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Low-density series expansions for directed percolation on square and triangular lattices

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Abstract. Greatly extended series have been derived for moments of the pair-connectedness for bond and site percolation on the directed square and triangular lattices. The length of the various series has been at least doubled to more than 110 (100) terms for the square-lattice bond (site) problem and more than 55 terms for the bond and site problems on the triangular lattice. Analysis of the series leads to very accurate estimates for the critical parameters and generally seems to rule out simple rational values for the critical exponents. The values of the critical exponents for the average cluster size, parallel and perpendicular connectedness lengths are estimated by $\gamma = 2.277\,69(4)$, $\nu_{\|} = 1.733\,825(25)$ and $\nu_{\perp} = 1.096\,844(14)$, respectively. An improved estimate for the percolation probability exponent is obtained from the scaling relation $\beta = (\nu_{\|} + \nu_{\perp} - \gamma)/2 = 0.276\,49(4)$. In all cases the leading correction to scaling term is analytic.

1. Introduction

Models exhibiting critical behaviour similar to directed percolation (DP) are encountered in a wide variety of problems such as fluid flow in porous media, Reggeon field theory, chemical reactions, population dynamics, catalysis, epidemics, forest fires, and even galactic evolution. Directed percolation is thus a model of relevance to a very diverse set of physical problems and it is therefore no wonder that it continues to attract a great deal of attention. Furthermore, two-dimensional directed percolation is one of the simplest models which is not translationally invariant and therefore cannot be treated in the framework of conformal field theory [1]. This leaves open a number of fundamental questions about this model. What should one expect an exact solution to look like and more concretely are the critical exponents rational?

In the absence of an exact solution the most powerful method for studying lattice-statistics models is probably that of series expansions. The method of exact series expansions consists of calculating the first few coefficients in the Taylor expansion of various thermodynamic functions, or, in more abstract terms, various moments of some appropriate generating function. Given such a series, highly accurate estimates can be obtained for the critical parameters using differential approximants [2]. In the most favourable cases one can even find an exact expression for the generating function from the first-series coefficients.

Low-density series in the variable p, which is the probability that bonds or sites are present, were first derived by Blease [3], who used a transfer-matrix method to calculate series for the cluster size and other moments of the pair-connectedness of bond percolation

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on directed square and triangular lattices. These series were greatly extended by Essam *et al* [4], who also studied site percolation. They devised a non-nodal graph expansion, which enabled them to calculate twice as many terms correctly from the basic transfer-matrix calculation, and derived the series to order 49 (48) for the square bond (site) problem and to order 25 (26) for the triangular bond (site) problem. These long series resulted in accurate exponent estimates and led to the conjectured critical exponents $\gamma = 41/18$, $\nu_\perp = 79/72$, $\nu_\parallel = 26/15$, and $\beta = 199/720$ [4].

High-density series for the percolation probability were derived by Blease [3]. The square bond series was greatly extended by Baxter and Guttmann [5] using a superior transfer-matrix method and an extrapolation procedure based on predicting correction terms from successive calculations on finite lattices of increasing size. The analysis of the resulting series conformed to the conjectured fraction for β. This series and the one for the square site problem were recently extended by Jensen and Guttmann [6] who also studied the triangular bond and site problems [7]. The analysis of these extended series yielded more precise exponent estimates. From these estimates they concluded that there are no simple rational fractions whose decimal expansion agrees with the highly accurate estimates of β obtained from the square bond and triangular site series. In particular, the rational fraction suggested by Essam *et al* [4] is incompatible with the estimates.

In this paper I combine an efficient transfer-matrix calculation with the non-nodal graph expansion and the above-mentioned extrapolation method and have been able to more than double the number of series terms for moments of the pair-connectedness. Most of the series have been extended to order 112 for the square bond problem, 106 for the square site problem, 57 for the triangular bond problem and 56 for the triangular site problem. The series were analysed using differential approximants which can accommodate a wide variety of functional features and certainly should be appropriate in this case. The major result of the analysis is that the exact exponent values conjectured by Essam *et al* [4] generally seems to be incompatible with the numerical estimates from the differential approximant analysis.

The remainder of the article is organized as follows. In section 2 I will give further details of the models studied in this paper. Section 3 contains a description of the series-expansion technique with special emphasis on the transfer-matrix calculation (section 3.1) and the extrapolation procedure for the square bond case (section 3.3). Details of the extrapolation procedure for the remaining problems are given in the appendix. Details of the series analysis are given in section 4 and the results are discussed and summarized in section 5.

2. Specification of the models

Domany and Kinzel [8] demonstrated that site and bond percolation on the directed square lattice are special cases of a one-dimensional stochastic cellular automaton in which the preferred direction t is time. DP is thus a model for a simple branching process in which a site x occupied at time t may give rise to zero or one offspring on each of the sites $x \pm 1$ at time $t+1$. Whether a site (x, t) is occupied or not depends only on the state of its nearest neighbours in the row above. The evolution of the model on the square lattice is therefore governed by the conditional probabilities $P(\sigma_x|\sigma_l, \sigma_r)$, with $\sigma_i = 1$ if site i is occupied and 0 otherwise. These transition probabilities are the probabilities of finding the site (x, t) in state σ_x given that the sites $(x-1, t-1)$ and $(x+1, t-1)$ were in states σ_l and σ_r, respectively. One has a very free hand in choosing the transition probabilities as long as one respects conservation of probability, $P(1|\sigma_l, \sigma_r) = 1 - P(0|\sigma_l, \sigma_r)$. In addition studies have generally been limited to cases in which the transition probabilities are independent of both x and t. In this paper I restrict my study to the following two cases corresponding to bond and site percolation:

$$P(0|\sigma_l, \sigma_r) = \begin{cases} (1-p)^{\sigma_l+\sigma_r} & \text{bond} \\ (1-p)^{1-(1-\sigma_l)(1-\sigma_r)} & \text{site.} \end{cases} \tag{2.1}$$

On the triangular lattice the model is described by the probabilities $P(\sigma_x|\sigma_l, \sigma_t, \sigma_r)$ of finding the site (x, t) in state σ_x given that the sites $(x-1, t-1)$, $(x, t-2)$, and $(x+1, t-1)$ were in states σ_l, σ_t and σ_r, respectively, and I study the two cases

$$P(0|\sigma_l, \sigma_t, \sigma_r) = \begin{cases} (1-p)^{\sigma_l+\sigma_t+\sigma_r} & \text{bond} \\ (1-p)^{1-(1-\sigma_1)(1-\sigma_t)(1-\sigma_r)} & \text{site.} \end{cases} \tag{2.2}$$

The behaviour of the model is controlled by the branching probability p. When p is smaller than a critical value p_c the branching process eventually dies out and all space–time clusters remain finite. For $p > p_\mathrm{c}$ there is a non-zero probability $P(p)$ that the branching process will survive indefinitely. This percolation probability is the order parameter of the process, and close to p_c it vanishes as a power-law:

$$P(p) \propto (p - p_c)^\beta \qquad p \to p_c^+. \tag{2.3}$$

In the low-density phase $(p < p_c)$ many quantities of interest can be derived from the pair-connectedness $C_{x,t}(p)$, which is the probability that the site x is occupied at time t given that the origin was occupied at $t = 0$. The moments of the pair-connectedness may be written as

$$\mu_{n,m}(p) = \sum_{t=0}^{\infty} \sum_x x^n t^m C_{x,t}(p). \tag{2.4}$$

Due to symmetry, moments involving odd powers of x vanish. The remaining moments diverge as p approaches the critical point from below:

$$\mu_{n,m}(p) \propto (p_c - p)^{-(\gamma+n\nu_\perp+m\nu_\parallel)} \qquad p \to p_c^-. \tag{2.5}$$

One generally only studies the lower-order moments such as the mean cluster size $S(p) = \mu_{0,0}(p)$, the first parallel moment $\mu_{0,1}(p)$, the second perpendicular moment $\mu_{2,0}(p)$, and the second parallel moment $\mu_{0,2}(p)$.

3. Series expansions

From (2.4) it follows that the first and second moments can be derived from the quantities

$$S(t) = \sum_x C_{x,t}(p) \qquad \text{and} \qquad X(t) = \sum_x x^2 C_{x,t}(p) \tag{3.1}$$

as

$$S = \sum_{t=0}^{\infty} S(t) \qquad \mu_{0,1} = \sum_{t=1}^{\infty} tS(t) \qquad \mu_{0,2} = \sum_{t=1}^{\infty} t^2 S(t) \qquad \mu_{2,0} = \sum_{t=0}^{\infty} X(t). \tag{3.2}$$

$S(t)$ and $X(t)$ are polynomials in p obtained by summing the pair-connectedness over all lattice sites whose parallel distance from the origin is t. As shown by Essam [9] the pair-connectedness can be expressed as a sum over all graphs formed by taking unions of directed paths connecting the origin to the site (x, t),

$$C_{x,t}(p) = \sum_g d(g) p^e \tag{3.3}$$

where e is the number of random elements (bonds or sites) in the graph g. Any directed path to a site whose parallel distance from the origin is t contains at least $m(t)$ steps with $m(t) = t$ for the square lattice and $m(t) = \lfloor (t+1)/2 \rfloor$ (integer division) for the triangular lattice. From this it follows that if $S(t)$ and $X(t)$ have been calculated for $t \leqslant t_{\max}$ then one can determine the moments to order $m(t_{\max}+1) - 1$. One can, however, do much better, as demonstrated by Essam *et al* [4]. They used a non-nodal graph expansion, based on work by Bhatti and Essam [10], to extend the series to order $n(t_{\max})$ approximately equal to $2m(t_{\max})$ (the actual order varies a little from problem to problem). Details of this expansion will be given below, but here it will suffice to note that it works by calculating the contributions $S^N(t)$ and $X^N(t)$ (correct to order $n(t)$) of non-nodal graphs to $S(t)$ and $X(t)$ and using the non-nodal expansions to calculate the final series for $S(p)$ and the various moments. Further extensions of the series can be obtained by using a procedure similar to that of Baxter and Guttmann [5]. One looks at correction terms to the series and tries to identify extrapolation formulae for the first n_r correction terms allowing one to derive a further n_r series terms correctly.

The series expansions for moments of the pair-connectedness is thus obtained as follows:

(i) Calculate the polynomials $S(t)$ and $X(t)$ for $t \leqslant t_{\max}$ using the transfer-matrix technique to an order greater than $n(t_{\max}) + n_r$.

(ii) For each t use the non-nodal graph expansion to calculate $S_t^N = \sum_{t' \leqslant t} S^N(t')$ and $X_t^N = \sum_{t' \leqslant t} X^N(t')$ correct to order $n(t)$.

