)$, we have*

$$u(x) = \int e^{2i\pi x\xi}\hat u(\xi)d\xi. \tag{4.1.3}$$

Proof. Using (4.1.2) we calculate for $u\in\mathscr{S}(\mathbb{R}^n)$ and $\epsilon>0$, dealing with absolutely converging integrals,

$$\begin{aligned}u_\epsilon(x) &= \int e^{2i\pi x\xi}\hat u(\xi)e^{-\pi\epsilon^2|\xi|^2}d\xi\\ &= \iint e^{2i\pi x\xi}e^{-\pi\epsilon^2|\xi|^2}u(y)e^{-2i\pi y\xi}dyd\xi\\ &= \int u(y)e^{-\pi\epsilon^{-2}|x-y|^2}\epsilon^{-n}dy\\ &= \int \underbrace{\big(u(x+\epsilon y)-u(x)\big)}_{\text{with absolute value}\le\epsilon|y|\|u'\|_{L^\infty}}e^{-\pi|y|^2}dy + u(x).\end{aligned}$$

Taking the limit when ϵ goes to zero, we get the Fourier inversion formula

$$u(x) = \int e^{2i\pi x\xi}\hat u(\xi)d\xi. \tag{4.1.4}$$

We have also proven for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\check{u}(x) = u(-x)$

$$u = \check{\hat{\check{\hat{u}}}}. \tag{4.1.5}$$

Since $u \mapsto \hat{u}$ and $u \mapsto \check{u}$ are continuous homomorphisms of $\mathscr{S}(\mathbb{R}^n)$, this completes the proof of the proposition. □

Proposition 4.1.5. *Using the notation*

$$D_{x_j} = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}, \quad D_x^\alpha = \prod_{j=1}^n D_{x_j}^{\alpha_j} \quad \text{with } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \tag{4.1.6}$$

we have, for $u \in \mathscr{S}(\mathbb{R}^n)$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \qquad (D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha u(x)}(\xi) \tag{4.1.7}$$

Proof. We have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x) dx$ and thus

$$(D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|} \int e^{-2i\pi x\cdot\xi} x^\alpha u(x) dx,$$

$$\xi^\alpha \hat{u}(\xi) = \int (-2i\pi)^{-|\alpha|} \partial_x^\alpha \bigl(e^{-2i\pi x\cdot\xi}\bigr) u(x) dx = \int e^{-2i\pi x\cdot\xi} (2i\pi)^{-|\alpha|} (\partial_x^\alpha u)(x) dx,$$

proving both formulas. □

N.B. The normalization factor $\frac{1}{2i\pi}$ leads to a simplification in the formulas (4.1.7), but the most important aspect of these formulas is certainly that the Fourier transformation exchanges the operation of derivation against the operation of multiplication. For instance if $P(D)$ is given by a formula (3.6.1), we have

$$\widehat{Pu}(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha \hat{u}(\xi) = P(\xi)\hat{u}(\xi).$$

Remark 4.1.6. We have the following continuous inclusions[2]

$$\mathscr{D}(\mathbb{R}^n) \hookrightarrow \mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathscr{E}(\mathbb{R}^n), \tag{4.1.8}$$

triggering the (continuous) inclusions of topological duals,

$$\mathscr{E}'(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n). \tag{4.1.9}$$

The space $\mathscr{S}'(\mathbb{R}^n)$ is the topological dual of the Fréchet space $\mathscr{S}(\mathbb{R}^n)$ and is called the space of *tempered distributions on* $\mathbb{R}^n$. We shall sometimes omit the "$\mathbb{R}^n$" in $\mathscr{S}(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n)$, at least when it is clear that the dimension is fixed equal to n.

The Fourier transformation can be extended to $\mathscr{S}'(\mathbb{R}^n)$.

[2]The first inclusion is certainly sequentially continuous according to the definition 3.1.9 and the second is an inclusion of Fréchet spaces: for each semi-norm p on $\mathscr{E}(\mathbb{R}^n)$, there exists a semi-norm q on $\mathscr{S}(\mathbb{R}^n)$ such that for all $u \in \mathscr{S}(\mathbb{R}^n)$, $p(u) \le q(u)$.

4.1.2 The Fourier transformation on $\mathscr{S}'(\mathbb{R}^n)$

Definition 4.1.7. *Let T be a tempered distribution ; the Fourier transform $\hat{T}$ of T is the tempered distibution defined by the formula*

$$\langle \hat{T}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}}. \tag{4.1.10}$$

The linear form $\hat{T}$ is obviously a tempered distribution since the Fourier transformation is continuous on $\mathscr{S}$. Thanks to the lemma 3.1.7, if $T \in \mathscr{S}$, the present definition of $\hat{T}$ and (4.1.1) coincide.

Note that for $T, \varphi \in \mathscr{S}$, we have $\langle \hat{T}, \varphi \rangle = \iint T(x) e^{-2i\pi x\cdot\xi} \varphi(\xi) dx d\xi = \langle T, \hat{\varphi} \rangle$. This definition gives that

$$\widehat{\delta_0} = 1, \tag{4.1.11}$$

since $\langle \widehat{\delta_0}, \varphi \rangle = \langle \delta_0, \widehat{\varphi} \rangle = \widehat{\varphi}(0) = \int \varphi(x) dx = \langle 1, \varphi \rangle$.

Theorem 4.1.8. *The Fourier transformation is an isomorphism of $\mathscr{S}'(\mathbb{R}^n)$. Let T be a tempered distribution. Then we have*[3]

$$T = \check{\hat{\hat{T}}}. \tag{4.1.12}$$

With obvious notations, we have the following extensions of (4.1.7),

$$\widehat{D_x^\alpha T}(\xi) = \xi^\alpha \hat{T}(\xi), \qquad (D_\xi^\alpha \hat{T})(\xi) = (-1)^{|\alpha|} \widehat{x^\alpha T(x)}(\xi). \tag{4.1.13}$$

Proof. Using the notation $(\check{\varphi})(x) = \varphi(-x)$ for $\varphi \in \mathscr{S}$, we define $\check{S}$ for $S \in \mathscr{S}'$ by (see the remark 3.4.4), $\langle \check{S}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle S, \check{\varphi} \rangle_{\mathscr{S}',\mathscr{S}}$ and we obtain for $T \in \mathscr{S}'$

$$\langle \check{\hat{\hat{T}}}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{\hat{T}}, \check{\varphi} \rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \hat{\check{\varphi}} \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\hat{\check{\varphi}}} \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \varphi \rangle_{\mathscr{S}',\mathscr{S}},$$

where the last equality is due to the fact that $\varphi \mapsto \check{\varphi}$ commutes[4] with the Fourier transform and (4.1.4) means $\check{\hat{\hat{\varphi}}} = \varphi$, a formula also proven true on $\mathscr{S}'$ by the previous line of equality. The formula (4.1.7) is true as well for $T \in \mathscr{S}'$ since, with $\varphi \in \mathscr{S}$ and $\varphi_\alpha(\xi) = \xi^\alpha \varphi(\xi)$, we have

$$\langle \widehat{D^\alpha T}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, (-1)^{|\alpha|} D^\alpha \hat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \widehat{\varphi_\alpha} \rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \varphi_\alpha \rangle_{\mathscr{S}',\mathscr{S}},$$

and the other part is proven the same way. □

The following lemma will be useful.

Lemma 4.1.9. *Let $T \in \mathscr{S}'(\mathbb{R}^n)$ be a homogeneous distribution of degree m. Then its Fourier transform is a homogeneous distribution of degree $-m-n$*

Proof. We check

$$(\xi \cdot D_\xi)\hat{T} = -\xi \cdot \widehat{xT} = -(\widehat{D_x \cdot xT}) = -\frac{n}{2i\pi}\hat{T} - \frac{1}{2i\pi}(\widehat{x \cdot \partial_x T}) = -\frac{(n+m)}{2i\pi}\hat{T},$$

so that the Euler equation (3.4.6) $\xi \dot{\partial}_\xi \hat{T} = -(n+m)\hat{T}$ is satisfied. □

[3] According to the remark 3.4.4, $\check{T}$ is the distribution defined by $\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle$ and if $T \in \mathscr{S}'$, $\check{T}$ is also a tempered distribution since $\varphi \mapsto \check{\varphi}$ is an involutive isomorphism of $\mathscr{S}$.

[4] If $\varphi \in \mathscr{S}$, we have $\widehat{\check{\varphi}}(\xi) = \int e^{-2i\pi x\cdot\xi} \varphi(-x) dx = \int e^{2i\pi x\cdot\xi} \varphi(x) dx = \hat{\varphi}(-\xi) = \check{\hat{\varphi}}(\xi)$.

4.1.3 The Fourier transformation on $L^1(\mathbb{R}^n)$

Theorem 4.1.10. *The Fourier transformation is linear continuous from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and for $u \in L^1(\mathbb{R}^n)$, we have*

$$\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx, \quad \|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}. \tag{4.1.14}$$

Proof. The formula (4.1.1) can be used to define directly the Fourier transform of a function in $L^1(\mathbb{R}^n)$ and this gives an $L^\infty(\mathbb{R}^n)$ function which coincides with the Fourier transform: for a test function $\varphi \in \mathscr{S}(\mathbb{R}^n)$, and $u \in L^1(\mathbb{R}^n)$, we have by the definition (4.1.10) above and the Fubini theorem

$$\langle \hat{u}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int u(x)\hat{\varphi}(x)dx = \iint u(x)\varphi(\xi)e^{-2i\pi x\cdot\xi}dxd\xi = \int \widetilde{u}(\xi)\varphi(\xi)d\xi$$

with $\widetilde{u}(\xi) = \int e^{-2i\pi x\cdot\xi}u(x)dx$ which is thus the Fourier transform of u. □

4.1.4 The Fourier transformation on $L^2(\mathbb{R}^n)$

We refer the reader to the section 5.3 in Chapter 5.

4.1.5 Some standard examples of Fourier transform

Let us consider the Heaviside function defined on $\mathbb{R}$ by $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \le 0$; it is obviously a tempered distribution, so that we can compute its Fourier transform. With the notation of this section, we have, with δ_0 the Dirac mass at 0, $\check{H}(x) = H(-x)$,

$$\widehat{H} + \widehat{\check{H}} = \hat{1} = \delta_0, \quad \widehat{H} - \widehat{\check{H}} = \widehat{\operatorname{sign}}, \qquad \frac{1}{i\pi} = \frac{1}{2i\pi}2\widehat{\delta_0}(\xi) = \widehat{D\operatorname{sign}}(\xi) = \xi\widehat{\operatorname{sign}}\xi$$

so that $\xi\big(\widehat{\operatorname{sign}}\xi - \frac{1}{i\pi}pv(1/\xi)\big) = 0$ and from the theorem 3.2.8, we get

$$\widehat{\operatorname{sign}}\xi - \frac{1}{i\pi}pv(1/\xi) = c\delta_0,$$

with $c = 0$ since the lhs is odd. We obtain

$$\widehat{\operatorname{sign}}(\xi) = \frac{1}{i\pi}pv\frac{1}{\xi}, \tag{4.1.15}$$

$$\widehat{pv(\frac{1}{\pi x})} = -i\operatorname{sign}\xi, \tag{4.1.16}$$

$$\hat{H} = \frac{\delta_0}{2} + \frac{1}{2i\pi}pv(\frac{1}{\xi}) = \frac{1}{(x - i0)}\frac{1}{2i\pi}. \tag{4.1.17}$$

Let us consider now for $0 < \alpha < n$ the $L^1_{\text{loc}}(\mathbb{R}^n)$ function $u_\alpha(x) = |x|^{\alpha-n}$ ($|x|$ is the Euclidean norm of x); since u_α is also bounded for $|x| \ge 1$, it is a tempered distribution. Let us calculate its Fourier transform v_α. Since u_α is homogeneous of degree $\alpha - n$, we get from the lemma 4.1.9 that v_α is a homogeneous distribution of degree $-\alpha$. On the other hand, if $S \in O(\mathbb{R}^n)$ (the orthogonal group), we have in the distribution sense (see the definition 3.4.3), since u_α is a radial function,

$$v_\alpha(S\xi) = v_\alpha(\xi). \tag{4.1.18}$$

The distribution $|\xi|^\alpha v_\alpha(\xi)$ is homogeneous of degree 0 on $\mathbb{R}^n \backslash \{0\}$ and is also "radial", i.e. satisfies (4.1.18). Moreover on $\mathbb{R}^n \backslash \{0\}$, the distribution v_α is a C^1 function which coincides with

$$\int e^{-2i\pi x\cdot\xi} \chi_0(x)|x|^{\alpha-n} dx + |\xi|^{-2N} \int e^{-2i\pi x\cdot\xi} |D_x|^{2N} \big(\chi_1(x)|x|^{\alpha-n}\big) dx,$$

where $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ is 1 near 0 and $\chi_1 = 1 - \chi_0$, $N \in \mathbb{N}, \alpha + 1 < 2N$. As a result $|\xi|^\alpha v_\alpha(\xi) = c_\alpha$ on $\mathbb{R}^n \backslash \{0\}$ and the distribution on $\mathbb{R}^n$ (note that $\alpha < n$)

$$T = v_\alpha(\xi) - c_\alpha |\xi|^{-\alpha}$$

is supported in $\{0\}$ and homogeneous (on $\mathbb{R}^n$) with degree $-\alpha$. From the theorem 3.3.4 and the lemma 3.4.8, the condition $0 < \alpha < n$ gives $v_\alpha = c_\alpha |\xi|^{-\alpha}$. To find c_α, we compute

$$\int |x|^{\alpha-n} e^{-\pi x^2} dx = \langle u_\alpha, e^{-\pi x^2} \rangle = c_\alpha \int |\xi|^{-\alpha} e^{-\pi \xi^2} d\xi$$

which yields

$$2^{-1}\Gamma(\frac{\alpha}{2})\pi^{-\frac{\alpha}{2}} = \int_0^{+\infty} r^{\alpha-1} e^{-\pi r^2} dr = c_\alpha \int_0^{+\infty} r^{n-\alpha-1} e^{-\pi r^2} dr = c_\alpha 2^{-1}\Gamma(\frac{n-\alpha}{2})\pi^{-\frac{n-\alpha}{2}}.$$

We have proven the following lemma.

Lemma 4.1.11. *Let $n \in \mathbb{N}^*$ and $\alpha \in]0, n[$. The function $u_\alpha(x) = |x|^{\alpha-n}$ is $L^1_{loc}(\mathbb{R}^n)$ and also a temperate distribution on $\mathbb{R}^n$. Its Fourier transform v_α is also $L^1_{loc}(\mathbb{R}^n)$ and given by*

$$v_\alpha(\xi) = |\xi|^{-\alpha} \pi^{\frac{n}{2}-\alpha} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

4.2 The Poisson summation formula

4.2.1 Wave packets

We define for $x \in \mathbb{R}^n$, $(y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-y)\cdot\eta} = 2^{n/4} e^{-\pi(x-y-i\eta)^2} e^{-\pi\eta^2}, \tag{4.2.1}$$

$$\text{where for } \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n, \quad \zeta^2 = \sum_{1\le j\le n} \zeta_j^2. \tag{4.2.2}$$

We note that the function $\varphi_{y,\eta}$ is in $\mathcal{S}(\mathbb{R}^n)$ and with L^2 norm 1. In fact, $\varphi_{y,\eta}$ appears as a *phase translation* of a normalized Gaussian. The following lemma introduces the wave packets transform as a Gabor wavelet.

Lemma 4.2.1. *Let u be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We define*

$$(Wu)(y,\eta) = (u, \varphi_{y,\eta})_{L^2(\mathbb{R}^n)} = 2^{n/4}\int u(x)e^{-\pi(x-y)^2}e^{-2i\pi(x-y)\cdot\eta}dx \tag{4.2.3}$$

$$= 2^{n/4}\int u(x)e^{-\pi(y-i\eta-x)^2}dxe^{-\pi\eta^2}. \tag{4.2.4}$$

For $u \in L^2(\mathbb{R}^n)$, the function Tu defined by

$$(Tu)(y+i\eta) = e^{\pi\eta^2}Wu(y,-\eta) = 2^{n/4}\int u(x)e^{-\pi(y+i\eta-x)^2}dx \tag{4.2.5}$$

is an entire function. The mapping $u \mapsto Wu$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula

$$u(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} Wu(y,\eta)\varphi_{y,\eta}(x)dyd\eta. \tag{4.2.6}$$

Proof. For u in $\mathcal{S}(\mathbb{R}^n)$, we have

$$Wu(y,\eta) = e^{2i\pi y\eta}\widehat{\Omega}^1(\eta, y)$$

where $\widehat{\Omega}^1$ is the Fourier transform with respect to the first variable of the $\mathcal{S}(\mathbb{R}^{2n})$ function $\Omega(x,y) = u(x)e^{-\pi(x-y)^2}2^{n/4}$. Thus the function Wu belongs to $\mathcal{S}(\mathbb{R}^{2n})$. It makes sense to compute

$$2^{-n/2}(Wu,Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\overline{u}(x_2)e^{-\pi[(x_1-y)^2+(x_2-y)^2+2i(x_1-x_2)\eta+\epsilon^2\eta^2]}dyd\eta dx_1dx_2. \tag{4.2.7}$$

Now the last integral on $\mathbb{R}^{4n}$ converges absolutely and we can use the Fubini theorem. Integrating with respect to η involves the Fourier transform of a Gaussian function and we get $\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}$. Since

$$2(x_1-y)^2 + 2(x_2-y)^2 = (x_1+x_2-2y)^2 + (x_1-x_2)^2,$$

integrating with respect to y yields a factor $2^{-n/2}$. We are left with

$$(Wu,Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\,\overline{u}(x_2)e^{-\pi(x_1-x_2)^2/2}\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}dx_1dx_2. \tag{4.2.8}$$

Changing the variables, the integral is

$$\lim_{\epsilon\to 0_+}\int u(s+\epsilon t/2)\,\overline{u}(s-\epsilon t/2)e^{-\pi\epsilon^2t^2/2}e^{-\pi t^2}dtds = \|u\|^2_{L^2(\mathbb{R}^n)}$$

by Lebesgue's dominated convergence theorem: the triangle inequality and the estimate $|u(x)| \le C(1+|x|)^{-n-1}$ imply, with $v = u/C$,

$$\begin{aligned}|v(s+\epsilon t/2)\,\overline{v}(s-\epsilon t/2)| &\le (1+|s+\epsilon t/2|)^{-n-1}(1+|s+\epsilon t/2|)^{-n-1}\\ &\le (1+|s+\epsilon t/2|+|s-\epsilon t/2|)^{-n-1}\\ &\le (1+2|s|)^{-n-1}.\end{aligned}$$

Eventually, this proves that

$$\|Wu\|^2_{L^2(\mathbb{R}^{2n})} = \|u\|^2_{L^2(\mathbb{R}^n)} \tag{4.2.9}$$

i.e.

$$W : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}) \quad \text{with} \quad W^*W = \mathrm{id}_{L^2(\mathbb{R}^n)}. \tag{4.2.10}$$

Noticing first that $\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$ belongs to $L^2(\mathbb{R}^n)$ (with a norm smaller than $\|Wu\|_{L^1(\mathbb{R}^{2n})}$) and applying Fubini's theorem, we get from the polarization of (4.2.9) for $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}(u,v)_{L^2(\mathbb{R}^n)} &= (Wu, Wv)_{L^2(\mathbb{R}^{2n})}\\ &= \iint Wu(y,\eta)(\varphi_{y,\eta}, v)_{L^2(\mathbb{R}^n)}dyd\eta\\ &= (\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta, v)_{L^2(\mathbb{R}^n)},\end{aligned}$$

yielding the result of the lemma $u = \iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$. □

4.2.2 Poisson's formula

The following lemma is in fact the Poisson summation formula for Gaussian functions in one dimension.

Lemma 4.2.2. *For all complex numbers z, the following series are absolutely converging and*

$$\sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi m^2} e^{2i\pi mz}. \tag{4.2.11}$$

Proof. We set $\omega(z) = \sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2}$. The function ω is entire and 1-periodic since for all $m \in \mathbb{Z}$, $z \mapsto e^{-\pi(z+m)^2}$ is entire and for $R > 0$

$$\sup_{|z|\le R} |e^{-\pi(z+m)^2}| \le \sup_{|z|\le R} |e^{-\pi z^2}| e^{-\pi m^2} e^{2\pi|m|R} \in l^1(\mathbb{Z}).$$

Consequently, for $z \in \mathbb{R}$, we obtain, expanding ω in Fourier series[5],

$$\omega(z) = \sum_{k\in\mathbb{Z}} e^{2i\pi kz} \int_0^1 \omega(x) e^{-2i\pi kx} dx.$$

[5] Note that we use this expansion only for a C^∞ 1-periodic function. The proof is simple and requires only to compute $1 + 2\,\mathrm{Re}\sum_{1\le k\le N} e^{2i\pi kx} = \frac{\sin\pi(2N+1)x}{\sin\pi x}$. Then one has to show that for a smooth 1-periodic function ω such that $\omega(0) = 0$,

$$\lim_{\lambda\to+\infty} \int_0^1 \frac{\sin\lambda x}{\sin\pi x}\omega(x)dx = 0,$$

which is obvious since for a smooth ν (here we take $\nu(x) = \omega(x)/\sin\pi x$), $|\int_0^1 \nu(x)\sin\lambda x dx| = O(\lambda^{-1})$ by integration by parts.

We also check, using Fubini's theorem on $L^1(0,1) \times l^1(\mathbb{Z})$

$$\begin{aligned}\int_0^1 \omega(x) e^{-2i\pi kx} dx &= \sum_{m\in\mathbb{Z}} \int_0^1 e^{-\pi(x+m)^2} e^{-2i\pi kx} dx \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} e^{-\pi t^2} e^{-2i\pi kt} dt \\ &= \int_{\mathbb{R}} e^{-\pi t^2} e^{-2i\pi kt} = e^{-\pi k^2}.\end{aligned}$$

So the lemma is proven for real z and since both sides are entire functions, we conclude by analytic continuation. □

It is now straightforward to get the n-th dimensional version of the previous lemma: for all $z \in \mathbb{C}^n$, using the notation (4.2.2), we have

$$\sum_{m\in\mathbb{Z}^n} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}^n} e^{-\pi m^2} e^{2i\pi m\cdot z}. \tag{4.2.12}$$

Theorem 4.2.3 (The Poisson summation formula)**.** *Let n be a positive integer and u be a function in $\mathcal{S}(\mathbb{R}^n)$. Then we have*

$$\sum_{k\in\mathbb{Z}^n} u(k) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k), \tag{4.2.13}$$

where $\hat{u}$ stands for the Fourier transform of u. In other words the tempered distribution $D_0 = \sum_{k\in\mathbb{Z}^n} \delta_k$ is such that $\widehat{D_0} = D_0$.

Proof. We write, according to (4.2.6) and to Fubini's theorem

$$\begin{aligned}\textstyle\sum_{k\in\mathbb{Z}^n} u(k) &= \sum_{k\in\mathbb{Z}^n} \iint Wu(y,\eta)\varphi_{y,\eta}(k) dy d\eta \\ &= \iint Wu(y,\eta) \sum_{k\in\mathbb{Z}^n} \varphi_{y,\eta}(k) dy d\eta.\end{aligned}$$

Now, (4.2.12), (4.2.1) give $\sum_{k\in\mathbb{Z}^n} \varphi_{y,\eta}(k) = \sum_{k\in\mathbb{Z}^n} \widehat{\varphi}_{y,\eta}(k)$, so that (4.2.6) and Fubini's theorem imply the result. □

4.3 Fourier transformation and convolution

4.3.1 Fourier transformation on $\mathscr{E}'(\mathbb{R}^n)$

Theorem 4.3.1. *Let $u \in \mathscr{E}'(\mathbb{R}^n)$. Then $\hat{u}$ is an entire function on $\mathbb{C}^n$.*

Proof. We have for $\varphi \in \mathscr{D}(\mathbb{R}^n)$, according to the definition (3.4.14),

$$\begin{aligned}\langle \hat{u}, \varphi\rangle = \langle u, \hat{\varphi}\rangle = \langle u(x), \int e^{-2i\pi x\cdot\xi}\varphi(\xi) d\xi\rangle &= \langle u(x)\otimes\varphi(\xi), e^{-2i\pi x\cdot\xi}\rangle_{\mathscr{E}'(\mathbb{R}^{2n}),\mathscr{E}(\mathbb{R}^{2n})} \\ &= \langle \varphi(\xi), \underbrace{\langle u(x), e^{-2i\pi x\cdot\xi}\rangle}_{\tilde{u}(\xi)}\rangle,\end{aligned}$$

an identity which implies $\hat{u} = \tilde{u}$ and moreover the function $\tilde{u}$ is indeed entire, since with $\zeta \in \mathbb{C}^n$, and $\tilde{u}(\zeta) = \langle u(x), e^{-2i\pi x\cdot\zeta}\rangle$ the function $\tilde{u}$ is $C^\infty(\mathbb{C}^n)$ from the corollary 3.4.2, and we can check that $\bar\partial\tilde{u} = 0$ (a direct computation of $\tilde{u}(\zeta+h)-u(\zeta)$ provides elementarily the holomorphy of $\tilde{u}$). $\square$

Definition 4.3.2. *The space $\mathscr{O}_M(\mathbb{R}^n)$ of multipliers of $\mathscr{S}(\mathbb{R}^n)$ is the subspace of the functions $f \in \mathscr{E}(\mathbb{R}^n)$ such that,*

$$\forall\alpha \in \mathbb{N}^n, \exists C_\alpha > 0, \exists N_\alpha \in \mathbb{N}, \quad \forall x \in \mathbb{R}^n, \quad |(\partial_x^\alpha f)(x)| \le C_\alpha(1+|x|)^{N_\alpha}. \tag{4.3.1}$$

It is easy to check that, for $f \in \mathscr{O}_M(\mathbb{R}^n)$, the operator $u \mapsto fu$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself, and by transposition from $\mathscr{S}'(\mathbb{R}^n)$ into itself: we have for $T \in \mathscr{S}'(\mathbb{R}^n)$, $f \in \mathscr{O}_M(\mathbb{R}^n)$,

$$\langle fT, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, f\varphi\rangle_{\mathscr{S}',\mathscr{S}},$$

and if p is a semi-norm of $\mathscr{S}$, the continuity on $\mathscr{S}$ of the multiplication by f implies that there exists a semi-norm q on $\mathscr{S}$ such that for all $\varphi \in \mathscr{S}$, $p(f\varphi) \le q(\varphi)$. A typical example of a function in $\mathscr{O}_M(\mathbb{R}^n)$ is $e^{iP(x)}$ where P is a real-valued polynomial: in fact the derivatives of $e^{iP(x)}$ are of type $Q(x)e^{iP(x)}$ where Q is a polynomial so that (4.3.1) holds.

Lemma 4.3.3. *Let $u \in \mathscr{E}'(\mathbb{R}^n)$. Then $\hat{u}$ belongs to $\mathscr{O}_M(\mathbb{R}^n)$.*

Proof. We have already seen that $\hat{u}(\xi) = \langle u(x), e^{-2i\pi x\cdot\xi}\rangle$ is a smooth function so that

$$(D_\xi^\alpha u)(\xi) = \langle u(x), e^{-2i\pi x\cdot\xi}x^\alpha\rangle(-1)^{|\alpha|}$$

which implies $|(D_\xi^\alpha u)(\xi)| \le C_0 \sup_{\substack{|\beta|\le N_0\\ x\in K_0}} |\partial_x^\beta(e^{-2i\pi x\cdot\xi}x^\alpha)| \le C_1(1+|\xi|)^{N_0}$, proving the sought result. $\square$

4.3.2 Convolution and Fourier transformation

Theorem 4.3.4. *Let $u \in \mathscr{S}'(\mathbb{R}^n), v \in \mathscr{E}'(\mathbb{R}^n)$. Then the convolution $u * v$ belongs to $\mathscr{S}'(\mathbb{R}^n)$ and*

$$\widehat{u * v} = \hat{u}\hat{v}. \tag{4.3.2}$$

N.B. We note that both sides of the equality (4.3.2) make sense since the lhs is the Fourier transform of $u*v$ which belongs to $\mathscr{S}'$(this has to be proven) and $\hat{v}$ belongs to $\mathscr{O}_M(\mathbb{R}^n)$ so that the product of $\hat{u} \in \mathscr{S}'$ with $\hat{v}$ makes sense.

Proof. Let us prove first that $u * v$ belongs to $\mathscr{S}'$. We have for $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $\chi \in \mathscr{D}(\mathbb{R}^n)$ equal to 1 near the support of v,

$$\langle u * v, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^n),\mathscr{D}(\mathbb{R}^n)} = \langle u(x) \otimes v(y), \varphi(x+y)\chi(y)\rangle_{\mathscr{D}'(\mathbb{R}^{2n}),\mathscr{D}(\mathbb{R}^{2n})}.$$

Now if $\varphi \in \mathscr{S}(\mathbb{R}^n)$ the function $(x,y) \mapsto \varphi(x+y)\chi(y) = \Phi(x,y)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$: it is a smooth function and $x^\alpha y^\beta \partial_x^\gamma \partial_y^\rho \Phi$ is a linear combination of terms of type

$$(x+y)^\omega(\partial^\nu\varphi)(x+y)y^\lambda(\partial^\mu\chi)(y)$$

which are bounded as product of bounded terms. Moreover, if $\Phi \in \mathscr{S}(\mathbb{R}^{2n})$, the function $\psi(x) = \langle v(y), \Phi(x,y)\rangle$ is smooth (see the corollary 3.4.2(2)) and belongs to $\mathscr{S}(\mathbb{R}^n)$ since $x^\alpha(\partial_x^\beta \psi)(x) = \langle v(y), x^\alpha \partial_x^\beta \Phi(x,y)\rangle$ and for some compact subset K_0 of $\mathbb{R}^n$,

$$|x^\alpha(\partial_x^\beta \psi)(x)| = |\langle v(y), x^\alpha \partial_x^\beta \Phi(x,y)\rangle| \le C \sup_{\substack{|\gamma|\le N_0\\ y\in K_0}} |x^\alpha \partial_x^\beta \partial_y^\gamma \Phi(x,y)| = p(\Phi),$$

where p is a semi-norm on $\mathscr{S}(\mathbb{R}^{2n})$. As a result, we can extend $u * v$ to a continuous linear form on $\mathscr{S}(\mathbb{R}^n)$ so that $u * v \in \mathscr{S}'(\mathbb{R}^n)$. Let $w \in \mathscr{S}'$ such[6] that $\hat w = \hat u \hat v$. For $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\langle w, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat u \hat v, \check{\hat\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat u, \hat v \check{\hat\varphi}\rangle_{\mathscr{S}',\mathscr{S}}.$$

On the other hand, we have

$$\begin{aligned}\hat v(\xi)\check{\hat\varphi}(\xi) &= \langle v(x), e^{-2i\pi x\cdot\xi}\rangle \int \varphi(y) e^{2i\pi y\cdot\xi} dy = \langle v(x)\otimes\varphi(y), e^{2i\pi(y-x)\cdot\xi}\rangle \\ &= \langle v(x), \langle \varphi(y), e^{2i\pi(y-x)\cdot\xi}\rangle\rangle = \langle v(x), \langle \check\varphi(y), e^{-2i\pi(y+x)\cdot\xi}\rangle\rangle = \widehat{(v*\check\varphi)}(\xi),\end{aligned}$$

so that

$$\begin{aligned}\langle w, \varphi\rangle = \langle \hat u, \widehat{(v*\check\varphi)}\rangle = \langle \check u, v*\check\varphi\rangle &= \langle u(-x), \langle v(x-y), \varphi(-y)\rangle\rangle \\ &= \langle u(x), \langle v(y-x), \varphi(y)\rangle\rangle = \langle (u*v), \varphi\rangle,\end{aligned}$$

which gives $w = u * v$ and (4.3.2). □

4.3.3 The Riemann-Lebesgue lemma

Lemma 4.3.5. *Let $u \in L^1(\mathbb{R}^n)$. Then from (4.1.14) $\hat u(\xi) = \int e^{-2i\pi x\cdot\xi} u(x) dx$; moreover $\hat u$ belongs to $C^0_{(0)}(\mathbb{R}^n)$, where $C^0_{(0)}(\mathbb{R}^n)$ stands for the space of continuous functions on $\mathbb{R}^n$ tending to 0 at infinity. In particular $\hat u$ is uniformly continuous.*

Proof. This follows from the Riemann-Lebesgue lemma (see e.g. the lemma 3.4.4 in [9]); moreover,

$$|\hat u(\xi + h) - \hat u(\xi)| = \int |u(x)||e^{-2i\pi x\cdot h} - 1| dx = \sigma_u(h),$$

and the Lebesgue dominated convergence theorem implies that $\lim_{h\to 0}\sigma_u(h) = 0$, implying as well the uniform continuity. □

4.4 Some fundamental solutions

4.4.1 The heat equation

The heat operator is the following constant coefficient differential operator on $\mathbb{R}_t \times \mathbb{R}^n_x$

$$\partial_t - \Delta_x, \tag{4.4.1}$$

where the Laplace operator Δ_x on $\mathbb{R}^n$ is defined by (3.6.3).

