The Hunting Group

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The early composers of change ringing music for English church bells were not mathematicians, yet they developed intricate algebraic ideas more than a century before mathematicians independently discovered them. We introduce the mathematical concepts of permutation group and symmetry group by means of elementary change ringing compositions of the 17th and 18th centuries.

Campanalogia Improved:

OR, THE

ART *of* RINGING

MADE EASY,

By Plain and Methodical Rules and Directions, whereby the Ingenious Practitioner may, with a little Practice and Care, attain to the Knowledge of Ringing all Manner of *Double*, *Tripple*, and *Quadruple Changes*.

With Variety of *New Peals* upon Five, Six, Seven, Eight, and Nine Bells. As alſo the Method of calling *Bobs* for any *Peal* of *Tripples* from 168 to 2520 (being the *Half Peal*:) Alſo for any *Peal* of *Quadruples*, or *Cators* from 324 to 1140.

Never before Publiſhed.

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FIGURE 1

Introduction

The mathematical concept of a group, arising out of work of Joseph Louis Lagrange in 1770, was made explicit in the 19th century by Evariste Galois and Augustin-Louis Cauchy. But in 1668 *Tintinnalogia — or the Art of Change Ringing* had been published, followed in 1677 by *Campanalogia* (see Box 1 and Figure 1); Fabian Stedman was involved with both publications. In 1715 the first peal of 'Plain Bob' was rung. All these events involve something called the *hunting group*, and although some bell ringers use the term *group* in a non-technical sense (Stedman used *course*), the technical and non-technical meanings coincide.

The Elements of Change Ringing

We denote the bells in a church tower by **1**, **2**, . . . , **n**, arranged in descending order of pitch from bell 1(the *treble*) to bell **n** (the *tenor*); *n* usually lies between 3 and 10 (inclusive). A *row* is a ringing of these *n* bells, once each, in some order; a *change* is the transition from one row to the next.

BOX 1

Fabian Stedman (1641 - 1713)

Fabian Stedman, the 'father of modern bell ringing', was a printer who lived, worked, composed, and rang, possibly in Cambridge and later in London. In 1668 *Tintinnalogia — or the Art of Change Ringing* was written by Richard Duckworth and printed for Stedman. In 1677 Stedman wrote *Campanalogia, or the Art of Ringing Improved.* Both books were dedicated to the Society of College Youths, the oldest society to promote bell ringing as an aristocratic pastime. In publishing them, Stedman's object was to formalize and set down for posterity the rules and compositions (many of them his own) that had evolved. In dedicating the *Campanalogia*, he wrote:

> The countenance you shew it will silence Detractors, and be Armour of proof against the fools bolts which may happen to be soon shot at the Author, who is *Gentlemen*,
>
> A constant Well-wisher to the
> Prosperity (though an unwor-
> thy member) of your So-
> ciety
>
> F. S.

Both books display striking mathematical insight. At the beginning of *Campanalogia* Stedman referred to "the Art of *Changes*, its Invention being Mathematical and produceth incredible effects, as hereafter will appear".

The central problem in change ringing is to ring an *extent* on the n bells; this is a sequence of $n!$ ($= 1 \times 2 \times ... \times n$) changes connecting $n! + 1$ rows, in such a way that:

(a) the first and last rows are both 1 2 3 . . . n (called *rounds*);

(b) no other row appears more than once;

(c) each change moves every bell by at most one position.

Condition (c) is due to mechanical considerations arising from the mounting of each bell on a wheel (see Figure 2).

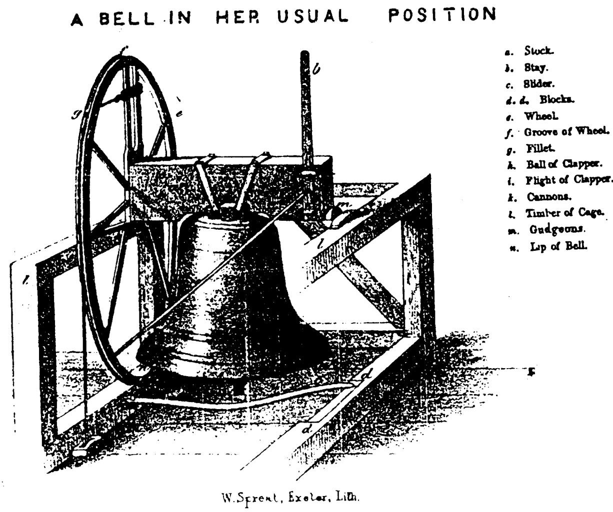


FIGURE 2

Examples of extents are given below. Each extent has a name, part of which (*Singles*, *Minimus*, etc.) reveals the value of n, as indicated in Box 2.