(iii) From the sequences obtained from $S_t^N - S_{t+1}^N = -S^N(t+1)$ and $X_t^N - X_{t+1}^N = -X^N(t+1)$ for $t < t_{\max}$ identify the first n_r correction terms.

(iv) Use these correction terms to extend the series for S^N and X^N to order $n(t_{\max}) + n_r$.

(v) Finally calculate the series for S, $\mu_{0,1}$, $\mu_{0,2}$ and $\mu_{2,0}$ correct to order $n(t_{\max}) + n_r$.

Details of the transfer-matrix technique, non-nodal graph expansion and extrapolation procedure are given in the following sections.

3.1. Transfer-matrix technique

Figure 1 shows the part of the square and triangular lattices which can be reached from the origin O using no more than five steps. Note that, in keeping with the prescription used by Essam *et al* [4], vertical steps on the triangular lattice correspond to incrementing t by two. The calculation of the pair-connectedness is readily turned into an efficient computer algorithm by use of the transfer-matrix technique. From (2.1) and (2.2) one sees that the evaluation of the pair-connectedness involves only local 'interactions' since the

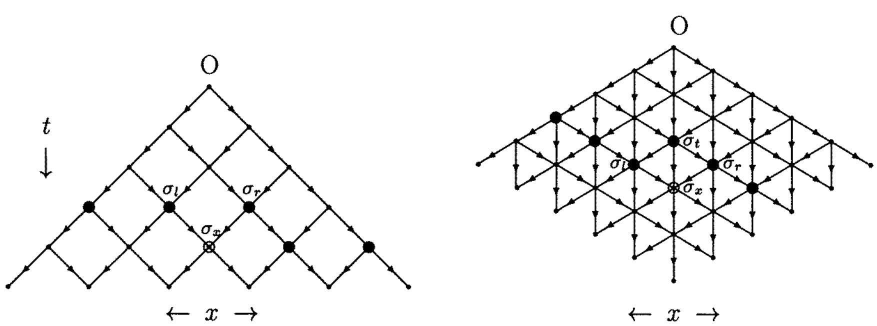

Figure 1. Directed square and triangular lattices with orientation given by the arrows.

transition probabilities depend on neighbouring sites only. The probability of finding a given configuration can therefore be calculated by moving a boundary through the lattice one site at a time. At any given stage this line cuts through a number of, say k, lattice sites thus leading to a total of 2^k possible configurations along this line. Configurations along the boundary line are trivially represented as binary numbers, and the probability of each configuration is represented by a truncated polynomial in p.

Figure 1 shows how the boundary (marked by large filled circles) is moved in order to pick up the weight associated with a given 'face' of the lattice at a position x along the boundary line. On the square lattice the boundary site at σ_r is moved to σ_x and the weight $P(\sigma_x|\sigma_l,\sigma_r)$ is picked up. Similarly on the triangular lattice the boundary site at σ_t is moved to σ_x while picking up the weight $P(\sigma_x|\sigma_l,\sigma_t,\sigma_r)$. In more detail, let $S0=(\sigma_1,\ldots,\sigma_{x-1},0,\sigma_{x+1},\ldots,\sigma_k)$ be the configuration of sites along the boundary with 0 at position x and similarly $S1=(\sigma_1,\ldots,\sigma_{x-1},1,\sigma_{x+1},\ldots,\sigma_k)$ the configuration with 1 at position x. Then in moving the x'th site as just described the boundary line polynomials are updated as follows on the square lattice

$$P(S0)=W(0|0,\sigma_l)P(S0)+W(0|1,\sigma_l)P(S1)$$
$$P(S1)=W(1|0,\sigma_l)P(S0)+W(1|1,\sigma_l)P(S1)$$

and as follows on the triangular lattice

$$P(S0)=W(0|\sigma_r,0,\sigma_l)P(S0)+W(0|\sigma_r,1,\sigma_l)P(S1)$$
$$P(S1)=W(1|\sigma_r,0,\sigma_l)P(S0)+W(1|\sigma_r,1,\sigma_l)P(S1).$$

The pair-connectedness is calculated from the boundary polynomials before the boundary leaves the site by summing over all configurations with a 1 at that site. In practise the data was collected when the boundary reached a horizontal position on the square lattice and a position parallel to the right edge of the triangular lattice. The pair-connectedness is obviously symmetrical in x, $C_{x,t}(p)=C_{-x,t}(p)$, so it suffices to calculate the pair-connectedness for $x\geqslant 0$. More importantly, due to the directedness of the lattices, if one looks at sites (x,t) with $x\geqslant 0$ they can never be reached by paths extending onto points (x',t') in the part of the lattice for which $t'>\lfloor t/2\rfloor$, $x'<-\lfloor t/2\rfloor$. This effectively means that the pair-connectedness at points with parallel distance t from the origin can be calculated using a boundary which cuts through at most $\lfloor t/2\rfloor+1$ sites. Thus the memory (and time) required to derive $S(t)$ and $X(t)$ grows like $2^{\lfloor t/2\rfloor+1}$.

For the bond and site problems on the square lattice I was able to calculate the pair-connectedness up to $t_{\max}=47$ and for the triangular lattice up to $t_{\max}=45$. Since the integer coefficients occurring in the series expansion become very large the calculation was performed using modular arithmetic [11]. Each run for $t_{\max}$, using a different prime number, took approximately 12 hours using 64 nodes on an Intel Paragon, and up to eight primes were needed to represent the coefficients correctly. The major limitation of the present calculation was available computer memory rather than time.

3.2. Non-nodal graph expansion

The non-nodal graph expansion has been described in detail in [4] and here I will only summarise the main points and introduce some notation. A graph g is nodal if there is a point (other than the terminal point) through which all paths pass. It is clear that each such nodal point effectively works as a new origin for the cluster growth. This is the essential idea behind the non-nodal graph expansion. $S^N(t)$ is the contribution to $S(t)$ obtained by restricting the sum in (3.3) to non-nodal graphs. The non-nodal expansions are obtained recursively from the polynomials $S(t)$ and $X(t)$. First one sets $S^N(1) = S(1)$ and $X^N(1) = X(1)$ and then for $2 \leqslant t \leqslant t_{\max}$ one calculates $S^N(t)$ and $X^N(t)$ from

$$S^N(t) = S(t) - \sum_{t'=1}^{t-1} S^N(t')S(t-t') \tag{3.4}$$

and

$$X^N(t) = X(t) - \sum_{t'=1}^{t-1}[S^N(t')X(t-t') + X^N(t')S(t-t')]. \tag{3.5}$$

Next form the sums of (3.2) using the truncated non-nodal polynomials $S^N(t)$ and $X^N(t)$ instead of $S(t)$ and $X(t)$. The final series are then obtained from the formulae

$$S = 1/(1 - S^N) \tag{3.6}$$

$$\mu_{0,1} = \mu_{0,1}^N S^2 \tag{3.7}$$

$$\mu_{0,2} = [\mu_{0,2}^N + 2(\mu_{0,1}^N)^2 S]S^2 \tag{3.8}$$

$$\mu_{2,0} = \mu_{2,0}^N S^2. \tag{3.9}$$

3.3. *Extrapolation procedure*

When forming the sums (3.2) one could have stopped the summation at any t prior to reaching $t_{\max}$ and used the formulae above to derive the series correct to order $n(t)$. Let S_t^N and X_t^N denote the non-nodal expansions obtained in this fashion. As observed by Baxter and Guttmann [5] one can often extend the series considerably by looking at correction terms to such series. The polynomials $S(t)$ and $X(t)$, and thus likewise the non-nodal expansions, will obviously contain terms of much higher order than that to which the final series is correct. One can therefore look at the difference between successive expansions, e.g.

$$S_t^N - S_{t+1}^N = -S^N(t+1) = p^{n(t+1)} \sum_{r\geqslant 0} s_{t,r} p^r \tag{3.10}$$

which yields sequences of numbers $s_{t,r}$ with $t < t_{\max}$. As observed in [5] the first sequence of numbers $s_{t,0}$ is often quite simple and can readily be conjectured so that a closed form expression or a simple recurrence relation can be found. In the following I will give the details of how this is done in the square bond case. The treatment of the other problems are detailed in the appendix. Note, that if one can find the first n_r correction terms one can use $S_{t_{\max}}^N = \sum_{m\geqslant 0} a_{N,m} p^m$ to extend the series $S^N = \sum_{m\geqslant 0} a_m p^m$ to order $n(t_{\max}) + n_r$, via

$$a_{n(t_{\max})+1+k} = a_{N,n(t_{\max})+1+k} - \sum_{m=0}^{\lfloor k/2 \rfloor} s_{t_{\max}+m,k-2m}. \tag{3.11}$$

So in order to find the correct series term $a_{n(t_{\max})+1+k}$ from the 'partial' term $a_{N,n(t_{\max})+1+k}$ one first subtracts $s_{t_{\max},k}$ which yields correctly the term $a_{N+1,n(t_{\max}+1)-1+k}$. One continues this process until arriving at $a_{N+\lfloor k/2\rfloor+1,n(t_{\max}+\lfloor k/2\rfloor+1)-q}$, where $q = 1(0)$ if k is even (odd), which is the correct term in the series for S^N.

In the square bond case the first sequence of correction terms start out as

$$s_{t,0} = 1, 2, 5, 14, 42, 132, 429, \ldots$$

which is immediately recognizable as the Catalan numbers $C_t = (2t)!/(t!(t+1)!)$. These also occurred as the first correction term for the percolation probability series [5]. There is a very simple combinatorial proof for the first correction term. The first correction term arises from the simplest (containing the minimum number of random elements) non-nodal graphs terminating at level $t+1$. These graphs are also the ones giving the first term of $S^N(t+1)$. It is obvious that these graphs are composed of two paths of length $t+1$ each, which meet at level $t+1$ but does not cross earlier. These graphs are in one-to-one correspondence with *staircase polyominoes* (or *polygons*) and it is well known that the latter are enumerated by the Catalan numbers [12, 13].

As was the case for the percolation probability series the higher-order correction terms can be expressed as rational functions of $s_{t,0}$. For S^N these extrapolation formulae are

$$s_{t,r} = \frac{2^r}{16\lfloor r/2\rfloor !} \sum_{k=1}^{\lfloor r/2\rfloor} b_{r,k}(2t)^k C_{t-r+2} + \sum_{j=1}^{2r} a_{r,j} C_{t-r+j} \qquad t \geqslant r \tag{3.12}$$

which are very similar to the formulae found in the percolation probability case [5]. The extrapolation formulae for $\mu^N_{0,1}$ and $\mu^N_{0,2}$ are simply $(t+1)s_{t,r}$ and $(t+1)^2 s_{t,r}$, respectively.