[6] Take $w = \check{\widehat{\hat u \hat v}}$.

Theorem 4.4.1. *We define on $\mathbb{R}_t \times \mathbb{R}^n_x$ the L^1_{loc} function*

$$E(t,x) = (4\pi t)^{-n/2} H(t) e^{-\frac{|x|^2}{4t}}. \tag{4.4.2}$$

The function E is C^∞ on the complement of $\{(0,0)\}$ in $\mathbb{R} \times \mathbb{R}^n$. The function E is a fundamental solution of the heat equation, i.e. $\partial_t E - \Delta_x E = \delta_0(t) \otimes \delta_0(x)$.

Proof. To prove that $E \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$, we calculate for $T \geq 0$,

$$\int_0^T \int_0^{+\infty} t^{-n/2} r^{n-1} e^{-\frac{r^2}{4t}} dt dr \underbrace{=}_{r=2t^{1/2}\rho} \int_0^T \int_0^{+\infty} t^{-n/2} 2^{n-1} t^{(n-1)/2} \rho^{n-1} e^{-\rho^2} 2 t^{1/2} dt d\rho$$
$$= 2^n T \int_0^{+\infty} \rho^{n-1} e^{-\rho^2} d\rho < +\infty.$$

Moreover, the function E is obviously analytic on the open subset of $\mathbb{R}^{1+n}$ $\{(t,x) \in \mathbb{R} \times \mathbb{R}^n, t \neq 0\}$. Let us prove that E is C^∞ on $\mathbb{R} \times (\mathbb{R}^n \backslash \{0\})$. With ρ_0 defined in (3.1.1), the function ρ_1 defined by $\rho_1(t) = H(t) t^{-n/2} \rho_0(t)$ is also C^∞ on $\mathbb{R}$ and

$$E(t,x) = H(\frac{|x|^2}{4t}) \big(\frac{|x|^2}{4t}\big)^{n/2} e^{-\frac{|x|^2}{4t}} |x|^{-n} \pi^{-n/2} = |x|^{-n} \pi^{-n/2} \rho_1\big(\frac{4t}{|x|^2}\big),$$

which is indeed smooth on $\mathbb{R}_t \times (\mathbb{R}^n_x \backslash \{0\})$. We want to solve the equation $\partial_t u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{n+1})$, we can consider its Fourier transform v with respect to x (well-defined by transposition as the Fourier transform in (4.1.10)), and we end-up with the simple ODE with parameters on v,

$$\partial_t v + 4\pi^2 |\xi|^2 v = \delta_0(t). \tag{4.4.3}$$

It remains to determine a fundamental solution of that ODE: we have

$$\frac{d}{dt} + \lambda = e^{-t\lambda} \frac{d}{dt} e^{t\lambda}, \quad \big(\frac{d}{dt} + \lambda\big)(e^{-t\lambda} H(t)) = \big(e^{-t\lambda} \frac{d}{dt} e^{t\lambda}\big)(e^{-t\lambda} H(t)) = \delta_0(t), \tag{4.4.4}$$

so that we can take $v = H(t) e^{-4\pi^2 t |\xi|^2}$, which belongs to $\mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^n_\xi)$. Taking the inverse Fourier transform with respect to ξ of both sides of (4.4.3) gives[7] with $u \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^n_\xi)$

$$\partial_t u - \Delta_x u = \delta_0(t) \otimes \delta_0(x). \tag{4.4.5}$$

To compute u, we check with $\varphi \in \mathscr{D}(\mathbb{R}), \psi \in \mathscr{D}(\mathbb{R}^n)$,

$$\langle u, \varphi \otimes \check{\psi} \rangle = \langle \widehat{v}^x, \varphi \otimes \psi \rangle = \langle v, \varphi \otimes \hat{\psi} \rangle = \int_0^{+\infty} \int_{\mathbb{R}^n} \varphi(t) \hat{\psi}(\xi) e^{-4\pi^2 t |\xi|^2} dt d\xi.$$

We can use the Fubini theorem in that absolutely converging integral and use (4.1.2) to get

$$\langle u, \varphi \otimes \check{\psi} \rangle = \int_0^{+\infty} \varphi(t) \left(\int_{\mathbb{R}^n} (4\pi t)^{-n/2} e^{-\pi \frac{|x|^2}{4\pi t}} \psi(x) dx \right) dt = \langle E, \varphi \otimes \check{\psi} \rangle,$$

where the last equality is due to the Fubini theorem and the local integrability of E. We have thus $E = u$ and E satisfies (4.4.5). The proof is complete. □

[7] The Fourier transformation obviously respects the tensor products.

Corollary 4.4.2. *The heat equation is C^∞ hypoelliptic (see the definition 3.6.4) , in particular for $w \in \mathscr{D}'(\mathbb{R}^{1+n})$,*

$$\operatorname{singsupp} w \subset \operatorname{singsupp}(\partial_t w - \Delta_x w),$$

where singsupp *stands for the C^∞ singular support as defined by* (3.1.9).

Proof. It is an immediate consequence of the theorem 3.6.5, since E is C^∞ outside zero from the previous theorem. □

Remark 4.4.3. It is also possible to define the *analytic singular support* of a distribution T in an open subset Ω of $\mathbb{R}^n$: we define

$$\operatorname{singsupp}_{\mathcal{A}} T = \{x \in \Omega, \forall U \text{open} \in \mathscr{V}_x,\ T_{|U} \notin \mathcal{A}(U)\}, \tag{4.4.6}$$

where $\mathcal{A}(U)$ stands for the analytic[8] functions on the open set U. It is a consequence[9] of the proof of theorem 4.4.1 that

$$\operatorname{singsupp}_{\mathcal{A}} E = \{0\} \times \mathbb{R}^n_x. \tag{4.4.7}$$

In particular this implies that the heat equation is *not* analytic-hypoelliptic since

$$\{0\} \times \mathbb{R}^n_x = \operatorname{singsupp}_{\mathcal{A}} E \not\subset \operatorname{singsupp}_{\mathcal{A}}(\partial_t E - \Delta_x E) = \operatorname{singsupp}_{\mathcal{A}} \delta_0 = \{0_{\mathbb{R}^{1+n}}\}.$$

4.4.2 The Schrödinger equation

We move forward now with the Schrödinger equation,

$$\frac{1}{i}\frac{\partial}{\partial t} - \Delta_x \tag{4.4.8}$$

which looks similar to the heat equation, but which is in fact drastically different.

Lemma 4.4.4.

$$\mathscr{D}(\mathbb{R}^{n+1}) \mapsto \int_0^{+\infty} e^{-i(n-2)\frac{\pi}{4}} (4\pi t)^{-n/2} \left(\int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx\right) dt = \langle E, \Phi\rangle \tag{4.4.9}$$

is a distribution in $\mathbb{R}^{n+1}$ of order $\le n+2$.

---

[8]A function f is said to be analytic on an open subset U of $\mathbb{R}^n$ if it is $C^\infty(U)$, and for each $x_0 \in U$ there exists $r_0 > 0$ such that $\bar{B}(x_0, r_0) \subset U$ and

$$\forall x \in \bar{B}(x_0, r_0), \quad f(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_x^\alpha f(x_0)(x - x_0)^\alpha.$$

[9]In fact, in the theorem, we have noted the obvious inclusion $\operatorname{singsupp}_{\mathcal{A}} E \subset \{0\} \times \mathbb{R}^n_x$, but since E is C^∞ in $t \neq 0$, vanishes identically on $t < 0$, is positive (it means > 0) on $t > 0$, it cannot be analytic near any point of $\{0\} \times \mathbb{R}^n_x$.

Proof. Let $\Phi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n)$; for $t > 0$ we have, using (4.6.7),

$$e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx = i \int_{\mathbb{R}^n} \hat{\Phi}^x(t,\xi) e^{-4i\pi^2 t|\xi|^2} d\xi,$$

so that with $\mathbb{N} \ni \tilde{n}$ even $> n$, using (4.1.7) and (4.1.14),

$$\sup_{t>0} \left| e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx \right| \le \sup_{t>0} \int_{\mathbb{R}^n} |\hat{\Phi}^x(t,\xi)| d\xi$$
$$\le \sup_{t>0} \int (1+|\xi|^2)^{-\tilde{n}/2} | \underbrace{(1+|\xi|^2)^{\tilde{n}/2}}_{\text{polynomial}} \hat{\Phi}(t,\xi)| d\xi \le C_n \max_{|\alpha|\le\tilde{n}} \|\partial_x^\alpha \Phi\|_{L^\infty(\mathbb{R}^{n+1})}.$$

As a result the mapping

$$\mathscr{D}(\mathbb{R}^{n+1}) \mapsto \int_0^{+\infty} e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2} \left(\int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx \right) dt = \langle E, \Phi \rangle$$

is a distribution of order $\le n+2$. □

Theorem 4.4.5. *The distribution E given by (4.4.9) is a fundamental solution of the Schrödinger equation, i.e. $\frac{1}{i}\partial_t E - \Delta_x E = \delta_0(t) \otimes \delta_0(x)$. Moreover, E is smooth on the open set $\{t \ne 0\}$ and equal there to*

$$e^{-i(n-2)\frac{\pi}{4}} H(t)(4\pi t)^{-n/2} e^{i\frac{|x|^2}{4t}}. \tag{4.4.10}$$

The distribution E is the partial Fourier transform with respect to the variable x of the $L^\infty(\mathbb{R}^{n+1})$ function

$$\tilde{E}(t,\xi) = iH(t) e^{-4i\pi^2 t|\xi|^2}. \tag{4.4.11}$$

Proof. We want to solve the equation $-i\partial_t u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{n+1})$, we can consider its Fourier transform v with respect to x (well-defined by transposition as the Fourier transform in (4.1.10)), and we end-up with the simple ODE with parameters on v,

$$\partial_t v + i4\pi^2|\xi|^2 v = i\delta_0(t). \tag{4.4.12}$$

Using the identity (4.4.4), we see that we can take $v = iH(t)e^{-i4\pi^2 t|\xi|^2}$, which belongs to $\mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^n_\xi)$. Taking the inverse Fourier transform with respect to ξ of both sides of (4.4.12) gives with $u \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^n_\xi)$

$$\partial_t u - i\Delta_x u = i\delta_0(t) \otimes \delta_0(x) \quad \text{i.e.} \quad \frac{1}{i}\partial_t u - \Delta_x u = \delta_0(t) \otimes \delta_0(x). \tag{4.4.13}$$

To compute u, we check with $\varphi \in \mathscr{D}(\mathbb{R}), \psi \in \mathscr{D}(\mathbb{R}^n)$,

$$\langle u, \varphi \otimes \psi \rangle = \langle \widehat{v}^x, \varphi \otimes \check{\psi} \rangle = \langle v, \varphi \otimes \check{\hat{\psi}} \rangle = i \int_0^{+\infty} \varphi(t) \left(\int_{\mathbb{R}^n} \hat{\psi}(\xi) e^{i\pi(-4\pi t)|\xi|^2} d\xi \right) dt. \tag{4.4.14}$$

We note now that, using (4.6.7) and (4.1.10), for $t > 0$,

$$i\int_{\mathbb{R}^n} \hat{\psi}(\xi) e^{i\pi(-4\pi t)|\xi|^2} d\xi = i\int_{\mathbb{R}^n} \psi(x)(4\pi t)^{-n/2} e^{i\frac{|x|^2}{4t}} dx e^{-n\frac{i\pi}{4}}$$
$$= e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{i\frac{|x|^2}{4t}}\psi(x)dx.$$

As a result, u is a distribution on $\mathbb{R}^{n+1}$ defined by

$$\langle u, \Phi\rangle = e^{-i(n-2)\frac{\pi}{4}}(4\pi)^{-n/2}\int_0^{+\infty} t^{-n/2}\left(\int_{\mathbb{R}^n}\Phi(t,x)e^{i\frac{|x|^2}{4t}}dx\right)dt$$

and coincides with E, so that E satisfies (4.4.13). The identity (4.4.14) is proving (4.4.11). The proof of the theorem is complete. □

Remark 4.4.6. The fundamental solution of the Schrödinger equation is unbounded near $t = 0$ and, since E is smooth on $t \neq 0$, its C^∞ singular support is equal to $\{0\} \times \mathbb{R}^n_x$. In particular, the Schrödinger equation is *not* hypoelliptic. We shall see that it looks like a propagation equation with an infinite speed, or more precisely with a speed depending on the frequency of the wave.

4.4.3 The wave equation

Presentation

The wave equation in d dimensions with speed of propagation $c > 0$, is given by the operator on $\mathbb{R}_t \times \mathbb{R}^d_x$

$$\square_c = c^{-2}\partial_t^2 - \Delta_x. \tag{4.4.15}$$

We want to solve the equation $c^{-2}\partial_t^2 u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{d+1})$, we can consider its Fourier transform v with respect to x, and we end-up with the ODE with parameters on v,

$$c^{-2}\partial_t^2 v + 4\pi^2|\xi|^2 v = \delta_0(t), \quad \partial_t^2 v + 4\pi^2 c^2|\xi|^2 v = c^2\delta_0(t). \tag{4.4.16}$$

Lemma 4.4.7. *Let $\lambda, \mu \in \mathbb{C}$. A fundamental solution of $P_{\lambda,\mu} = (\frac{d}{dt} - \lambda)(\frac{d}{dt} - \mu)$ (on the real line) is*

$$\begin{cases} (\dfrac{e^{t\lambda} - e^{t\mu}}{\lambda - \mu})H(t) & \textit{for } \lambda \neq \mu, \\ te^{t\lambda}H(t) & \textit{for } \lambda = \mu. \end{cases} \tag{4.4.17}$$

Proof. If $\lambda \neq \mu$, to solve $(\frac{d}{dt} - \lambda)(\frac{d}{dt} - \mu) = \delta_0(t)$, the method of variation of parameters gives a solution $a(t)e^{\lambda t} + b(t)e^{\mu t}$ with

$$\begin{pmatrix} e^{t\lambda} & e^{t\mu} \\ \lambda e^{t\lambda} & \mu e^{t\mu}\end{pmatrix}\begin{pmatrix}\dot a \\ \dot b\end{pmatrix} = \begin{pmatrix} 0 \\ \delta\end{pmatrix} \Longrightarrow \begin{pmatrix}\dot a \\ \dot b\end{pmatrix} = \frac{1}{\lambda - \mu}\begin{pmatrix}\delta \\ -\delta\end{pmatrix} \Longrightarrow \text{(4.4.17) for } \lambda \neq \mu,$$

which gives also the result for $\lambda = \mu$ by differentiation with respect to λ of the identity $P_{\lambda,\mu}(e^{t\lambda} - e^{t\mu}) = (\lambda - \mu)\delta$. □

Going back to the wave equation, we can take v as the temperate distribution[10] given by

$$v(t,\xi) = c^2 H(t) \frac{e^{2i\pi ct|\xi|} - e^{-2i\pi ct|\xi|}}{4i\pi c|\xi|} = c^2 H(t) \frac{\sin\bigl(2\pi ct|\xi|\bigr)}{2\pi c|\xi|}. \tag{4.4.18}$$

Taking the inverse Fourier transform with respect to ξ of both sides of (4.4.16) gives with $u \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^d_\xi)$

$$c^{-2}\partial_t^2 u - \Delta_x u = \delta_0(t) \otimes \delta_0(x). \tag{4.4.19}$$

To compute u, we check with $\Phi \in \mathscr{D}(\mathbb{R}^{1+d})$,

$$\langle u, \Phi\rangle = \langle \widehat{v}^x(t,\xi), \Phi(t,-\xi)\rangle = \int_0^{+\infty}\int_{\mathbb{R}^n} \widehat{\Phi}^x(t,\xi) c \frac{\sin\bigl(2\pi ct|\xi|\bigr)}{2\pi|\xi|} d\xi dt. \tag{4.4.20}$$

We have found an expression for a fundamental solution of the wave equation in d space dimensions and proven the following proposition.

Proposition 4.4.8. *Let E_+ be the temperate distribution on $\mathbb{R}^{d+1}$ such that*

$$\widehat{E_+}^x(t,\xi) = cH(t)\frac{\sin\bigl(2\pi ct|\xi|\bigr)}{2\pi|\xi|}. \tag{4.4.21}$$

Then E_+ is a fundamental solution of the wave equation (4.4.15), i.e. satisfies $\square_c E_+ = \delta_0(t) \otimes \delta_0(x)$.

Remark 4.4.9. Defining the forward-light-cone $\Gamma_{+,c}$ as

$$\Gamma_{+,c} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^d, ct \geq |x|\}, \tag{4.4.22}$$

one can prove more precisely that E_+ is the only fundamental solution with support in $\{t \geq 0\}$ and that

$$\operatorname{supp} E_+ = \Gamma_+, \text{ when } d = 1 \text{ and } d \geq 2 \text{ is even}, \tag{4.4.23}$$

$$\operatorname{supp} E_+ = \partial\Gamma_+, \text{ when } d \geq 3 \text{ is odd}, \tag{4.4.24}$$

$$\operatorname{singsupp} E_+ = \partial\Gamma_+, \text{ in any dimension}. \tag{4.4.25}$$

Lemma 4.4.10. *Let E_1, E_2 be fundamental solutions of the wave equation such that $\operatorname{supp} E_1 \subset \Gamma_{+,c}, \operatorname{supp} E_2 \subset \{t \geq 0\}$. Then $E_1 = E_2$.*

Proof. Defining $u = E_1 - E_2$, we have $\operatorname{supp} u \subset \{t \geq 0\}$ and the mapping

$$\{t \geq 0\} \times \Gamma_{+,c} \ni \bigl((t,x),(s,y)\bigr) \mapsto (t+s, x+y) \in \mathbb{R}^{d+1}$$

is proper since

$$t, s \geq 0, cs \geq |y|, |t+s| \leq T, |x+y| \leq R \Longrightarrow t, s \in [0,T], |x| \leq R + cT, |y| \leq cT,$$

so that the section 3.5.3 allows to perform the following calculations

$$u = u * \delta_0 = u * \square_c E_1 = \square_c u * E_1 = 0.$$ □

[10] The function $\mathbb{R} \ni s \mapsto \frac{\sin s}{s} = \sum_{k\geq 0}(-1)^k \frac{s^{2k}}{(2k+1)!} = S(s^2)$ is a smooth bounded function of s^2, so that $v(t,\xi) = c^2 H(t) t S(4\pi^2 c^2 t^2 |\xi|^2)$ is continuous and such that $|v(t,\xi)| \leq CtH(t)$, thus a tempered distribution.

The wave equation in one space dimension

Theorem 4.4.11. *On $\mathbb{R}_t \times \mathbb{R}_x$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$E_+(t,x) = \frac{c}{2} H(ct - |x|). \tag{4.4.26}$$

where E_+ is defined in (4.4.21). That fundamental solution is bounded and the properties (4.4.23), (4.4.25) are satisfied.

Proof. We have $c^{-2}\partial_t^2 - \partial_x^2 = (c^{-1}\partial_t - \partial_x)(c^{-1}\partial_t + \partial_x)$ and changing (linearly) the variables with $x_1 = ct + x, x_2 = ct - x$, we have $t = \frac{1}{2c}(x_1 + x_2), x = \frac{1}{2}(x_1 - x_2)$, using the notation

$$(x_1, x_2) \mapsto (t,x) \mapsto u(t,x) = v(x_1, x_2),$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x_1} c + \frac{\partial v}{\partial x_2} c, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x_1} - \frac{\partial v}{\partial x_2}, \quad c^{-1}\partial_t - \partial_x = 2\partial_{x_2}, c^{-1}\partial_t + \partial_x = 2\partial_{x_1},$$

and thus $\square_c = 4\frac{\partial^2}{\partial x_1 \partial x_2}$, so that a fundamental solution is $v = \frac{1}{4}H(x_1)H(x_2)$. We have now to pull-back this distribution by the linear mapping $(t,x) \mapsto (x_1, x_2)$: we have the formula

$$\varphi(0,0) = \langle 4\frac{\partial^2 v}{\partial x_1 \partial x_2}(x_1,x_2), \varphi(x_1,x_2)\rangle = \langle (\square_c u)(t,x), \varphi(ct+x, ct-x)\rangle 2c$$

which gives the fundamental solution $\frac{2c}{4}H(ct+x)H(ct-x) = \frac{c}{2}H(ct-|x|)$. Moreover that fundamental solution is supported in $\Gamma_{+,c}$ and since E_+ is supported in $\{t \geq 0\}$, we can apply the lemma 4.4.10 to get their equality. $\square$

The wave equation in two space dimensions

We consider (4.4.15) with $d = 2$, i.e. $\square_c = c^{-2}\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2$.

Theorem 4.4.12. *On $\mathbb{R}_t \times \mathbb{R}_x^2$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$E_+(t,x) = \frac{c}{2\pi} H(ct - |x|)(c^2t^2 - |x|^2)^{-1/2}, \tag{4.4.27}$$

where E_+ is defined in (4.4.21). That fundamental solution is L^1_{loc} and the properties (4.4.23), (4.4.25) are satisfied.

Proof. From the lemma 4.4.10, it is enough to prove that the rhs of (4.4.27) is indeed a fundamental solution. The function $E(t,x) = \frac{c}{2\pi}H(ct-|x|)(c^2t^2-|x|^2)^{-1/2}$ is locally integrable in $\mathbb{R} \times \mathbb{R}^2$ since

$$\int_0^T \int_0^{ct} (c^2t^2 - r^2)^{-1/2} r dr dt = \int_0^T [(c^2t^2 - r^2)^{1/2}]_{r=ct}^{r=0} dt = cT^2/2 < +\infty.$$

Moreover E is homogeneous of degree -1, so that $\square_c E$ is homogeneous with degree -3 and supported in $\Gamma_{+,c}$. We use now the independently proven three-dimensional case (theorem 4.4.13). We define with $E_{+,3}$ given by (4.4.29), $\varphi \in \mathscr{D}(\mathbb{R}^3_{t,x_1,x_2}), \chi \in \mathscr{D}(\mathbb{R})$ with $\chi(0)=1$,

$$\begin{aligned}
\langle u, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^3),\mathscr{D}(\mathbb{R}^3)} &= \lim_{\epsilon\to 0}\langle E_{+,3}, \varphi(t,x_1,x_2)\otimes\chi(\epsilon x_3)\rangle_{\mathscr{D}'(\mathbb{R}^4),\mathscr{D}(\mathbb{R}^4)}\\
&= \lim_{\epsilon\to 0}\frac{1}{4\pi}\iiint_{\mathbb{R}^3}\frac{\varphi(c^{-1}\sqrt{x_1^2+x_2^2+x_3^2}, x_1, x_2)}{\sqrt{x_1^2+x_2^2+x_3^2}}\chi(\epsilon x_3)dx_1dx_2dx_3\\
&= \frac{1}{4\pi}2\iiint_{\mathbb{R}^2_{x_1,x_2}\times\{x_3\ge 0\}}\frac{\varphi(c^{-1}\sqrt{x_1^2+x_2^2+x_3^2}, x_1, x_2)}{\sqrt{x_1^2+x_2^2+x_3^2}}dx_1dx_2dx_3 \quad (t = c^{-1}\sqrt{x_1^2+x_2^2+x_3^2})\\
&= \frac{1}{2\pi}\iiint_{\mathbb{R}^2_{x_1,x_2}\times\{ct\ge\sqrt{x_1^2+x_2^2}\}}\frac{\varphi(t,x_1,x_2)}{ct}\frac{1}{2}(c^2t^2-x_1^2-x_2^2)^{-1/2}2c^2t\,dx_1dx_2dt\\
&= \frac{c}{2\pi}\iiint_{\mathbb{R}^2_{x_1,x_2}\times\{ct\ge\sqrt{x_1^2+x_2^2}\}}\varphi(t,x_1,x_2)(c^2t^2-x_1^2-x_2^2)^{-1/2}dx_1dx_2dt\\
&= \langle E,\varphi\rangle_{\mathscr{D}'(\mathbb{R}^3),\mathscr{D}(\mathbb{R}^3)}, \qquad \text{so that } E = u.
\end{aligned}$$

With $\square_{c,d}$ standing for the wave operator in d dimensions with speed c, we have, since

$$\square_{c,3}\big(\varphi(t,x_1,x_2)\otimes\chi(\epsilon x_3)\big) = \square_{c,2}\big(\varphi(t,x_1,x_2)\big)\otimes\chi(\epsilon x_3) - \varphi(t,x_1,x_2)\epsilon^2\chi''(\epsilon x_3)$$

$$\begin{aligned}
\langle\square_{c,2}u,\varphi\rangle &= \lim_{\epsilon\to 0}\langle E_{+,3}, (\square_{c,2}\varphi)(t,x_1,x_2)\otimes\chi(\epsilon x_3)\rangle\\
&= \lim_{\epsilon\to 0}\Big(\langle E_{+,3}, \square_{c,3}\big(\varphi(t,x_1,x_2)\otimes\chi(\epsilon x_3)\big)\rangle + \langle E_{+,3}, \varphi(t,x_1,x_2)\epsilon^2\chi''(\epsilon x_3)\rangle\Big)\\
&= \varphi(0,0,0),
\end{aligned}$$

which gives $\square_{c,2}E = \square_{c,2}u = \delta_{0,\mathbb{R}^3}$ and the result. □

The wave equation in three space dimensions

We consider (4.4.15) with $d=3$, i.e. $\square_c = c^{-2}\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2$.

Theorem 4.4.13. *On $\mathbb{R}_t\times\mathbb{R}^3_x$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$E_+(t,x) = \frac{1}{4\pi|x|}\delta_{0,\mathbb{R}}(t - c^{-1}|x|), \tag{4.4.28}$$

$$\text{i.e. for } \Phi\in\mathscr{D}(\mathbb{R}_t\times\mathbb{R}^3_x), \quad \langle E_+,\Phi\rangle = \int_{\mathbb{R}^3}\frac{1}{4\pi|x|}\Phi(c^{-1}|x|,x)dx. \tag{4.4.29}$$

where E_+ is defined in (4.4.21). The properties (4.4.24), (4.4.25) are satisfied.