BOX 2

number of bells (n)	*name*	*number of rows (n! + 1)*	*approximate ringing time*
3	Singles	7	6 seconds
4	Minimus	25	30 seconds
5	Doubles	121	3 minutes
6	Minor	721	22 minutes
7	Triples	5041	3 hours
8	Major	40321	24 hours
9	Caters	362881	9 days
10	Royal	3628801	3 months

Singles Extents (3 bells)

Consider the seven rows: **1 2 3**
2 1 3
2 3 1
3 2 1
3 1 2
1 3 2
\- - -
1 2 3

You can easily check conditions (a) and (b) for an extent; for condition (c), note that, to get each successive row, either the first two bells or the last two bells of the previous row are exchanged. This extent is known informally as *Slow Six.* You may like to check that there is only one other Singles extent possible; it consists of the same seven rows, but in the reverse order, and is known as *Quick Six.*

We can represent this extent by a picture, known as a *Cayley diagram* (named after the 19th-century English algebraist Arthur Cayley). The vertices are labelled with the rows, and the connecting lines are labelled with the changes; solid lines correspond to interchanging the first two bells, and dashed lines correspond to interchanging the last two bells (see Figure 3). Traversing the lines in a clockwise direction gives Slow Six; for Quick Six, proceed anti-clockwise.

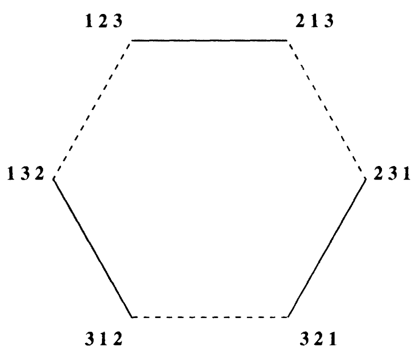
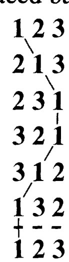


FIGURE 3

Note the path of bell **1,** the treble, in Slow Six; it 'hunts up from position 1 to position 3, 'makes third place', and 'hunts back' to position 1. This process is known as *plain hunting*, and the block of six rows (three positions for bell **1** going up, and three more coming down) is called the *hunting group.* The concept of a hunting group can be traced back to the 17th century.



Using Permutations

We now investigate alternative interpretations of the rows of Slow Six. In Box 3 we describe the properties of permutations, which are relevant to change ringing since we can regard each change as a permutation of the n positions. For example, we can denote the change between the first two rows as (12), since we exchange the bells in positions 1 and 2, ignoring the bell in position 3; similarly, we write (23) (and not (13)) for the change between the second and third rows, since we exchange the bells in positions 2 and 3, ignoring the bell in position 1.

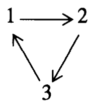
BOX 3

Permutations

A *permutation* of a set S is a one-to-one function from S to itself.

For example, if $S = \{1, 2, 3\}$, then one permutation of S is the function which maps 1 to 2, 2 to 3, and 3 to 1; we denote this by (123), with each number mapped to its successor.

Thus (1 2 3) means



or $\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{array}$.

Similarly, the permutation which maps 1 to 2, 2 to 1, and 3 to 3 is denoted by (12)(3), often abbreviated to (12). If S has n elements, then there are exactly $n!$ permutations; for example, there are 3! permutations of $\{1, 2, 3\}$ — namely, e (the identity permutation), (123), (132), (12), (13) and (23). We combine permutations by function composition, reading from right to left. Thus, to form the 'product' (123) . (12), we note that the right-hand bracket maps 1 to 2, and the left-hand bracket then maps 2 to 3 — so the product maps 1 to 3. Similarly, the product maps 2 (via 1) to 2, and 3 (via 3) to 1. Thus,

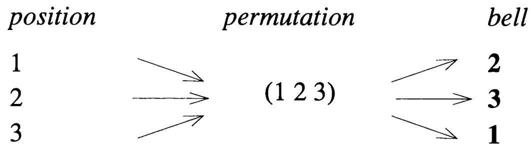
$$(123).(12) = (13).$$

You may like to check similarly that (13).(23) = (132).