The factor in front of the first sum has been chosen so as to make the leading coefficients particularly simple. I was able to find formulae for all correction terms up to $r = 16$. The coefficients in the extrapolation formulae are listed in table 1.

From (3.12) it is clear that the $t_{\max} - r$ terms available in the sequences for the correction terms are not sufficient to determine all the $2r + \lfloor r/2\rfloor$ unknown coefficients of the extrapolation formulae for large r. However, from table 1 one immediately sees that the leading coefficients $a_{r,2r}$ and $b_{r,\lfloor r/2\rfloor}$ in the extrapolation formulae are very simple In particular one has, $(-1)^r a_{r,2r} = 2$, and

$$b_{r,\lfloor r/2\rfloor} = \begin{cases} (-1)^{\lfloor r/2\rfloor}(r-9) & r \text{ odd} \\ (-1)^{\lfloor r/2\rfloor} & r \text{ even.} \end{cases}$$

Likewise, $a_{r,1}$ is zero for $r > 2$. In general I find that the leading coefficients $a_{r,2r-m}$ are expressible as polynomials in r of order m:

$$(-1)^r a_{r,2r-m} = \begin{cases} -4r & r > 0, m = 1 \\ 4r^2 - 10 & r > 2, m = 2 \\ -8r^3/3 + 80r/3 - 40 & r > 4, m = 3 \\ 4r^4/3 - 100r^2/3 + 86r - 48 & r > 6, m = 4 \\ -8r^5/15 + 80r^3/3 - 92r^2 - 62r/15 + 350 & r > 8, m = 5. \end{cases}$$

So when calculating the coefficients listed in table 1 I first used the sequences for the correction terms to predict as many of the extrapolation formulae (3.12) as possible. Then I predicted as many of the leading coefficients as possible. This in turn allowed me to find more extrapolation formulae, which I used to find more of the formulae for the leading coefficients $a_{r,2r-m}$. I repeated this until the process stopped with the extrapolation formulae listed in table 1.

For X^N the sequence determining the first correction formula starts out as

$$x_{t,0} = 0, 2, 8, 30, 112, 420, 1584, 6006, 22\,880, \ldots$$

from which one sees that $x_{t,0} = 2(t-1)C_{t-1}$. The proof of this formula is a little more involved. First one needs the number of configurations, $w(t,x)$, of two non-crossing paths terminating at (x,t). Essam and Guttmann [14] gives a formula for the number of non-crossing watermelon configurations with p chains which join s steps and at height q from the origin

$$w_s(0) = 1 \qquad w_s(s-q) = w_s(q)$$

Table 1. The coefficients $a_{r,j}$ and $b_{r,k}$ in the extrapolation formulae for S^N in the square bond problem.

	$a_{r,j}$															
r/j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	4	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	−2	$7\frac{1}{2}$	−14	52	−772	8396	−104 432	1 087 360	−11 675 848	117 363 472	−1 206 800 312	12 123 014 088	−122 433 360 736	1 219 851 293 868 $\frac{4}{5}$	−12 122 717 004 480	119 511 222 145 568
3		−8	24	−48	−60	282	−2588	22 370	−193 832	1 695 688	−15 086 824	135 051 980	−1 214 979 776	10 953 873 032	−9 899 560 396	895 836 211 184
4		2	−26	110	−140	−30	−858	5455	−49 296	428 060	−3 811 900	34 074 102	−306 557 008	2 761 883 296	−24 957 565 640	225 709 412 166
5			12	−102	352	−316	−132	806	−13 960	105 294	−970 136	8 576 744	−77 439 024	691 132 356	−6 293 972 216	56 869 637 848
6			−2	54	−392	1090	−1582	732	−3504	23 242	−256 572	2 132 332	−19 644 478	175 222 526	−1 588 285 518	14 327 820 664
7				−16	240	−1368	4072	−4980	−1404	5678	−65 048	526 194	−5 002 064	44 102 064	−400 539 832	3 613 132 526
8				2	−90	998	−5092	14 364	−12 582	7548	−25 206	120 268	−1 287 920	10 991 864	−101 688 652	907 714 555
9					20	−456	3992	−19 094	44 144	−48 124	44 728	72 284	−316 072	2 784 916	−25 377 216	230 241 986
10					−2	134	−2122	16 004	−67 204	150 272	−211 740	69 021	66 134	1 052 566	−6 237 298	57 545 347
11						−24	768	−9392	62 452	−236 716	551 884	−620 424	36 808	−220 248	−1 052 472	17 113 494
12						2	−186	3968	−40 390	237 804	−861 506	1 851 702	−1 856 298	1 529 676	−1 552 350	458 569
13							28	−1192	19 192	−168 552	903 432	−3 068 002	6 065 788	−7 534 902	5 408 144	5 296 402
14							−2	246	−6774	88 790	−687 670	3 402 918	−10 712 930	21 555 948	−27 034 302	7 985 377
15								−32	1744	−35 558	396 164	−2 761 138	12 625 252	−38 084 370	75 851 240	−80 523 984
16								2	−314	10 812	−176 520	1 717 822	−10 913 378	46 662 216	−135 911 846	253 167 388
17									36	−2440	61 228	−839 098	7 280 308	−42 583 794	172 181 172	−476 657 120
18									−2	390	−16 386	324 908	−3 853 692	30 262 590	−164 637 226	630 128 108
19										−40	3296	−99 578	1 639 988	−17 211 632	123 782 008	−630 765 580
20										2	−474	23 832	−562 680	7 955 196	−75 101 678	499 439 206
21											44	−4328	154 688	−3 006 340	37 359 148	−321 314 788
22											−2	566	−33 518	927 428	−15 361 860	170 769 484
23												−48	5552	−231 406	5 229 684	−75 687 360
24												2	−666	45 844	−1 467 208	28 080 208
25													52	−6984	335 412	−8 708 108
26													−2	774	−61 242	2 242 268
27														−56	8640	−473 282
28														2	−890	80 176
29															60	−10 536
30															−2	1014
31																−64
32																2

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1		−1	6	−37	176	−1955	9898	−166 600	796 424	−17 178 537	79 858 876	−2 144 369 019	9 971 872 629	−318 781 058 794 $\frac{1}{2}$	1 476 852 478 036 $\frac{1}{2}$	−54 261 648 660 378
2				1	−4	30	−150	1766	−10 896	183 506	−951 764	20 902 321	−101 995 034	2 914 183 463	−13 744 040 652 $\frac{1}{2}$	472 954 813 060
3						−1	2	−20	112	−1142	11 614	−134 103	1 076 040	−17 820 094	121 408 458	−3 073 413 820
4								1	0	19	−64	547	−12 680	−3508	−1 366 634	1 944 614
5										−1	−2	−39	−24	−928	12 624	228 338
6												1	4	92	214	4475
7														−1	−6	−190
8																1

and

$$w_s(q) = \prod_{i=1}^{q} \frac{(p+i)_{s-2i+1}}{(i)_{s-2i+1}} \qquad 1 \leqslant q \leqslant \lfloor s/2 \rfloor \tag{3.13}$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$, is Pochhammer's symbol. A watermelon configuration with two chains is in one-to-one correspondence with the configuration obtained from the two non-crossing paths by deleting the two bonds connected to the origin and the two bonds connected to the terminal point, so that $w(t,x) = w_{t-2}(x)$. In the case $p = 2$ (3.13) reduces to a simple product of binomial coefficients,

$$\begin{aligned} w_s(q) &= \prod_{i=1}^{q} \frac{(s-i+2)(s-i+1)}{i(i+1)} = \frac{s!(s+1)!}{(s-q)!q!(s+1-q)!(q+1)!} \\ &= \frac{1}{s+2}\binom{s}{q}\binom{s+2}{q+1}. \end{aligned} \tag{3.14}$$

The correction term $s_{t,0}$ can easily be derived from (3.14) as (remembering that $s_{t,0}$ arises from paths terminating at level $t+1$)

$$\begin{aligned} s_{t,0} &= \sum_{q=0}^{t-1} w_{t-1}(q) = \frac{1}{t+1}\sum_{q=0}^{t-1}\binom{t-1}{q}\binom{t+1}{q+1} \\ &= \frac{1}{t+1}\sum_{q=0}^{t}\binom{t-1}{q}\binom{t+1}{t-q} = \frac{1}{t+1}\binom{2t}{t} = C_t. \end{aligned}$$

In this derivation I have used only standard properties of binomial coefficients, the main one being the formula

$$\sum_{q=0}^{p}\binom{m}{q}\binom{n}{p-q} = \binom{m+n}{p}. \tag{3.15}$$

After this little diversion I return to the calculation of $x_{t,0}$. From (3.1) and the measurement of x with respect to the centre line it is clear that

$$x_{t,0} = \sum_{q=0}^{s}(s-2q)^2 w_s(q) \tag{3.16}$$

where $s = t-1$. By simple expansion of the square and insertion of $w_s(q)$ one finds

$$\begin{aligned} x_{t,0} = \frac{1}{s+2}\Bigg[& s^2\sum_{q=0}^{s+1}\binom{s}{q}\binom{s+2}{q+1} - 4s\sum_{q=0}^{s+1} q\binom{s}{q}\binom{s+2}{q+1} \\ & +4\sum_{q=0}^{s+1} q(q+1)\binom{s}{q}\binom{s+2}{q+1} - 4\sum_{q=0}^{s+1} q\binom{s}{q}\binom{s+2}{q+1}\Bigg] \\ = & \frac{1}{s+2}\left[s^2\binom{2s+2}{s+1} - 4s^2\binom{2s+1}{s+1} - 4s(s+2)\binom{2s}{s} - 4s\binom{2s+1}{s+1}\right] \\ = & \frac{1}{s+2}\left[-\frac{2s^2(2s+1)}{s+1}\binom{2s}{s} + 4s(s+2)\binom{2s}{s} - \frac{4s(2s+1)}{s+1}\binom{2s}{s}\right] \\ = & \frac{1}{(s+2)(s+1)}\binom{2s}{s}[2s^2+4s] = \frac{2s}{(s+1)}\binom{2s}{s} \\ = & 2sC_s = 2(t-1)C_{t-1}. \end{aligned}$$

The major step was the use of (3.15) to get rid of the sum over q. For the rest of the calculations I only used the definition and well known properties of the binomial coefficients.