Proof. The formula (4.4.29) is defining a Radon measure E with support $\partial\Gamma_{+,c}$, so that the last statements of the lemmas are clear. From the lemma 4.4.10, it is enough to prove that (4.4.29) defines indeed a fundamental solution. We check for $\varphi\in\mathscr{D}(\mathbb{R}), \psi\in\mathscr{D}(\mathbb{R}^3)$

$$\begin{aligned}
\langle\square_c E, \varphi(t)\otimes\psi(x)\rangle &= \langle E, \square_c(\varphi\otimes\psi)\rangle\\
&= \frac{1}{4\pi}\int_{\mathbb{R}^3}|x|^{-1}\Big(c^{-2}\varphi''(c^{-1}|x|)\psi(x) - \varphi(c^{-1}|x|)(\Delta\psi)(x)\Big)dx.
\end{aligned}$$

If we assume that $\operatorname{supp}\varphi \subset \mathbb{R}_+^*$, we get

$$\begin{aligned}\int_{\mathbb{R}^3} |x|^{-1}\varphi(c^{-1}|x|)(\Delta\psi)(x)dx &= \int_{\mathbb{R}^3} \Delta\big(|x|^{-1}\varphi(c^{-1}|x|)\big)\psi(x)dx \\ &= \int_{\mathbb{R}^3}\Big(\big(r^{-1}\varphi(c^{-1}r)\big)'' + 2r^{-1}\big(r^{-1}\varphi(c^{-1}r)\big)'\Big)\psi(x)dx \qquad (r=|x|) \\ &= \int \psi(x)\Big(r^{-1}\varphi''(c^{-1}r)c^{-2} + 2(-r^{-2})\varphi'(c^{-1}r)c^{-1} + 2r^{-3}\varphi(c^{-1}r) \\ &\qquad + 2r^{-1}r^{-1}\varphi'(c^{-1}r)c^{-1} + 2r^{-1}(-r^{-2})\varphi(c^{-1}r)\Big)dx,\end{aligned}$$

which gives $\langle \square_c E, \varphi(t)\otimes\psi(x)\rangle = 0$. As a result,

$$\operatorname{supp}(\square_c E) \subset \partial\Gamma_{+,c} \cap \{t \le 0\} = \{(0_{\mathbb{R}}, 0_{\mathbb{R}^3})\},$$

and since E is homogeneous with degree -2, the distribution $\square_c E$ is homogeneous with degree -4 with support at the origin of $\mathbb{R}^4$: the lemma 3.4.8 and the theorem 3.3.4 imply that $\square_c E = \kappa\delta_{0,\mathbb{R}^4}$. To check that $\kappa = 1$, we calculate for $\varphi \in \mathscr{D}(\mathbb{R})$ (noting that $|t| \le C$ and $|x| \le c|t| + 1$ implies $|x| \le cC + 1$)

$$\begin{aligned}\langle \square_c E, \varphi(t)\otimes 1\rangle = \frac{1}{4\pi}\int_0^{+\infty} r^{-1}c^{-2}\varphi''(c^{-1}r)r^2dr4\pi &= \int_0^{+\infty}\varphi''(r)rdr \\ &= [\varphi'(r)r]_0^{+\infty} - \int_0^{+\infty}\varphi'(r)dr = \varphi(0),\end{aligned}$$

so that $\kappa = 1$ and the theorem is proven. □

4.5 Periodic distributions

4.5.1 The Dirichlet kernel

For $N \in \mathbb{N}$, the Dirichlet kernel D_N is defined on $\mathbb{R}$ by

$$\begin{aligned}D_N(x) &= \sum_{-N\le k\le N} e^{2i\pi kx} = 1 + 2\operatorname{Re}\sum_{1\le k\le N} e^{2i\pi kx} \underbrace{=}_{x\notin\mathbb{Z}} 1 + 2\operatorname{Re}\Big(e^{2i\pi x}\frac{e^{2i\pi Nx}-1}{e^{2i\pi x}-1}\Big) \\ &= 1 + 2\operatorname{Re}\big(e^{2i\pi x - i\pi x + i\pi Nx}\big)\frac{\sin(\pi Nx)}{\sin(\pi x)} = 1 + 2\cos(\pi(N+1)x)\frac{\sin(\pi Nx)}{\sin(\pi x)} \\ &= 1 + \frac{1}{\sin(\pi x)}\Big(\sin\big(\pi x(2N+1)\big) - \sin(\pi x)\Big) = \frac{\sin\big(\pi x(2N+1)\big)}{\sin(\pi x)},\end{aligned}$$

and extending by continuity at $x \in \mathbb{Z}$ that 1-periodic function, we find that

$$D_N(x) = \frac{\sin\big(\pi x(2N+1)\big)}{\sin(\pi x)}. \tag{4.5.1}$$

Now, for a 1-periodic $v \in C^1(\mathbb{R})$, with

$$(D_N \star u)(x) = \int_0^1 D_N(x-t)u(t)dt, \tag{4.5.2}$$

we have

$$\lim_{N\to+\infty}\int_0^1 D_N(x-t)v(t)dt = v(x) + \lim_{N\to+\infty}\int_0^1 \sin(\pi t(2N+1))\frac{\big(v(x-t)-v(x)\big)}{\sin(\pi t)}dt,$$

and the function θ_x given by $\theta_x(t) = \frac{v(x-t)-v(x)}{\sin(\pi t)}$ is continuous on $[0,1]$, and from the Riemann-Lebesgue lemma 4.3.5, we obtain

$$\lim_{N\to+\infty}\sum_{-N\le k\le N} e^{2i\pi kx}\int_0^1 e^{-2i\pi kt}v(t)dt = \lim_{N\to+\infty}\int_0^1 D_N(x-t)v(t)dt = v(x).$$

On the other hand if v is 1-periodic and C^{1+l}, the Fourier coefficient

$$c_k(v) = \int_0^1 e^{-2i\pi kt}v(t)dt \overbrace{=}^{\text{for } k\neq 0} \frac{1}{2i\pi k}[e^{-2i\pi kt}v(t)]_{t=1}^{t=0} + \int_0^1 \frac{1}{2i\pi k}e^{-2i\pi kt}v'(t)dt, \quad (4.5.3)$$

and iterating the integration by parts, we find $c_k(v) = O(k^{-1-l})$ so that for a 1-periodic C^2 function v, we have

$$\sum_{k\in\mathbb{Z}} e^{2i\pi kx}c_k(v) = v(x). \quad (4.5.4)$$

4.5.2 Pointwise convergence of Fourier series

Lemma 4.5.1. *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a 1-periodic $L^1_{loc}(\mathbb{R})$ function and let $x_0 \in [0,1]$. Let us assume that there exists $w_0 \in \mathbb{R}$ such that the Dini condition is satisfied, i.e.*

$$\int_0^{1/2} \frac{|u(x_0+t)+u(x_0-t)-2w_0|}{t}dt < +\infty. \quad (4.5.5)$$

Then, $\lim_{N\to+\infty}\sum_{|k|\le N} c_k(u)e^{2i\pi kx_0} = w_0$ with $c_k(u) = \int_0^1 e^{-2i\pi tk}u(t)dt$.

Proof. Using the calculations of the previous section 4.5.1, we find

$$\sum_{|k|\le N} c_k(u)e^{2i\pi kx_0} = (D_N * u)(x_0) = w_0 + \int_0^1 \frac{\sin\big(\pi t(2N+1)\big)}{\sin(\pi t)}\big(u(x_0-t)-w_0\big)dt,$$

so that, using the periodicity of u and the fact that D_N is an even function , we get

$$(D_N * u)(x_0) - w_0 = \int_0^{1/2} \frac{\sin\big(\pi t(2N+1)\big)}{\sin(\pi t)}\big(u(x_0-t)+u(x_0+t)-2w_0\big)dt.$$

Thanks to the hypothesis (4.5.5), the function $t \mapsto \mathbf{1}_{[0,1]}(t)\dfrac{u(x_0-t)+u(x_0+t)-2w_0}{\sin(\pi t)}$ belongs to $L^1(\mathbb{R})$ and the Riemann-Lebesgue lemma 4.3.5 gives the conclusion. □

Theorem 4.5.2. *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a 1-periodic L^1_{loc} function.*
(1) *Let $x_0 \in [0,1], w_0 \in \mathbb{R}$. We define $\omega_{x_0,w_0}(t) = |u(x_0+t)+u(x_0-t)-2w_0|$ and we assume that*

$$\int_0^{1/2} \omega_{x_0,w_0}(t)\frac{dt}{t} < +\infty. \tag{4.5.6}$$

Then the Fourier series $(D_N * u)(x_0)$ converges with limit w_0. In particular, if (4.5.6) is satisfied with $w_0 = u(x_0)$, the Fourier series $(D_N * u)(x_0)$ converges with limit $u(x_0)$. If u has a left and right limit at x_0 and is such that (4.5.6) is satisfied with $w_0 = \frac{1}{2}\big(u(x_0+0)+u(x_0-0)\big)$, the Fourier series $(D_N * u)(x_0)$ converges with limit $\frac{1}{2}\big(u(x_0-0)+u(x_0+0)\big)$.
(2) *If the function u is Hölder-continuous*[11]*, the Fourier series $(D_N * u)(x)$ converges for all $x \in \mathbb{R}$ with limit $u(x)$.*
(3) *If u has a left and right limit at each point and a left and right derivative at each point, the Fourier series $(D_N * u)(x)$ converges for all $x \in \mathbb{R}$ with limit $\frac{1}{2}\big(u(x-0)+u(x+0)\big)$.*

Proof. (1) follows from the lemma 4.5.1; to obtain (2), we note that for a Hölder continuous function of index $\theta \in]0,1]$, we have for $t \in]0,1/2]$

$$t^{-1}\omega_{x,u(x)}(t) \le Ct^{\theta-1} \in L^1([0,1/2]).$$

If u has a right-derivative at x_0, it means that

$$u(x_0+t) = u(x_0+0) + u'_r(x_0)t + t\epsilon_0(t), \qquad \lim_{t\to 0_+} \epsilon_0(t) = 0.$$

As a consequence, for $t \in]0,1/2]$, $t^{-1}|u(x_0+t)-u(x_0+0)| \le |u'_r(x_0)+\epsilon_0(t)|$. Since $\lim_{t\to 0_+}\epsilon_0(t) = 0$, there exists $T_0 \in]0,1/2]$ such that $|\epsilon_0(t)| \le 1$ for $t \in [0,T_0]$. As a result, we have

$$\int_0^{1/2} t^{-1}|u(x_0+t)-u(x_0+0)|dt \\ \le \int_0^{T_0}(|u'_r(x_0)|+1)dt + \int_{T_0}^{1/2}|u(x_0+t)-u(x_0+0)|dt T_0^{-1} < +\infty,$$

since u is also L^1_{loc}. The integral $\int_0^{1/2} t^{-1}|u(x_0-t)-u(x_0-0)|dt$ is also finite and the condition (4.5.6) holds with $w_0 = \frac{1}{2}\big(u(x_0-0)+u(x_0+0)\big)$. The proof of the lemma is complete. □

4.5.3 Periodic distributions

We consider now a distribution u on $\mathbb{R}^n$ which is periodic with periods $\mathbb{Z}^n$. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi = 1$ on $[0,1]^n$. Then the function χ_1 defined by

$$\chi_1(x) = \sum_{k\in\mathbb{Z}^n} \chi(x-k)$$

[11] Hölder-continuity of index $\theta \in]0,1]$ means that $\exists C > 0, \forall t,s,\ |u(t)-u(s)| \le C|t-s|^\theta$.

is C^∞ periodic[12] with periods $\mathbb{Z}^n$. Moreover since $\mathbb{R}^n \ni x \in \prod_{1\le j\le n}[E(x_j), E(x_j)+1[$, the bounded function χ_1 is also bounded from below and such that $1 \le \chi_1(x)$. With $\chi_0 = \chi/\chi_1$, we have

$$\sum_{k\in\mathbb{Z}^n} \chi_0(x-k) = 1, \quad \chi_0 \in C_c^\infty(\mathbb{R}^n).$$

For $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have from the periodicity of u

$$\langle u, \varphi\rangle = \sum_{k\in\mathbb{Z}^n} \langle u(x), \varphi(x)\chi_0(x-k)\rangle = \sum_{k\in\mathbb{Z}^n} \langle u(x), \varphi(x+k)\chi_0(x)\rangle,$$

where the sums are finite. Now if $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have, since χ_0 is compactly supported in $|x| \le R_0$,

$$\begin{aligned}|\langle u(x), \varphi(x+k)\chi_0(x)\rangle| &\le C_0 \sup_{|\alpha|\le N_0, |x|\le R_0} |\varphi^{(\alpha)}(x+k)| \\ &\le C_0 \sup_{|\alpha|\le N_0, |x|\le R_0} |(1+R_0+|x+k|)^{n+1}\varphi^{(\alpha)}(x+k)|(1+|k|)^{-n-1} \\ &\le p_0(\varphi)(1+|k|)^{-n-1},\end{aligned}$$

where p_0 is a semi-norm of φ (independent of k). As a result u is a tempered distribution and we have for $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \langle u(x), \sum_{k\in\mathbb{Z}^n} \underbrace{\varphi(x+k)\chi_0(x)}_{\psi_x(k)}\rangle = \langle u(x), \sum_{k\in\mathbb{Z}^n} \widehat{\psi_x}(k)\rangle.$$

Now we see that $\widehat{\psi_x}(k) = \int_{\mathbb{R}^n} \varphi(x+t)\chi_0(x)e^{-2i\pi kt}dt = \chi_0(x)e^{2i\pi kx}\hat\varphi(k)$, so that $\langle u, \varphi\rangle = \sum_{k\in\mathbb{Z}^n}\langle u(x), \chi_0(x)e^{2i\pi kx}\rangle\hat\varphi(k)$ which means

$$u(x) = \sum_{k\in\mathbb{Z}^n} \langle u(t), \chi_0(t)e^{2i\pi kt}\rangle e^{-2i\pi kx} = \sum_{k\in\mathbb{Z}^n} \langle u(t), \chi_0(t)e^{-2i\pi kt}\rangle e^{2i\pi kx}.$$

Theorem 4.5.3. *Let u be a periodic distribution on $\mathbb{R}^n$ with periods $\mathbb{Z}^n$. Then u is a tempered distribution and if χ_0 is a $C_c^\infty(\mathbb{R}^n)$ function such that $\sum_{k\in\mathbb{Z}^n}\chi_0(x-k) = 1$, we have*

$$u = \sum_{k\in\mathbb{Z}^n} c_k(u)e^{2i\pi kx}, \tag{4.5.7}$$

$$\hat u = \sum_{k\in\mathbb{Z}^n} c_k(u)\delta_k, \quad \textit{with} \quad c_k(u) = \langle u(t), \chi_0(t)e^{-2i\pi kt}\rangle, \tag{4.5.8}$$

and convergence in $\mathscr{S}'(\mathbb{R}^n)$. If u is in $C^m(\mathbb{R}^n)$ with $m > n$, the previous formulas hold with uniform convergence for (4.5.7) and

$$c_k(u) = \int_{[0,1]^n} u(t)e^{-2i\pi kt}dt. \tag{4.5.9}$$

[12] Note that the sum is locally finite since for K compact subset of $\mathbb{R}^n$, $(K-k)\cap \operatorname{supp}\chi_0 = \emptyset$ except for a finite subset of $k \in \mathbb{Z}^n$.

Proof. The first statements are already proven and the calculation of $\hat{u}$ is immediate. If u belongs to L^1_{loc} we can redo the calculations above choosing $\chi_0 = \mathbf{1}_{[0,1]^n}$ and get (4.5.7) with c_k given by (4.5.9). Moreover, if u is in C^m with $m > n$, we get by integration by parts that $c_k(u)$ is $O(|k|^{-m})$ so that the series (4.5.7) is uniformly converging. □

Theorem 4.5.4. *Let u be a periodic distribution on $\mathbb{R}^n$ with periods $\mathbb{Z}^n$. If $u \in L^2_{loc}$ (i.e. $u \in L^2(\mathbb{T}^n)$ with $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$), then*

$$u(x) = \sum_{k \in \mathbb{Z}^n} c_k(u) e^{2i\pi kx}, \quad \textit{with} \quad c_k(u) = \int_{[0,1]^n} u(t) e^{-2i\pi kt} dt, \tag{4.5.10}$$

and convergence in $L^2(\mathbb{T}^n)$. Moreover $\|u\|^2_{L^2(\mathbb{T}^n)} = \sum_{k \in \mathbb{Z}^n} |c_k(u)|^2$. Conversely, if the coefficients $c_k(u)$ defined by (4.5.8) are in $\ell^2(\mathbb{Z}^n)$, the distribution u is $L^2(\mathbb{T}^n)$

Proof. As said above the formula for the $c_k(u)$ follows from changing the choice of χ_0 to $\mathbf{1}_{[0,1]^n}$ in the discussion preceding the theorem 4.5.3. The formula (4.5.7) gives the convergence in $\mathscr{S}'(\mathbb{R}^n)$ to u. Now, since $\int_{[0,1]^n} e^{2i\pi(k-l)t} dt = \delta_{k,l}$ we see from the theorem 4.5.3 that for $u \in C^{n+1}(\mathbb{T}^n)$, $\langle u, u \rangle_{L^2(\mathbb{T}^n)} = \sum_{k \in \mathbb{Z}^n} |c_k(u)|^2$. As a consequence the mapping $L^2(\mathbb{T}^n) \ni u \mapsto (c_k(u))_{k \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$ is isometric with a range containing the dense subset $\ell^1(\mathbb{Z}^n)$ (if $(c_k(u))_{k \in \mathbb{Z}^n} \in \ell^1(\mathbb{Z}^n)$, u is a continuous function); since the range is closed, the mapping is onto and is an isometric isomorphism from the open mapping theorem. □

4.6 Appendix

4.6.1 The logarithm of a nonsingular symmetric matrix

The set $\mathbb{C}\backslash\mathbb{R}_-$ is star-shaped with respect to 1, so that we can define the principal determination of the logarithm for $z \in \mathbb{C}\backslash\mathbb{R}_-$ by the formula

$$\operatorname{Log} z = \oint_{[1,z]} \frac{d\zeta}{\zeta}. \tag{4.6.1}$$

The function Log is holomorphic on $\mathbb{C}\backslash\mathbb{R}_-$ and we have $\operatorname{Log} z = \ln z$ for $z \in \mathbb{R}_+^*$ and by analytic continuation $e^{\operatorname{Log} z} = z$ for $z \in \mathbb{C}\backslash\mathbb{R}_-$. We get also by analytic continuation, that $\operatorname{Log} e^z = z$ for $|\operatorname{Im} z| < \pi$.

Let Υ_+ be the set of symmetric nonsingular $n \times n$ matrices with complex entries and nonnegative real part. The set Υ_+ is star-shaped with respect to the Id: for $A \in \Upsilon_+$, the segment $[1, A] = \big((1-t)\operatorname{Id} + tA\big)_{t\in[0,1]}$ is obviously made with symmetric matrices with nonnegative real part which are invertible[13], since for $0 \le t < 1$, $\operatorname{Re}\big((1-t)\operatorname{Id} + tA\big) \ge (1-t)\operatorname{Id} > 0$ and for $t = 1$, A is assumed to be invertible. We can now define for $A \in \Upsilon_+$

$$\operatorname{Log} A = \int_0^1 (A - I)\big(I + t(A - I)\big)^{-1} dt. \tag{4.6.2}$$

We note that A commutes with $(I + sA)$ (and thus with $\operatorname{Log} A$), so that, for $\theta > 0$,

$$\begin{aligned}\frac{d}{d\theta}\operatorname{Log}(A + \theta I) = \int_0^1 &\big(I + t(A + \theta I - I)\big)^{-1} dt \\ &- \int_0^1 (A + \theta I - I)t\big(I + t(A + \theta I - I)\big)^{-2} dt,\end{aligned}$$

and since $\frac{d}{dt}\Big\{\big(I+t(A+\theta I-I)\big)^{-1}\Big\} = -\big(I+t(A+\theta I-I)\big)^{-2}(A+\theta I-I)$, we obtain by integration by parts $\frac{d}{d\theta}\operatorname{Log}(A + \theta I) = (A + \theta I)^{-1}$. As a result, we find that for $\theta > 0, A \in \Upsilon_+$, since all the matrices involved are commuting,

$$\frac{d}{d\theta}\left((A + \theta I)^{-1} e^{\operatorname{Log}(A+\theta I)}\right) = 0,$$

so that, using the limit $\theta \to +\infty$, we get that $\forall A \in \Upsilon_+, \forall \theta > 0,\ e^{\operatorname{Log}(A+\theta I)} = (A+\theta I)$, and by continuity

$$\forall A \in \Upsilon_+, \quad e^{\operatorname{Log} A} = A, \quad \text{which implies} \quad \det A = e^{\operatorname{trace} \operatorname{Log} A}. \tag{4.6.3}$$

Using (4.6.3), we can define for $A \in \Upsilon_+$, using (4.6.2)

$$(\det A)^{-1/2} = e^{-\frac{1}{2}\operatorname{trace} \operatorname{Log} A} = |\det A|^{-1/2} e^{-\frac{i}{2}\operatorname{Im}(\operatorname{trace} \operatorname{Log} A)}. \tag{4.6.4}$$

[13] Note that a symmetric matrix B with a positive-definite real part is indeed invertible since for $u \in \mathbb{C}^n$, $Bu = 0$ implies $0 = \operatorname{Re}\langle Bu, \bar{u}\rangle = \langle (\operatorname{Re} B)u, \bar{u}\rangle \ge c_0\|u\|^2$ with $c_0 > 0$ and thus $u = 0$.

- When A is a positive definite matrix, $\operatorname{Log} A$ is real-valued and $(\det A)^{-1/2} = |\det A|^{-1/2}$.

- When $A = -iB$ where B is a real nonsingular symmetric matrix, we note that $B = PD^tP$ with $P \in O(n)$ and D diagonal. We see directly on the formulas (4.6.2),(4.6.1) that
$$\operatorname{Log} A = \operatorname{Log}(-iB) = P(\operatorname{Log}(-iD))^tP, \quad \operatorname{trace} \operatorname{Log} A = \operatorname{trace} \operatorname{Log}(-iD)$$
and thus, with (μ_j) the (real) eigenvalues of B, we have $\operatorname{Im}(\operatorname{trace} \operatorname{Log} A) = \operatorname{Im} \sum_{1 \le j \le n} \operatorname{Log}(-i\mu_j)$, where the last Log is given by (4.6.1). Finally we get,
$$\operatorname{Im}(\operatorname{trace} \operatorname{Log} A) = -\frac{\pi}{2} \sum_{1 \le j \le n} \operatorname{sign} \mu_j = -\frac{\pi}{2} \operatorname{sign} B$$
where $\operatorname{sign} B$ is the signature of B. As a result, we have when $A = -iB$, B real symmetric nonsingular matrix
$$(\det A)^{-1/2} = |\det A|^{-1/2} e^{i\frac{\pi}{4} \operatorname{sign}(iA)} = |\det B|^{-1/2} e^{i\frac{\pi}{4} \operatorname{sign} B}. \tag{4.6.5}$$

4.6.2 Fourier transform of Gaussian functions

Proposition 4.6.1. *Let A be a symmetric nonsingular $n \times n$ matrix with complex entries such that $\operatorname{Re} A \ge 0$. We define the Gaussian function v_A on $\mathbb{R}^n$ by $v_A(x) = e^{-\pi\langle Ax, x\rangle}$. The Fourier transform of v_A is*
$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi, \xi\rangle}, \tag{4.6.6}$$
where $(\det A)^{-1/2}$ is defined according to the formula (4.6.4). In particular, when $A = -iB$ with a symmetric real nonsingular matrix B, we get
$$\operatorname{Fourier}(e^{i\pi\langle Bx, x\rangle})(\xi) = \widehat{v_{-iB}}(\xi) = |\det B|^{-1/2} e^{i\frac{\pi}{4} \operatorname{sign} B} e^{-i\pi\langle B^{-1}\xi, \xi\rangle}. \tag{4.6.7}$$

Proof. Let us define Υ_+^* as the set of symmetric $n \times n$ complex matrices with a positive definite real part (naturally these matrices are nonsingular since $Ax = 0$ for $x \in \mathbb{C}^n$ implies $0 = \operatorname{Re}\langle Ax, \bar{x}\rangle = \langle(\operatorname{Re} A)x, \bar{x}\rangle$, so that $\Upsilon_+^* \subset \Upsilon_+$).

Let us assume first that $A \in \Upsilon_+^*$; then the function v_A is in the Schwartz class (and so is its Fourier transform). The set Υ_+^* is an open convex subset of $\mathbb{C}^{n(n+1)/2}$ and the function $\Upsilon_+^* \ni A \mapsto \widehat{v_A}(\xi)$ is holomorphic and given on $\Upsilon_+^* \cap \mathbb{R}^{n(n+1)/2}$ by (4.6.6). On the other hand the function $\Upsilon_+^* \ni A \mapsto e^{-\frac{1}{2} \operatorname{trace} \operatorname{Log} A} e^{-\pi\langle A^{-1}\xi, \xi\rangle}$ is also holomorphic and coincides with previous one on $\mathbb{R}^{n(n+1)/2}$. By analytic continuation this proves (4.6.6) for $A \in \Upsilon_+^*$.

If $A \in \Upsilon_+$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have $\langle \widehat{v_A}, \varphi\rangle_{\mathscr{S}', \mathscr{S}} = \int v_A(x)\hat{\varphi}(x)dx$ so that $\Upsilon_+ \ni A \mapsto \langle \widehat{v_A}, \varphi\rangle$ is continuous and thus (note that the mapping $A \mapsto A^{-1}$ is an homeomorphism of Υ_+), using the previous result on Υ_+^*,
$$\langle \widehat{v_A}, \varphi\rangle = \lim_{\epsilon \to 0_+} \langle \widehat{v_{A+\epsilon I}}, \varphi\rangle = \lim_{\epsilon \to 0_+} \int e^{-\frac{1}{2} \operatorname{trace} \operatorname{Log}(A+\epsilon I)} e^{-\pi\langle (A+\epsilon I)^{-1}\xi, \xi\rangle} \varphi(\xi) d\xi$$
$$\text{(by continuity of Log on } \Upsilon_+ \text{ and domin. cv.)} \qquad = \int e^{-\frac{1}{2} \operatorname{trace} \operatorname{Log} A} e^{-\pi\langle A^{-1}\xi, \xi\rangle} \varphi(\xi) d\xi,$$
which is the sought result. □

Chapter 5

Analysis on Hilbert spaces

5.1 Hilbert spaces

5.1.1 Definitions and characterization

The definition and basic examples of Hilbert spaces were given in the section 1.4.1 and in the definition 1.3.7. Some important properties, such as the Cauchy-Schwarz inequality (1.3.3) were derived above. We shall always deal with complex Hilbert spaces and derive in this section a few more general properties for these spaces.

Theorem 5.1.1 (Jordan – von Neumann theorem)**.** *Let E be a Banach space, such that the parallelogram identity holds, i.e. for all $u, v \in E$,*

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2. \tag{5.1.1}$$

Then E is a Hilbert space with the scalar product

$$\langle u, v\rangle = \frac{1}{4}\big(\|u+v\|^2 - \|u-v\|^2\big) + \frac{i}{4}\big(\|u+iv\|^2 - \|u-iv\|^2\big). \tag{5.1.2}$$

Conversely both properties hold for a Hilbert space.

Proof. Let us first check the last statement: in a Hilbert space

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \|u\|^2 + 2\operatorname{Re}\langle u, v\rangle + \|v\|^2, \tag{5.1.3}$$

which implies readily (5.1.1), (5.1.2). Conversely, if E is a Banach space satisfying (5.1.1), the formula (5.1.2) defines a sesquilinear Hermitian form: it satisfies

$$\langle u, u\rangle = \|u\|^2 + \frac{i}{4}(2\|u\|^2 - 2\|u\|^2) = \|u\|^2 \quad \text{and } \langle u, v\rangle = \overline{\langle v, u\rangle} \tag{5.1.4}$$

since

$$\begin{aligned}\overline{\langle v, u\rangle} &= \frac{1}{4}\big(\|v+u\|^2 - \|v-u\|^2\big) - \frac{i}{4}\big(\|v+iu\|^2 - \|v-iu\|^2\big) \\ &= \frac{1}{4}\big(\|u+v\|^2 - \|u-v\|^2\big) - \frac{i}{4}\big(\|u-iv\|^2 - \|u+iv\|^2\big).\end{aligned}$$

It is linear with respect to u since, using (5.1.1),

$$\begin{aligned}4\operatorname{Re}\langle u_1, v\rangle + 4\operatorname{Re}\langle u_2, v\rangle &= \|u_1+v\|^2 - \|u_1-v\|^2 + \|u_2+v\|^2 - \|u_2-v\|^2\\ &= \frac{1}{2}\|u_1+u_2+2v\|^2 + \frac{1}{2}\|u_1-u_2\|^2 - \frac{1}{2}\|u_1+u_2-2v\|^2 - \frac{1}{2}\|u_1-u_2\|^2\\ &= \frac{1}{2}\|u_1+u_2+2v\|^2 - \frac{1}{2}\|u_1+u_2-2v\|^2 = \frac{1}{2}4\operatorname{Re}\langle u_1+u_2, 2v\rangle\end{aligned}$$

and thus, using the obvious identity $\langle u, 0\rangle = 0$,

$$\operatorname{Re}\langle u_1, v\rangle + \operatorname{Re}\langle u_2, v\rangle = \frac{1}{2}\operatorname{Re}\langle u_1+u_2, 2v\rangle, \tag{5.1.5}$$

$$\text{which implies for } u_2 = 0, \quad 2\operatorname{Re}\langle u_1, v\rangle = \operatorname{Re}\langle u_1, 2v\rangle, \tag{5.1.6}$$

so that using (5.1.6) in (5.1.5), we obtain

$$\operatorname{Re}\langle u_1, v\rangle + \operatorname{Re}\langle u_2, v\rangle = \operatorname{Re}\langle u_1+u_2, v\rangle. \tag{5.1.7}$$

We have similarly $4\operatorname{Im}\langle u_1, v\rangle + 4\operatorname{Im}\langle u_2, v\rangle = \|u_1+iv\|^2 - \|u_1-iv\|^2 + \|u_2+iv\|^2 - \|u_2-iv\|^2 = \frac{1}{2}\|u_1+u_2+2iv\|^2 - \frac{1}{2}\|u_1+u_2-2iv\|^2 = \frac{1}{2}4\operatorname{Im}\langle u_1+u_2, 2v\rangle$ and thus, $\operatorname{Im}\langle u_1, v\rangle + \operatorname{Im}\langle u_2, v\rangle = \frac{1}{2}\operatorname{Im}\langle u_1+u_2, 2v\rangle$, which implies for $u_2 = 0$, $2\operatorname{Im}\langle u_1, v\rangle = \operatorname{Im}\langle u_1, 2v\rangle$ and we obtain

$$\operatorname{Im}\langle u_1, v\rangle + \operatorname{Im}\langle u_2, v\rangle = \operatorname{Im}\langle u_1+u_2, v\rangle, \tag{5.1.8}$$

finally getting from (5.1.7), (5.1.8), (5.1.4)

$$\langle u_1+u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle, \quad \langle u, v_1+v_2\rangle = \langle u, v_1\rangle + \langle u, v_2\rangle. \tag{5.1.9}$$

The identity (5.1.9) implies

$$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle \tag{5.1.10}$$

for $\lambda \in \mathbb{Q}$ and we have also

$$4\langle iu, v\rangle = \frac{1}{4}\big(\|u-iv\|^2 - \|u+iv\|^2\big) + \frac{i}{4}\big(\|u+v\|^2 - \|u-v\|^2\big) = i4\langle u, v\rangle,$$

so that (5.1.10) holds as well for $\lambda \in \mathbb{Q} + i\mathbb{Q}$. Now the function $\mathbb{C} \ni \lambda \mapsto \langle \lambda u, v\rangle$ is continuous since

$$\langle \lambda u, v\rangle = \frac{1}{4}\big(\|\lambda u+v\|^2 - \|\lambda u-v\|^2\big) + \frac{i}{4}\big(\|\lambda u+iv\|^2 - \|\lambda u-iv\|^2\big),$$

and for $\lambda, h \in \mathbb{C}$, the triangle inequality and the homogeneity of the norm imply $\big|\|(\lambda+h)u+v\| - \|\lambda u+v\|\big| \le \|hu\| = |h|\|u\|$. The continuous function $\mathbb{C} \ni \lambda \mapsto \langle \lambda u, v\rangle - \lambda\langle u, v\rangle$ vanishes on the dense subset $\mathbb{Q} + i\mathbb{Q}$ and thus everywhere. The proof of the theorem is complete. □

5.1.2 Projection on a closed convex set. Orthogonality

We shall now prove a theorem of projection on closed convex subsets of a Hilbert space $\mathbb{H}$. We recall that a subset M of a vector space is said to be convex whenever

$$\forall u, v \in M, \quad \forall \theta \in [0,1], \quad (1-\theta)u + \theta v \in M. \tag{5.1.11}$$

Theorem 5.1.2. *Let $\mathbb{H}$ be a Hilbert space and M a (non-empty) convex closed subset of $\mathbb{H}$. Then for all $u \in \mathbb{H}$, there exists a unique $v_u \in M$ such that*

$$\inf_{w \in M} \|u - w\| = \|u - v_u\|.$$

We shall note $v_u = p_M(u)$ and call it the projection of u on M. The mapping $p_M : \mathbb{H} \to M$ is the identity on M and $p_M^2 = p_M$.