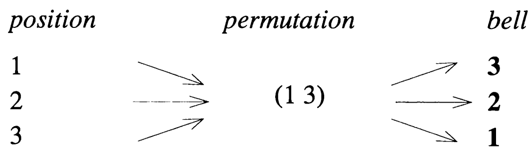
Alternating these two changes gives us the rows of the extent:

row	*change*	*permutation*
1 2 3		e
	(12)	
2 1 3		(12)
	(23)	
2 3 1		(123)
	(12)	
3 2 1		(13)
	(23)	
3 1 2		(132)
	(12)	
1 3 2		(23)
-------	(23)	
1 2 3		

In the last column of this table, we associate a permutation with each row of the extent. For example, row three **2 3 1** yields the permutation (123), since bell **2** is in position 1, bell **3** is in position 2, and bell **1** is in position 3.



Similarly, row four **3 2 1** yields the permutation (13), since bell **3** is in position 1, bell **1** is in position 3, and in bell **2** stays in its original position.



Note that the change (12) links these rows, as can be seen by combining the corresponding permutations:

$$(123)\,.\,(12) = (13).$$

(Remember to multiply from right to left.) Similarly, we get from row 4 with permutation (13) to row 5 with permutation (132) by using the change (23):

$$(13)\,.\,(23) = (132).$$

We can also consider these permutations geometrically as the symmetries (rotations and reflections) of an equilateral triangle with vertices 1, 2, 3. For example, the permutation (123), which maps 1 to 2, 2 to 3, and 3 to 1, corresponds to a clockwise rotation through 120°, whereas the permutation (12), which interchanges 1 and 2 and fixes 3, corresponds to a reflection. The six symmetries of the triangle, together with their permutations, are shown below. We call this set of symmetries S(Δ), the *symmetry group of the triangle*; the term *group* is explained in Box 4.

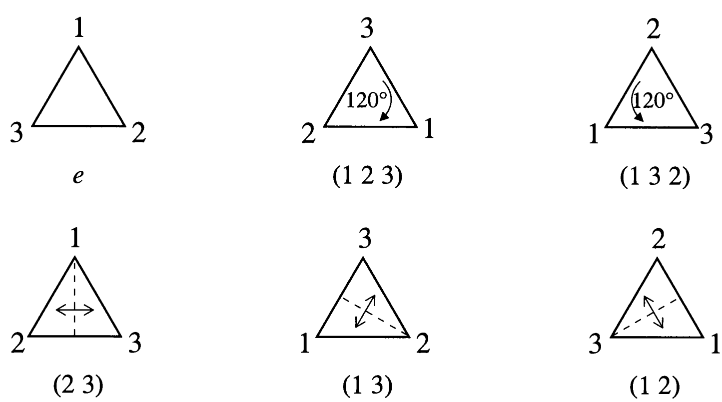


FIGURE 4 Symmetries of a triangle

BOX 4

Groups

A *group* G consists of a set of elements, and a way of combining them, called a *binary operation* (denoted by .), satisfying the following four conditions:

(a) CLOSURE: if x and y are any elements of G, then so is $x.y$;

(b) ASSOCIATIVITY: if x, y and z are any elements of G, then $(x.y).z = x.(y.z)$;

(c) IDENTITY: G has an element e such that, for each x in G, $e.x = x.e = x$;

(d) INVERSES: if x is any element of G, then there is an element y of G such that $x.y = y.x = e$.

Examples of groups are:

(i) the set of positive real numbers with binary operation ×: the identity element is 1 and the inverse of x is $1/x$;

(ii) the set of integers with binary operation + : the identity element is 0 and the inverse of x is $-x$;

(iii) the set of all permutations of $\{1, 2, \ldots, n\}$ with the binary operation of product as described in Box 3: this group is denoted by S_n;

(iv) the set of all symmetries of the equilateral triangle, combined in the obvious way: this group is the same as the group S_3.