In this case I find that the general extrapolation formulae can be written as

$$x_{t,r} = \frac{2^r}{16\lfloor r/2\rfloor!}\sum_{k=1}^{\lfloor r/2\rfloor+1} b_{r,k}(2t)^k C_{t-r+2} + \sum_{j=0}^{2r} a_{r,j} C_{t-r+j} \qquad t \geqslant r. \tag{3.17}$$

The coefficients are not reproduced here due to the excessive length of this material, but are available from the author (please see end of article for details). Again I found that the leading coefficients are very simple, so a procedure similar to that used to find more extrapolation formulae for S^N was applied for X^N also. Though in this case it is slightly more complicated because different polynomials are found for $a_{r,2r-m}$ depending on whether r is odd or even. I was able to find the extrapolation formulae for $r \leqslant 15$.

From the polynomials for $S^N(t_{\max})$ and $X^N(t_{\max})$, using the extrapolation formulae given above, I extended the series for $S(p)$, $\mu_{0,1}(p)$ and $\mu_{0,2}(p)$ to order 112 and the series for $\mu_{2,0}(p)$ to order 111. The new series terms are listed in table 2, while the terms for $n \leqslant 49$ can be found in [4]. The full series are available from the author via e-mail or can be retrieved from the authors homepage on the world wide web (see later for details).

For the square site problem I have identified the first 12 extrapolation formulae for S^N and the first nine for X^N. This allowed me to derive the series correctly to order 106 and 103, respectively. For the triangular bond and site cases the first 10–12 extrapolation formulae were found and the series calculated to orders 55–57 depending on the particular problem. Details of the extrapolation formulae and lists of the new series coefficients can be found in the appendix. The full series and tables of the coefficients in the extrapolation formulae can be obtained from the author.

4. Analysis of the series

In the vicinity of the critical point one expects the moments of the pair-connectedness to have the functional form

$$f(p) \propto A(p_c - p)^{\lambda}[1 + a_1(p_c - p)^{\Delta_1} + b_1(p_c - p)\ldots] \tag{4.1}$$

where λ is the critical exponent, Δ_1 the leading confluent exponent and the ... represents higher-order correction terms. By universality we expect λ to be the same for all the percolation problems. In addition to the physical singularity, the series may have non-physical singularities for other values (real or complex) of p.

The series for moments of the pair-connectedness were analysed using inhomogeneous first- and second-order differential approximants. A comprehensive review of these and other techniques for series analysis may be found in [2]. Here it suffices to say that a Kth-order differential approximant to a function f is formed by matching the earliest series coefficients to an inhomogeneous differential equation of the form (see [2] for details)

$$\sum_{i=0}^{K} Q_i(x)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^i f(x) = P(x) \tag{4.2}$$

where Q_i and P are polynomials of order N_i and L, respectively. First- and second-order approximants are denoted by $[L/N_0; N_1]$ and $[L/N_0; N_1; N_2]$, respectively.

Table 2. New series terms for the directed square lattice bond problem.

n	$S(p)$	$\mu_{0,1}(p)$	$\mu_{0,2}(p)$	$\mu_{2,0}(p)$
50	−48 816 119 038	11 801 670 105 578	1 619 393 474 185 766	27 794 063 081 342
51	507 516 102 724	33 055 165 149 064	3 063 931 985 169 024	54 920 977 045 280
52	−288 652 716 240	27 869 200 356 228	4 530 325 110 201 816	73 258 860 229 496
53	1 605 880 660 392	96 170 461 301 080	8 892 704 619 221 536	154 245 664 038 528
54	−1 407 950 918 758	58 847 785 748 014	12 476 033 918 538 246	189 153 100 033 446
55	5 398 489 609 494	288 365 269 158 218	25 899 537 405 464 346	436 835 649 689 930
56	−6 021 475 295 246	97 008 272 891 722	33 711 579 420 868 182	474 309 443 770 870
57	17 915 929 204 078	876 853 221 827 434	75 639 045 971 965 390	1 248 873 201 075 582
58	−23 161 191 351 438	50 270 991 328 638	89 157 533 500 835 018	1 142 258 426 763 018
59	61 169 203 195 260	2 742 424 862 540 904	222 615 251 058 740 148	362 093 554 078 700
60	−91 439 492 617 463	−723 012 645 772 984	226 410 239 178 311 060	2 540 682 041 470 492
61	218 285 935 121 478	8 945 610 206 297 122	665 257 166 510 500 110	10 729 171 422 690 574
62	−347 041 940 934 654	−5 091 807 702 556 172	541 873 450 068 575 656	4 813 181 710 705 328
63	75 582 536 721 926	29 441 230 893 756 258	2 010 803 687 079 582 486	32 414 565 156 737 718
64	−1 261 522 730 127 947	−24 604 605 804 865 004	1 176 137 623 037 120 136	5 583 933 472 771 488
65	2 689 697 586 459 424	100 083 593 993 221 016	6 208 781 157 063 955 092	100 528 453 740 276 036
66	−4 794 978 299 078 876	−111 027 801 572 997 440	1 897 872 187 352 474 044	−13 398 245 182 310 812
67	9 873 705 455 451 962	353 256 305 942 487 862	19 749 039 440 486 959 110	322 040 908 558 415 270
68	−17 606 769 359 805 002	−459 124 803 459 589 112	105 921 802 167 944 744	−141 155 953 736 298 432
69	34 685 584 933 271 312	1 234 649 044 784 083 520	63 823 159 209 011 263 356	1 053 196 692 821 964 284
70	−63 346 329 725 838 982	−1 803 990 875 049 717 410	−18 787 876 064 221 921 686	−760 886 807 616 650 166
71	126 576 386 179 363 762	4 457 869 599 502 824 958	213 199 421 030 557 203 290	3 546 162 218 978 100 650
72	−238 791 893 310 090 455	−7 204 198 205 577 806 878	−130 294 082 472 485 176 236	−3 521 825 272 581 984 064
73	467 217 890 189 754 678	16 419 837 227 409 088 034	735 449 584 170 612 356 710	12 284 194 787 984 123 846
74	−865 360 273 580 474 576	−27 618 071 407 049 240 332	−648 894 890 087 745 222 380	−14 870 112 157 423 507 452
75	1 655 020 489 419 904 522	59 215 007 852 286 252 798	2 558 081 110 403 875 257 118	42 945 484 977 991 237 294
76	−3 119 681 720 859 651 798	−104 269 518 320 642 632 294	−2 872 616 792 616 193 864 740	−59 746 354 645 402 475 464
77	6 112 229 358 703 831 342	220 308 940 252 364 053 854	9 196 867 775 386 117 146 210	153 618 586 695 190 985 346
78	−11 754 183 721 345 954 258	−404 350 946 017 058 554 676	−12 360 072 007 022 536 761 656	−237 460 263 100 122 622 008
79	22 597 239 603 197 843 510	825 284 068 524 839 748 354	33 739 282 207 965 719 236 902	558 339 048 175 747 451 206
80	−42 254 381 215 339 002 849	−1 512 209 154 805 053 886 454	−50 141 941 500 909 233 943 898	−917 262 309 953 119 861 762
81	80 119 633 205 161 441 704	3 008 597 412 927 623 407 944	123 045 017 109 279 192 315 256	2 023 409 847 652 367 652 792
82	−153 436 526 269 872 506 166	−5 661 354 126 139 476 495 002	−199 404 059 032 602 758 054 790	−3 497 872 081 717 444 367 406
83	299 742 770 352 697 886 058	11 376 987 638 602 404 205 186	462 621 148 772 929 893 023 982	7 468 540 111 543 307 136 334
84	−578 005 275 119 339 317 137	−21 748 117 195 128 953 695 678	−799 093 581 590 590 191 030 632	−13 423 468 537 971 564 505 180
85	1 100 376 713 100 175 425 834	42 666 489 134 233 272 441 382	1 745 525 522 573 249 273 895 934	27 702 351 077 321 825 033 806
86	−2 066 519 690 614 778 360 502	−80 452 042 465 425 106 274 566	−3 091 636 958 239 698 242 569 606	−50 534 526 731 865 521 375 910
87	3 925 426 563 659 158 745 246	156 482 448 914 584 874 236 898	6 508 930 727 244 009 005 368 374	101 918 197 947 493 977 841 846
88	−7 570 287 289 675 980 312 099	−301 476 120 742 919 711 572 632	−11 977 317 344 882 408 349 739 708	−190 086 876 471 603 772 883 468
89	14 770 206 114 483 585 630 780	595 928 021 892 468 292 003 228	24 964 572 420 431 393 417 069 916	381 418 739 444 284 933 316 252
90	−28 497 105 408 805 663 534 168	−1 153 611 368 793 245 492 912 948	−46 972 868 730 035 864 908 413 600	−721 567 973 001 941 669 604 264
91	54 009 873 057 404 488 263 124	2 230 172 227 389 674 189 568 676	95 014 521 823 798 847 778 682 224	1 423 864 146 941 093 990 943 952
92	−101 365 149 804 767 013 432 178	−4 238 623 392 738 344 821 239 874	−178 540 440 608 962 610 328 762 650	−2 690 665 938 166 916 889 494 170
93	193 074 470 702 855 611 598 800	8 194 551 426 018 374 360 988 480	357 068 364 097 506 426 338 276 644	5 267 639 463 907 912 905 453 732
94	−375 465 307 728 947 308 049 038	−15 966 490 078 269 042 239 928 778	−687 579 693 042 527 922 973 062 762	−10 093 017 091 775 195 821 161 034
95	733 587 080 957 649 030 952 780	31 419 252 633 404 837 133 144 864	1 382 496 577 727 085 057 659 832 724	19 846 440 181 751 841 888 348 276
96	−1 407 768 320 341 892 431 455 597	−60 854 835 125 808 366 150 603 264	−2 671 473 287 753 792 887 166 131 898	−38 165 367 702 608 542 662 852 262
97	2 652 453 424 628 111 858 120 636	116 710 563 412 971 455 236 833 084	5 252 303 428 933 538 484 371 763 852	74 090 674 609 046 304 004 729 820
98	−4 994 997 189 815 654 309 285 716	−222 700 149 867 630 979 856 555 884	−10 070 886 824 842 706 773 951 858 088	−141 654 537 468 965 192 312 439 616
99	9 582 116 900 498 277 108 211 678	431 233 968 917 196 829 158 559 222	19 830 249 071 310 192 476 250 960 630	274 936 632 555 726 697 937 295 702
100	−18 695 928 027 022 491 233 374 283	−844 128 796 374 222 622 704 429 022	−38 800 460 642 107 452 115 957 062 464	−532 127 310 827 238 890 555 549 348
101	36 447 747 150 709 546 344 466 562	1 656 288 019 513 322 011 385 494 706	77 005 465 410 557 975 976 009 279 278	1 039 178 034 809 963 112 448 701 838
102	−69 751 142 361 738 816 206 100 980	−3 203 512 282 371 391 967 581 738 048	−149 882 328 321 413 389 925 795 707 828	−2 007 176 273 662 763 046 778 255 692
103	131 093 981 974 714 374 047 295 196	6 119 858 028 807 821 975 019 141 708	291 426 808 439 994 369 970 860 912 820	3 872 225 062 856 423 137 844 592 132
104	−246 912 382 538 210 356 128 227 719	−11 686 384 832 378 651 095 002 246 250	−562 231 983 817 634 749 999 483 821 520	−7 431 921 350 302 739 474 421 979 228
105	476 269 621 022 570 452 892 936 354	22 695 145 396 282 470 359 093 537 754	1 102 918 492 775 428 103 319 535 730 226	14 398 589 108 038 667 956 855 208 338
106	−93 592 019 593 491 769 435 721 118	−44 650 872 404 026 806 263 517 170 226	−2 173 673 482 315 575 515 684 047 562 710	−27 996 025 755 985 946 126 762 621 566
107	1 822 666 367 955 366 954 226 762 322	87 475 964 091 663 148 670 303 074 082	4 292 233 544 563 601 832 800 911 011 570	54 550 461 477 489 119 415 528 104 754
108	−3 457 328 237 704 527 379 069 614 957	−168 248 261 202 406 774 396 896 258 028	−8 344 957 626 439 769 709 378 845 906 902	−105 346 747 734 498 192 654 664 703 386
109	6 468 620 061 451 349 324 632 525 978	320 141 983 848 608 665 933 961 797 186	16 131 167 712 769 833 203 421 125 262 258	202 541 716 970 409 485 480 895 800 850
110	−12 274 653 298 845 615 056 223 573 114	−613 827 858 236 772 855 763 836 272 346	−31 237 041 224 868 511 036 559 013 283 630	−389 508 487 345 526 842 037 950 714 262
111	23 895 824 638 927 458 824 334 426 734	1 198 741 273 733 824 166 575 265 793 142	61 365 773 705 437 232 962 411 451 241 006	755 678 002 297 838 255 419 395 153 550
112	−46 949 709 528 735 587 230 164 873 730	−2 360 701 178 771 867 028 398 496 651 684	−121 215 418 908 857 920 604 650 167 026 140	