Proof. Let $u \in \mathbb{H}$ and $(w_k)_{k \geq 1}$ a sequence in M such that $\lim_k \|u - w_k\| = d(u, M) = \inf_{w \in M} \|u - w\|$. We have from (5.1.1)

$$\begin{aligned} 2\|u - w_k\|^2 + 2\|u - w_l\|^2 &= 4\|u - \overbrace{\frac{1}{2}(w_k + w_l)}^{\in M}\|^2 + \|w_k - w_l\|^2 \\ &\geq 4d(u,M)^2 + \|w_k - w_l\|^2, \end{aligned} \tag{5.1.12}$$

so that $\|w_k - w_l\|^2 \leq 2\|u - w_k\|^2 + 2\|u - w_l\|^2 - 4d(u,M)^2$ and $(w_k)_{k \geq 1}$ is a Cauchy sequence, thus converging to a point v, which is in M since M is closed. We have thus by the continuity of the norm (see e.g. the footnote 2 in the section 2.1.2)

$$d(u, M) = \lim_k \|u - w_k\| = \|u - v\|.$$

Now if $w \in M$ also satisfies $d(u, M) = \|u - w\|$, the inequality (5.1.12) with w_k, w_l replaced by v, w gives $4d(u,M)^2 \geq 4d(u,M)^2 + \|v - w\|^2$ and $v = w$, proving the uniqueness, which implies also that p_M is the identity on M and $p_M^2 = p_M$. The proof is complete. □

Definition 5.1.3. *Let $\mathbb{H}$ be a Hilbert space and $u, v \in \mathbb{H}$. The vectors u, v are said to be orthogonal when $\langle u, v \rangle = 0$. Let F be a subset of $\mathbb{H}$: we define $F^\perp$, the orthogonal of F, as*

$$F^\perp = \{u \in \mathbb{H}, \forall v \in F, \langle u, v \rangle = 0\} \tag{5.1.13}$$

Theorem 5.1.4. *Let $u_1, \dots, u_m$ be pairwise orthogonal vectors in $\mathbb{H}$, then the Pythagorean identity holds:*

$$\|\sum_{1 \leq j \leq m} u_j\|^2 = \sum_{1 \leq j \leq m} \|u_j\|^2. \tag{5.1.14}$$

Let F be a subset of $\mathbb{H}$. Then $F^\perp$ is a closed subspace of $\mathbb{H}$ and if F is a closed subspace of $\mathbb{H}$, we have

$$F \oplus F^\perp = \mathbb{H}. \tag{5.1.15}$$

Proof. When $m = 2$, the first part is (5.1.1) ; an induction on m gives the result. $F^\perp$ is closed as an intersection of closed sets (each linear form $u \mapsto \langle u, v\rangle$ is continuous from (1.3.3)) and $F^\perp$ is obviously stable by linear combination. We postpone the proof of (5.1.15) to the end of the proof of the next theorem. □

Theorem 5.1.5. *Let $\mathbb{H}$ be a Hilbert space and F be a closed subspace of $\mathbb{H}$ (thus F is closed, convex and... not empty). The mapping $p_F : \mathbb{H} \to F$ defined in the theorem 5.1.2 is a bounded linear operator, such that $\|p_F\| = 1$ (if F is not reduced to $\{0\}$) and*

$$p_F^2 = p_F, \quad \ker p_F = F^\perp, \quad \operatorname{ran}(p_F) = F, \tag{5.1.16}$$

$$p_F \text{ is selfadjoint, i.e. } \quad \forall u, v \in \mathbb{H}, \quad \langle p_F u, v\rangle = \langle u, p_F v\rangle. \tag{5.1.17}$$

Proof. Let $u \in \mathbb{H}$. We have for $w \in F$

$$d(u, F)^2 \le \|u - p_F(u) + w\|^2 = \|u - p_F(u)\|^2 + \|w\|^2 + 2\operatorname{Re}\langle u - p_F(u), w\rangle$$

so that

$$\forall w \in F, \quad 0 \le 2\operatorname{Re}\langle u - p_F(u), w\rangle + \|w\|^2.$$

If $\langle u-p_F(u), w\rangle = \rho e^{i\theta}, \rho \ge 0, \theta \in \mathbb{R}$, we shall get for all $t \in \mathbb{R}$, $\langle u-p_F(u), twe^{i\theta}\rangle = t\rho$ and $twe^{i\theta} \in F$ so that

$$\forall t \in \mathbb{R}, \quad 0 \le 2\operatorname{Re}\langle u - p_F(u), twe^{i\theta}\rangle + \|twe^{i\theta}\|^2 = t\rho + t^2\|w\|^2 \Longrightarrow \rho = 0,$$

i.e. $\langle u - p_F(u), w\rangle = 0$ for all $w \in F$, giving

$$u - p_F(u) \in F^\perp \quad \text{and, with (5.1.14)}, \quad \|u\|^2 = \|u - p_F u\|^2 + \|p_F u\|^2. \tag{5.1.18}$$

Claim: If $u \in \mathbb{H}, v \in F$ are such that $u - v \in F^\perp$, then $v = p_F(u)$: (5.1.19)

we have indeed from (5.1.18) and (5.1.14),

$$\begin{aligned} 2\|v - p_F(u)\|^2 + 2\|u\|^2 &= \|v - u - p_F(u)\|^2 + \|v + u - p_F(u)\|^2 \\ &= \|v - u\|^2 + \|p_F u\|^2 + \|v\|^2 + \|u - p_F u\|^2 = \|v - u\|^2 + \|v\|^2 + \|u\|^2, \end{aligned}$$

so that $2\|v - p_F(u)\|^2 + 2\|u\|^2 = \|u\|^2 + \|u\|^2$ and $v = p_F(u)$, proving the claim. With this characterization of $p_F(u)$, we get immediately that p_F is linear since, for $u_1, u_2 \in \mathbb{H}, \lambda_1, \lambda_2 \in \mathbb{C}$,

$$\lambda_1 p_F(u_1) + \lambda_2 p_F(u_2) \in F, \quad \lambda_1 u_1 + \lambda_2 u_2 - \lambda_1 p_F(u_1) - \lambda_2 p_F(u_2) \in F^\perp \text{ (a vector space)},$$

so that (5.2.5) implies $\lambda_1 p_F(u_1) + \lambda_2 p_F(u_2) = p_F(\lambda_1 u_1 + \lambda_2 u_2)$. The identity (5.1.18) implies $\|p_F\| \le 1$ and if F is not reduced to zero, we have with $0 \ne v \in F, \|v\| = \|p_F v\|$, giving also $\|p_F\| = 1$. The first equality of (5.1.16) is already proven, while the second follows from (5.2.5), (5.1.18): if $u \in F^\perp, u - 0 \in F^\perp$ and $0=p_F(u)$, whereas if $p_F u = 0$, we have $u = u - p_F u \in F^\perp$. The third equality follows from $\operatorname{ran} p_F \subset F$ and from the fact that for $v \in F$, $p_F v = v$. To get (5.1.17), we note that for $u, v \in \mathbb{H}$, from (5.1.18), $\langle p_F u, v\rangle = \langle p_F u, p_F v\rangle = \langle u, p_F v\rangle$. The proof of the theorem 5.1.5 is complete. Let us now check (5.1.15). Let F be a closed subspace of $\mathbb{H}$; from (5.1.18), we have $F + F^\perp = \mathbb{H}$ since $u = u - p_F u + p_F u$ and moreover $F \cap F^\perp = \{0\}$ since $u \in F \cap F^\perp$ implies $u = p_F u = 0$. This completes as well the proof of the theorem 5.1.4. □

Proposition 5.1.6. *Let $\mathbb{H}$ be a Hilbert space and F be a subset of $\mathbb{H}$. Then $(F^\perp)^\perp$ is the closed linear span of F. If F is a closed subspace of $\mathbb{H}$, then $(F^\perp)^\perp = F$.*

Proof. We have always $F \subset (F^\perp)^\perp$ since for $u \in F, v \in F^\perp$, $\langle u, v\rangle = 0$. If F is a closed subspace of $\mathbb{H}$, then $F, F^\perp$ are both closed subspace of $\mathbb{H}$ and $(F^\perp)^\perp = \ker p_{F^\perp}$. Now we have $p_{F^\perp} = \mathrm{Id} - p_F$ since for $u \in \mathbb{H}$,

$$u - p_F u \in F^\perp, \quad u - (u - p_F u) \in F \subset (F^\perp)^\perp \Longrightarrow_{(5.2.5)} p_{F^\perp} u = u - p_F u.$$

As a result, if $u \in (F^\perp)^\perp$, $p_{F^\perp} u = 0$ and thus $u = p_F u \in F$. We assume now that F is a subset of $\mathbb{H}$; the closed linear span $\widetilde{F}$ of F is defined as

$$\widetilde{F} = \bigcap_{\substack{E \text{ closed subspace} \\ E \supset F}} E. \tag{5.1.20}$$

It is easy to verify that $\widetilde{F}$ is a closed subspace of $\mathbb{H}$ and that

$$\widetilde{F} = \text{closure}\{\sum_{1 \le k \le m} \lambda_k u_k, \lambda_k \in \mathbb{C}, u_k \in F\}. \tag{5.1.21}$$

Since $F \subset (F^\perp)^\perp$, we get that $\widetilde{F} \subset (F^\perp)^\perp$. On the other hand, we have $F \subset \widetilde{F}$ and thus $(\widetilde{F})^\perp \subset F^\perp$ so that, using the already proven part of the theorem, we get

$$(F^\perp)^\perp \subset \big((\widetilde{F})^\perp\big)^\perp = \widetilde{F} \subset (F^\perp)^\perp \Longrightarrow (F^\perp)^\perp = \widetilde{F}. \qquad \square$$

Remark 5.1.7. Let $\mathbb{H}$ be a Hilbert space and F be a subspace of $\mathbb{H}$. The subspace F is dense in $\mathbb{H}$ if and only if $F^\perp = \{0\}$: F is dense means that $\widetilde{F} = \overline{F} = \mathbb{H}$, which is equivalent (from (5.1.15)) to $(\widetilde{F})^\perp = \{0\}$. Now if $F^\perp = \{0\}$, we have $(\widetilde{F})^\perp \subset F^\perp = \{0\}$ and conversely if $(\widetilde{F})^\perp = \{0\}$, F is dense and for $u \in \mathbb{H}$ if $\forall v \in F, \langle u, v\rangle = 0 \Longrightarrow u \in \mathbb{H}^\perp = \{0\}$ so that $F^\perp = \{0\}$.

5.1.3 The Riesz representation theorem

Theorem 5.1.8. *[Riesz representation theorem[1]] Let $\mathbb{H}$ be a Hilbert space and $\xi \in \mathbb{H}^*$. Then there exists a unique $u \in \mathbb{H}$ such that $\forall v \in \mathbb{H}$, $\langle v, u\rangle = \xi(v)$. Moreover $\|\xi\|_{\mathbb{H}^*} = \|u\|_{\mathbb{H}}$.*

Proof. The uniqueness is obvious since for $u \in \mathbb{H}$, $\langle v, u\rangle = 0$ for all $v \in \mathbb{H}$ implies $u = 0$. Since ξ is a continuous linear form, $\ker \xi$ is a closed linear subspace and $\ker \xi \oplus (\ker \xi)^\perp = \mathbb{H}$. If $\xi \ne 0$, $(\ker \xi)^\perp$ is not reduced to $\{0\}$: let us take $u_0 \in (\ker \xi)^\perp$ such that $\xi(u_0) \ne 0$. We have for $v \in \mathbb{H}$,

$$\xi\big(v - \xi(v)\xi(u_0)^{-1} u_0\big) = 0 \Longrightarrow v - \xi(v)\xi(u_0)^{-1} u_0 \in \ker \xi \Longrightarrow \langle v - \xi(v)\xi(u_0)^{-1} u_0, u_0\rangle = 0,$$

so that $\xi(v) = \langle v, u_0\rangle \|u_0\|^{-2} \xi(u_0)$ and the result with $u = u_0 \|u_0\|^{-2} \xi(u_0)$. The norm of ξ is defined as $\|\xi\|_{\mathbb{H}^*} = \sup_{\|v\|=1} |\xi(v)| = \|u\|_{\mathbb{H}}$ (from (1.3.3)). This implies that

[1] Biographical details on Frigyes Riesz (1880-1956) can be found on the website http://www-history.mcs.st-and.ac.uk/history/Biographies/Riesz.html

there is an isometric (anti)linear[2] mapping κ from $\mathbb{H} \ni u \mapsto \kappa(u) \in \mathbb{H}^*$, given by $\kappa(u)(v) = \langle v, u\rangle$ which is also bijective; we have also proven κ is an isometric isomorphism identifying $\mathbb{H}$ with $\mathbb{H}^*$. Moreover, looking at the mapping $j : \mathbb{H} \to \mathbb{H}^{**}$ defined in the proposition 2.5.13, we consider $U_0 \in \mathbb{H}^{**}$, we have that $\overline{U_0 \circ \kappa} \in \mathbb{H}^*$, so that $\overline{U_0 \circ \kappa} = \kappa(u_0)$; but for $\xi \in \mathbb{H}^*, \exists w \in \mathbb{H}, \xi = \kappa(w)$,

$$j(u_0)(\xi) = \xi(u_0) = (\kappa(w))(u_0) = \langle u_0, w\rangle, \qquad (U_0 \circ \kappa)(w) = \overline{\kappa(u_0)(w)} = \langle u_0, w\rangle_{\mathbb{H}},$$

so that $j(u_0) = U_0$ and j is onto. □

We have also proven the following result.

Theorem 5.1.9. *Let $\mathbb{H}$ be a Hilbert space. Then the mapping $\kappa : \mathbb{H} \to \mathbb{H}^*$ defined by $\kappa(u)(v) = \langle v, u\rangle_{\mathbb{H}}$ is an isometric antilinear isomorphism and $\mathbb{H}$ is reflexive.*

5.1.4 Hilbert basis

Definition 5.1.10. *Let $\mathbb{H}$ be a Hilbert space.*
(1) *Let S be a subset of $\mathbb{H}$. The subset S is said to be an orthonormal subset of $\mathbb{H}$ if $\forall e \in S, \|e\|_{\mathbb{H}} = 1$ and for $e_1 \neq e_2 \in S$, $\langle e_1, e_2\rangle_{\mathbb{H}} = 0$.*
(2) *A Hilbert basis of $\mathbb{H}$ is a maximal orthornormal subset.*

Theorem 5.1.11. *Let $\mathbb{H}$ be a Hilbert space. If S is an orthonormal subset of $\mathbb{H}$, there exists a Hilbert basis containing S. In particular, in every Hilbert space there exists a Hilbert basis.*

Proof. The proof follows from a simple *Zornification* (see the lemma 2.2.2). Given an orthonormal subset S_0 of $\mathbb{H}$, we consider the set $\mathcal{X} = \{S \subset \mathbb{H}, S \supset S_0, S \text{ orthonormal}\}$, ordered by the inclusion. $\mathcal{X}$ is not empty (it contains S_0) and is inductive: if $(S_j)_{j\in J}$ is a totally ordered family in $\mathcal{X}$, we consider $S = \cup_{j\in J} S_j$, an obvious upper bound in $\mathcal{X}$ (note that for $e_1, e_2 \in S, e_k \in S_{j_k}$ but $S_{j_1} \subset S_{j_2}$ or $S_{j_2} \subset S_{j_1}$). □

Theorem 5.1.12 (Gram-Schmidt orthonormalization process)**.** *Let $\mathbb{H}$ be a Hilbert space and $\{u_k\}_{1\le k\le N}$ be a linearly independent subset of $\mathbb{H}$. Then there exists an orthonormal subset $\{e_k\}_{1\le k\le N}$ of $\mathbb{H}$ such that $\operatorname{Vect}\{e_k\}_{1\le k\le N} = \operatorname{Vect}\{u_k\}_{1\le k\le N}$ (it means that the vector spaces generated by the two families are the same).*

Proof. Obvious for $N = 1$: take $e_1 = u_1/\|u_1\|$. Induction: if $N \ge 1$ and $\{u_k\}_{1\le k\le N+1}$ is a linearly independent subset of $\mathbb{H}$, we consider the orthonormal subset $\{e_k\}_{1\le k\le N}$ obtained inductively such that $E = \operatorname{Vect}\{e_k\}_{1\le k\le N} = \operatorname{Vect}\{u_k\}_{1\le k\le N}$, and we define

$$e_{N+1} = \frac{u_{N+1} - \operatorname{pr}_E(u_{N+1})}{\|u_{N+1} - \operatorname{pr}_E(u_{N+1})\|},$$

which makes sense since $u_{N+1} \notin \operatorname{Vect}\{u_k\}_{1\le k\le N} = E$; moreover e_{N+1} is a unit vector orthogonal to E. □

[2]The mapping κ satisfies $\kappa(\lambda u) = \bar\lambda\kappa(u)$ for $\lambda \in \mathbb{C}, u \in \mathbb{H}$ and also $\kappa(u + v) = \kappa(u) + \kappa(v)$ for $u, v \in \mathbb{H}$.

Remark 5.1.13. Note that this is a constructive process since with $E = \mathrm{Vect}\{e_k\}_{1\le k\le N}$ when $\{e_k\}_{1\le k\le N}$ is an orthonormal family, we have

$$\mathrm{pr}_E(u) = \sum_{1\le j\le N} \langle u, e_k\rangle e_k. \tag{5.1.22}$$

In fact according to the theorem 5.1.5, since E is a closed subspace, writing $\mathbb{H} = E \oplus E^\perp$, pr_E is the (unique) linear map given by the identity on E and 0 on $E^\perp$, which is exactly the case for the mapping defined by (5.1.22): since $E^\perp = \{u \in \mathbb{H}, \forall k \in \{1, \dots, N\}, \langle u, e_k\rangle = 0\}$, we have indeed $\mathrm{pr}_E E^\perp = 0$ and also $\mathrm{pr}_E(e_k) = e_k$.

Theorem 5.1.14. *Let $\mathbb{H}$ be a Hilbert space and $\{e_k\}_{k\in\mathbb{N}^*}$ be an orthonormal subset of $\mathbb{H}$. Then Bessel's inequality holds:*

$$\forall u \in \mathbb{H}, \quad \sum_{k\ge 1} |\langle u, e_k\rangle|^2 \le \|u\|^2. \tag{5.1.23}$$

Moreover if $\mathbb{H}$ is separable and infinite-dimensional [3]*, there exists a countable Hilbert basis $\{e_k\}_{k\in\mathbb{N}^*}$ such that $\forall u \in \mathbb{H}$,*

$$u = \lim_n \sum_{1\le k\le n} \langle u, e_k\rangle e_k, \qquad \sum_{k\ge 1} |\langle u, e_k\rangle|^2 = \|u\|^2. \tag{5.1.24}$$

Proof. To prove (5.1.23), we may assume that $\{e_k\}_{1\le k\le N}$ is finite; from the remark 5.1.23, with $E = \mathrm{Vect}\,\{e_k\}_{1\le k\le N}$, the formulas (5.1.22) and (5.1.14) give

$$\|u\|^2 = \|\,\mathrm{pr}_E(u)\|^2 + \|u - \mathrm{pr}_E(u)\|^2 \ge \|\,\mathrm{pr}_E(u)\|^2 = \sum_{1\le k\le N} |\langle u, e_k\rangle|^2.$$

Let $\mathbb{H}$ be a separable infinite-dimensional Hilbert space and $\{u_k\}_{k\in\mathbb{N}^*}$ be a dense countable subset of $\mathbb{H}$. We define $E_n = \mathrm{Vect}\,\{u_k\}_{1\le k\le n}$ and we note that $\cup_{n\ge 1} E_n$ is a dense vector subspace of $\mathbb{H}$. The finite dimensional E_n has dimension d_n (the sequence $(d_n)_{n\ge 1}$ is non-decreasing with limit $+\infty$ since $\mathbb{H}$ is not finite-dimensional). We may assume that $d_1 = 1$. We claim that we can find a sequence $(v_j)_{j\ge 1}$ such that for each $n \ge 1$, $\mathrm{Vect}\{v_j\}_{1\le j\le d_n} = E_n$. Since we have assumed $d_1 = 1$, we define $v_1 = u_1$; inductively, assuming that $\mathrm{Vect}\{v_j\}_{1\le j\le d_n} = E_n$, we look at $d_n = d_{n+p_n-1} < d_n + 1 = d_{n+p_n}$ and

$$E_n = \cdots = E_{n+p_n-1} = \mathrm{Vect}\,\{v_j\}_{1\le j\le d_n}, \quad E_{n+p_n} = \mathrm{Vect}\{v_j\}_{1\le j\le d_{n+p_n}}, v_{d_n+1} = u_{n+p_n}.$$

Using the theorem 5.1.12, we can find an orthonormal subset $\{e_j\}_{j\ge 1}$, such that $F = \mathrm{Vect}\,\{e_j\}_{j\ge 1}$ is dense in $\mathbb{H}$ so that from the remark 5.1.7, $F^\perp = \{0\}$ and $\mathrm{pr}_{\bar F} = \mathrm{Id}$. Let $u \in \mathbb{H}$: we consider the sequence $w_n = \sum_{1\le j\le n} \langle u, e_j\rangle e_j$. We have for $n \le m$

$$w_m - w_n = \sum_{n<j\le m} \langle u, e_j\rangle e_j, \quad \|w_m - w_n\|^2 = \sum_{n<j\le m} |\langle u, e_j\rangle|^2$$

[3] If $\mathbb{H}$ is finite dimensional, it is isomorphic to $\mathbb{C}^N$ with the standard scalar product.

and since the series $\sum_j |\langle u, e_j\rangle|^2$ is converging from the already proven (5.1.23), we get that (w_n) is a Cauchy sequence, thus a converging one with limit w. Now for each $k \geq 1$,

$$\langle u - w, e_k\rangle = \lim_n \langle u - w_n, e_k\rangle = \langle u, e_k\rangle - \langle u, e_k\rangle = 0,$$

so that $u - w \in F^\perp$ and thus $u = w$, i.e. $u = \lim_n \sum_{1\leq j\leq n} \langle u, e_j\rangle e_j$, and taking the norms of both sides, we get (5.1.24). □

Corollary 5.1.15. *All separable infinite dimensional Hilbert spaces are isomorphic.*

Proof. Let $\mathbb{H}$ be a separable Hilbert space. According to the previous theorem, we can find on $\mathbb{H}$ a countable Hilbert basis $(e_j)_{j\geq 1}$. Let us now consider now the linear mapping Φ

$$\mathbb{H} \ni u \mapsto (\langle u, e_k\rangle)_{k\in\mathbb{N}^*} \in \ell^2(\mathbb{N}).$$

This mapping is obviously one-to-one and also onto: if $(x_k)_{k\in\mathbb{N}^*} \in \ell^2(\mathbb{N})$, the sequence $(\sum_{1\leq k\leq n} x_k e_k)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathbb{H}$ with limit u (same proof as above) and $\langle u, e_k\rangle = x_k$. Moreover, Φ is isometric as well as its inverse so that

$$4\operatorname{Re}\langle \Phi u, \Phi v\rangle_{\ell^2(\mathbb{N}^*)} = \|\Phi(u+v)\|^2_{\ell^2(\mathbb{N}^*)} - \|\Phi(u-v)\|^2_{\ell^2(\mathbb{N}^*)} = \|u+v\|^2_{\mathbb{H}} - \|u-v\|^2_{\mathbb{H}} = 4\operatorname{Re}\langle u, v\rangle_{\mathbb{H}}$$

and using also (5.1.2), we get $\langle \Phi u, \Phi v\rangle_{\ell^2(\mathbb{N}^*)} = \langle u, v\rangle_{\mathbb{H}}$. □

Remark 5.1.16. Except for the finite dimensional case, a Hilbert basis of a Hilbert space is never an algebraic basis (also called Hamel basis). A Hamel basis of a vector space is a linearly independent and generating family $(e_j)_{j\in J}$. It is also a maximal linearly independent family; using Zorn's lemma, it can be proven that every vector space has a Hamel basis. If $(e_j)_{j\in J}$ is a Hamel basis of a vector space E, every vector $u \in E$ can be written is a unique way as a *finite* linear combination of the e_j. For instance, looking at $\ell^2(\mathbb{N})$ with the Hilbert basis $(e_j)_{j\in\mathbb{N}}$ defined by $e_j = (\delta_{j,k})_{k\in\mathbb{N}}$, it is clear that $(e_j)_{j\in\mathbb{N}}$ is not a Hamel basis: for instance, $u = (\frac{1}{1+k})_{k\geq 0}$ belongs to $\ell^2(\mathbb{N})$ and is not a finite linear combination of the e_j.

Remark 5.1.17. Let J be an uncountable set. We define $\ell^2(J)$ as the set of mappings x from J to $\mathbb{C}$, $x = (x_j)_{j\in J}$, such that

$$N(x) = \sup_{L \text{ finite } \subset J} \sum_{j\in L} |x_j|^2 < +\infty.$$

It is possible to prove that $\ell^2(J)$ equipped with the norm N is actually a Hilbert space which is nonseparable since J is uncountable.

5.2 Bounded operators on a Hilbert space

Let $\mathbb{H}_1, \mathbb{H}_2$ be Hilbert spaces; the first properties of the Banach space $\mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$ are given in the proposition 2.1.5. If $\mathbb{H}_1 = \mathbb{H}_2$, we shall use the notation $\mathcal{L}(\mathbb{H}_1)$ for that space.

Definition 5.2.1. *Let $\mathbb{H}$ be a Hilbert space and $\mathcal{L}(\mathbb{H})$ the Banach algebra*[4] *of the bounded linear maps from $\mathbb{H}$ to $\mathbb{H}$. For $A \in \mathcal{L}(\mathbb{H})$, we define the adjoint of A, denoted by A^* as the unique operator in $\mathcal{L}(\mathbb{H})$ such that*

$$\forall u, v \in \mathbb{H}, \quad \langle A^* u, v\rangle_{\mathbb{H}} = \langle u, Av\rangle_{\mathbb{H}}. \tag{5.2.1}$$

Remark 5.2.2. For $A \in \mathcal{L}(\mathbb{H}), u_0 \in \mathbb{H}$, the mapping $\mathbb{H} \ni v \mapsto \langle Av, u_0\rangle \in \mathbb{C}$ is linear continuous ($|\langle Av, u_0\rangle| \le \|Av\|\|u_0\| \le \|A\|\|v\|\|u_0\|$) and thus an element of $\mathbb{H}^*$. From the Riesz representation theorem 5.1.8,

$$\forall u_0 \in \mathbb{H}, \exists! w(u_0) \in \mathbb{H}, \forall v \in \mathbb{H}, \quad \langle Av, u_0\rangle = \langle v, w(u_0)\rangle.$$

The uniqueness of w implies that it depends linearly on u_0: take $u_0, u_1 \in \mathbb{H}, \lambda_0, \lambda_1 \in \mathbb{C}$, then

$$\begin{aligned}\forall v \in \mathbb{H}, \quad \langle Av, \lambda_0 w(u_0) + \lambda_1 w(u_1)\rangle &= \overline{\lambda_0}\langle Av, u_0\rangle + \overline{\lambda_1}\langle Av, u_1\rangle \\ &= \langle Av, \lambda_0 u_0 + \lambda_1 u_1\rangle = \langle v, w(\lambda_0 u_0 + \lambda_1 u_1)\rangle, \end{aligned} \tag{5.2.2}$$

so that $\lambda_0 w(u_0) + \lambda_1 w(u_1) - w(\lambda_0 u_0 + \lambda_1 u_1) \in \mathbb{H}^\perp = \{0\}$. Moreover we have from (1.3.4),

$$\|w(u_0)\| = \sup_{\|v\|=1} |\langle v, w(u_0)\rangle| = \sup_{\|v\|=1} |\langle Av, u_0\rangle| \le \|A\|\|u_0\|.$$

We define then A^* by $A^* u = w(u)$ and we have proven $A^* \in \mathcal{L}(\mathbb{H})$ as well as (5.2.1). Moreover if (5.2.1) is satisfied, this implies $A^* u = w(u)$. As a result, the previous definition is consistent.

N.B. There is of course a close relationship between ${}^tA$, the transposed operator of A, as given by the definition 2.5.22, and its adjoint. For a Hilbert space $\mathbb{H}$, we have the following characterization of the transposed operator

$$\forall \eta \in \mathbb{H}^*, \forall x \in \mathbb{H}, \quad \langle\!\langle {}^tA\eta, x\rangle\!\rangle_{\mathbb{H}^*,\mathbb{H}} = \langle\!\langle \eta, Ax\rangle\!\rangle_{\mathbb{H}^*,\mathbb{H}},$$

where the brackets here are brackets of duality that we have denoted by $\langle\!\langle, \rangle\!\rangle_{\mathbb{H}^*,\mathbb{H}}$. Using the isometric antilinear map κ of the theorem 5.1.11, we get that this is equivalent to require

$$\forall y \in \mathbb{H}, \forall x \in \mathbb{H}, \quad \langle\!\langle {}^tA\kappa(y), x\rangle\!\rangle_{\mathbb{H}^*,\mathbb{H}} = \langle\!\langle \kappa(y), Ax\rangle\!\rangle_{\mathbb{H}^*,\mathbb{H}}$$

and since[5] $\langle\!\langle \kappa(y), Ax\rangle\!\rangle_{\mathbb{H}^*,\mathbb{H}} = \langle Ax, y\rangle_{\mathbb{H}} = \langle x, A^*y\rangle_{\mathbb{H}} = \langle\!\langle \kappa(A^*y), x\rangle\!\rangle_{\mathbb{H}^*,\mathbb{H}}$ we find that

$${}^tA\kappa = \kappa A^*, \quad \text{i.e.} \quad A^* = \kappa^{-1}({}^tA)\kappa, \quad {}^tA = \kappa A^* \kappa^{-1}. \tag{5.2.3}$$

Since κ, κ^{-1} are isometric, we get from (2.5.20), (5.2.3) that

$$\|A\|_{\mathcal{L}(\mathbb{H})} = \|A^*\|_{\mathcal{L}(\mathbb{H})}, \tag{5.2.4}$$

although we have in the sequel more informations on this topic and also a simpler proof in the Hilbertian case.

[4]We have seen in the proposition 2.1.5 that $\mathcal{L}(\mathbb{H})$ is a Banach space; a Banach algebra is a Banach space which is also an associative algebra and such that the multiplication (here the composition of maps) satisfy $\|AB\| \le \|A\|\|B\|$.

[5]Here $\langle, \rangle_{\mathbb{H}}$ is the scalar product on $\mathbb{H}$.