Minimus Extents (4 bells)

In extending the above ideas to four bells, we first consider the rows of the hunting group:

1 2 3 4
2 1 4 3
2 4 1 3
4 2 3 1
4 3 2 1
3 4 1 2
3 1 4 2
1 3 2 4
1 2 3 4

To obtain this list, we exchange either the first pair and last pair of bells, or the middle pair of bells. The corresponding Cayley diagram is shown in Figure 5; solid lines correspond to exchanging the outer pairs of bells, and dashed lines correspond to exchanging the middle two bells.

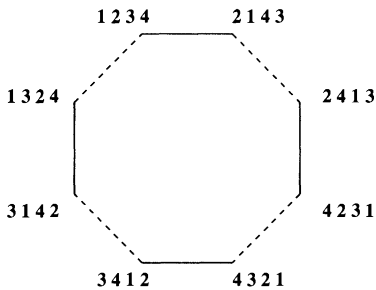


FIGURE 5

We can represent this hunting group using permutations, as above. We get the following table:

row	*change*	*permutation*
1 2 3 4		*e*
	(12)(34)	
2 1 4 3		(12)(34)
	(23)	
2 4 1 3		(1243)
	(12)(34)	
4 2 3 1		(14)
	(23)	
4 3 2 1		(14)(23)
	(12)(34)	
3 4 1 2		(13)(24)
	(23)	
3 1 4 2		(1342)
	(12)(34)	
1 3 2 4		(23)
--------	(23)	
1 2 3 4		

The hunting group permutations on the right also have a geometrical interpretation — as the symmetries of a square with vertices 1, 2, 4, 3 (in that order). The eight symmetries of the square, together with their permutations, are shown in Figure 6. We call this set of symmetries S(□), the *symmetry group of the square.*

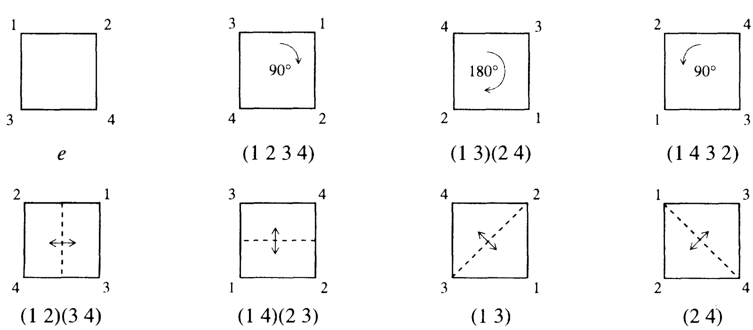
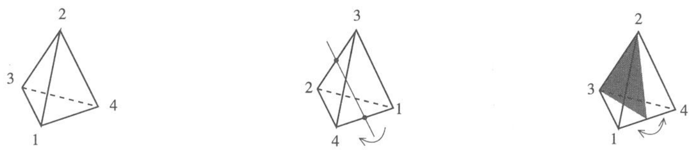


FIGURE 6 Symmetries of a square

Unfortunately, this gives us only eight of the 24 possible rows. In order to complete the extent, we replace the last change (23) by a new change (34) which exchanges only the last two bells. Repeating the earlier changes gives eight more rows, and using (34) again gives the final eight rows. Finally, using (34) a third time brings us back to rounds. The result is as follows:

row	*change*	*permutation*
1 2 3 4		*e*
	(12)(34)	
2 1 4 3		(12)(34)
	(23)	
2 4 1 3		(1243)
	(12)(34)	
4 2 3 1		(14)
	(23)	
4 3 2 1		(14)(23)
	(12)(34)	
3 4 1 2		(13)(24)
	(23)	
3 1 4 2		(1342)
	(12)(34)	
1 3 2 4		(23)
--------	(34)	--------
1 3 4 2		(234)
	(12)(34)	
3 1 2 4		(132)
	(23)	
3 2 1 4		(13)
	(12)(34)	
2 3 4 1		(1234)
	(23)	
2 4 3 1		(124)
	(12)(34)	
4 2 1 3		(143)
	(23)	
4 1 2 3		(1432)
	(12)(34)	
1 4 3 2		(24)
--------	(34)	--------
1 4 2 3		(243)
	(12)(34)	
4 1 3 2		(142)
	(23)	
4 3 1 2		(1423)
	(12)(34)	
3 4 2 1		(1324)
	(23)	
3 2 4 1		(134)
	(12)(34)	
2 3 1 4		(123)
	(23)	
2 1 3 4		(12)
	(12)(34)	
1 2 4 3		(34)
--------	(34)	
1 2 3 4		