4.1. The square bond series

In this section I will give a detailed account of the analysis of the square bond series which leads to the most accurate estimates. The analysis of the series for the other problems are described summarily in the following sections. In addition to the moment series I have also analysed the series $\mu_{0,2}(p)/\mu_{0,1}(p) \sim (p_c - p)^{-\nu_\parallel}$ and the series $\mu_{2,0}(p)\mu_{0,2}(p)/(\mu_{0,1}(p))^2 \sim (p_c - p)^{-2\nu_\perp}$.

In order to locate the singularities of the series in a systematic fashion I used the following procedure: I calculate all $[L/N; M]$ and $[L/N; M; M]$ first- and second-order inhomogeneous differential approximants with $|N - M| \leqslant 1$ and $L \leqslant 35$, which use more than 95 or 90 terms, respectively. Each approximant yields M possible singularities and associated exponents from the M zeroes of Q_1 or Q_2, respectively (many of these are of course not actual singularities of the series but merely spurious zeros.) Next these zeroes are sorted into equivalence classes by the criterion that they lie at most a distance 2^{-k} apart. An equivalence class is accepted as a singularity if it contains more than N_c approximants, and an estimate for the singularity and exponent is obtained by averaging over the approximants (the spread among the approximants is also calculated). I used $N_c = 20$ (15) for first-order (second-order) approximants, which means that at least two-thirds to three-quarters of all approximants had to be included before an equivalence class was accepted. The calculation was then repeated for $k - 1$, $k - 2, \ldots$ until a minimal value of 8 or so was reached. To avoid outputting well-converged singularities at every level, once an equivalence class has been accepted, the approximants which are members of it are removed, and the subsequent analysis is carried out on the remaining data only. One advantage of this method is that spurious outliers, a few of which will almost always be present when so many approximants are generated, are discarded systematically and automatically.

In table 3 I have listed the estimates for the physical critical point p_c and the associated exponents obtained from the six series that I studied. The errors listed in the parentheses are calculated from the spread among the approximants and equals one standard deviation. Note that these error estimates should *not* be seen as accurately representing the true errors. N_a is the number of approximants included in the estimates.

Generally the estimates for various orders L of the inhomogeneous polynomial are exceptionally well converged and excellent agreement is observed both between the various estimates for each series as well as between the p_c-estimates from the different series. Apart from the first-order approximants for small L to $\mu_{2,0}(p)\mu_{0,2}(p)/(\mu_{0,1}(p))^2$ all estimates for p_c are consistent with the highly accurate value $p_c = 0.644\,700\,15(15)$. This slight discrepancy is not important since one generally would expect large L first-order approximants and second-order approximants to yield more reliable estimates. These approximants are better at dealing with analytic background terms or other features which might possibly slow down the convergence of the estimates to the true critical values. Further note that N_a generally is well above the cut-off N_c showing that in most cases only a few approximants are discarded. The uncertainty in the last digits of the p_c-estimate, given in parentheses, is probably on the conservative side, and is mostly due to the tendency of $\mu_{0,1}$ and $\mu_{0,2}$ to favour a somewhat lower estimate for the critical point.

Before proceeding I will consider possible sources of systematic errors. First and foremost the possibility that the estimates might display a systematic drift as the number of terms used is increased and secondly the possibility of numerical errors. The latter possibility is quickly dismissed. The calculations were performed using 128-bit real numbers (REAL*16 on an IBM RISC work station). The estimates from a few approximants were compared to values obtained using MAPLE with up to 100 digits accuracy and this clearly

Table 3. Estimates of p_c and critical exponents for the square bond problem.

	First-order DA			Second-order DA		
L	p_c	γ	N_a	p_c	γ	N_a
0	0.644 700 51(60)	2.278 32(77)	25	0.644 700 181(37)	2.277 716(30)	22
5	0.644 700 18(72)	2.278 07(71)	25	0.644 700 169(26)	2.277 708(23)	18
10	0.644 700 04(13)	2.277 602(93)	26	0.644 700 158(41)	2.277 703(34)	23
15	0.644 700 136(29)	2.277 665(56)	23	0.644 700 146(29)	2.776 90(23)	20
20	0.644 700 102(21)	2.277 649(21)	24	0.644 700 146(17)	2.277 689(14)	18
25	0.644 700 097(49)	2.277 646(42)	23	0.644 700 149(20)	2.277 693(15)	21
30	0.644 700 108(29)	2.277 659(24)	26	0.644 700 162(12)	2.277 704(11)	16
35	0.644 700 129(21)	2.277 678(15)	21	0.644 700 29(22)	2.277 92(42)	22
L	p_c	$\nu_\parallel$	N_a	p_c	$\nu_\parallel$	N_a
0	0.644 700 153(12)	1.733 818 4(50)	22	0.644 700 169(97)	1.733 845(45)	19
5	0.644 700 154(31)	1.733 818(12)	27	0.644 700 178(50)	1.733 846(28)	16
10	0.644 700 115(11)	1.733 807 1(35)	22	0.644 700 171 8(88)	1.733 836 2(42)	20
15	0.644 700 142(33)	1.733 819(21)	22	0.644 700 136(50)	1.733 813(34)	18
20	0.644 700 162(14)	1.733 831 9(78)	25	0.644 700 154(23)	1.733 827(11)	19
25	0.644 700 149(24)	1.733 824(11)	25	0.644 700 142(13)	1.733 821 3(67)	18
30	0.644 700 155 7(63)	1.733 827 9(31)	23	0.644 700 122(34)	1.733 806(25)	21
35	0.644 700 150 3(61)	1.733 825 4(32)	22	0.644 700 164(20)	1.733 831 2(92)	20
L	p_c	$2\nu_\perp$	N_a	p_c	$2\nu_\perp$	N_a
0	0.644 700 40(13)	2.193 828(55)	22	0.644 700 196(17)	2.193 711(11)	17
5	0.644 700 438(94)	2.193 843(36)	22	0.644 700 192(18)	2.193 708(10)	18
10	0.644 700 41(17)	2.193 826(95)	22	0.644 700 174(47)	2.193 703(29)	17
15	0.644 700 147(17)	2.193 685 2(79)	22	0.644 700 163(23)	2.193 693(12)	18
20	0.644 700 201(17)	2.193 712 6(82)	23	0.644 700 217(40)	2.193 722(22)	16
25	0.644 700 200(10)	2.193 713 2(54)	23	0.644 700 192(28)	2.193 708(13)	16
30	0.644 700 196(10)	2.193 710 7(51)	23	0.644 700 183(12)	2.193 703 9(64)	17
35	0.644 700 195(14)	2.193 711 0(69)	23	0.644 700 182(15)	2.193 703 1(84)	18
L	p_c	$\gamma + \nu_\parallel$	N_a	p_c	$\gamma + \nu_\parallel$	N_a
0	0.644 700 091(76)	4.011 423(76)	24	0.644 700 091(32)	4.011 434(35)	18
5	0.644 700 042(74)	4.011 375(65)	25	0.644 700 095(20)	4.011 440(23)	18
10	0.644 700 023(97)	4.011 361(79)	25	0.644 700 079(37)	4.011 413(44)	20
15	0.644 700 071(72)	4.011 403(73)	24	0.644 700 105(47)	4.011 455(50)	20
20	0.644 700 015(66)	4.011 350(57)	26	0.644 700 096(32)	4.011 443(34)	18
25	0.644 700 04(15)	4.011 39(15)	21	0.644 700 096(63)	4.011 440(73)	19
30	0.644 700 037(68)	4.011 370(59)	24	0.644 700 101(21)	4.011 448(22)	19
35	0.644 700 038(54)	4.011 369(49)	23	0.644 700 090(20)	4.011 438(22)	18
L	p_c	$\gamma + 2\nu_\parallel$	N_a	p_c	$\gamma + 2\nu_\parallel$	N_a
0	0.644 700 043(87)	5.745 15(10)	24	0.644 700 079(19)	5.745 208(29)	18
5	0.644 700 079(96)	5.745 20(13)	24	0.644 700 084(25)	5.745 224(35)	16
10	0.644 700 05(11)	5.745 17(13)	21	0.644 700 075(29)	5.745 208(37)	17
15	0.644 700 11(10)	5.745 25(17)	22	0.644 700 075(17)	5.745 213(25)	22
20	0.644 700 051(27)	5.745 156(34)	24	0.644 700 087(38)	5.745 232(51)	17
25	0.644 700 13(17)	5.745 31(32)	25	0.644 700 082(22)	5.745 225(32)	18
30	0.644 700 068(45)	5.745 180(57)	21	0.644 700 082(25)	5.745 231(50)	18
35	0.644 699 99(10)	5.745 10(11)	25	0.644 700 091(45)	5.745 231(75)	19