Proposition 5.2.3. *Let $\mathbb{H}$ be a Hilbert space and $A, B \in \mathcal{L}(\mathbb{H}), \lambda, \mu \in \mathbb{C}$. Then we have*

$$(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*, \quad (AB)^* = B^* A^*, \quad (A^*)^* = A. \tag{5.2.5}$$

If $A \in \mathcal{L}(\mathbb{H})$ is invertible[6] *with inverse A^{-1}, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$. For $A \in \mathcal{L}(\mathbb{H})$, we have*

$$\|A\|_{\mathcal{L}(\mathbb{H})} = \|A^*\|_{\mathcal{L}(\mathbb{H})} = \|A^* A\|_{\mathcal{L}(\mathbb{H})}^{1/2} \tag{5.2.6}$$

Proof. The properties (5.2.5) are trivial consequences of (5.2.1). For the next property, we see from (5.2.5) that $(A^{-1})^* A^* = (AA^{-1})^* = \mathrm{Id}^* = \mathrm{Id} = (A^{-1}A)^* = A^*(A^{-1})^*$. The first equality in (5.2.6) follows from (5.2.4), but can be proven directly with

$$\|A\|_{\mathcal{L}(\mathbb{H})} = \sup_{\|u\|_{\mathbb{H}}=1} \|Au\|_{\mathbb{H}} = \sup_{\|u\|_{\mathbb{H}}=1=\|v\|_{\mathbb{H}}} |\langle Au, v\rangle_{\mathbb{H}}| = \sup_{\|u\|_{\mathbb{H}}=1=\|v\|_{\mathbb{H}}} |\langle u, A^* v\rangle_{\mathbb{H}}| = \|A^*\|_{\mathcal{L}(\mathbb{H})},$$

and we have also

$$\begin{aligned}\|A\|_{\mathcal{L}(\mathbb{H})}^2 = \sup_{\|u\|_{\mathbb{H}}=1} |\langle Au, Au\rangle_{\mathbb{H}}| &= \sup_{\|u\|_{\mathbb{H}}=1} |\langle A^* Au, u\rangle_{\mathbb{H}}| \\ &\le \|A^* A\|_{\mathcal{L}(\mathbb{H})} \le \|A^*\|_{\mathcal{L}(\mathbb{H})} \|A\|_{\mathcal{L}(\mathbb{H})} = \|A\|_{\mathcal{L}(\mathbb{H})}^2, \quad \text{proving (5.2.6).} \quad \square\end{aligned}$$

Definition 5.2.4. *Let $\mathbb{H}$ be a Hilbert space and $A \in \mathcal{L}(\mathbb{H})$. The operator A is said to be selfadjoint (resp. normal) if $A = A^*$ (resp. $A^* A = AA^*$).*

Proposition 5.2.5. *Let $\mathbb{H}$ be a complex*[7] *Hilbert space and $A \in \mathcal{L}(\mathbb{H})$. The operator A is selfadjoint if and only if $\forall u \in \mathbb{H}, \ \langle Au, u\rangle \in \mathbb{R}$.*

Proof. If A is selfadjoint we have $\langle Au, u\rangle = \langle u, Au\rangle = \overline{\langle Au, u\rangle}$ and thus $\langle Au, u\rangle \in \mathbb{R}$. Conversely, if $\langle Au, u\rangle \in \mathbb{R}$ for all $u \in \mathbb{H}$, since

$$\langle A(u + iv), u + iv\rangle = \langle Au, u\rangle + \langle Av, v\rangle - i\langle Au, v\rangle + i\langle Av, u\rangle,$$

we have $\operatorname{Im}\big(-i\langle Au, v\rangle + i\langle Av, u\rangle\big) = 0$ so that

$$\operatorname{Re}(\langle Av, u\rangle) = \operatorname{Re}(\langle Au, v\rangle).$$

Changing u in iu, we get $\operatorname{Re}(-i\langle Av, u\rangle) = \operatorname{Re}(i\langle Au, v\rangle)$, which is $\operatorname{Im}(\langle Av, u\rangle) = -\operatorname{Im}(\langle Au, v\rangle)$, so that

$$\forall u, v \in \mathbb{H}, \quad \langle u, Av\rangle = \overline{\langle Av, u\rangle} = \langle Au, v\rangle \Longrightarrow A^* = A. \qquad \square$$

Proposition 5.2.6. *Let $\mathbb{H}$ be a Hilbert space and A be a selfadjoint bounded operator. Then $\|A\| = \sup_{\|u\|=1} |\langle Au, u\rangle|$.*

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[6] It means that there exists $A' \in \mathcal{L}(\mathbb{H})$ such that $AA' = A'A = \mathrm{Id}$; in that case A' is uniquely determined, since $AA'' = \mathrm{Id}$ implies $A'' = A'AA'' = A'$. We denote the inverse by A^{-1}. The open mapping theorem (theorem 2.1.10) shows that if $A \in \mathcal{L}(\mathbb{H})$ is only bijective, it is invertible.

[7] It was already said on page 127 that we dealt with complex Hilbert spaces, but we emphasize this here since the result is not true for a real Hilbert space (exercise).

Proof. We have $T = \sup_{\|u\|=1} |\langle Au, u\rangle| \le \|A\|$ and also

$$\langle A(u+v), u+v\rangle - \langle A(u-v), u-v\rangle = 2\langle Au, v\rangle + 2\langle Av, u\rangle \\ = 2\langle Au, v\rangle + 2\langle v, Au\rangle = 4\operatorname{Re}\langle Au, v\rangle,$$

so that for $\|u\| = \|v\| = 1$,

$$4|\langle Au, v\rangle| \le T(\|u+v\|^2 + \|u-v\|^2) = T(2\|u\|^2 + 2\|v\|^2) = 4T,$$

which gives $\|A\| = \sup_{\|u\|=\|v\|=1} |\langle Au, v\rangle| \le T$ and the result. □

Theorem 5.2.7. *Let $\mathbb{H}$ be a Hilbert space and $A \in \mathcal{L}(\mathbb{H})$. Then*

$$\ker A = (\operatorname{ran} A^*)^\perp. \tag{5.2.7}$$

Proof. $u \in \ker A$ means $Au = 0$, which is equivalent to $\forall v \in \mathbb{H}, \langle Au, v\rangle = 0$, i.e. $\forall v \in \mathbb{H}, \langle u, A^*v\rangle = 0$, i.e. $u \in (\operatorname{ran} A^*)^\perp$. □

Remark 5.2.8. The property above implies $\ker A^* = (\operatorname{ran} A)^\perp$, and from the proposition 5.1.6,

$$(\ker A)^\perp = \overline{\operatorname{ran} A^*}. \tag{5.2.8}$$

5.3 The Fourier transform on $L^2(\mathbb{R}^n)$

5.3.1 Plancherel formula

Theorem 5.3.1. *The Fourier transformation can be extended into a unitary operator of $L^2(\mathbb{R}^n)$, i.e. there exists a unique linear operator $F : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, such that for $u \in \mathscr{S}(\mathbb{R}^n)$, $Fu = \hat{u}$ and we have $F^*F = FF^* = \operatorname{Id}_{L^2(\mathbb{R}^n)}$. Moreover*

$$F^* = CF = FC, \quad F^2C = \operatorname{Id}_{L^2(\mathbb{R}^n)}. \tag{5.3.1}$$

where C is the involutive isomorphism of $L^2(\mathbb{R}^n)$ defined by $(Cu)(x) = u(-x)$. This gives the Plancherel formula: for $u, v \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi = \int u(x)\overline{v(x)}dx. \tag{5.3.2}$$

Proof. For the test functions $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$, using the Fubini theorem and (4.1.4), we get[8]

$$(\hat{\psi}, \hat{\varphi})_{L^2(\mathbb{R}^n)} = \int \hat{\psi}(\xi)\overline{\hat{\varphi}(\xi)}d\xi = \iint \hat{\psi}(\xi)e^{2i\pi x\cdot\xi}\overline{\varphi(x)}dxd\xi = (\psi, \varphi)_{L^2(\mathbb{R}^n)}.$$

Next, the density of $\mathscr{S}$ in L^2 shows that there is a unique continuous extension F of the Fourier transform to L^2 and that extension is an isometric operator (i.e.

[8]We have to pay attention to the fact that the scalar product $(u, v)_{L^2}$ in the complex Hilbert space $L^2(\mathbb{R}^n)$ is linear with respect to u and antilinear with respect to v: for $\lambda, \mu \in \mathbb{C}, (\lambda u, \mu v)_{L^2} = \lambda\bar{\mu}(u, v)_{L^2}$.

satisfying for all $u \in L^2(\mathbb{R}^n)$, $\|Fu\|_{L^2} = \|u\|_{L^2}$, i.e. $F^*F = \mathrm{Id}_{L^2}$). We note that the operator C defined by $Cu = \check{u}$ is an involutive isomorphism of $L^2(\mathbb{R}^n)$ and that for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$CF^2u = u = FCFu = F^2Cu.$$

By the density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, the bounded operators $CF^2, \mathrm{Id}_{L^2(\mathbb{R}^n)}, FCF, F^2C$ are all equal. On the other hand for $u, \varphi \in \mathscr{S}(\mathbb{R}^n)$

$$(F^*u, \varphi)_{L^2} = (u, F\varphi)_{L^2} = \int u(x)\overline{\hat{\varphi}(x)}dx = \iint u(x)\bar{\varphi}(\xi)e^{2i\pi x\cdot\xi}dxd\xi = (CFu, \varphi)_{L^2},$$

so that $F^*u = CFu$ for all $u \in \mathscr{S}$ and by continuity $F^* = CF$ as bounded operators on $L^2(\mathbb{R}^n)$, thus $FF^* = FCF = \mathrm{Id}$. The proof is complete. □

5.3.2 Convolution of L^2 functions

Let $u, v \in L^2(\mathbb{R}^n)$. We consider $\int u(y)v(x-y)dy = \omega(u,v)(x)$, which makes sense since $\int |u(y)v(x-y)|dy \le \|u\|_{L^2}\|v\|_{L^2} < +\infty$, so that $\omega(u,v) \in L^\infty(\mathbb{R}^n)$. Moreover $\omega(u,v) \in C^0(\mathbb{R}^n)$ since, with $(\tau_h w)(x) = w(x-h)$, we have

$$\omega(u,v)(x+h) - \omega(u,v)(x) = \int u(y)\big((\tau_{-h}v)(x-y) - v(x-y)\big)dy,$$

and thus

$$|\omega(u,v)(x+h) - \omega(u,v)(x)| \le \|u\|_{L^2(\mathbb{R}^n)}\|\tau_{-h}v - v\|_{L^2(\mathbb{R}^n)},$$

and since[9] $\lim_{h\to 0}\|\tau_h v - v\|_{L^2(\mathbb{R}^n)} = 0$, we get the uniform continuity of $\omega(u,v)$. The reader may check the chapter 6 in [9] to see that $\omega(u,v)$ is the convolution of u with v and that $\omega(u,v) = \omega(v,u)$ by a change of variables. However, we have to pay attention to the fact that we have given earlier in the section 3.5 another definition of the convolution when $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$, and we have to verify that these definitions coincide when $u \in L^2_{\mathrm{comp}}(\mathbb{R}^n), v \in L^2(\mathbb{R}^n)$. In fact, for $u, v \in L^2(\mathbb{R}^n), \varphi \in C^0_c(\mathbb{R}^n)$ we have from the Fubini theorem

$$\int \omega(u,v)(x)\varphi(x)dx = \iint u(x)v(y)\varphi(x+y)dxdy, \tag{5.3.3}$$

since with $w(x) = \int |v(y)||\varphi(x+y)|dy = \omega(|\varphi|, |\check{v}|)(x)$, we have[10]

$$\|\omega(|\varphi|, |\check{v}|)\|_{L^2} \le \|v\|_{L^2}\|\varphi\|_{L^1},$$

[9] For $v \in L^2(\mathbb{R}^n), \varphi \in C^0_c(\mathbb{R}^n)$, $\tau_h v - v = \tau_h(v-\varphi) + \tau_h(\varphi) - \varphi + \varphi - v$, and thus

$$\|\tau_h v - v\|_{L^2} \le 2\|v-\varphi\|_{L^2} + \|\tau_h(\varphi) - \varphi\|_{L^2} \Longrightarrow \limsup_{h\to 0}\|\tau_h v - v\|_{L^2} \le 2\|v-\varphi\|_{L^2},$$

and since $C^0_c(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ this implies $\lim_{h\to 0}\|\tau_h v - v\|_{L^2} = 0$.

[10] This follows from Young's inequality (see e.g. the Théorème 6.2.1 in [9]) but there is a simpler argument: for $w_1 \in L^1, w_2 \in L^2$, then $w_1 * w_2 \in L^2$ with $\|w_1 * w_2\|_{L^2} \le \|w_1\|_{L^1}\|w_2\|_{L^2}$: we have

$$\int \left|\int w_1(y)w_2(x-y)dy\right|^2 dx \le \int \||w_1|^{1/2}\|^2_{L^2}\int |w_1(y)||w_2(x-y)|^2dydx = \|w_1\|^2_{L^1}\|w_2\|^2_{L^2}.$$

$$\iint |u(x)||v(y)||\varphi(x+y)|dxdy \le \|u\|_{L^2}\|w\|_{L^2} \le \|u\|_{L^2}\|v\|_{L^2}\|\varphi\|_{L^1} < +\infty,$$

and (5.3.3) gives $\omega(u,v) = u * v$, where the convolution is taken in the distribution sense. We have proven the first part of the following lemma.

Lemma 5.3.2.
(1) *The mapping* $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \ni (u,v) \mapsto u * v \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *as defined above is symmetric and*

$$\|u * v\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} \tag{5.3.4}$$

and coincides with the convolution in the distribution sense when u *(or* v*) is compactly supported.*
(2) *For* $u, v \in L^2(\mathbb{R}^n)$*, we have* $\widehat{u * v} = \hat{u}\hat{v}$.

N.B. The formula (2) was proven for $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$ in (4.3.2); here, we know that both sides of the equality makes sense, since $u*v \in L^\infty(\mathbb{R}^n)$ and thus is a tempered distribution whose Fourier transform has a meaning. On the other hand, $\hat{u}\hat{v}$ is a product of L^2 functions and thus is a L^1 function.

Proof. We shall see that an approximation argument, the continuity property expressed by the inequality (5.3.4) and (4.3.2) will imply the result. For $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have with $\chi \in C_c^\infty(\mathbb{R}^n)$, equal to 1 near 0 and $\chi_k(x) = \chi(x/k)$,

$$\langle \widehat{u * v}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle u * v, \hat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}} = \int (u * v)(x)\hat{\varphi}(x)dx = \lim_{k\to+\infty} \int (\chi_k u * v)(x)\hat{\varphi}(x)dx,$$

since $\chi_k u$ tends to u in $L^2(\mathbb{R}^n)$ and thus

$$\int |\big((\chi_k u - u) * v\big)(x)\hat{\varphi}(x)|dx \le \int |\hat{\varphi}(x)|dx \|\chi_k u - u\|_{L^2}\|v\|_{L^2}.$$

On the other hand, using (4.3.2), we get, since $\chi_k u, v \in L^2(\mathbb{R}^n)$,

$$\int (\chi_k u * v)(x)\hat{\varphi}(x)dx = \langle \widehat{\chi_k u * v}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle \widehat{\chi_k u}\hat{v}, \varphi \rangle_{\mathscr{S}',\mathscr{S}}$$

$$= \int (F\chi_k u)(x)(Fv)(x)\varphi(x)dx = \langle F(\chi_k u), \overline{\varphi F v} \rangle_{L^2} \underset{k\to+\infty}{\longrightarrow} \langle Fu, \overline{\varphi F v} \rangle_{L^2},$$

a limit which is equal to $\int (Fu)(x)(Fv)(x)\varphi(x)dx$. This completes the proof of (2) in the lemma. □

5.4 Sobolev spaces

5.4.1 Definitions, Injections

For $\xi \in \mathbb{R}^n$, we define

$$\langle \xi \rangle = \sqrt{1 + |\xi|^2}. \tag{5.4.1}$$

It is easy to see that this function as well as all functions $\xi \mapsto \langle\xi\rangle^s$ when $s \in \mathbb{R}$ are elements of the space of multipliers $\mathscr{O}_M$ as given by the definition 4.3.2. In particular, it means that for $u \in \mathscr{S}'(\mathbb{R}^n)$, the product $\langle\xi\rangle^s \hat{u}(\xi)$ makes sense and belongs to $\mathscr{S}'(\mathbb{R}^n)$.

Definition 5.4.1. *Let $s \in \mathbb{R}$. We define the Sobolev space $H^s(\mathbb{R}^n)$ as*

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n), \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}. \tag{5.4.2}$$

Proposition 5.4.2. *Let $s \in \mathbb{R}$. The space $H^s(\mathbb{R}^n)$ equipped with the scalar product*

$$\langle u, v\rangle_{H^s(\mathbb{R}^n)} = \int \langle\xi\rangle^{2s} \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi = \langle \hat{u}(\xi)\langle\xi\rangle^s, \hat{v}(\xi)\langle\xi\rangle^s\rangle_{L^2(\mathbb{R}^n)}, \tag{5.4.3}$$

is a Hilbert space. The space $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Proof. It is obvious that $\langle u, v\rangle_{H^s(\mathbb{R}^n)}$ is a sesquilinear Hermitian and positive-definite form: note in particular that $0 = \langle u, u\rangle_{H^s(\mathbb{R}^n)} = \|\hat{u}(\xi)\langle\xi\rangle^s\|^2_{L^2(\mathbb{R}^n)}$ implies $\hat{u}(\xi)\langle\xi\rangle^s = 0$ in $L^2(\mathbb{R}^n)$ and thus in $\mathscr{S}'(\mathbb{R}^n)$, so that we can muliply that identity by the multiplier $\langle\xi\rangle^{-s}$, get $\hat{u} = 0$ and thus $u = 0$. On the other hand, if $(u_k)_{k\geq 1}$ is a Cauchy sequence in $H^s(\mathbb{R}^n)$, the sequence $(v_k)_{k\geq 1}$, $v_k(\xi) = \hat{u}_k(\xi)\langle\xi\rangle^s$ converges in $L^2(\mathbb{R}^n)$. Let $v \in L^2$ be its limit; the tempered distribution w defined by the product $w(\xi) = \langle\xi\rangle^{-s} v(\xi)$ is such that $u = \check{\hat{w}} \in H^s(\mathbb{R}^n)$ since $\langle\xi\rangle^s w(\xi) \in L^2$: we have

$$\|u_k - u\|_{H^s} = \|\langle\xi\rangle^s \widehat{u_k}(\xi) - \langle\xi\rangle^s w(\xi)\|_{L^2} = \|v_k - v\|_{L^2} \longrightarrow 0,$$

and the result that H^s is complete. Next we see that, since $\xi \mapsto \langle\xi\rangle^s \hat{u}(\xi)$ is in $\mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, when $u \in \mathscr{S}(\mathbb{R}^n)$, each $H^s(\mathbb{R}^n)$ contains $\mathscr{S}(\mathbb{R}^n)$. To prove the density of $\mathscr{S}(\mathbb{R}^n)$, we note that if $u \in (\mathscr{S}(\mathbb{R}^n))^{\perp_s}$, i.e.

$$u \in H^s(\mathbb{R}^n), \forall\varphi \in \mathscr{S}(\mathbb{R}^n), \quad \int \langle\xi\rangle^{2s} \hat{u}(\xi)\overline{\hat{\varphi}(\xi)} d\xi = 0,$$

this[11] implies $\forall\psi \in \mathscr{S}(\mathbb{R}^n)$, $\langle \hat{u}, \psi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = 0$, i.e. $\hat{u} = 0$ as a tempered distribution, thus $u = 0$. □

Theorem 5.4.3. . *Let $s_1 \leq s_2$ be real numbers. Then $H^{s_2}(\mathbb{R}^n) \subset H^{s_1}(\mathbb{R}^n)$ with a continuous injection: for $u \in H^{s_2}(\mathbb{R}^n)$ we have*

$$\|u\|_{H^{s_1}(\mathbb{R}^n)} \leq \|u\|_{H^{s_2}(\mathbb{R}^n)}. \tag{5.4.4}$$

For a multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, the operator ∂_x^α is continuous from $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$.

Proof. The inequality (5.4.4) holds true for $u \in \mathscr{S}(\mathbb{R}^n)$. Now if $u \in H^{s_2}$, $u = \lim_k u_k$ in H^{s_2} with $u_k \in \mathscr{S}(\mathbb{R}^n)$; from (5.4.4) on $\mathscr{S}(\mathbb{R}^n)$, we see that (u_k) is a Cauchy sequence in H^{s_1}, thus converges to $v \in H^{s_1}$. Now the convergence in H^s implies the weak-dual convergence in $\mathscr{S}'(\mathbb{R}^n)$, since for $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $\exists\psi \in \mathscr{S}(\mathbb{R}^n)$ with

$$\langle u_k, \varphi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle \widehat{u_k}, \check{\hat{\varphi}}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle\langle\xi\rangle^s \widehat{u_k}(\xi), \underbrace{\langle\xi\rangle^{-s}\overline{\check{\hat{\varphi}}(\xi)}}_{\hat{\psi}(\xi)\langle\xi\rangle^s}\rangle_{L^2} = \langle u_k, \psi\rangle_{H^s}.$$

[11] The mapping $\chi \mapsto \tilde{\chi}$ given by $\tilde{\chi}(\xi) = \langle\xi\rangle^s \chi(\xi)$ is an isomorphism of $\mathscr{S}(\mathbb{R}^n)$.

As a result, the sequence (u_k) converges in the weak-dual topology on $\mathscr{S}'(\mathbb{R}^n)$ with limit u (convergence in H^{s_2}) and limit v (convergence in H^{s_1}), thus $u = v$ and the injection property. The inequality (5.4.4) follows from its version with $u \in \mathscr{S}(\mathbb{R}^n)$ and the density, and it implies the continuity. The last property follows from (4.1.7), the density of $\mathscr{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ and the inequality for $m \geq 0$, $|\xi|^m \langle\xi\rangle^{s-m} \leq \langle\xi\rangle^s$. $\square$

5.4.2 Identification of $(H^s)^*$ with H^{-s}

Let $s \in \mathbb{R}$. We consider now the following pairing

$$\begin{array}{ccc} H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) & \longrightarrow & \mathbb{C} \\ (u,v) & \mapsto & \langle \langle\xi\rangle^s \hat{u}(\xi), \langle\xi\rangle^{-s}\hat{v}(\xi)\rangle_{L^2(\mathbb{R}^n)} = T(u,v) \end{array} \tag{5.4.5}$$

so that

$$|T(u,v)| \leq \|u\|_{H^s}\|v\|_{H^{-s}}. \tag{5.4.6}$$

We see that it gives a mapping

$$\Phi : H^{-s}(\mathbb{R}^n) \longrightarrow (H^s(\mathbb{R}^n))^* \tag{5.4.7}$$

defined by

$$\langle \Phi(v), u\rangle_{(H^s)^*,H^s} = T(u,v), \quad \text{with} \quad \|\Phi(v)\|_{(H^s)^*} = \sup_{\|u\|_{H^s}=1} |T(u,v)| = \|v\|_{H^{-s}},$$

since the inequality $\sup_{\|u\|_{H^s}=1} |T(u,v)| \leq \|v\|_{H^{-s}}$ follows from (5.4.6) and, for $v \neq 0$, taking u such that $\hat{u}(\xi) = \langle\xi\rangle^{-2s}\hat{v}(\xi)\|v\|_{H^{-s}}^{-1}$, we see that $u \in H^s$ with $\|u\|_{H^s} = 1$ so that $T(u,v) = \|v\|_{H^{-s}}$, providing the equality. The mapping Φ is isometric (thus injective) and to prove that it is an isometric isomorphism, using the open mapping theorem 2.1.10, it is enough to prove that Φ is onto. Let us take $L_0 \in (H^s)^*$: according to the Riesz representation theorem 5.1.8, there exists $u_0 \in H^s$ such that

$$\langle L_0, u\rangle_{(H^s)^*,H^s} = \langle u, u_0\rangle_{H^s} = \langle \langle\xi\rangle^s\hat{u}(\xi), \langle\xi\rangle^s\widehat{u_0}(\xi)\rangle_{L^2} = \langle \langle\xi\rangle^s\hat{u}(\xi), \langle\xi\rangle^{-s} \underbrace{\langle\xi\rangle^{2s}\widehat{u_0}(\xi)}_{\widehat{v_0}(\xi)}\rangle_{L^2},$$

with $v_0 \in H^{-s}$ since $\langle\xi\rangle^{-s}\widehat{v_0}(\xi) = \langle\xi\rangle^s\widehat{u_0}(\xi) \in L^2$, and this gives

$$\langle L_0, u\rangle_{(H^s)^*,H^s} = T(u, v_0) = \Phi(v_0),$$

and the surjectivity of Φ_0. We have proven the following theorem

Theorem 5.4.4. *The pairing (5.4.5) gives a canonical isometric isomorphism Φ (5.4.7) from $H^{-s}(\mathbb{R}^n)$ onto the dual of $H^s(\mathbb{R}^n)$.*

5.4.3 Continuous functions and Sobolev spaces

Theorem 5.4.5. *Let $m \in \mathbb{N}$. Then*

$$H^m(\mathbb{R}^n) = \{u \in \mathscr{D}'(\mathbb{R}^n), \forall\alpha \in \mathbb{N}^n \text{ such that } |\alpha| \leq m, \quad \partial_x^\alpha u \in L^2(\mathbb{R}^n)\}. \tag{5.4.8}$$

Moreover, $H^m(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm

$$\Big(\sum_{|\alpha|\leq m} \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^n)}^2\Big)^{1/2}. \tag{5.4.9}$$

Proof. Taking $u \in H^m(\mathbb{R}^n)$ in the sense of the definition 5.4.1, we get that $u \in \mathscr{S}'(\mathbb{R}^n)$, $\langle\xi\rangle^m \hat{u}(\xi) \in L^2(\mathbb{R}^n)$ and as a consequence $\hat{u} \in L^2_{\text{loc}}$, $\widehat{D_x^\alpha u} = \xi^\alpha \hat{u}(\xi)$ belongs to $L^2(\mathbb{R}^n)$ if $|\alpha| \le m$ since

$$\int |\xi^\alpha \hat{u}(\xi)|^2 d\xi \le \int \langle\xi\rangle^{2m} |\hat{u}(\xi)|^2 d\xi < +\infty.$$

Conversely, if u satisfies (5.4.8), u belongs to $L^2(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, and $\xi^\alpha \hat{u}(\xi)$ is in $L^2(\mathbb{R}^n)$ for $|\alpha| \le m$. We have also from Hölder's inequality

$$\langle\xi\rangle^{2m} = (1 + \sum_{1\le j\le n} \xi_j^2)^m \le \big(1 + \sum_{1\le j\le n} \xi_j^{2m}\big)(n+1)^{m-1}, \tag{5.4.10}$$

so that $\int \langle\xi\rangle^{2m} |\hat{u}(\xi)|^2 d\xi \le \big(\|u\|^2_{L^2(\mathbb{R}^n)} + \sum_{1\le j\le n} \|D_j^m u\|^2_{L^2(\mathbb{R}^n)}\big)(n+1)^{m-1} < +\infty$. We have thus proven the first statement of the theorem and also that the Hilbertian norms of $H^m(\mathbb{R}^n)$ and (5.4.9) are equivalent. We have already seen in the proposition 5.4.2 that $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$, with a continuous injection since for $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$\|\varphi\|^2_{H^s} = \int \langle\xi\rangle^{2s+n+1} |\hat{\varphi}(\xi)|^2 \langle\xi\rangle^{-n-1} d\xi \le C(n) p_s(\varphi), \tag{5.4.11}$$

where p_s is a semi-norm on $\mathscr{S}(\mathbb{R}^n)$.

Lemma 5.4.6. *$C_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$.*

Proof of the lemma. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $\chi \in C_c^\infty(\mathbb{R}^n; [0,1])$ equal to 1 on the unit ball of $\mathbb{R}^n$, the sequence of functions $\varphi_k \in C_c^\infty(\mathbb{R}^n)$ defined by $\varphi_k(x) = \chi(x/k)\varphi(x)$ has limit φ in $\mathscr{S}(\mathbb{R}^n)$: we calculate with the standard Leibniz formula

$$\frac{1}{\alpha!}(\partial_x^\alpha \varphi_k)(x) = \sum_{\beta+\gamma=\alpha} \frac{1}{\beta!\gamma!} k^{-|\beta|} (\partial_x^\beta \chi)(x/k)(\partial_x^\gamma \varphi)(x)$$

so that

$$|x^\lambda \big(\partial_x^\alpha(\varphi_k - \varphi)\big)(x)| \le |x^\lambda \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\ge 1}} \frac{\alpha!}{\beta!\gamma!} k^{-|\beta|} (\partial_x^\beta \chi)(x/k)(\partial_x^\gamma \varphi)(x)| + |x^\lambda \underbrace{\big(\chi(x/k) - 1\big)}_{\substack{|x|\ge k \\ \text{on its support}}} (\partial_x^\alpha \varphi)(x)|$$

and

$$\sup_{x\in\mathbb{R}^n} |x^\lambda \big(\partial_x^\alpha(\varphi_k - \varphi)\big)(x)| \le k^{-1} p(\varphi) C(\chi, \alpha) + \frac{2}{k+1} \sup_{|x|\ge k} |(1+|x|) x^\lambda (\partial_x^\alpha \varphi)(x)|,$$

proving that the sequence (φ_k) converges to φ in $\mathscr{S}(\mathbb{R}^n)$ and the lemma. □

The inequality (5.4.11) and the lemma give the density of $C_c^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$: for $\epsilon > 0$ and $u \in H^s$, there exists $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\|u - \varphi\|_{H^s} < \epsilon/2$ and for that φ there exists $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $p_s(\varphi - \psi) < \frac{\epsilon}{2C(n)+1}$, implying $\|\varphi - \psi\|_{H^s} < \epsilon/2$ and then $\|u - \psi\|_{H^s} < \epsilon$. □

If $f \in \mathscr{O}_M(\mathbb{R}^n)$ (see the definition 4.3.2), we define the operator, called a *Fourier multiplier*, $f(D)$ on $\mathscr{S}'(\mathbb{R}^n)$ by $\widehat{f(D)u} = f(\xi)\hat{u}(\xi)$ and we note that $f(D)$ is an endomorphism of $\mathscr{S}(\mathbb{R}^n)$. The notation is consistent with the fact that for a polynomial P on $\mathbb{R}^n$, the differential operator $P(D)$ is indeed the Fourier multiplier $P(D)$.