The rows on the left give the extent known as 'Plain Bob Minimus'. The permutations on the right are all the possible permutations of 1, 2, 3, 4, and form a group called the *symmetric group* S_4. This also has a geometrical interpretation, as the symmetries of a regular tetrahedron. For example, the permutation (14)(23) is a rotation about a line joining midpoints of opposite edges, and the permutation (14) is a reflection about the plane through the line joining the vertices 2 and 3 (see Figure 7).



tetrahedron (14)(23) (14)

FIGURE 7

There is an interesting connection between the three sets of eight permutations on page 13. If we combine the permutation (234) at the top of the second set with each permutation in the first set, we get the corresponding permutations in the second set; for example,

$$(234)\,.\,(12)(34) = (132).$$

Similarly, if we combine the permutation (243) at the top of the third set with each permutation in the first set, we get the corresponding permutation in the third set; for example,

$$(243)\,.\,(1243) = (1423).$$

These sets are called *cosets*. The first coset is the hunting group S(□). The second coset is obtained by combining each element of S(□) with (234), and is written (234)S(□). The third coset is obtained by combining each element of S(□) with (243), and is written (243)S(□). Between them, these cosets give all 24 elements of the symmetric group S_4. (You might like to investigate what you get if you form cosets from other permutations; try, for example, (132)S(□), (1324)S(□), and (14)S(□).) This decomposition of S_4 into cosets was known to bell ringers more than a century before mathematicians discovered it.

We conclude by drawing the Cayley diagram for Plain Bob Minimus. Here, in Figure 8, solid lines correspond, as before, to the change (12)(34), dotted lines correspond to the change (23), and dashed lines correspond to the change (34). The shaded octagons correspond to the three cosets. To simplify the picture, we have used half-edges, joined as indicated by the Greek letters. You might like to trace the extent in this picture.

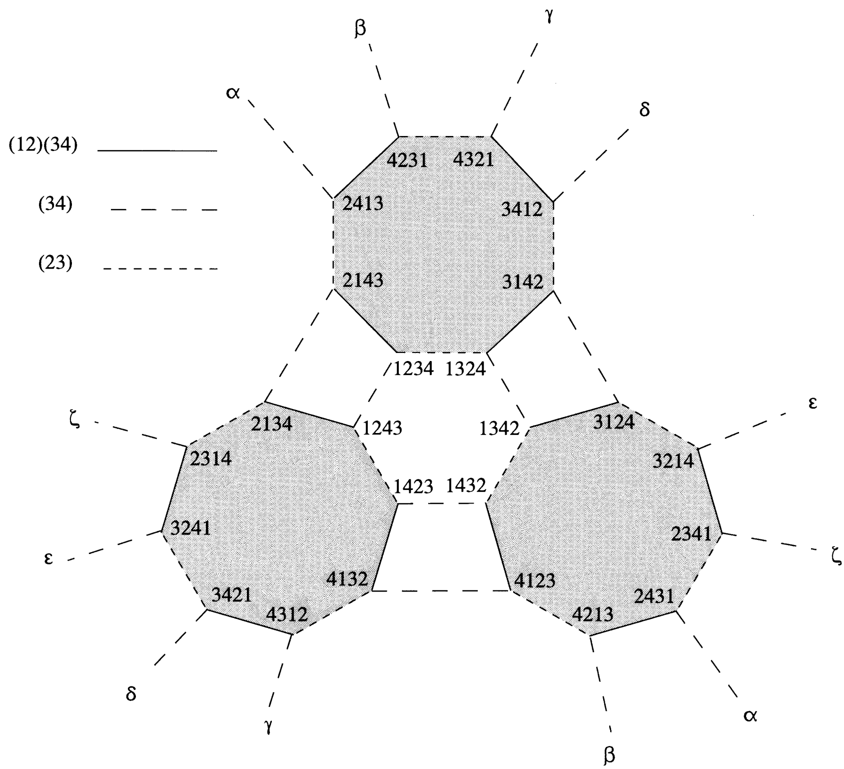


FIGURE 8

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And then you go through the 720 elements of S_6.