Table 3. (Continued)

	First-order DA			Second-order DA		
L	p_c	$\gamma + 2\nu_\perp$	N_a	p_c	$\gamma + 2\nu_\perp$	N_a
0	0.644 700 081 9(37)	4.471 298 8(18)	22	0.644 700 119(52)	4.471 341(57)	20
5	0.644 700 080 6(26)	4.471 298 1(13)	23	0.644 700 117(21)	4.471 329(20)	17
10	0.644 700 085 7(78)	4.471 301 7(62)	24	0.644 700 115(46)	4.471 332(46)	16
15	0.644 700 138(69)	4.471 36(10)	21	0.644 700 094(68)	4.471 319(50)	16
20	0.644 700 101(24)	4.471 315(21)	23	0.644 700 132(40)	4.471 351(42)	16
25	0.644 700 101(29)	4.471 316(25)	25	0.644 700 101(16)	4.471 314(14)	16
30	0.644 700 112(21)	4.471 324(19)	21	0.644 700 121(42)	4.471 340(46)	19
35	0.644 700 119(17)	4.471 330(16)	21	0.644 700 114(41)	4.471 334(44)	18

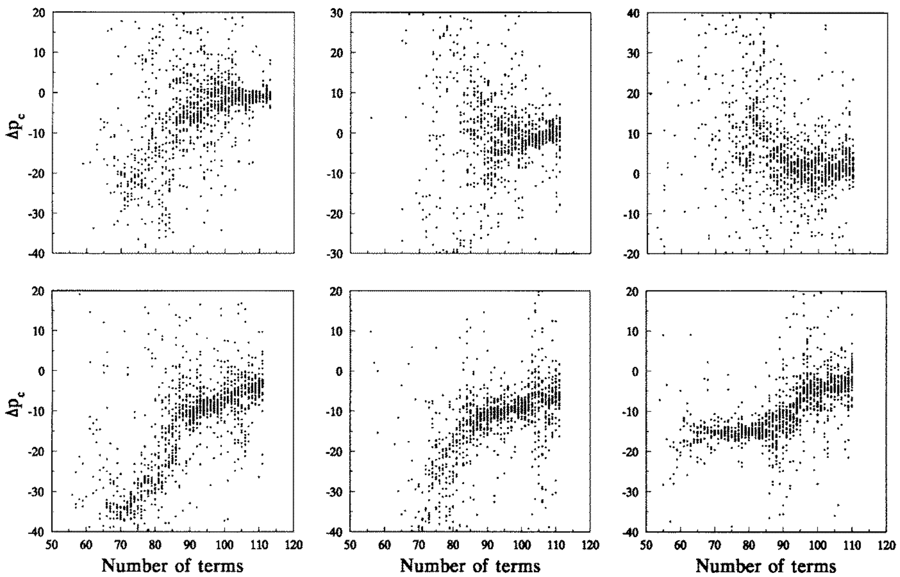

Figure 2. The deviation in the last two digits, $10^8\Delta p_c$, from the central estimate of the critical point $p_c = 0.644\,700\,15$, of the estimates for the critical point by second-order differential approximants. Shown is (from left to right and top to bottom) estimates from the series $S(p)$, $\mu_{0,2}(p)/\mu_{0,1}(p)$, $\mu_{2,0}(p)\mu_{0,2}(p)/(\mu_{0,1}(p))^2$, $\mu_{0,1}(p)$, $\mu_{0,2}(p)$, and $\mu_{2,0}(p)$.

showed that the program was numerically stable and rounding errors were negligible. In order to address the possibility of systematic drift and lack of convergence to the true critical values I refer to figure 2. In this figure I have plotted the deviation in the last two digits, $10^8\Delta p_c$, from the critical point $p_c = 0.644\,700\,15$. Included in the figure are estimates from inhomogeneous second-order differential approximants with $L \leqslant 35$ to the six series that I have studied. From this figure it is evident that the series estimates displayed on the top row are well converged once the number of terms exceeds 90 or so, while the series on the bottom row still show evidence of a systematic drift and the estimates have not yet converged to their asymptotic value. This is particularly manifest for the series $\mu_{0,1}$ and $\mu_{0,2}$ shown in the bottom left and central panels. Since these series were the ones responsible for most of the error on the estimate for p_c, and given the very good convergence of the estimates from the series shown in the top row, it does not seem overly optimistic to adopt the tighter estimate $p_c = 0.647\,700\,15(5)$. Clearly the large majority of estimates for the first three series lie well within this error-bound as the number of terms increase and likewise the estimates from the remaining series clearly seem to converge towards this value.

Next I turn my attention to the estimates for the critical exponents. Very precise estimates for γ, $\nu_\parallel$, and $2\nu_\perp$ can be obtained by examining table 3. I have used a slightly more systematic and enlightening procedure. Close to the critical point there is an apparent linear dependence of the estimates for critical exponents on the estimates for p_c. One can use this to obtain improved estimates for the exponents by performing a linear fit of the exponent estimates as a function of Δp_c (the distance from the critical point). The result of such linear fits is listed below. In these fits I used the same set of approximants as those on which the estimates in the tables above were based. But I discarded any approximant for which $|\Delta p_c| = |p_c - 0.644\,700\,15| > 0.000\,000\,15$. The error on the 'pure' exponent part of the estimates mainly reflects the slight difference between the first- and second-order approximants (the errors as listed are approximately twice this difference). In the estimates for γ and $\gamma + 2\nu_\perp$ I used only the first-order approximants with $L \geqslant 15$.

$$
\begin{aligned}
\gamma &= 2.277\,690(10) \pm 750\Delta p_c \\
\nu_\parallel &= 1.733\,824(3) \pm 500\Delta p_c \\
2\nu_\perp &= 2.193\,687(2) \pm 500\Delta p_c \\
\gamma + \nu_\parallel &= 4.011\,495(15) \pm 1150\Delta p_c \\
\gamma + 2\nu_\parallel &= 5.745\,308(15) \pm 1400\Delta p_c \\
\gamma + 2\nu_\perp &= 4.471\,368(3) \pm 1000\Delta p_c.
\end{aligned}
\tag{4.3}
$$

As can be seen the exponent estimates are very precise. Even with the very small error in the p_c-estimate, this is still the major source of error (by an order of magnitude) in the exponent estimates. As previously noted [6], there is no simple rational fraction whose decimal expansion agrees with the estimate of β obtained from the percolation-probability series. The same is true for the estimates of $\nu_\parallel$ and $2\nu_\perp$ listed above. In particular note that the rational fraction suggested by Essam *et al* [4], $\nu_\parallel = 26/15 = 1.733\,333\ldots$, and $2\nu_\perp = 79/36 = 2.194\,44\ldots$, is incompatible with the estimates. The rational fraction suggested for $\gamma = 41/18 = 2.277\,777\ldots$ lies within the error bounds for the exponent estimate if the error on p_c exceeds 10^{-7}. So the more conservative error estimate listed earlier would just include the suggested value of γ. However, most of the estimates in table 3 clearly exclude the exact fraction as does the more narrow error estimate on p_c. Finally I note that the better converged estimates for $\gamma + 2\nu_\perp$ and $2\nu_\perp$ yields the estimate $\gamma = 2.277\,681(5)$, which, within the error, agrees with the direct estimate but points to a possibly slightly lower value of γ.

The estimate for p_c advocated above lies within the error-bounds of that obtained from the percolation probability series [6] $p_c = 0.644\,700\,6(10)$, though a lower central value is favoured by the series analysed in this paper. From the scaling relation $\beta = (\nu_\parallel + \nu_\perp - \gamma)/2$ I obtain the estimate $\beta = 0.276\,489(7) \pm 750\Delta p_c$, which is consistent with the direct estimate $\beta = 0.276\,43(10)$. It is quite likely that the minor discrepancies between the central values would disappear if the percolation probability series could be extended from the 55 terms in [6] to an order comparable to the series analysed here. Evidence to this effect is provided by the biased estimate $\beta = 0.276\,483(14)$ calculated at $p_c = 0.644\,700\,15$ using Dlog Padé approximants utilizing at least 45 terms of the percolation-probability series.

I also analysed the series in order to estimate the leading confluent exponents Δ_1. As was the case for the percolation-probability series both the Baker–Hunter transformation and the method of Adler, Moshe and Privman (see [6] and references therein for details regarding these methods) yielded estimates consistent with $\Delta_1 = 1$. So there are no signs of non-analytic corrections to scaling.

Finally I looked for non-physical singularities of the series. The series have a singularity on the negative axis closer to the origin than p_c. This singularity is quite weak and consequently the estimates for its location and the associated exponents are quite inaccurate. The singularity is located at $p_- = -0.5168(5)$ and the associated exponents are $\gamma = 0.065(15)$, $\nu_\| = 0.97(3)$ and $2\nu_\perp = 0.90(15)$. It is quite possible that the divergence of the cluster length series at p_- is logarithmic and the estimates are certainly consistent with $\gamma = 0$, $\nu_\| = 1$ and $\nu_\perp = \frac{1}{2}$. Finally there is some weak evidence of a pair of singularities in the complex p-plane at $p_\pm = -0.2255(15) \pm 0.440(1)\mathrm{i}$. Note that this singularity pair also lies within the physical disc. The exponent estimates at $p_\pm$ are not very accurate. The cluster size series seems to *converge* with exponent $\gamma \simeq -3$, while $\nu_\| \simeq 1$ and $\nu_\perp \simeq \frac{1}{2}$, but the error on these estimates are as large as 25–50%.

4.2. The square site series

In table 4 I have listed some of the estimates for p_c and critical exponents obtained from an analysis of the square site series. The estimates are based on approximants using at least 85–90 terms with $N_c = 15$. Though the length of the series is comparable to the bond case the estimates are generally less accurate. In particular it should be noted that the p_c-estimates obtained from different series are only marginally consistent leading to the rather poor estimate, $p_c = 0.705\,485\,0(15)$, which is at least an order of magnitude less accurate than in the bond case. Some exponent estimates differ significantly from those of the bond case. Particularly γ and $\gamma + 2\nu_\|$ are generally quite a bit smaller than the bond estimates. However, due to the discrepancy between the various site series, the importance of this deviation is questionable. If the error-bar on p_c is accepted, the resulting exponent estimates from the site series will agree with the bond estimates.