Lemma 5.4.7. *Let $s, t \in \mathbb{R}$. Then the Fourier multiplier $\langle D\rangle^s$ is an isomorphism from $H^{s+t}(\mathbb{R}^n)$ onto $H^t(\mathbb{R}^n)$ whose inverse is $\langle D\rangle^{-s}$. If $f \in \mathscr{O}_M$ is bounded, then $f(D)$ is an endomorphism of $H^s(\mathbb{R}^n)$. If $m \in \mathbb{N}$, $H^{-m}(\mathbb{R}^n)$ is the set of linear combinations of derivatives of order $\le m$ of functions of $L^2(\mathbb{R}^n)$.*

Proof. We assume first $t = 0$; we have indeed for $u \in H^s$, $\|u\|_{H^s} = \|\langle D\rangle^s u\|_{L^2}$, and for $u \in L^2$, $\|u\|_{L^2} = \|\langle D\rangle^{-s}u\|_{H^s}$, with $\langle D\rangle^s\langle D\rangle^{-s} = \langle D\rangle^{-s}\langle D\rangle^s = \mathrm{Id}_{\mathscr{S}'(\mathbb{R}^n)}$. If $t \ne 0$, we use the identity $\langle D\rangle^s = \langle D\rangle^{-t}\langle D\rangle^{s+t}$, (valid on $\mathscr{S}'(\mathbb{R}^n)$), so that

$$H^{s+t} \underset{\approx}{\xrightarrow{\langle D\rangle^{s+t}}} H^0 \underset{\approx}{\xrightarrow{\langle D\rangle^{-t}}} H^t.$$

Now if $f \in \mathscr{O}_M$ is bounded, $f(D)$ is bounded on H^0 and the identity $f(D) = \langle D\rangle^{-s} f(D)\langle D\rangle^s$ (valid on $\mathscr{S}'(\mathbb{R}^n)$) proves the boundedness on H^s. For the second part, we consider for a multi-index α with $|\alpha| \le m$, the Fourier multiplier D^α is bounded from L^2 into H^{-m} from the theorem 5.4.3. With $\chi_j(\xi) = \xi_j\langle\xi\rangle^{-1}$, the Fourier multiplier

$$(1 + \sum_{1\le j\le n} \chi_j(D)D_j)^m$$

is an isomorphism from H^0 onto H^{-m}. This implies that for $u \in H^{-m}$,$\exists v \in L^2$ such that

$$u = (1 + \sum_{1\le j\le n} \chi_j(D)D_j)^m v = \sum_{|\alpha|\le m} D^\alpha \psi_\alpha(D)v$$

with each $\psi_\alpha(D)$ bounded on L^2 as a product of $\chi_j(D)$. □

Theorem 5.4.8. *Let $s > n/2$. Then $H^s(\mathbb{R}^n) \subset C^0_{(0)}(\mathbb{R}^n)$ with continuous injection (see the lemma 4.3.5 for the definition of that space).*

Proof. For $u \in H^s(\mathbb{R}^n)$, we have $\hat{u} \in L^2(\mathbb{R}^n)$ and $\hat{u}(\xi) = \langle\xi\rangle^{-s}\langle\xi\rangle^s\hat{u}(\xi)$ with $\langle\xi\rangle^{-s} \in L^2(\mathbb{R}^n), \langle\xi\rangle^s\hat{u}(\xi) \in L^2(\mathbb{R}^n)$ so that $\hat{u} \in L^1(\mathbb{R}^n)$ and we can apply the lemma 4.3.5. The injection is continuous since (4.1.14) applied to the L^1 function $\hat{u}$ gives

$$\|u\|_{L^\infty} \le \|\hat{u}\|_{L^1} \le \left(\int \langle\xi\rangle^{-2s}d\xi\right)^{1/2}\left(\int \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2 d\xi\right)^{1/2} = c(s,n)\|u\|_{H^s}. \quad (5.4.12)$$

□

5.5 The Littlewood-Paley decomposition

Let $\varphi_0 \in C_c^\infty(\mathbb{R}^n)$, $1 \ge \varphi_0(\xi) \ge 0$ such that

$$\varphi_0(\xi) = 1 \quad \text{if } |\xi| \le 1 \text{ and } \varphi_0(\xi) = 0 \text{ if } |\xi| \ge 2,\ \varphi_0 \text{ radial decreasing of } |\xi|.$$

We set
$$\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi).$$
The function φ is supported in the ring $1/2 \le |\xi| \le 2$: if $|\xi| \ge 2$, $\varphi(\xi) = 0$ and if $|\xi| \le 1/2$, $\varphi_0(\xi) = 1 = \varphi_0(2\xi)$ so that $\varphi(\xi) = 0$. We have also $0 \le \varphi(\xi) \le 1$. We define, for a positive integer ν, φ_ν to be
$$\varphi_\nu(\xi) = \varphi(\frac{\xi}{2^\nu})$$
which is supported in the ring $\{2^{\nu-1} \le |\xi| \le 2^{\nu+1}\}$. We have then
$$\varphi_\nu(\xi)\varphi_\mu(\xi) = 0 \quad \text{if } |\nu - \mu| \ge 2.$$
We set, for $\nu \in \mathbb{N}$,
$$S_\nu(\xi) = \sum_{0\le\mu\le\nu} \varphi_\mu(\xi).$$
and we have
$$S_\nu(\xi) = \varphi_0(\xi) + \sum_{1\le\mu\le\nu} \varphi_0(\frac{\xi}{2^\mu}) - \varphi_0(\frac{\xi}{2^{\mu-1}}),$$
so that
$$S_\nu(\xi) = \varphi_0(\frac{\xi}{2^\nu}) = 1 \quad \text{if } |\xi| \le 2^\nu \text{ and } 0 \text{ if } |\xi| \ge 2^{\nu+1}.$$
Consequently, we obtain
$$1 = \sum_{\mu=0}^{+\infty} \varphi_\mu(\xi).$$
Moreover, we get (with $\varphi_{-1} \equiv 0$)
$$1 = \sum_{\mu,\nu} \varphi_\mu(\xi)\varphi_\nu(\xi) = \sum_{\mu\ge0} \varphi_\mu\varphi_{\mu-1} + \varphi_\mu^2 + \varphi_\mu\varphi_{\mu+1}$$
and thus
$$\frac{1}{3} \le \sum_{\mu=0}^{+\infty} \varphi_\mu(\xi)^2 \le 1,$$
the last inequality follows from $0 \le \varphi_\mu(\xi) \le 1$. We'll use that $\varphi_\nu(D_x)$ is the convolution with $\hat{\varphi}(2^\nu x)2^{\nu n}$.

Theorem 5.5.1. *Let $s \in \mathbb{R}$. Then there exists $C_s > c_s > 0$ such that*
$$\forall u \in H^s(\mathbb{R}^n), \quad c_s\|u\|_{H^s}^2 \le \sum_{\mu=0}^{+\infty} \|\varphi_\mu(D_x)u\|_{L^2(\mathbb{R}^n)}^2 2^{2\mu s} \le C_s\|u\|_{H^s}^2.$$
Let $\rho \in (0,1)$. We define the space
$$C^\rho(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n), \sup_{x'\ne x''} \frac{|u(x') - u(x'')|}{|x' - x''|^\rho} < +\infty\}, \tag{5.5.1}$$
$$\|u\|_{C^\rho(\mathbb{R}^n)} = \|u\|_{L^\infty(\mathbb{R}^n)} + \sup_{x'\ne x''} \frac{|u(x') - u(x'')|}{|x' - x''|^\rho}. \tag{5.5.2}$$

For $\rho \in (0,1)$, $C^\rho(\mathbb{R}^n)$ equipped with the above norm is a Banach space; moreover, there exists $C > c > 0$ such that

$$\forall u \in C^\rho(\mathbb{R}^n), \quad c\|u\|_{C^\rho(\mathbb{R}^n)} \le \sup_{\mu \ge 0} \|\varphi_\mu(D_x)u\|_{L^\infty(\mathbb{R}^n)} 2^{\mu\rho} \le C\|u\|_{C^\rho(\mathbb{R}^n)}.$$

Proof. Defining the Besov space $B^s_{p,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}, p, q \ge 1$ by

$$B^s_{p,q}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n), \quad \bigl(2^{\nu s}\|\varphi_\nu(D)u\|_{L^p(\mathbb{R}^n)}\bigr)_{\nu \ge 0} \in \ell^q(\mathbb{N})\}, \tag{5.5.3}$$

the theorem is stating that

$$\forall s \in \mathbb{R}, B^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n), \quad \forall \rho \in (0,1), B^\rho_{\infty,\infty}(\mathbb{R}^n) = C^\rho(\mathbb{R}^n).$$

The first statement is quite obvious since for $\xi \in \operatorname{supp} \varphi_0$, we have $1 \le \langle \xi \rangle \le 5^{1/2}$, and for

$$\xi \in \operatorname{supp} \varphi_\nu, \quad \nu \ge 1, \quad 2^{-1} \le 2^{-\nu}(1 + 2^{2\nu-2})^{1/2} \le \frac{\langle \xi \rangle}{2^\nu} \le 2^{-\nu}(1 + 2^{2\nu+2})^{1/2} \le 5^{1/2},$$

so that

$$\frac{1}{2^{3|s|}3}\langle \xi \rangle^{2s} \le 2^{-3|s|} \sum_{\nu \ge 0} \langle \xi \rangle^{2s} \varphi_\nu(\xi)^2 \le \sum_{\nu \ge 0} 2^{2\nu s} \varphi_\nu(\xi)^2 \le 2^{3|s|} \sum_{\nu \ge 0} \langle \xi \rangle^{2s} \varphi_\nu(\xi)^2 \le 2^{3|s|} \langle \xi \rangle^{2s}.$$

Let us now assume that $u \in C^\rho(\mathbb{R}^n)$, i.e. u is a continuous bounded function on $\mathbb{R}^n$ such that $\|u\|_{\Lambda^\rho} < +\infty$. Then, with $\|u\|_{C^\rho} = \|u\|_{L^\infty} + \|u\|_{\Lambda^\rho}$, we have

$$\|\varphi_\nu(D)u\|_{L^\infty(\mathbb{R}^n)} = \|\widehat{\varphi}(2^\nu \cdot)2^{\nu n} * u\|_{L^\infty(\mathbb{R}^n)} \le 2^{-\nu\rho}\|u\|_{C^\rho} C(\varphi_0),$$

since it is obvious for $\nu = 0$ and for $\nu \ge 1$, since $\varphi(0) = 0$ (thus $\int \hat{\varphi} = 0$), we have

$$(\widehat{\varphi}(2^\nu \cdot)2^{\nu n} * u)(x) = \int \widehat{\varphi}(2^\nu y)2^{\nu n}\bigl(u(x-y) - u(x)\bigr)dy,$$

which implies $\|\varphi_\nu(D)u\|_{L^\infty(\mathbb{R}^n)} \le \int |\hat{\varphi}(2^\nu y)|2^{\nu n}\|u\|_{\Lambda^\rho}|y|^\rho dy = C(\varphi_0)\|u\|_{\Lambda^\rho}2^{-\nu\rho}$. Conversely if $u \in B^\rho_{\infty,\infty}$, then $u = \sum_{\nu \ge 0} \varphi_\nu(D)u$ and

$$\|u\|_{L^\infty} \le \sum_{\nu \ge 0} \|\varphi_\nu(D)u\|_{L^\infty} \le \sum_{\nu \ge 0} 2^{-\nu\rho}\|u\|_{B^\rho_{\infty,\infty}},$$

so that $u \in L^\infty$. Moreover for $x, h \in \mathbb{R}^n$, we have

$$|u(x+h) - u(x)| \le \underbrace{\sum_{\substack{\nu \\ |h| \le 2^{-\nu}}} |(\varphi_\nu(D)u)(x+h) - (\varphi_\nu(D)u)(x)|}_{=A(h)} + 2\underbrace{\sum_{\substack{\nu \\ |h| > 2^{-\nu}}} 2^{-\nu\rho}}_{\le C|h|^\rho} \|u\|_{B^\rho_{\infty,\infty}}.$$

On the other hand, with $\psi \in C^\infty_c(\mathbb{R}^n), \psi = 1$ on the support of φ, $\psi = 0$ near 0, so that with $\nu \ge 1$, $\varphi_\nu(\xi) = \varphi_\nu(\xi)\psi_\nu(\xi)$ with $\psi_\nu(\xi) = \psi(\xi 2^{-\nu})$, $\psi_0 \in C^\infty_c(\mathbb{R}^n)$, $\psi_0 = 1$ on the support of φ_0, we have

$$\begin{aligned} A(h) &\le \sum_{\substack{\nu \\ |h| \le 2^{-\nu}}} 2\pi|h|\|D\varphi_\nu(D)\psi_\nu(D)u\|_{L^\infty} \le 2\pi|h| \sum_{\substack{\nu \\ |h| \le 2^{-\nu}}} 2^\nu\|2^{-\nu}D\psi_\nu(D)\varphi_\nu(D)u\|_{L^\infty} \\ &\le 2\pi|h| \sum_{\substack{1 \le \nu \\ |h| \le 2^{-\nu}}} 2^\nu\|\varphi_\nu(D)u\|_{L^\infty} \le 2\pi|h| \sum_{\substack{1 \le \nu \\ |h| \le 2^{-\nu}}} 2^{\nu(1-\rho)}\|u\|_{B^\rho_{\infty,\infty}} \\ &\le C\|u\|_{B^\rho_{\infty,\infty}}|h|\bigl(|h|^{-1}\bigr)^{1-\rho}, \end{aligned}$$

so that $|u(x+h) - u(x)| \le C'|h|^\rho\|u\|_{B^\rho_{\infty,\infty}}$ and the sought result $u \in C^\rho$. $\square$

Theorem 5.5.2. *The space $B^1_{\infty,\infty}(\mathbb{R}^n)$ given by (5.5.3) has the following characterization: $u \in B^1_{\infty,\infty}(\mathbb{R}^n)$ if and only if $u \in L^\infty(\mathbb{R}^n)$ and*

$$|||u|||_1 = \sup_{x\in\mathbb{R}^n, 0\neq h\in\mathbb{R}^n} |u(x+h)+u(x-h)-2u(x)||h|^{-1} < +\infty. \tag{5.5.4}$$

There exists $C > c > 0$ such that, $\forall u \in B^1_{\infty,\infty}(\mathbb{R}^n)$,

$$c\|u\|_{B^1_{\infty,\infty}(\mathbb{R}^n)} \le \|u\|_{L^\infty(\mathbb{R}^n)} + |||u|||_1 \le C\|u\|_{B^1_{\infty,\infty}(\mathbb{R}^n)}. \tag{5.5.5}$$

Moreover, if $u \in B^1_{\infty,\infty}(\mathbb{R}^n)$, $\exists C > 0$ such that

$$\forall x \in \mathbb{R}^n, \forall h \in \mathbb{R}^n, \quad |u(x+h)-u(x)| \le C|h|\bigl(1+\ln(|h|^{-1})\bigr). \tag{5.5.6}$$

We define $\operatorname{Lip}(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n), \nabla u \in L^\infty(\mathbb{R}^n)\}$; this is a Banach space for the norm $\|u\|_{L^\infty(\mathbb{R}^n)} + \|\nabla u\|_{L^\infty(\mathbb{R}^n)}$. The inclusion $\operatorname{Lip}(\mathbb{R}^n) \subset B^1_{\infty,\infty}$ is continuous and strict.

Proof. Let us consider $u \in L^\infty(\mathbb{R}^n)$ such that $|||u|||_1 < +\infty$. Then we have

$$\|\varphi_\nu(D)u\|_{L^\infty(\mathbb{R}^n)} = \|\widehat{\varphi}(2^\nu\cdot)2^{\nu n} * u\|_{L^\infty(\mathbb{R}^n)} \le 2^{-\nu}(\|u\|_{L^\infty} + |||u|||_1)C(\varphi_0),$$

since it is obvious for $\nu = 0$ and for $\nu \ge 1$, since $\varphi(0) = 0$ (thus $\int \hat\varphi = 0$), we have, using that φ is even,

$$2(\widehat{\varphi}(2^\nu\cdot)2^{\nu n} * u)(x) = \int \widehat{\varphi}(2^\nu y)2^{\nu n}\bigl(u(x-y)+u(x+y)-2u(x)\bigr)dy,$$

which implies $2\|\varphi_\nu(D)u\|_{L^\infty(\mathbb{R}^n)} \le \int |\hat\varphi(2^\nu y)|2^{\nu n}|||u|||_1|y|dy = 2C(\varphi_0)|||u|||_1 2^{-\nu}$, and the first inequality in (5.5.5). Conversely if $u \in B^1_{\infty,\infty}$, then $u = \sum_{\nu\ge 0}\varphi_\nu(D)u$ and

$$\|u\|_{L^\infty} \le \sum_{\nu\ge 0}\|\varphi_\nu(D)u\|_{L^\infty} \le \sum_{\nu\ge 0} 2^{-\nu}\|u\|_{B^1_{\infty,\infty}} = 2\|u\|_{B^1_{\infty,\infty}},$$

so that $u \in L^\infty$. Moreover for $x, h \in \mathbb{R}^n$, we have

$$|u(x+h)+u(x-h)-2u(x)| \le \underbrace{\sum_{\substack{\nu \\ |h|\le 2^{-\nu}}} |(\varphi_\nu(D)u)(x+h)+(\varphi_\nu(D)u)(x-h)-2(\varphi_\nu(D)u)(x)|}_{=A(h)} + 4\underbrace{\sum_{\substack{\nu \\ |h|>2^{-\nu}}} 2^{-\nu}}_{\le C|h|}\|u\|_{B^1_{\infty,\infty}}.$$

We set $v_\nu(x) = (\varphi_\nu(D))u(x)$ and we note that v_ν is a C^∞ function; we have

$$v_\nu(x+h) = v_\nu(x) + v_\nu'(x)h + \int_0^1 (1-\theta)v_\nu''(x+\theta h)d\theta h^2$$

and thus $v_\nu(x+h)+v_\nu(x-h)-2v_\nu(x) = \int_{-1}^1(1-|\theta|)v_\nu''(x+\theta h)d\theta h^2$. As a result, we have

$$A(h) \le |h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\le 2^{-\nu}}} \|D^2\varphi_\nu(D)u\|_{L^\infty}.$$

We consider $\psi \in C_c^\infty(\mathbb{R}^n), \psi = 1$ on the support of φ, $\psi = 0$ near 0, and ψ even, so that with $\nu \geq 1$, $\varphi_\nu(\xi) = \varphi_\nu(\xi)\psi_\nu(\xi)$ with $\psi_\nu(\xi) = \psi(\xi 2^{-\nu})$ and $\psi_0 \in C_c^\infty(\mathbb{R}^n)$, $\psi_0 = 1$ on the support of φ_0. We have

$$\begin{aligned} A(h) \leq |h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} \|D^2\varphi_\nu(D)u\|_{L^\infty} &= |h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} \|D^2\varphi_\nu(D)\psi_\nu(D)u\|_{L^\infty} \\ &= |h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} 2^{2\nu}\|2^{-2\nu}D^2\psi_\nu(D)\varphi_\nu(D)u\|_{L^\infty} \\ &\leq C|h|^2 4\pi^2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} 2^{2\nu}\|\varphi_\nu(D)u\|_{L^\infty} \\ &\leq C|h|^2 4\pi^2 \|u\|_{B^1_{\infty,\infty}} \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} 2^{\nu} \leq C_1|h|^2\|u\|_{B^1_{\infty,\infty}}|h|^{-1}, \end{aligned}$$

so that $|u(x+h)+u(x-h)-2u(x)| \leq C'|h|\|u\|_{B^1_{\infty,\infty}}$ and the second inequality in (5.5.5). Let us consider now $u \in B^1_{\infty,\infty}$. Moreover for $x, h \in \mathbb{R}^n$, with $h \neq 0$, we have

$$|u(x+h)-u(x)| \leq \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} |(\varphi_\nu(D)u)(x+h) - (\varphi_\nu(D)u)(x)| + 2\underbrace{\sum_{\substack{\nu \\ |h|>2^{-\nu}}} 2^{-\nu}}_{\leq C|h|}\|u\|_{B^1_{\infty,\infty}}.$$

With the same ψ as above, we have

$$\begin{aligned} |u(x+h)-u(x)| &\leq |h|C_1\|u\|_{B^1_{\infty,\infty}} + \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} |h|2\pi\|D\psi_\nu(D)\varphi_\nu(D)u\|_{L^\infty} \\ &\leq |h|C_1\|u\|_{B^1_{\infty,\infty}} + \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} |h|2\pi 2^{\nu}\|2^{-\nu}D\psi_\nu(D)\varphi_\nu(D)u\|_{L^\infty} \\ &\leq |h|C_1\|u\|_{B^1_{\infty,\infty}} + |h|C_2 \sum_{\substack{\nu \\ |h|\leq 2^{-\nu}}} 2^{\nu}\|\varphi_\nu(D)u\|_{L^\infty} \\ &\leq |h|C_1\|u\|_{B^1_{\infty,\infty}} + |h|C_2\|u\|_{B^1_{\infty,\infty}} \underbrace{\operatorname{Card}\{\nu \in \mathbb{N}, 2^\nu \leq |h|^{-1}\}}_{\leq \log_2(|h|^{-1})}, \end{aligned}$$

which gives (5.5.6). We consider now $u \in \operatorname{Lip}(\mathbb{R}^n)$. We have $\|\varphi_0(D)u\|_{L^\infty} \leq C\|u\|_{L^\infty}$ and for $\nu \geq 1$,

$$(\varphi_\nu(D)u)(x) = (\widehat{\varphi}(2^\nu \cdot)2^{\nu n} * u)(x) = \int \widehat{\varphi}(2^\nu y)2^{\nu n}\big(u(x-y)-u(x)\big)dy.$$

We have also in the distribution sense

$$u(x-y)-u(x) = \int_0^1 u'(x-\theta y)d\theta y \Longrightarrow |u(x-y)-u(x)| \leq \|u'\|_{L^\infty}|y|,$$

so that $\|\varphi_\nu(D)u\|_{L^\infty} \leq \int |\varphi(2^\nu y)|2^{\nu n}|y|dy\|u'\|_{L^\infty} \leq C\|u'\|_{L^\infty}2^{-\nu}$, proving the continuous inclusion $\operatorname{Lip}(\mathbb{R}^n) \subset B^1_{\infty,\infty}(\mathbb{R}^n)$. Let us prove finally that this inclusion is strict: we consider

$$T(x) = \int_1^{+\infty} e^{2i\pi x\xi}\xi^{-2}d\xi.$$

The Fourier transform of T belongs to $L^1(\mathbb{R})$ and thus T is a continuous bounded function. We have also

$$(\varphi_\nu(D)T)(x) = \int_1^{+\infty} e^{2i\pi x\xi}\xi^{-2}\varphi_\nu(\xi)d\xi.$$

and for $\nu \geq 1$,

$$(\varphi_\nu(D)T)(x) = \int_1^{+\infty} e^{2i\pi x\xi}\xi^{-2}\varphi(2^{-\nu}\xi)d\xi = 2^{-2\nu}\int_{2^{-\nu}}^{+\infty} e^{2i\pi x 2^\nu \xi}\xi^{-2}\varphi(\xi)d\xi 2^\nu.$$

Since the function φ is (non-negative and) supported in $1/2 \leq |\xi| \leq 2$, we get for $\nu \geq 1$ that

$$2^\nu(\varphi_\nu(D)T)(x) = \int_{1/2}^{2} e^{2i\pi x 2^\nu\xi}\xi^{-2}\varphi(\xi)d\xi \Longrightarrow \|2^\nu\varphi_\nu(D)T\|_{L^\infty(\mathbb{R})} \leq \int_{1/2}^{2}\xi^{-2}\varphi(\xi)d\xi < +\infty.$$

On the other hand $(\varphi_0(D)T)(x) = \int_1^{+\infty} e^{2i\pi x\xi}\xi^{-2}\varphi_0(\xi)d\xi$ is a bounded function ; we have proven that $T \in B^1_{\infty,\infty}(\mathbb{R})$. Let us prove that T is not in $\operatorname{Lip}(\mathbb{R}^n)$. We calculate for $\epsilon > 0$,

$$\langle T', \epsilon^{-1}e^{-\pi\epsilon^{-2}x^2}\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = 2i\pi\langle\xi\hat{T}, e^{-\pi\epsilon^2\xi^2}\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = 2i\pi\int_1^{+\infty}\xi^{-1}e^{-\pi\epsilon^2\xi^2}d\xi \underset{\epsilon\to 0_+}{\longrightarrow} +\infty,$$

say from the Fatou theorem, and if T' were a bounded function, we would have

$$|\langle T', \epsilon^{-1}e^{-\pi\epsilon^{-2}x^2}\rangle| \leq \|T'\|_{L^\infty(\mathbb{R})}\|\epsilon^{-1}e^{-\pi\epsilon^{-2}x^2}\|_{L^1(\mathbb{R})} = \|T'\|_{L^\infty(\mathbb{R})} < +\infty.$$

The proof of the theorem is complete. □

Chapter 6

Fourier Analysis, continued

6.1 Paley – Wiener's theorem

Lemma 6.1.1. *For $u \in \mathscr{S}'(\mathbb{R}^n)$ the following properties are equivalent.*

(*i*) $u \in C_c^\infty(\mathbb{R}^n), \quad \operatorname{supp} u \subset \{x \in \mathbb{R}^n, |x| \le R\}$.

(*ii*) *$\hat{u}$ can be extended to $\mathbb{C}^n$ as an entire function such that*

$$\forall N \in \mathbb{N}, \exists C_N > 0, \quad |\hat{u}(\zeta)| \le C_N(1+|\zeta|)^{-N} e^{2\pi R|\operatorname{Im}\zeta|}. \tag{6.1.1}$$

Proof. Let us assume (i). Using the notation $\mathbb{C}^n \ni \zeta = \xi + i\eta, \xi, \eta \in \mathbb{R}^n$, the Fourier transform of u can be extended to $\mathbb{C}^n$ as an entire function, simply with the formula

$$\hat{u}(\xi + i\eta) = \int e^{-2i\pi x\cdot(\xi+i\eta)} u(x)dx \qquad (\text{note} \quad x\cdot(\xi+i\eta) = x\cdot\xi + ix\cdot\eta).$$

As a result, for a polynomial P on $\mathbb{R}^n$, we have $(\widehat{P(D)u})(\zeta) = P(\zeta)\hat{u}(\zeta)$ and thus

$$|P(\zeta)\hat{u}(\zeta)| \le \|P(D)u\|_{L^1(\mathbb{R}^n)} e^{2\pi R|\operatorname{Im}\zeta|},$$

implying for all multi-indices $\alpha \in \mathbb{N}^n$, $|\zeta^\alpha \hat{u}(\zeta)| \le \|D^\alpha u\|_{L^1(\mathbb{R}^n)} e^{2\pi R|\operatorname{Im}\zeta|}$, i.e.

$$|\zeta_1|^{\alpha_1} \dots |\zeta_n|^{\alpha_n} |\hat{u}(\zeta)| \le \|D^\alpha u\|_{L^1(\mathbb{R}^n)} e^{2\pi R|\operatorname{Im}\zeta|}.$$

As a consequence, for $m \in 2\mathbb{N}$, we have with $\|u\|_{W^{m,1}} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^1(\mathbb{R}^n)}$,

$$(1+|\zeta|^2)^{m/2} |\hat{u}(\zeta)| \le C_m \|u\|_{W^{m,1}} e^{2\pi R|\operatorname{Im}\zeta|} \Longrightarrow (ii).$$

Conversely, if (ii) holds, the function $\hat{u}$ is C^∞ on $\mathbb{R}^n$ and for all $N \in \mathbb{N}$, $|\hat{u}(\xi)| \le C_N \langle\xi\rangle^{-N}$. Thus $\hat{u} \in L^1(\mathbb{R}^n)$ and one can apply the theorem 4.1.10, so that $u(x) = \int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} \hat{u}(\xi)d\xi$. Now we have also for all $\eta \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} \hat{u}(\xi)d\xi = \int_{\mathbb{R}^n} e^{2i\pi x\cdot(\xi+i\eta)} \hat{u}(\xi+i\eta)d\xi,$$

where both sides make sense thanks to the estimate (6.1.1), which also allow to shift integration of the entire function $\zeta \mapsto \hat{u}(\zeta)e^{2i\pi x\cdot\zeta}$ from $\mathbb{R}^n$ to $\mathbb{R}^n + i\eta$. Now if $|x| > R$, we obtain for all $\eta \in \mathbb{R}^n$,

$$|u(x)| \le C_N e^{2\pi(R|\eta| - x\cdot\eta)} \int_{\mathbb{R}^n} (1+|\xi|)^{-N} d\xi$$

and in particular choosing $\eta = \lambda x/|x|$, $N = n+1$, we get for all $\lambda > 0$, $|u(x)| \le C'_n e^{2\pi(R\lambda - \lambda|x|)}$, so that for $|x| > R$ we obtain $u(x) = 0$ and (i). □

Lemma 6.1.2. *Let Ω be an open set of $\mathbb{R}^n$, $x_0 \in \Omega$ and $u \in \mathscr{D}'(\Omega)$. The following properties are equivalent.*

(i) $x_0 \notin \operatorname{singsupp} u$,

(ii) $\exists V_0 \in \mathscr{V}_{x_0}$ *such that for all* $\chi \in C_c^\infty(V_0)$, *for all* $N \in \mathbb{N}$, $\exists C$ *such that*

$$|\widehat{\chi u}(\xi)| \le C(1+|\xi|)^{-N}.$$

(iii) $\exists V_0 \in \mathscr{V}_{x_0}$, $\exists \chi_0 \in C_c^\infty(V_0)$, *such that* $\chi_0(x_0) \ne 0$, *for all* $N \in \mathbb{N}$, $\exists C$ *such that*

$$|\widehat{\chi_0 u}(\xi)| \le C(1+|\xi|)^{-N}.$$

Proof. If (i) holds, $\exists V_0 \in \mathscr{V}_{x_0}$ such that for all $\chi \in C_c^\infty(V_0)$, $\chi u \in C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ and thus $\widehat{\chi u} \in \mathscr{S}(\mathbb{R}^n)$, implying (ii). If (ii) holds, then it is the case of the weaker (iii); we take $\chi_0 \in C_c^\infty(V_0)$, different from 0 on a compact neighborhood V_1 of x_0, and we get $\widehat{\chi_0 u} \in L^1(\mathbb{R}^n)$, so that

$$(\chi_0 u)(x) = \int e^{2i\pi x\cdot\xi} \widehat{\chi_0 u}(\xi) d\xi$$

and the estimate of (iii) gives $\chi_0 u \in C_c^\infty(\mathbb{R}^n)$ and $u_{|V_1} = \frac{1}{\chi_{0|V_1}} (\chi_0 u)_{|V_1} \in C^\infty(V_1)$, implying (i). □

Lemma 6.1.3. *For $u \in \mathscr{S}'(\mathbb{R}^n)$ the following properties are equivalent.*

(i) $u \in \mathscr{E}'(\mathbb{R}^n)$, $\quad \operatorname{supp} u \subset \{x \in \mathbb{R}^n, |x| \le R_0\}$, $\operatorname{order} u = N_0$.