If one accepts the exponent estimates from the bond series one can use the linear dependence between p_c and exponent estimates to obtain improved estimates for p_c. (This is just the reverse of the method used in the previous section to obtain the exponent estimates.) By performing a linear fit of the p_c-estimates as a function of the deviation of the exponent estimate from the central values listed in the previous section I obtain the estimate $p_c = 0.705\,485\,3(5)$. In these fits I used the approximants whose exponent estimates differ by less than 0.001 from the central values. This estimate agrees with that obtained from the percolation-probability series [6] $p_c = 0.705\,485(5)$.

The square site series have a singularity on the negative axis closer to the origin then p_c. In this case the singularity appears to be stronger than in the bond case, i.e. the various estimates are better converged. The singularity is located at $p_- = -0.451\,952\,2(3)$ and the associated exponents are quite possibly consistent with $\gamma = -\frac{1}{2}$ (i.e. the cluster-size series *converges*), $\nu_\| = 1$ and $\nu_\perp = \frac{1}{2}$. There is firm evidence of a pair of singularities in the complex p-plane at $p_\pm = -0.2263(1) \pm 0.3847(1)\mathrm{i}$, which is within the physical disc. The exponent estimates at this pair of singularities are quite accurate. The cluster-size series seems to *converge*, with $\gamma \simeq -3$, while $\nu_\| \simeq 1$ and $\nu_\perp \simeq \frac{1}{2}$, where errors on the estimates are only a few per cent.

4.3. The triangular bond series

Table 5 lists a selection of estimates for p_c and critical exponents obtained from the analysis of the triangular bond series. The estimates are based on approximants using at least 45 or

Table 4. Estimates of p_c and critical exponents for the square site problem.

	First-order DA			Second-order DA		
L	p_c	γ	N_a	p_c	γ	N_a
0	0.705 483 90(20)	2.276 850(66)	19	0.705 485 00(26)	2.277 51(15)	17
5	0.705 484 09(20)	2.276 924(88)	23	0.705 485 16(28)	2.277 60(18)	18
10	0.705 484 41(35)	2.277 21(30)	24	0.705 484 72(19)	2.277 334(95)	17
15	0.705 484 594(68)	2.277 232(33)	23	0.705 484 71(14)	2.277 314(74)	19
20	0.705 484 805(72)	2.277 364(39)	24	0.705 484 86(36)	2.277 42(25)	20
25	0.705 484 723(82)	2.277 319(46)	20	0.705 484 671(58)	2.277 295(35)	16
30	0.705 484 811(34)	2.277 367(18)	21	0.705 484 689(29)	2.277 306(16)	16
35	0.705 484 850(62)	2.277 389(31)	21	0.705 484 713(83)	2.277 313(39)	17
L	p_c	$\nu_\parallel$	N_a	p_c	$\nu_\parallel$	N_a
0	0.705 484 49(93)	1.733 47(25)	19	0.705 484 96(30)	1.733 70(10)	16
5	0.705 484 27(28)	1.733 416(72)	23	0.705 484 91(23)	1.733 686(84)	16
10	0.705 484 85(36)	1.733 66(14)	20	0.705 485 020(95)	1.733 729(25)	16
15	0.705 485 13(26)	1.733 763(88)	23	0.705 484 91(34)	1.733 69(12)	18
20	0.705 485 65(53)	1.733 97(20)	22	0.705 484 80(17)	1.733 650(66)	19
25	0.705 485 75(33)	1.734 03(12)	23	0.705 484 70(21)	1.733 608(93)	17
30	0.705 485 60(63)	1.733 96(28)	19	0.705 484 43(26)	1.733 50(11)	16
35	0.705 485 45(43)	1.733 88(17)	24	0.705 484 52(21)	1.733 548(84)	16
L	p_c	$2\nu_\perp$	N_a	p_c	$2\nu_\perp$	N_a
0	0.705 486 9(13)	2.194 45(46)	19	0.705 486 50(23)	2.194 33(21)	19
5	0.705 486 87(57)	2.194 47(16)	19	0.705 486 47(23)	2.194 34(13)	16
10	0.705 485 1(15)	2.193 97(33)	21	0.705 486 49(12)	2.194 254(51)	16
15	0.705 485 7(10)	2.194 00(39)	19	0.705 485 77(24)	2.194 033(76)	20
20	0.705 486 6(16)	2.194 34(53)	19	0.705 485 89(42)	2.194 06(13)	21
25	0.705 486 0(10)	2.194 12(42)	19	0.705 485 85(24)	2.194 048(81)	17
30	0.705 486 0(12)	2.194 10(45)	20	0.705 485 60(65)	2.193 91(28)	18
35	0.705 486 2(13)	2.194 08(53)	20	0.705 485 15(78)	2.193 76(31)	17
L	p_c	$\gamma + \nu_\parallel$	N_a	p_c	$\gamma + \nu_\parallel$	N_a
0	0.705 483 65(38)	4.009 89(23)	19	0.705 484 03(70)	4.010 23(58)	18
5	0.705 483 81(17)	4.010 00(12)	23	0.705 484 38(33)	4.010 47(39)	16
10	0.705 483 85(42)	4.010 05(29)	25	0.705 484 41(34)	4.010 55(30)	16
15	0.705 483 62(55)	4.009 94(38)	24	0.705 484 30(51)	4.010 46(44)	21
20	0.705 483 49(30)	4.009 79(20)	19	0.705 484 24(34)	4.010 41(28)	18
25	0.705 483 80(43)	4.010 06(30)	22	0.705 484 50(65)	4.010 67(65)	21
30	0.705 483 80(21)	4.009 99(14)	21	0.705 484 28(21)	4.010 43(18)	16
35	0.705 483 78(61)	4.010 02(43)	23	0.705 484 47(33)	4.010 61(32)	19
L	p_c	$\gamma + 2\nu_\parallel$	N_a	p_c	$\gamma + 2\nu_\parallel$	N_a
0	0.705 483 58(35)	5.743 11(21)	19	0.705 484 60(45)	5.744 20(51)	19
5	0.705 483 55(20)	5.743 07(14)	19	0.705 484 43(18)	5.744 00(20)	17
10	0.705 484 04(60)	5.743 58(65)	23	0.705 484 34(18)	5.743 92(21)	17
15	0.705 483 82(10)	5.743 299(94)	19	0.705 484 31(52)	5.743 90(62)	20
20	0.705 483 79(15)	5.743 27(14)	22	0.705 484 15(22)	5.743 69(24)	18
25	0.705 483 75(16)	5.743 21(13)	22	0.705 484 00(10)	5.743 52(10)	16
30	0.705 483 68(16)	5.743 17(14)	19	0.705 484 22(25)	5.743 77(30)	16
35	0.705 483 87(24)	5.743 34(22)	25	0.705 484 74(65)	5.744 49(85)	19

Table 4. (Continued)

	First-order DA			Second-order DA		
L	p_c	$\gamma+2\nu_\perp$	N_a	p_c	$\gamma+2\nu_\perp$	N_a
0	0.705 483 8(33)	4.472 9(94)	19	0.705 484 57(13)	4.470 71(10)	20
5	0.705 484 58(16)	4.470 69(11)	19	0.705 484 60(10)	4.470 740(93)	16
10	0.705 484 63(16)	4.470 72(10)	20	0.705 484 57(11)	4.470 695(93)	19
15	0.705 484 77(19)	4.470 84(15)	19	0.705 484 73(27)	4.470 84(25)	21
20	0.705 484 43(43)	4.470 61(26)	20	0.705 484 72(17)	4.470 81(15)	17
25	0.705 484 49(47)	4.470 66(30)	20	0.705 484 80(49)	4.470 89(45)	19
30	0.705 484 75(42)	4.470 87(37)	19	0.705 484 2(13)	4.470 4(11)	17
35	0.705 484 69(22)	4.470 78(18)	19	0.705 485 1(13)	4.471 3(12)	20

40 terms with $N_c = 15$ or 10 for first and second order, respectively. As one would expect, due to the shorter series, the estimates are generally encumbered with larger errors than was the case for the square bond series. The estimates for $\nu_\parallel$ and $2\nu_\perp$ are generally consistent with those from the square bond series, while the remaining exponent estimates exceeds those from the square bond case. The linear fit of p_c to the deviation of the exponent estimates from the values favoured by the square bond series yields $p_c = 0.478\,025(1)$, which is in excellent agreement with the estimate $p_c = 0.478\,02(1)$ from the percolation-probability series [7]. The triangular bond series does not appear to have any non-physical singularities.

4.4. The triangular site series

In table 6 I have listed some estimates for p_c and critical exponents obtained from an analysis of the triangular site series similar to that for the bond problem. In this case all exponent estimates are consistent with the square bond case. The biased estimate for p_c based on the usual fitting procedure is $p_c = 0.595\,646\,8(5)$ in excellent agreement with the estimate $p_c = 0.595\,647\,2(10)$ from the percolation probability series [7]. Again there is no compelling evidence for non-physical singularities.

5. Summary and discussion

From the analysis presented in the previous section it was clear that the square bond series yield by far the most accurate p_c-estimates which in turn enables one to obtain very precise estimates for the critical exponents. The remaining cases yielded less accurate estimates. Though the square site and triangular bond cases tended to yield exponent estimates only marginally consistent with the square bond estimates, the p_c estimates showed less consistency among the various series. In the square site case this could possibly be caused by the presence of rather strong non-physical singularities closer to the origin than p_c. The triangular site estimates, though marred by larger error-bars, were fully consistent with the square bond estimates. I have therefore chosen to base my final exponent estimates mainly on the square bond series.

From figure 2 it would appear that the estimate $p_c = 0.644\,700\,15(5)$ is fully consistent with the data and not overly optimistic. With this highly accurate p_c value one can obtain very accurate exponent estimates using the values listed in (4.3). The values of the critical exponents for the average cluster size, parallel and perpendicular connectedness lengths are

Table 5. Estimates of p_c and critical exponents for the triangular bond problem.