(ii) $\hat{u}$ *can be extended to* $\mathbb{C}^n$ *as an entire function such that*

$$|\hat{u}(\zeta)| \le C_0(1+|\zeta|)^{N_0} e^{2\pi R_0|\operatorname{Im}\zeta|}. \tag{6.1.2}$$

Proof. If (i) holds, the theorem 4.3.1 gives that $\hat{u}$ is the entire function $\hat{u}(\zeta) = \langle u(x), e^{-2i\pi x\cdot\zeta}\rangle_{\mathscr{E}',\mathscr{E}}$. Moreover, since u is compactly supported in $\bar{B}(0, R_0)$, we have for all $\epsilon > 0$ and $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on $B(0,1)$,

$$\hat{u}(\zeta) = \langle u(x), \chi_0\Big(\frac{x}{R_0+\epsilon}\Big) e^{-2i\pi x\cdot\zeta}\rangle_{\mathscr{E}',\mathscr{E}}.$$

This implies $|\hat{u}(\zeta)| \le C_{\chi_0,N_0} \sup_{\substack{|x|\le R_0+\epsilon \\ |\alpha|+|\beta|\le N_0}} |e^{-2i\pi x\cdot\zeta}\zeta^\alpha (\partial^\beta\chi_0)\big(\frac{x}{R_0+\epsilon}\big)(R_0+\epsilon)^{-|\beta|}|$ and thus

$$\forall \epsilon > 0, \quad |\hat{u}(\zeta)| \le C_{\chi_0,N_0} e^{2\pi(R_0+\epsilon)|\operatorname{Im}\zeta|} \sup_{|\alpha|+|\beta|\le N_0} |\zeta^\alpha (R_0+\epsilon)^{-|\beta|}| \sup_{|\beta|\le N_0} \|\partial^\beta\chi_0\|_{L^\infty}.$$

We choose now, assuming $R_0 > 0$ (otherwise the implication follows from the theorem 3.3.4) $\epsilon = \frac{R_0}{1+|\zeta|}$. We get then

$$|\hat{u}(\zeta)| \le C'_{\chi_0,N_0} e^{2\pi R_0|\operatorname{Im}\zeta|} e^{2\pi \frac{R_0|\operatorname{Im}\zeta|}{1+|\zeta|}} (R_0^{-1} + |\zeta|)^{N_0} \Longrightarrow (ii).$$

Conversely, if (ii) holds, we consider a standard mollifier ρ_ϵ given with $\epsilon > 0$ by $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$, $\rho \in C_c^\infty(\mathbb{R}^n), \int \rho = 1$, ρ supported in the unit ball. We have from (4.3.2) $\widehat{u * \rho_\epsilon} = \hat{u}\hat{\rho}(\epsilon\cdot)$ and the function $\hat{u}\hat{\rho}(\epsilon\cdot)$ is entire with

$$|\hat{u}(\zeta)\hat{\rho}(\epsilon\zeta)| \le C_{N,\epsilon}(1+|\zeta|)^{-N} e^{2\pi(R_0+\epsilon)|\operatorname{Im}\zeta|}.$$

From the first lemma 6.1.1, we have $\operatorname{supp}(u * \rho_\epsilon) \subset \bar{B}(0, R_0+\epsilon)$. For $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have from the proposition 3.1.1

$$\langle u * \rho_\epsilon, \varphi\rangle = \langle u, \check{\rho}_\epsilon * \varphi\rangle \underset{\epsilon\to 0_+}{\longrightarrow} \langle u, \varphi\rangle,$$

and thus if $\operatorname{supp}\varphi \subset \big(\bar{B}(0,R_0+\epsilon)\big)^c$, we get $\langle u * \rho_\epsilon, \varphi\rangle = 0 = \langle u, \varphi\rangle$, so that $\operatorname{supp} u \subset \bar{B}(0, R_0+\epsilon)$ for all $\epsilon > 0$ and eventually

$$\operatorname{supp} u \subset \cap_{\epsilon>0}\bar{B}(0, R_0+\epsilon) = \bar{B}(0, R_0),$$

yielding the conclusion. □

Remark 6.1.4. Let us recall the expression of E_+, fundamental solution of the wave equation, given by (4.4.21):

$$\widehat{E_+}^x(t,\xi) = cH(t)\frac{\sin\big(2\pi ct|\xi|\big)}{2\pi|\xi|} = c^2 H(t)\int_0^t \cos(2\pi cs|\xi|)ds. \tag{6.1.3}$$

Since $\cos(2\pi cs|\xi|) = \sum_{k\ge 0}\frac{(-1)^k(2\pi cs)^{2k}}{(2k)!}(\sum_{1\le j\le d}\xi_j^2)^k$ the function $\widehat{E}(t,\cdot)$ is entire on $\mathbb{C}^d$ and we have for $\zeta \in \mathbb{C}^d$, using the notation $\zeta^2 = \sum_{1\le j\le d}\zeta_j^2$,

$$\widehat{E_+}^x(t,\zeta) = c^2 H(t)\int_0^t \sum_{k\ge 0}\frac{(-1)^k(2\pi cs)^{2k}}{(2k)!}(\zeta^2)^k ds = c^2 H(t)\int_0^t \cos(2\pi cs(\zeta^2)^{1/2})ds.$$

We have also for $z \in \mathbb{C}$

$$2|\cos z|^2 = 2(\cos z)(\cos\bar{z}) = \cos(2\operatorname{Re} z) + \cos(2i\operatorname{Im} z) \le 1 + e^{2|\operatorname{Im} z|} \le 2e^{2|\operatorname{Im} z|},$$

and as a consequence

$$\text{for } 0 \le s \le t, \quad |\cos(2\pi cs(\zeta^2)^{1/2})| \le \exp 2\pi ct|\operatorname{Im}\big((\zeta^2)^{1/2}\big)|. \tag{6.1.4}$$

We note that with $\zeta = \xi + i\eta, \xi, \eta \in \mathbb{R}^n$,

$$\zeta^2 = |\xi|^2 - |\eta|^2 + 2i\langle\xi,\eta\rangle = |\xi|^2 - |\eta|^2 + 2i\sigma|\xi||\eta|, \quad \text{with } \sigma \in \mathbb{R}, |\sigma| \le 1.$$

So if $z = a + ib \in \mathbb{C}, a, b \in \mathbb{R}$ is such that $z^2 = \zeta^2$, we have

$$a^2 - b^2 = |\xi|^2 - |\eta|^2, \quad |ab| \le |\xi||\eta|.$$

If we had $|b| > |\eta|$, that would imply from the first equation that $|a| > |\xi|$ and $|ab| > |\xi||\eta|$, which contradicts the second equation; as a result we have $|b| \le |\eta|$ and $|\operatorname{Im}\big((\zeta^2)^{1/2}\big)| \le |\operatorname{Im}\zeta|$, implying

$$|\widehat{E_+}^x(t,\zeta)| \le ctH(t)\exp 2\pi ct|\operatorname{Im}\zeta|,$$

which gives from the Paley-Wiener theorem 6.1.3 that

$$\operatorname{supp} E_+(t,\cdot) \subset \{x \in \mathbb{R}^n, |x| \le ct\}. \tag{6.1.5}$$

6.2 Stationary phase method

6.2.1 Preliminary remarks

It is well-known that

$$\int_{\mathbb{R}} \frac{\sin x}{x} dx = \pi, \quad \text{although} \quad \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right| dx = +\infty. \tag{6.2.1}$$

To get this, we integrate the function e^{iz}/z on the following path: the segment $[\epsilon, R]$, the half-circle $(R, iR, -R)$, the segment $[-R, -\epsilon]$, the half-circle $(-\epsilon, i\epsilon, \epsilon)$. We get

$$0 = 2i \int_\epsilon^R \frac{\sin x}{x} dx + \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta - \int_0^\pi \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta.$$

The third integral has limit $i\pi$ for $\epsilon \to 0$. The absolute value of the second integral is bounded above by $\int_0^\pi e^{-R\sin\theta} d\theta$ which goes to zero when R goes[1] to infinity, yielding the value π in (6.2.1). On the other hand, for $n \in \mathbb{N}^*$, we have

$$\int_{2n\pi}^{(2n+1)\pi} \left|\frac{\sin x}{x}\right| dx \ge \frac{1}{(2n+1)\pi} \int_{2n\pi}^{(2n+1)\pi} \sin x dx = \frac{2}{(2n+1)\pi},$$

the general term of a diverging series, so that (6.2.1) is proven. In the integral $\int_{\mathbb{R}} \frac{\sin x}{x} dx$, the *amplitude* $1/x$ is too large at infinity to guarantee the absolute convergence of the integral, although the *oscillations* of the term $\sin x = \operatorname{Im} e^{ix}$ compensate the size of the amplitude and lead to some cancellation phenomena. We want to study this phenomenon more closely and in more geometrical terms. Although the

[1] One may apply Lebesgue's dominated convergence theorem, but it is way too much: it is enough to note that $0 \le \frac{2\theta}{\pi} \le \sin\theta$ for $\theta \in [0, \pi/2]$ and

$$\int_0^\pi e^{-R\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta \le 2\int_0^{\pi/2} e^{-2R\theta/\pi} d\theta \le \pi/R.$$

function $\sin x/x$ does not belong to $L^1(\mathbb{R}^n)$, we still[2] have in the sense of weak-dual convergence (see the definition 3.1.16)

$$\lim_{\lambda\to+\infty} \frac{1}{\pi}\frac{\sin(\lambda x)}{x} = \delta_0. \tag{6.2.2}$$

In fact for $\varphi \in C_c^1(\mathbb{R}), \operatorname{supp}\varphi \subset [-M_0, M_0]$, the function ψ defined by

$$\psi(x) = x^{-1}\bigl(\varphi(x) - \varphi(0)\bigr) = \int_0^1 \varphi'(\theta x)d\theta$$

is continuous and equal to $-\varphi(0)x^{-1}$ for $|x| \ge M_0(>0)$. As a consequence, we have

$$\int \frac{\sin(\lambda x)}{x}\varphi(x)dx = \int \underbrace{\psi(x)\mathbf{1}_{[-M_0,M_0]}(x)}_{\in L^1(\mathbb{R})}\sin(\lambda x)dx + \varphi(0)\int_{|x|\le M_0} x^{-1}\sin(\lambda x)dx.$$

The Riemann-Lebesgue lemma 4.3.5 implies that the first term in the rhs tends to 0 with $1/\lambda$, whereas

$$\int_{|x|\le M_0} x^{-1}\sin(\lambda x)dx = \int_{|y|\le \lambda M_0} x^{-1}\sin x dx \underset{\lambda\to+\infty}{\longrightarrow} \pi,$$

proving 6.2.2.

6.2.2 Non-stationary phase

Theorem 6.2.1. *Let $a \in C_c^\infty(\mathbb{R}^n)$ and ϕ be a real-valued C^∞ function defined on $\mathbb{R}^n$ such that $d\phi \ne 0$ on the support of a. We define for $\lambda \in \mathbb{R}$,*

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}a(x)dx. \tag{6.2.3}$$

Then for all $N \ge 0$, $\sup_{\lambda\in\mathbb{R}} |\lambda^N I(\lambda)| < +\infty$.

Proof. Since the support of a is compact, we know that $\inf_{x\in\operatorname{supp} a} |d\phi(x)| = c_0 > 0$. We define then the differential operator L on the open set $\Omega = \{x \in \mathbb{R}^n, d\phi(x) \ne 0\} \supset \operatorname{supp} a$ by

$$L = \frac{1}{i}\sum_{1\le j\le n} |d\phi|^{-2}\frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}. \tag{6.2.4}$$

On Ω, we have $L(e^{i\lambda\phi}) = \lambda e^{i\lambda\phi}\sum_{1\le j\le n}|d\phi|^{-2}\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j} = \lambda e^{i\lambda\phi}$, as well as for all $N \in \mathbb{N}$, $e^{i\lambda\phi} = (\lambda^{-N}L^N)(e^{i\lambda\phi})$, implying that, for $\lambda \ne 0$,

$$I(\lambda) = \lambda^{-N}\int_\Omega L^N(e^{i\lambda\phi})a(x)dx = \lambda^{-N}\int_{\operatorname{supp} a} e^{i\lambda\phi(x)}({}^tL^N a)(x)dx.$$

[2] If $u \in L^1(\mathbb{R}^n)$, $\varphi \in C^0(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, then with $\lambda > 0$, we have $\int u(\lambda x)\lambda^n\varphi(x)dx = \int u(x)\varphi(\lambda^{-1}x)dx$, and using the Lebesgue dominated convergence theorem, this gives

$$\lim_{\lambda\to+\infty}\int u(\lambda x)\lambda^n\varphi(x)dx = \varphi(0)\int u(x)dx.$$

As a result we get for $\lambda \in \mathbb{R}$, $|\lambda^N I(\lambda)| \le \|{}^tL^N a\|_{L^1(\mathbb{R}^n)} < +\infty$, since

$$ {}^tL = i \sum_{1\le j\le n} \frac{\partial}{\partial x_j} |d\phi|^{-2} \frac{\partial \phi}{\partial x_j}, \qquad {}^tL^N = \sum_{|\alpha|\le N} c_\alpha(x)\partial_x^\alpha, \quad c_\alpha \in C^\infty(\Omega). $$

□

This theorem means that the integral (6.2.3) is rapidly decreasing with respect to the large parameter λ, provided the real phase ϕ does not have stationary points on the support of the amplitude a. We shall now concentrate our attention on the case where the phase does have stationary points ; a first simple model is concerned with (real) quadratic phases.

6.2.3 Quadratic phase

We recall part of the proposition 4.6.1 as a lemma.

Lemma 6.2.2. *Let A be a real symmetric nonsingular $n \times n$ matrix. Then $x \mapsto e^{i\pi\langle Ax,x\rangle}$ is a bounded measurable function, thus a tempered distribution and we have*

$$ \text{Fourier}(e^{i\pi\langle Ax,x\rangle})(\xi) = |\det A|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} A} e^{-i\pi\langle A^{-1}\xi,\xi\rangle}. \tag{6.2.5} $$

Theorem 6.2.3. *Let $a \in \mathscr{S}(\mathbb{R}^n)$ and A be a real symmetric nonsingular $n \times n$ matrix. Defining $I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\langle Ax,x\rangle} a(x)dx$, we have for $\lambda > 0$,*

$$ I(\lambda) = \frac{\pi^{n/2} e^{\frac{i\pi}{4}\operatorname{sign} A}}{\lambda^{\frac{n}{2}} |\det A|^{1/2}} \Big(\sum_{0\le k<N} \lambda^{-k} \frac{\pi^{2k}}{i^k k!} \big(\langle A^{-1}D, D\rangle^k a\big)(0) + r_N(\lambda) \Big), \tag{6.2.6} $$

$$ |r_N(\lambda)| \le \lambda^{-N} \frac{\pi^{2N}}{N!} \|\langle A^{-1}D, D\rangle^N a\|_{FL^1}, \tag{6.2.7} $$

where $\|u\|_{FL^1} = \|\hat u\|_{L^1(\mathbb{R}^n)}$, so that $\|\langle A^{-1}D, D\rangle^N a\|_{FL^1} = \|\langle A^{-1}\xi, \xi\rangle^N \hat a\|_{L^1(\mathbb{R}^n)}$ (see also the notation (4.1.6)).

Proof. We write with $\lambda = \pi\mu$ that

$$ \begin{aligned} I(\lambda) = \langle e^{i\pi\langle \mu Ax,x\rangle}, a(x)\rangle_{\mathscr{S}',\mathscr{S}} &= \langle \text{Fourier}(e^{i\pi\langle \mu Ax,x\rangle}), \check{\hat a}\rangle_{\mathscr{S}',\mathscr{S}} \\ &= \mu^{-n/2} |\det A|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} A} \int e^{-i\pi\mu^{-1}\langle A^{-1}\xi,\xi\rangle} \hat a(\xi) d\xi, \end{aligned} $$

and since

$$ \begin{aligned} \int e^{-i\pi\mu^{-1}\langle A^{-1}\xi,\xi\rangle} \hat a(\xi) d\xi &= \sum_{0\le k<N} \frac{(-i\pi\mu^{-1})^k}{k!} \int \langle A^{-1}\xi,\xi\rangle^k \hat a(\xi) d\xi \\ &+ \int_0^1 \int e^{-i\theta\pi\mu^{-1}\langle A^{-1}\xi,\xi\rangle} \langle A^{-1}\xi,\xi\rangle^N \hat a(\xi) d\xi \frac{(1-\theta)^{N-1}}{(N-1)!} d\theta \left(\frac{-i\pi}{\mu}\right)^N, \end{aligned} $$

we get (6.2.6) with $|r_N(\lambda)| \le \|\langle A^{-1}\xi,\xi\rangle^N \hat a(\xi)\|_{L^1} \frac{\pi^{2N}}{N!\lambda^N}$. □

Remark 6.2.4. In particular, under the assumptions of the theorem, we have, if $a(0) \neq 0$,

$$\int_{\mathbb{R}^n} e^{i\lambda\langle Ax,x\rangle} a(x)dx = I(\lambda) \underset{\lambda\to+\infty}{\sim} \frac{\pi^{\frac{n}{2}} e^{\frac{i\pi}{4}\operatorname{sign} A}}{\lambda^{\frac{n}{2}} |\det A|^{1/2}} a(0), \tag{6.2.8}$$

a sharp contrast with the results of the previous subsection 6.2.2. Naturally, in this case, the phase has a (unique) stationary point at the origin. Note also that in one dimension, we can recover[3] the so-called Fresnel integrals

$$\int_{\mathbb{R}} e^{ix^2} dx = \pi^{1/2} e^{i\pi/4}, \quad \text{i.e.} \quad \int_{\mathbb{R}} \cos(x^2)dx = \int_{\mathbb{R}} \sin(x^2)dx = \sqrt{\frac{\pi}{2}}. \tag{6.2.9}$$

6.2.4 The Morse lemma

The most important step in the proof is the following lemma.

Lemma 6.2.5. *Let U be a neighborhood of 0 in $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ be a C^∞ function such that $df(0) = 0$, $\frac{\partial^2 f}{\partial x_1^2}(0) \neq 0$. Then there exists a local diffeomorphism ν of neighborhoods of 0 such that*

$$(f \circ \nu)(y_1, y') = g(y') + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0) y_1^2.$$

Proof. We may assume that $f(0) = 0$. Thanks to the implicit function theorem, we note that the equation $\frac{\partial f}{\partial x_1}(x_1, x') = 0$ has a unique solution $x_1 = \alpha(x')$ near the origin: there exists $r_0 > 0$, a neighborhood W of 0 in $\mathbb{R}^{n-1}$ and a C^∞ function $\alpha : W \to \mathbb{R}$ such that $\alpha(0) = 0$ and for $|x_1| < r_0, x' \in W$,

$$\frac{\partial f}{\partial x_1}(x_1, x') = 0 \Longleftrightarrow x_1 = \alpha(x').$$

As a result, we have for $|x_1| < r_0, x' \in W$,

$$f(x_1, x') = f\big(\alpha(x'), x'\big) + \int_0^1 (1-\theta)\frac{\partial^2 f}{\partial x_1^2}\big(\alpha(x') + \theta(x_1 - \alpha(x')), x'\big) d\theta \big(x_1 - \alpha(x')\big)^2,$$

i.e. with a C^∞ function e defined in $]-r_0, r_0[\times W$, a C^∞ function g defined in W,

$$f(x_1, x') = g(x') + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0) e(x)\big(x_1 - \alpha(x')\big)^2, \quad e(0) = 1.$$

Shrinking if necessary the neighborhoods, we define near 0 the local diffeomorphism κ by

$$\kappa(x_1, x') = \big(e(x)^{1/2}(x_1 - \alpha(x')), x'\big) = (y_1, y')$$

[3] We have with $\chi \in C_c^\infty(\mathbb{R})$ even, equal to 1 on $[-1, 1]$, supported in $[-2, 2]$,

$$2\int_0^T e^{ix^2} dx = \int e^{ix^2}\chi(\frac{x}{T})dx - 2\int_{x\geq T} e^{ix^2}\chi(\frac{x}{T})dx = \int e^{iT^2x^2}\chi(x)dxT - 2\int_{x\geq T} 2ixe^{ix^2}\chi(\frac{x}{T})(2ix)^{-1}dx.$$

From (6.2.8), $\lim_{T\to+\infty}\int e^{iT^2x^2}\chi(x)dxT = \pi^{1/2}e^{i\pi/4}$ and an integration by parts yields that the last term is $O(T^{-1})$.

and we have with $\nu = \kappa^{-1}$

$$(f \circ \nu)(y_1, y') = f(x_1, x') = g(y') + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0) y_1^2,$$

yielding the conclusion. □

Theorem 6.2.6. *Let $x_0 \in \mathbb{R}^n$, $U \in \mathscr{V}_{x_0}$ and $f : U \to \mathbb{R}$ be a C^∞ function such that $df(x_0) = 0, \det f''(x_0) \neq 0$. Then there exists an open neighborhood U_0 of x_0, an open neighborhood V_0 of 0 and a C^∞ diffeomorphism $\nu : V_0 \to U_0$ such that $U_0 \subset U$, $\det \nu'(0) = 1$, and for $y \in V_0$,*

$$(f \circ \nu)(y) - (f \circ \nu)(0) = \frac{1}{2} \sum_{1 \le j \le n} \mu_j y_j^2, \tag{6.2.10}$$

where $(\mu_1, \dots, \mu_n)$ are the eigenvalues of the symmetric matrix $f''(x_0)$.

Proof. We may assume for notational simplicity that $x_0 = 0$ and $f(0) = 0$. After composing f with a rotation, we may assume that e_1 is an eigenvector of $f''(0)$, so that in particular, the assumptions of the previous lemma are satisfied. Then we are reduced to tackle a function $g(x') + \frac{1}{2}\mu_1 x_1^2$. We have $dg(0) = 0$, the eigenvalues of $f''(0)$ are $\{\mu_1\} \cup \operatorname{spectrum}(g''(0))$. We get the conclusion by an induction on n. □

6.2.5 Stationary phase formula

We consider now, for $\lambda > 0$ and

$$I(\lambda) = \int e^{i\lambda\phi(x)} a(x) dx, \tag{6.2.11}$$

where the amplitude $a \in C_c^\infty(\mathbb{R}^n)$ and the phase function ϕ is a *Morse function*, i.e. a real-valued smooth function such that

$$\forall x \in \operatorname{supp} a, \quad d\phi(x) = 0 \Longrightarrow \det \phi''(x) \neq 0. \tag{6.2.12}$$

Using the Borel-Lebesgue property, we get that

$$\operatorname{supp} a \subset \underbrace{\{x \in \mathbb{R}^n, d\phi(x) \neq 0\}}_{=\Omega_0} \cup_{1 \le j \le N} \Omega_j$$

where Ω_j for $1 \le j \le N$ is an open set such that there exists a C^∞ diffeomorphism $\nu_j : V_j \to \Omega_j$, where V_j is a neighborhood of 0 in $\mathbb{R}^n$ with

$$(\phi \circ \nu_j)(y) = (\phi \circ \nu_j)(0) + \frac{1}{2}\phi''(\nu_j(0)) y^2.$$

Using the theorem 3.1.14, we are able to find $(\psi_j)_{0 \le j \le N}$ with $\psi_j \in C_c^\infty(\Omega_j)$, such that $\sum_{0 \le j \le N} \psi_j$ is 1 near $\operatorname{supp} a$. We obtain then that

$$I(\lambda) = \underbrace{\int e^{i\lambda\phi(x)} \psi_0(x) a(x) dx}_{=O(\lambda^{-\infty}) \text{ from Theorem 6.2.1}} + \sum_{1 \le j \le N} \int e^{i\lambda\phi(x)} \psi_j(x) a(x) dx,$$

i.e. $I(\lambda)=\sum_{1\le j\le N}\int_{V_j} e^{i\lambda(\phi\circ\nu_j)(y)}(\psi_j a)(\nu_j(y))|\det\nu_j'(y)|dy+O(\lambda^{-\infty})$. We note that, according to the theorem 6.2.3

$$\begin{aligned}\int_{V_j} & e^{i\lambda(\phi\circ\nu_j)(y)}(\psi_j a)(\nu_j(y))|\det\nu_j'(y)|dy\\ &= e^{i\lambda\phi(\nu_j(0))}\int_{V_j} e^{i\lambda\frac12\phi''(\nu_j(0))y^2}(\psi_j a)(\nu_j(y))|\det\nu_j'(y)|dy\\ &= \lambda^{-\frac n2}e^{i\lambda\phi(\nu_j(0))}\frac{(2\pi)^{n/2}e^{i\frac\pi4\operatorname{sign}\phi''(\nu_j(0))}}{|\det\phi''(\nu_j(0))|^{1/2}}(\psi_j a)(\nu_j(0))|\det\nu_j'(0)|+O(\lambda^{-\frac n2-1}).\end{aligned}$$

We note also that the stationary points of a Morse function are isolated, since for an invertible symmetric matrix Q, the only singular point of $y\mapsto\langle Qy,y\rangle$ is 0. In particular, there are only finitely many singular points of a Morse function in a compact set.

Theorem 6.2.7. *Let a be a $C_c^\infty(\mathbb{R}^n)$ function and ϕ be a Morse function (see (6.2.12)). We define $I(\lambda)$ by (6.2.11). We have for $\lambda\to+\infty$*

$$I(\lambda)=\lambda^{-\frac n2}(2\pi)^{n/2}\sum_{\substack{x,d\phi(x)=0\\ x\in\operatorname{supp}a}} e^{i\lambda\phi(x)}\frac{e^{i\frac\pi4\operatorname{sign}(\phi''(x))}}{|\det\phi''(x)|^{1/2}}a(x)+O(\lambda^{-\frac n2-1}). \tag{6.2.13}$$

Proof. We note that the determinant of $\nu'(0)$ is 1 in the theorem 6.2.6 and the formula of Theorem 6.2.3 gives the result if we replace $\psi_j a$ by a; it is indeed harmless to do this since we can assume that $x_1,\dots,x_N$ are the distinct singular points of ϕ in $\operatorname{supp}a$ and write, with $C_c^\infty(\mathbb{R}^n)\ni\widetilde{\psi}_j=1$ near x_j, $\widetilde{\psi}_j\widetilde{\psi}_k=0$ if $1\le j\ne k\le N$

$$a=\sum_{1\le j\le N}\widetilde{\psi}_j a+\underbrace{a-\sum_{1\le j\le N}\widetilde{\psi}_j a}_{\text{supported in }\Omega_0}. \qquad \square$$

6.3 The Wave-Front set of a distribution, the H^s wave-front set

Let Ω be an open subset of $\mathbb{R}^n$ and $u\in\mathscr{D}'(\Omega)$. Let us recall that the support and the singular support of u are defined by

$$\operatorname{supp}u=\{x\in\Omega,\text{there is no open }V\ni x\text{ with }u_{|V}=0\}, \tag{6.3.1}$$
$$\operatorname{singsupp}u=\{x\in\Omega,\text{there is no open }V\ni x\text{ with }u_{|V}\in C^\infty(V)\}. \tag{6.3.2}$$

Both sets are closed and we have obviously $\operatorname{singsupp}u\subset\operatorname{supp}u$. The Fourier transform allows a more refined analysis of singularities: first we notice that $x_0\notin\operatorname{singsupp}u$ iff there exists a neighborhood U of x_0 such that for all $\chi\in C_c^\infty(U)$,

$$\forall N\in\mathbb{N},\quad \sup_{\xi\in\mathbb{R}^n}|(\widehat{\chi u})(\xi)||\xi|^N<\infty. \tag{†}$$

This is obvious when we assume $x_0 \notin \operatorname{singsupp} u$ since there exists a neighborhood U of x_0 such that $\chi u \in C_c^\infty(\mathbb{R}^n)$ and thus $\widehat{\chi u} \in \mathscr{S}(\mathbb{R}^n)$. Conversely, since $\widehat{\chi u}$ is the Fourier transform of a compactly supported distribution, it is an entire function on $\mathbb{C}^n$, and assuming (†), we see that $(\chi u)(x) = \int e^{2i\pi x\cdot\xi}\widehat{\chi u}(\xi)d\xi$, and the rhs is a C^∞ function, qed.

We use the notation $\Omega \times \mathbb{R}^n\backslash\{0\} = \dot{T}^*(\Omega)$, the cotangent bundle minus the zero section.

Definition 6.3.1. *Let Ω be an open set of $\mathbb{R}^n$ and let $u \in \mathscr{D}'(\Omega)$. The wave-front-set of u, denoted by WFu, is defined as the complement in $\dot{T}^*(\Omega)$ of the set of points (x_0, ξ_0) such that there exist some neighborhoods U, V respectively of x_0, ξ_0 (with $U \times V \subset \dot{T}^*(\Omega)$) such that for all $\chi \in C_c^\infty(U)$,*

$$\forall N \in \mathbb{N}, \quad \sup_{\xi \in \widetilde{V}} |(\widehat{\chi u})(\xi)||\xi|^N < \infty, \quad \textit{with } \widetilde{V} = \cup_{\tau>0}\tau V. \tag{6.3.3}$$

Remark 6.3.2. Note that the wave-front-set is a closed (its complement is open) conic subset of $\dot{T}^*(\Omega)$: conic means here that for all $\tau > 0$, $(x,\xi) \in WFu \Longrightarrow (x, \tau\xi) \in WFu$. On the other hand, with $\operatorname{pr} : \dot{T}^*(\Omega) \to \Omega$ defined by $\operatorname{pr}((x,\xi)) = x$, we get that

$$\operatorname{pr} WFu = \operatorname{singsupp} u. \tag{6.3.4}$$

Let $x_0 \notin \operatorname{singsupp} u$. Then from (†), we see that for all $\xi \in \mathbb{S}^{n-1}$, $(x_0,\xi) \notin WFu$, so that $x_0 \notin \operatorname{pr} WFu$. Conversely, if $x_0 \notin \operatorname{pr} WFu$, for all $\eta \in \mathbb{S}^{n-1}$, there exists some neighborhoods U_η, V_η of x_0, η such that for all $\chi \in C_c^\infty(U_\eta)$,

$$\forall N \in \mathbb{N}, \quad \sup_{\xi \in \widetilde{V}_\eta} |(\widehat{\chi u})(\xi)||\xi|^N < \infty.$$

By compactness, we get $\mathbb{S}^{n-1} \subset \cup_{1\le j\le\nu} V_{\eta_j}$ and defining $U = \cap_{1\le j\le\nu} U_{\eta_j}$, we get that for all $\chi \in C_c^\infty(U)$,

$$\forall j \in \{1,\dots,\nu\}, \forall N \in \mathbb{N}, \quad \sup_{\xi \in \widetilde{V}_{\eta_j}} |(\widehat{\chi u})(\xi)||\xi|^N < \infty,$$

which gives the result (†) since $\cup_{1\le j\le\nu}\widetilde{V}_{\eta_j} = \mathbb{R}^n\backslash\{0\}$ and $\widehat{\chi u}$ is a smooth function.

Examples. It is easy to see that
(1) $WF(\delta_0) = \{0\} \times \mathbb{R}^n\backslash\{0\}$, δ_0 is the Dirac mass at zero in $\mathbb{R}^n$,
(2) $WF(\frac{1}{x+i0}) = \{0\} \times (0, +\infty)$, $\frac{1}{x+i0} = \frac{d}{dx}(\ln|x|) - i\pi\delta_0$, distribution on $\mathbb{R}$,
(3) and with $H = \mathbf{1}_{\mathbb{R}_+}$, considering the distribution on $\mathbb{R}^2$,

$$WF\big(H(x_1)H(x_2)\big) = \{(0, x_2, \xi_1, 0)\}_{x_2>0,\xi_1\ne0} \cup \{(x_1, 0, 0, \xi_2)\}_{x_1>0,\xi_2\ne0} \\ \cup \{(0,0)\} \times \mathbb{R}^2\backslash\{(0,0)\}.$$

(4) If u is a distribution, one can easily define the complex conjugate by duality[4] and we have

$$WF\bar{u} = \check{WF}u = \{(x,\xi) \text{ such that} (x,-\xi) \in WFu\}$$

[4]We define $\prec \bar{u}, \varphi \succ_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \overline{\prec u, \bar{\varphi} \succ}_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}$.

and in particular, a real-valued distribution (i.e. such that $\bar u = u$) has a *projective* wave-front-set, i.e. $(x,\xi) \in WFu \Longleftrightarrow (x,-\xi) \in WFu$, so that, instead of being included in the sphere fiber $S^*(\Omega)$ image of the fiber bundle $\dot T^*(\Omega)$ by the mapping $(x,\xi) \mapsto (x, \xi/|\xi|)$, the wave-front-set of a real-valued distribution can be seen as a part of the projective bundle for which the fibers are the quotient of the sphere $\mathbb{S}^{n-1}$ by $\{-1,1\}$, that is $\mathbb{P}^{n-1}(\mathbb{R})$. In particular for a real-valued distribution u on an open set Ω of the real line, then the wave-front-set does not carry more information than the singular support since $WFu = \operatorname{singsupp} u \times \mathbb{R}^*$.

The following lemma provides a characterization of the wave-front-set which is closer of the pseudodifferential approach.

Lemma 6.3.3. *Let $\theta_0 \in C_c^\infty(\mathbb{R}^n;[0,1]), \operatorname{supp}\theta_0 \subset B(0,1)$, $\theta_0 = 1$ on $B(0,1/2)$. Let Ω be an open set of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. The complement of WFu in $\dot T^*(\Omega)$ is the set of (x,ξ) such that there exists $r>0$ such that*

$$T_r(D)t_r u \quad \textit{belongs to } \mathscr{S}(\mathbb{R}^n),$$

where $T_r(\xi) = \theta_0\left(\frac{\xi}{r|\xi|} - \frac{\xi_0}{r|\xi_0|}\right)(1-\theta_0)(\frac{r\xi}{2}), \quad t_r(x) = \theta_0\left(\frac{x-x_0}{r}\right).$

Proof. Let us assume first that $\dot T^*(\Omega) \ni (x_0,\xi_0) \notin WFu$. Using the definition 6.3.1, we get that for some positive r, for all N, $T_r(\xi)\widehat{t_r u}(\xi) = O(\langle\xi\rangle^{-N})$ and since the functions $D_\xi^\alpha\big(\widehat{t_r u}\big) = (-1)^{|\alpha|}\widehat{x^\alpha t_r u}$ are also rapidly decreasing on the support of T_r (from the definition 6.3.1), we get that $\xi \mapsto T_r(\xi)\widehat{t_r u}(\xi)$ is in the Schwartz class as well as its inverse Fourier transform $T_r(D)t_r u$.

Conversely, if for $(x_0,\xi_0) \in \dot T^*(\Omega)$ (we may assume $|\xi_0| = 1$) and some positive r, $T_r(D)t_r u \in \mathscr{S}(\mathbb{R}^n)$, we get indeed as in (6.3.3)

$$\forall N \in \mathbb{N}, \quad \sup_{\xi \in \widetilde V} |\widehat{t_r u}(\xi)||\xi|^N < \infty, \quad \text{with } V \text{ neighborhood of } \xi_0.$$

Now if $\chi \in C_c^\infty(B(x_0, r/2)$, we have $\chi = \chi t_r$ and

$$T_{r/4}(\xi)\widehat{\chi u}(\xi) = T_{r/4}(\xi)\widehat{\chi t_r u}(\xi) = T_{r/4}(\xi)\int \underbrace{\hat\chi(\xi-\eta)}_{O(\langle\xi-\eta\rangle^{-N})}\underbrace{T_r(\eta)\widehat{t_r u}(\eta)}_{O(\langle\eta\rangle^{-2N})}d\eta$$
$$+ T_{r/4}(\xi)\int \hat\chi(\xi-\eta)\underbrace{(1-T_r(\eta))\widehat{t_r u}(\eta)}_{O(\langle\eta\rangle^{M_0})}d\eta.$$

Using the Peetre inequality[5], we get that the first term is $O(\langle\xi\rangle^{-N})$. To handle the next term we note that, on the support of $T_{r/4}$, we have

$$|\xi| \ge 4/r, \quad \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| \le r/4$$

[5]We use $\langle\xi+\eta\rangle \le 2^{1/2}\langle\xi\rangle\langle\eta\rangle$ so that, for all $s \in \mathbb{R}$,

$$\langle\xi+\eta\rangle^s \le 2^{|s|/2}\langle\xi\rangle^s\langle\eta\rangle^{|s|}, \tag{6.3.5}$$

a convenient inequality (to get it for $s \ge 0$, raise the first inequality to the power s, and for $s<0$, replace ξ by $-\xi-\eta$) a.k.a. Peetre's inequality.

and on the integrand we have either $|\eta| \le 1/r$ (harmless term since $\hat\chi \in \mathscr{S}$) or

$$|\eta| \ge 1/r \quad \text{and} \quad \left|\frac{\eta}{|\eta|} - \frac{\xi_0}{|\xi_0|}\right| \ge r/2 \quad \Longrightarrow \left|\frac{\eta}{|\eta|} - \frac{\xi}{|\xi|}\right| \ge r/4. \tag{$\star$}$$

Using the inequality[6]

$$\big||\eta|\xi - |\xi|\eta\big|\big(|\xi| + |\eta|\big) \le 4|\xi||\eta||\xi - \eta|, \tag{6.3.6}$$

we obtain here (for the nonzero vectors ξ, η satisfying ($\star$)), $4|\xi - \eta| \ge \frac{r}{4}\big(|\xi| + |\eta|\big)$, so that the rapid decay of $\hat\chi(\xi - \eta)$ gives the result of the lemma. □

The wave-front-set of a distribution depends only on the manifold structure of the open set Ω.

Theorem 6.3.4. *let $\kappa : \Omega_2 \longrightarrow \Omega_1$ a C^∞ diffeomorphism of open subsets of $\mathbb{R}^n$ and let $u_1 \in \mathscr{D}'(\Omega_1)$. Then we have*

$$WF\big(\kappa^*(u_1)\big) = \kappa^*\big(WFu_1\big) = \Big\{\Big(\kappa^{-1}(x_1), {}^t\kappa'\big(\kappa^{-1}(x_1)\big)\xi_1\Big)\Big\}_{(x_1,\xi_1)\in WFu_1}.$$

Proof. Let us define $u_2 = \kappa^*(u_1)$, so that for $\chi_2 \in C_c^\infty(\Omega_2)$, we have, for $\varphi_2 \in C_c^\infty(\Omega_2)$, with brackets of duality and $\nu = \kappa^{-1}$, $\chi_1(x_1) = \chi_2(\nu(x_1))|\det \nu'(x_1)|$ (note that χ_1 belongs to $C_c^\infty(\Omega_1)$ and $\chi_1|dx_1|$ is the κ-push-forward of the density $\chi_2|dx_2|$), $\psi_1 \in C_c^\infty(\Omega_1)$ equal to 1 on the support of χ_1,

$$\widehat{\chi_2 u_2}(\xi_2) = \int \chi_1(x_1)u_1(x_1)e^{-2i\pi\nu(x_1)\cdot\xi_2}dx_1$$
$$= \int \widehat{\chi_1 u_1}(\xi_1)\left(\int e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}\psi_1(x_1)dx_1\right)d\xi_1$$

where the integral with respect to ξ_1 is in fact a bracket of duality. We may thus consider the identity

$$\Big(1 + \big(\xi_1 - {}^t\nu'(x_1)\xi_2\big)\cdot D_{x_1}\Big)\big(e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}\big) = e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}\big(1 + \|\xi_1 - {}^t\nu'(x_1)\xi_2\|^2\big)$$

which gives with $L = \big(1 + \|\xi_1 - {}^t\nu'(x_1)\xi_2\|^2\big)^{-1}\Big(1 + \big(\xi_1 - {}^t\nu'(x_1)\xi_2\big)\cdot D_{x_1}\Big)$,

$$\forall N \in \mathbb{N}, \quad L^N\big(e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}\big) = e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}$$

so that $\widehat{\chi_2 u_2}(\xi_2) = \int \widehat{\chi_1 u_1}(\xi_1)\left(\int e^{2i\pi(\xi_1 x_1 - \xi_2\nu(x_1))}({}^tL)^N(\psi_1)(x_1)dx_1\right)d\xi_1$ and

$$|\widehat{\chi_2 u_2}(\xi_2)| \le C_N \iint |\widehat{\chi_1 u_1}(\xi_1)|\langle\xi_1 - {}^t\nu'(x_1)\xi_2\rangle^{-N}\mathbf{1}_{\operatorname{supp}\psi}(x_1)dx_1 d\xi_1. \tag{$\star$}$$

[6]The proof of (6.3.6) is the following: we have $\big||\eta|\xi - |\xi|\eta\big| \le |\eta||\xi - \eta| + |\eta|\big||\xi| - |\eta|\big| \le 2|\eta||\xi - \eta|$ and thus $\big||\eta|\xi - |\xi|\eta\big| \le 2|\xi - \eta|\min(|\xi|, |\eta|)$ which gives

$$\big||\eta|\xi - |\xi|\eta\big|\big(|\xi| + |\eta|\big) \le 2|\xi - \eta|\min(|\xi|, |\eta|)2\max(|\xi|, |\eta|) = 4|\xi||\eta||\xi - \eta|.$$

Let us assume that $\dot{T}^*(\Omega_1) \ni (x_{01}, \xi_{01}) \notin WFu_1$; the point (x_{02}, ξ_{02}) is defined as $(\nu(x_{01}), {}^t\nu'(x_{01})^{-1}\xi_{01})$. We assume that ξ_2 belongs to a conic neighborhood Γ_2 of ξ_{02}. We consider first for $r > 0$ the conic subset of $\mathbb{R}^n$ defined by

$$\Gamma_1(r) = \{\xi_1 \in \mathbb{R}^n, \forall \xi_2 \in \Gamma_2, \inf_{x_1 \in \operatorname{supp} \psi_1} |\xi_1 - {}^t\nu'(x_1)\xi_2| < r(|\xi_1| + |\xi_2|)\}.$$

The set $\Gamma_1(r)$ is also open and contains ξ_{01}. If r is small enough and the support of χ_2 is included in a small enough ball around x_{02}, we have from our assumption $|\widehat{\chi_1 u_1}(\xi_1)| = O(\langle \xi_1 \rangle^{-2N})$ on $\Gamma_1(r)$. When the integration in $(\star)$ takes place in $\Gamma_1(r)$, we estimate that part of the integral, using the footnote on page 159 by

$$C'_N \iint \langle \xi_1 \rangle^{-2N+N} \langle {}^t\nu'(x_1)\xi_2 \rangle^{-N} \mathbf{1}_{\operatorname{supp} \psi}(x_1) dx_1 d\xi_1 = O(\langle \xi_2 \rangle^{-N}).$$

When the integration in $(\star)$ takes place outside $\Gamma_1(r)$, we know that for some $r > 0$ and all $x_1 \in \operatorname{supp} \psi$, $|\xi_1 - {}^t\nu'(x_1)\xi_2| \geq r(|\xi_1| + |\xi_2|)$. We have thus the estimate, with a fixed M_0,

$$C''_N \iint \langle \xi_1 \rangle^{M_0} (\langle \xi_1 \rangle + \langle \xi_2 \rangle)^{-2N} \mathbf{1}_{\operatorname{supp} \psi}(x_1) dx_1 d\xi_1 = O(\langle \xi_2 \rangle^{-N}), \text{ for } N > M_0 + n.$$

The proof of the theorem is complete. □

Definition 6.3.5. *Let Ω be an open set of $\mathbb{R}^n$, let $u \in \mathscr{D}'(\Omega)$ and $s \in \mathbb{R}$. The H^s-wave-front-set of u, denoted by $WF_s u$, is defined as the complement in $\dot{T}^*(\Omega)$ of the set of points (x_0, ξ_0) such that there exist some neighborhoods U, V respectively of x_0, ξ_0 (with $U \times V \subset \dot{T}^*(\Omega)$) such that for all $\chi \in C_c^\infty(U)$,*

$$\int_{\widetilde{V} \cap \{|\xi| \geq 1\}} |(\widehat{\chi u})(\xi)|^2 |\xi|^{2s} d\xi < \infty, \quad \text{with } \widetilde{V} = \cup_{\tau > 0} \tau V.$$

6.4 Oscillatory Integrals

Definition 6.4.1. *Let Ω be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R}, N \in \mathbb{N}^*$. The space $S^m(\Omega \times \mathbb{R}^N)$ is defined as the set of functions $a \in C^\infty(\Omega \times \mathbb{R}^N; \mathbb{C})$ such that, for all K compact subset of Ω, for all $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^N$, there exists $C_{K,\alpha,\beta}$ such that*

$$\forall x \in K, \forall \theta \in \mathbb{R}^N, \quad |(\partial_x^\alpha \partial_\theta^\beta a)(x, \theta)| \leq C_{K,\alpha,\beta} \langle \theta \rangle^{m - |\beta|}. \tag{6.4.1}$$

It is a easy exercise left to the reader, consequence of the Leibniz formula, to prove that the space $S^m(\Omega \times \mathbb{R}^N)$ is a Fréchet space and that the mappings

$$S^{m_1}(\Omega \times \mathbb{R}^N) \times S^{m_2}(\Omega \times \mathbb{R}^N) \ni (a_1, a_2) \mapsto a_1 a_2 \in S^{m_1+m_2}(\Omega \times \mathbb{R}^N)$$

are continuous. Moreover for any multi-indices $\alpha, \beta \in \mathbb{N}^n \times \mathbb{N}^N$, the mapping

$$S^m(\Omega \times \mathbb{R}^N) \ni a \mapsto \partial_x^\alpha \partial_\theta^\beta a \in S^{m-|\beta|}(\Omega \times \mathbb{R}^N)$$

is continuous.

Definition 6.4.2. *Let Ω be an open subset of $\mathbb{R}^n$, $N \in \mathbb{N}^*$, $\phi \in S^1(\Omega \times \mathbb{R}^N)$. The function ϕ is called a standard phase function on $\Omega \times \mathbb{R}^N$ whenever $\phi \in S^1(\Omega \times \mathbb{R}^N)$ is real-valued and such that, for all K compact subset of Ω, there exists $c_K > 0$ such that*

$$\forall x \in K, \forall \theta \in \mathbb{R}^N \text{ with } |\theta| \geq 1, \quad \left|\frac{\partial \phi}{\partial x}(x,\theta)\right|^2 + |\theta|^2 \left|\frac{\partial \phi}{\partial \theta}(x,\theta)\right|^2 \geq c_K |\theta|^2. \tag{6.4.2}$$

For $a \in S^m(\Omega \times \mathbb{R}^N)$ with $m < -N$ and ϕ a standard phase function, we define

$$T_{a,\phi}(x) = \int e^{i\phi(x,\theta)} a(x,\theta) d\theta \tag{6.4.3}$$

which is a continuous function on Ω ; note also that if $m < -N - k$ with $k \in \mathbb{N}$, $T_{a,\phi}$ belongs to $C^k(\Omega)$.

Theorem 6.4.3. *Let Ω be an open subset of $\mathbb{R}^n$, $m \in \mathbb{R}, N \in \mathbb{N}^*, a \in S^m(\Omega \times \mathbb{R}^N)$ and ϕ be a standard phase function on $\Omega \times \mathbb{R}^N$. Then $T_{a,\phi}$ is a distribution on Ω with order $> m + N$ in the following sense. The mapping*

$$\begin{array}{ccc} C_c^\infty(\Omega) \times S^m(\Omega \times \mathbb{R}^N) & \longrightarrow & \mathbb{C} \\ (u,a) & \mapsto & \iint e^{i\phi(x,\theta)} a(x,\theta) u(x) dx d\theta \end{array} \tag{6.4.4}$$

extends the formula (6.4.3) defined for $m < -N$ in a unique way and continuously.

6.5 Singular integrals, examples

6.5.1 The Hilbert transform

A basic object in the classical theory of harmonic analysis is the Hilbert transform, given by the one-dimensional convolution with $pv(1/\pi x) = \frac{d}{\pi dx}(\ln|x|)$, where we consider here the distribution derivative of the $L^1_{\text{loc}}(\mathbb{R})$ function $\ln|x|$. We can also compute the Fourier transform of $pv(1/\pi x)$, which is given by $-i \operatorname{sign} \xi$. As a result the Hilbert transform $\mathscr{H}$ is a unitary operator on $L^2(\mathbb{R})$ defined by

$$\widehat{\mathscr{H}u}(\xi) = -i \operatorname{sign} \xi \hat{u}(\xi). \tag{6.5.1}$$

It is also given by the formula

$$(\mathscr{H}u)(x) = \lim_{\epsilon \to 0_+} \frac{1}{\pi} \int_{|x-y|\geq \epsilon} \frac{u(y)}{x-y} dy.$$

The Hilbert transform is certainly the first known example of a *Fourier multiplier* ($\mathscr{H}u = F^{-1}(a\hat{u})$ with a bounded a).

6.5.2 The Riesz operators, the Leray-Hopf projection

The Riesz operators are the natural multidimensional generalization of the Hilbert transform. We define for $u \in L^2(\mathbb{R}^n)$,

$$\widehat{R_j u}(\xi) = \frac{\xi_j}{|\xi|}\hat{u}(\xi), \quad \text{so that } R_j = D_j/|D| = (-\Delta)^{-1/2}\frac{\partial}{i\partial x_j}. \tag{6.5.2}$$

The R_j are selfadjoint bounded operators on $L^2(\mathbb{R}^n)$ with norm 1.

We can also consider the $n \times n$ matrix of operators given by $Q = R \otimes R = (R_j R_k)_{1 \le j,k \le n}$ sending the vector space of $L^2(\mathbb{R}^n)$ vector fields into itself. The operator Q is selfadjoint and is a projection since $\sum_l R_l^2 = \mathrm{Id}$ so that $Q^2 = (\sum_l R_j R_l R_l R_k)_{j,k} = Q$. As a result the operator

$$\mathbb{P} = \mathrm{Id} - R \otimes R = \mathrm{Id} - |D|^{-2}(D \otimes D) = \mathrm{Id} - \Delta^{-1}(\nabla \otimes \nabla) \tag{6.5.3}$$

is also an orthogonal projection, the Leray-Hopf projector (a.k.a. the Helmholtz-Weyl projector); the operator $\mathbb{P}$ is in fact the orthogonal projection onto the closed subspace of L^2 vector fields with null divergence. We have for a vector field $u = \sum_j u_j \partial_j$, the identities $\operatorname{grad}\operatorname{div} u = \nabla(\nabla \cdot u)$, $\operatorname{grad}\operatorname{div} = \nabla \otimes \nabla = (-\Delta)(iR \otimes iR)$, so that

$$Q = R \otimes R = \Delta^{-1}\operatorname{grad}\operatorname{div}, \quad \operatorname{div} R \otimes R = \operatorname{div},$$

which implies $\operatorname{div}\mathbb{P}u = \operatorname{div} u - \operatorname{div}(R \otimes R)u = 0$, and if $\operatorname{div} u = 0$, $\mathbb{P}u = u$. The Leray-Hopf projector is in fact the $(n \times n)$-matrix-valued Fourier multiplier given by $\mathrm{Id} - |\xi|^{-2}(\xi \otimes \xi)$. This operator plays an important role in fluid mechanics since the Navier-Stokes system for incompressible fluids can be written for a given divergence-free v_0,

$$\begin{cases} \partial_t v - \nu \triangle v = -\mathbb{P}\nabla(v \otimes v), \\ \mathbb{P}v = v, \\ v_{|t=0} = v_0. \end{cases}$$

As already said for the Riesz operators, $\mathbb{P}$ is not a classical pseudodifferential operator, because of the singularity at the origin: however it is indeed a Fourier multiplier with the same functional properties as those of R.

In three dimensions the curl operator is given by the matrix

$$\operatorname{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = \operatorname{curl}^* \tag{6.5.4}$$

so that $\operatorname{curl}^2 = -\Delta\,\mathrm{Id} + \operatorname{grad}\operatorname{div}$ and (the Biot-Savard law)

$$\mathrm{Id} = (-\Delta)^{-1}\operatorname{curl}^2 + \Delta^{-1}\operatorname{grad}\operatorname{div}, \quad \text{also equal to } \ (-\Delta)^{-1}\operatorname{curl}^2 + \mathrm{Id} - \mathbb{P},$$

which gives $\operatorname{curl}^2 = -\Delta\mathbb{P}$, so that

$$[\mathbb{P}, \operatorname{curl}] = \Delta^{-1}\big(\Delta\mathbb{P}\operatorname{curl} - \Delta\operatorname{curl}\mathbb{P}\big) = \Delta^{-1}\big(-\operatorname{curl}^3 + \operatorname{curl}(-\Delta\mathbb{P})\big) = 0,$$
$$\mathbb{P}\operatorname{curl} = \operatorname{curl}\mathbb{P} = \operatorname{curl}(-\Delta)^{-1}\operatorname{curl}^2 = \operatorname{curl}\big(\mathrm{Id} - \Delta^{-1}\operatorname{grad}\operatorname{div}\big) = \operatorname{curl}$$

since $\operatorname{curl}\operatorname{grad} = 0$ (note also that $\operatorname{div}\operatorname{curl} = 0$).

Theorem 6.5.1. *Let Ω be a function in $L^1(\mathbb{S}^{n-1})$ such that $\int_{\mathbb{S}^{n-1}} \Omega(\omega)d\sigma(\omega) = 0$. Then the following formula defines a tempered distribution T:*

$$\langle T, \varphi\rangle = \lim_{\epsilon\to 0_+} \int_{|x|\geq\epsilon} \Omega\big(\frac{x}{|x|}\big)|x|^{-n}\varphi(x)dx = -\int (x\cdot\partial_x\varphi(x))\Omega\big(\frac{x}{|x|}\big)|x|^{-n}\ln|x|dx.$$

The distribution T is homogeneous of degree $-n$ on $\mathbb{R}^n$ and, if Ω is odd, the Fourier transform of T is a bounded function.

N.B. We shall use the principal-value notation

$$T = pv\Big(|x|^{-n}\Omega\big(\frac{x}{|x|}\big)\Big).$$

When $n = 1$ and $\Omega = \text{sign}$, we recover the principal value $pv(1/x) = \frac{d}{dx}(\ln|x|)$ which is odd, homogeneous of degree -1, and whose Fourier transform is $-i\pi\,\text{sign}\,\xi$.

Proof. Let φ be in $\mathscr{S}(\mathbb{R}^n)$ and $\epsilon > 0$. Using polar coordinates, we check

$$\int_{\mathbb{S}^{n-1}} \Omega(\omega)\int_\epsilon^{+\infty} \varphi(r\omega)\frac{dr}{r}d\sigma(\omega)$$
$$= \int_{\mathbb{S}^{n-1}} \Omega(\omega)\Big[\varphi(\epsilon\omega)\ln(\epsilon^{-1}) - \int_\epsilon^{+\infty} \omega\cdot d\varphi(r\omega)\ln r dr\Big]d\sigma(\omega).$$

Since the mean value of Ω is 0, we get the first statement of the theorem, noticing that the function $x\mapsto \Omega(x/|x|)|x|^{-n+1}\ln(|x|)(1+|x|)^{-2}$ is in $L^1(\mathbb{R}^n)$. We have

$$\langle x\cdot\partial_x T, \varphi\rangle = -\langle T, x\cdot\partial_x\varphi\rangle - n\langle T, \varphi\rangle \tag{$\circledast$}$$

and we see that

$$\langle T, x\cdot\partial_x\varphi\rangle = \lim_{\epsilon\to 0_+}\int_{\mathbb{S}^{n-1}} \Omega(\omega)\int_\epsilon^{+\infty} r\omega\cdot(d\varphi)(r\omega)\frac{dr}{r}d\sigma(\omega)$$
$$= \int_{\mathbb{S}^{n-1}} \Omega(\omega)\int_0^{+\infty} \omega\cdot(d\varphi)(r\omega)dr d\sigma(\omega)$$
$$= \int_{\mathbb{S}^{n-1}} \Omega(\omega)\int_0^{+\infty} \frac{d}{dr}\big(\varphi(r\omega)\big)dr d\sigma(\omega) = -\varphi(0)\int_{\mathbb{S}^{n-1}} \Omega(\omega)d\sigma(\omega) = 0$$

so that ($\circledast$) implies that $x\cdot\partial_x T = -nT$ which is the homogeneity of degree $-n$ of T. As a result the Fourier transform of T is an homogeneous distribution with degree 0.

N.B. Note that the formula

$$-\int (x\cdot\partial_x\varphi(x))\Omega\big(\frac{x}{|x|}\big)|x|^{-n}\ln|x|dx$$

makes sense for $\Omega\in L^1(\mathbb{S}^{n-1}), \varphi\in\mathscr{S}(\mathbb{R}^n)$ and defines a tempered distribution. For instance, if $n = 1$ and $\Omega = 1$, we get the distribution derivative $\frac{d}{dx}\big(\text{sign}\,x\ln|x|\big)$. However, the condition of mean value 0 for Ω on the sphere is necessary to obtain T

as a principal value, since in the discussion above, the term factored out by $\ln(1/\epsilon)$ is $\int_{\mathbb{S}^{n-1}} \Omega(\omega)\varphi(\epsilon\omega)d\sigma(\omega)$ which has the limit $\varphi(0)\int_{\mathbb{S}^{n-1}} \Omega(\omega)d\sigma(\omega)$. On the other hand, from the defining formula of T, we get with $\Omega_j(\omega) = \frac{1}{2}(\Omega(\omega) + (-1)^j\Omega(-\omega))$ (Ω_1(resp.Ω_2) is the odd (resp. even) part of Ω)

$$\langle T, \varphi\rangle = \int_{\mathbb{S}^{n-1}} \Omega_1(\omega)\langle pv(\frac{1}{2t}), \varphi(t\omega)\rangle_{\mathscr{S}'(\mathbb{R}_t),\mathscr{S}(\mathbb{R}_t)}d\sigma(\omega) + \int_{\mathbb{S}^{n-1}} \Omega_2(\omega)\langle \frac{d}{dt}\big(H(t)\ln t\big), \varphi(t\omega)\rangle_{\mathscr{S}'(\mathbb{R}_t),\mathscr{S}(\mathbb{R}_t)}d\sigma(\omega). \quad (6.5.5)$$

Let us show that, when Ω is odd, the Fourier transform of T is bounded. We get

$$\langle \hat{T}, \psi\rangle = \int_{\mathbb{S}^{n-1}} \Omega(\omega)\langle pv(\frac{1}{2t}), \widehat{\psi}(t\omega)\rangle d\sigma(\omega) = -\frac{i\pi}{2}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} \Omega(\omega)\,\mathrm{sign}(\omega\cdot\xi)\varphi(\xi)d\xi d\sigma(\omega)$$

proving that

$$\hat{T}(\xi) = -\frac{i\pi}{2}\int_{\mathbb{S}^{n-1}} \Omega(\omega)\,\mathrm{sign}(\omega\cdot\xi)d\sigma(\omega) \quad (6.5.6)$$

which is indeed a bounded function. □

6.6 Appendix

6.6.1 On the Faà di Bruno formula

That formula[7] is dealing with the iterated derivative of a composition of functions. First of all, let us consider (smooth) functions of one real variable

$$U \xrightarrow{f} V \xrightarrow{g} W, \qquad U, V, W \text{ open sets of } \mathbb{R}.$$

With $g^{(r)}$ always evaluated at $f(x)$, we have

$$\begin{aligned}
(g\circ f)' &= g'f' \\
(g\circ f)'' &= g''f'^2 + g'f'' \\
(g\circ f)''' &= g'''f'^3 + g''3f''f' + g'f''' \\
(g\circ f)^{(4)} &= g^{(4)}(f')^4 + 6g^{(3)}f'^2f'' + g''(4f'''f' + 3f''^2) + g'f^{(4)}
\end{aligned}$$

i.e. $\frac{1}{4!}(g\circ f)^{(4)} = \frac{g^{(4)}}{4!}\left(\frac{f'}{1!}\right)^4 + 3\frac{g^{(3)}}{3!}\left(\frac{f''}{2!}\right)\left(\frac{f'}{1!}\right)^2 + \frac{g^{(2)}}{2!}\Big[\left(\frac{f''}{2!}\right)^2 + 2\frac{f'''}{3!}f'\Big] + \frac{g^{(1)}}{1!}\frac{f^{(4)}}{4!}.$

[7]Francesco Faà di Bruno (1825–1888) was an italian mathematician and priest, born at Alessandria. He was beatified in 1988, probably the only mathematician to reach sainthood so far. The "Chevalier François Faà di Bruno, Capitaine honoraire d'État-Major dans l'armée Sarde", defended his thesis in 1856, in the Faculté des Sciences de Paris in front of the following jury: Cauchy (chair), Lamé and Delaunay.

More generally we have the remarkably simple

$$\frac{(g\circ f)^{(k)}}{k!}=\sum_{1\le r\le k}\frac{g^{(r)}\circ f}{r!}\prod_{\substack{k_1+\dots+k_r=k\\ k_j\ge 1}}\frac{f^{(k_j)}}{k_j!}. \tag{6.6.1}$$

· There is only one multi-index $(1,1,1,1)\in\mathbb{N}^{*4}$ such that $\sum_{1\le j\le 4}k_j=4$.
· There are 3 multi-indices $(1,1,2),(1,2,1),(2,1,1)\in\mathbb{N}^{*3}$ with $\sum_{1\le j\le 3}k_j=4$.
· There is 1 multi-index $(2,2)\in\mathbb{N}^{*2}$ with $\sum_{1\le j\le 2}k_j=4$ and 2 multiindices $(1,3),(3,1)$ such that $\sum_{1\le j\le 2}k_j=4$.
· There is 1 index $4\in\mathbb{N}^*$ with $\sum_{1\le j\le 1}k_j=4$.

Usually the formula is written in a different way with the more complicated

$$\frac{(g\circ f)^{(k)}}{k!}=\sum_{\substack{l_1+2l_2+\dots+kl_k=k\\ r=l_1+\dots+l_k}}\frac{g^{(r)}\circ f}{l_1!\dots l_k!}\prod_{1\le j\le k}\left(\frac{f^{(j)}}{j!}\right)^{l_j}. \tag{6.6.2}$$

Let us show that the two formulas coincide. We start from (6.6.1)

$$\frac{(g\circ f)^{(k)}}{k!}=\sum_{1\le r\le k}\frac{g^{(r)}\circ f}{r!}\prod_{\substack{k_1+\dots+k_r=k\\ k_j\ge 1}}\frac{f^{(k_j)}}{k_j!}.$$

If we consider a multi-index

$$(k_1,\dots,k_r)=(\underbrace{1,\dots,1}_{l_1\text{times}},\underbrace{2,\dots,2}_{l_2\text{times}},\dots,\underbrace{j,\dots,j}_{l_j\text{times}},\dots,\underbrace{k,\dots,k}_{l_{k\text{times}}})$$

we get in factor of $g^{(r)}/r!$ the term $\prod_{1\le j\le k}\left(\frac{f^{(j)}}{j!}\right)^{l_j}$ with $l_1+2l_2+\dots+kl_k=k,\quad l_1+\dots+l_k=r$ and since we can permute the $(k_1,\dots,k_r)$ above, we get indeed a factor $\frac{r!}{l_1!\dots l_k!}$ which gives (6.6.2).

The proof above can easily be generalized to a multidimensional setting with

$$U\xrightarrow{f}V\xrightarrow{g}W,\qquad U,V,W\text{ open sets of }\mathbb{R}^m,\mathbb{R}^n,\mathbb{R}^p,\ f,g\text{ of class }C^k.$$

Since the derivatives are multilinear symmetric mappings, they are completely determined by their values on the "diagonal" $T\otimes\dots\otimes T$: the symmetrized products of $T_1\otimes\dots\otimes T_k$, noted as $T_1\dots T_k$, can be written as a linear combination of k-th powers. In fact, in a commutative algebra on a field with characteristic 0, using the polarization formula, the products $T_1\dots T_k$ are linear combination of k-th powers

$$T_1T_2\dots T_k=\frac{1}{2^kk!}\sum_{\epsilon_j=\pm1}\epsilon_1\dots\epsilon_k(\epsilon_1T_1+\dots+\epsilon_kT_k)^k. \tag{6.6.3}$$

For $T\in\mathcal{T}_x(U)$, we have

$$\frac{(g\circ f)^{(k)}}{k!}T^k=\sum_{1\le r\le k}\frac{g^{(r)}\circ f}{r!}\prod_{\substack{k_1+\dots+k_r=k\\ k_j\ge 1}}\frac{f^{(k_j)}}{k_j!}T^{k_j},$$

which is consistent with the fact that $f^{(k_j)}(x)T^{k_j}$ belongs to the tangent space $\mathcal{T}_{f(x)}(V)$ of V at $f(x)$ and $\otimes_{1\le j\le r}f^{(k_j)}(x)T^{k_j}$ is a tensor product in $\mathcal{T}^{r,0}(\mathcal{T}_{f(x)}(V))$ on which $g^{(r)}(f(x))$ acts to send it on $\mathcal{T}_{g(f(x))}(W)$.

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