	First-order DA			Second-order DA		
L	p_c	γ	N_a	p_c	γ	N_a
0	0.478 026 8(13)	2.278 50(35)	21	0.478 025 48(13)	2.277 976(80)	15
4	0.478 025 96(10)	2.278 170(47)	16	0.478 025 78(42)	2.278 09(21)	14
8	0.478 026 14(10)	2.278 242(64)	16	0.478 025 60(16)	2.278 054(48)	11
12	0.478 026 02(42)	2.278 19(14)	20	0.478 025 79(27)	2.278 093(91)	14
16	0.478 025 99(29)	2.278 19(10)	18	0.478 026 05(50)	2.278 20(19)	17
L	p_c	$\nu_\parallel$	N_a	p_c	$\nu_\parallel$	N_a
0	0.478 027 2(19)	1.734 35(30)	17	0.478 026 24(79)	1.734 13(18)	17
4	0.478 025 5(10)	1.734 04(33)	17	0.478 025 85(59)	1.734 04(17)	12
8	0.478 025 51(57)	1.733 98(16)	16	0.478 026 4(10)	1.734 17(30)	15
12	0.478 025 6(18)	1.734 03(53)	19	0.478 025 36(79)	1.733 92(22)	11
16	0.478 024 4(25)	1.733 65(65)	18	0.478 027 3(19)	1.734 41(52)	15
L	p_c	$2\nu_\perp$	N_a	p_c	$2\nu_\perp$	N_a
0	0.478 027 16(70)	2.194 29(16)	18	0.478 026 0(10)	2.193 89(23)	17
4	0.478 026 83(80)	2.194 20(15)	17	0.478 026 1(17)	2.193 95(54)	14
8	0.478 024 74(53)	2.193 55(15)	16	0.478 024 6(12)	2.193 55(33)	14
12	0.478 025 1(28)	2.193 67(71)	18	0.478 024 4(12)	2.193 49(36)	14
16	0.478 024 7(11)	2.193 54(35)	17	0.478 025 22(40)	2.193 69(11)	11
L	p_c	$\gamma+\nu_\parallel$	N_a	p_c	$\gamma+\nu_\parallel$	N_a
0	0.478 026 76(52)	4.012 59(28)	18	0.478 026 65(24)	4.012 624(79)	13
4	0.478 026 70(47)	4.012 61(14)	20	0.478 026 86(12)	4.012 693(33)	13
8	0.478 026 45(51)	4.012 51(22)	19	0.478 026 66(17)	4.012 649(45)	11
12	0.478 026 12(59)	4.012 36(30)	17	0.478 026 53(68)	4.012 44(54)	16
16	0.478 026 22(45)	4.012 43(21)	16	0.478 026 82(16)	4.012 688(36)	11
L	p_c	$\gamma+2\nu_\parallel$	N_a	p_c	$\gamma+2\nu_\parallel$	N_a
0	0.478 025 4(17)	5.7456(17)	17	0.478 026 4(16)	5.7464(14)	13
4	0.478 025 1(10)	5.745 66(95)	19	0.478 026 6(24)	5.7460(20)	13
8	0.478 025 2(11)	5.7457(11)	17	0.478 026 4(19)	5.7461(16)	17
12	0.478 025 66(33)	5.746 23(26)	16	0.478 025 4(10)	5.7457(11)	16
16	0.478 025 88(78)	5.746 33(52)	18	0.478 026 3(18)	5.7463(12)	17
L	p_c	$\gamma+2\nu_\perp$	N_a	p_c	$\gamma+2\nu_\perp$	N_a
0	0.478 026 16(38)	4.472 28(18)	16	0.478 025 85(24)	4.472 04(14)	13
4	0.478 026 32(82)	4.472 34(41)	17	0.478 025 70(52)	4.471 91(33)	14
8	0.478 025 89(47)	4.472 14(23)	17	0.478 026 37(54)	4.472 35(31)	11
12	0.478 025 66(48)	4.471 96(31)	18	0.478 026 24(50)	4.472 28(31)	13
16	0.478 026 18(31)	4.472 28(15)	17	0.478 026 10(42)	4.472 18(23)	12

estimated by $\gamma = 2.277\,69(4)$, $\nu_\parallel = 1.733\,825(25)$ and $\nu_\perp = 1.096\,844(14)$, respectively. An improved estimate for the percolation probability exponent is obtained from the scaling relation $\beta = (\nu_\parallel + \nu_\perp - \gamma)/2 = 0.276\,49(4)$. As already noted these estimates are generally incompatible with the exact fractions conjectured by Essam *et al* [4]. Only γ is marginally consistent with the suggested fraction, $\gamma = 41/18 = 2.77\,777\ldots$, if a larger error-bar were adopted for p_c.

Table 6. Estimates of p_c and critical exponents for the triangular site problem.

	First-order DA			Second-order DA		
L	p_c	γ	N_a	p_c	γ	N_a
0	0.595 647 31(31)	2.277 848(67)	16	0.595 645 98(71)	2.277 49(16)	18
4	0.595 646 41(30)	2.277 597(79)	18	0.595 646 5(13)	2.277 55(64)	16
8	0.595 646 64(41)	2.277 67(12)	18	0.595 646 81(10)	2.277 708(28)	12
12	0.595 646 53(27)	2.277 628(81)	16	0.595 646 67(20)	2.277 672(64)	13
16	0.595 646 84(78)	2.277 72(22)	18	0.595 646 59(32)	2.277 662(84)	12
L	p_c	$\nu_\parallel$	N_a	p_c	$\nu_\parallel$	N_a
0	0.595 646 56(15)	1.733 766(15)	16	0.595 646 75(45)	1.733 796(53)	15
4	0.595 645 4(11)	1.733 58(18)	16	0.595 646 62(60)	1.733 78(11)	11
8	0.595 645 9(88)	1.7336(17)	16	0.595 644 8(32)	1.733 44(74)	11
12	0.595 647 6(31)	1.734 07(68)	16	0.595 645 7(13)	1.733 61(29)	11
16	0.595 650 7(29)	1.734 77(65)	16	0.595 643 2(58)	1.7328(15)	15
L	p_c	$2\nu_\perp$	N_a	p_c	$2\nu_\perp$	N_a
0	0.595 650(12)	2.1943(37)	16	0.595 647 0(38)	2.1938(12)	14
4	0.595 655 5(49)	2.1958(11)	16	0.595 647 7(10)	2.193 97(25)	11
8	0.595 648 9(14)	2.194 25(30)	17	0.595 647 53(88)	2.193 97(24)	11
12	0.595 646 9(73)	2.1938(15)	16	0.595 645 7(22)	2.193 57(42)	12
16	0.595 647 3(10)	2.193 87(22)	16	0.595 648 5(18)	2.194 11(37)	16
L	p_c	$\gamma+\nu_\parallel$	N_a	p_c	$\gamma+\nu_\parallel$	N_a
0	0.595 643 5(26)	4.010 06(80)	18	0.595 645 3(22)	4.0108(10)	15
4	0.595 644 6(16)	4.010 36(54)	16	0.595 647 6(46)	4.0122(24)	17
8	0.595 645 42(67)	4.010 64(27)	17	0.595 647 29(73)	4.011 68(46)	11
12	0.595 644 89(48)	4.010 41(20)	16	0.595 647 19(88)	4.011 68(49)	11
16	0.595 644 95(28)	4.010 47(10)	17	0.595 645 0(12)	4.010 57(55)	11
L	p_c	$\gamma+2\nu_\parallel$	N_a	p_c	$\gamma+2\nu_\parallel$	N_a
0	0.595 648 4(66)	5.7469(60)	17	0.595 644 4(17)	5.743 86(91)	11
4	0.595 644 0(29)	5.7437(10)	16	0.595 644 2(28)	5.7438(16)	12
8	0.595 649 2(45)	5.7468(31)	18	0.595 643 2(32)	5.7433(12)	13
12	0.595 646 3(37)	5.7448(24)	17	0.595 646 2(20)	5.7448(13)	12
16	0.595 645 7(15)	5.744 40(85)	17	0.595 646 5(13)	5.745 02(80)	12
L	p_c	$\gamma+2\nu_\perp$	N_a	p_c	$\gamma+2\nu_\perp$	N_a
0	0.595 647 7(11)	4.471 67(39)	16	0.595 647 15(31)	4.471 61(13)	12
4	0.595 647 48(19)	4.471 776(73)	17	0.595 647 06(43)	4.471 56(17)	14
8	0.595 647 49(26)	4.471 770(98)	17	0.595 647 24(29)	4.471 64(12)	12
12	0.595 647 56(33)	4.471 79(12)	16	0.595 647 44(81)	4.471 70(29)	14
16	0.595 647 58(42)	4.471 80(15)	17	0.595 647 29(15)	4.471 670(61)	12

Below I have listed improved estimates for a number of critical exponents obtained using various scaling relations.

$$\Delta = \beta + \gamma = 2.554\,18(8)$$
$$\tau = \nu_\parallel - \beta = 1.457\,34(7)$$
$$z = \nu_\parallel / \nu_\perp = 1.580\,74(4)$$

$$\gamma' = \gamma - \nu_{\|} = 0.543\,86(7)$$
$$\delta = \beta/\nu_{\|} = 0.159\,47(3)$$
$$\eta = \gamma/\nu_{\|} - 1 = 0.313\,68(4).$$

Here Δ is the exponent characterizing the scale of the cluster size distribution, τ is the cluster length exponent, z is the dynamical critical exponent, γ' the exponent characterizing the steady-state fluctuations of the order parameter, while δ and η characterize the behaviour at p_c as $t \to \infty$ of the survival probability and average number of particles, respectively.

Assuming that the exponent estimates from the square bond case are correct, improved p_c-estimates were obtained for the three other problems studied in this paper. These are:

$$p_c = 0.705\,485\,3(5) \qquad \text{square site}$$
$$p_c = 0.478\,025(1) \qquad \text{triangular bond}$$
$$p_c = 0.595\,646\,8(5) \qquad \text{triangular site.}$$

Finally I note, that the analysis of the various series, in order to determine the value of the confluent exponent, yielded estimates consistent with $\Delta_1 \simeq 1$. Thus there is no evidence of non-analytic confluent correction terms. This provides a hint that the models might be exactly solvable.

E-mail or WWW retrieval of series

The series and the coefficients in the extrapolation formulae for the directed percolation problems on the various lattices can be obtained via e-mail by sending a request to iwan@maths.mu.oz.au or via the world wide web on the URL http://www.maths.mu.oz.au/˜iwan/ by following the relevant links.

Acknowledgments

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Appendix. The extrapolation formulae and series for the square site, triangular bond and triangular site problems

A.1. The square site problem

The sequence determining the first correction term for S^N starts out as

$$s_{t,0} = 1, 0, 1, 2, 6, 18, 57, 186, 622, 2120, 7338, \ldots$$

from which one sees that $2s_{t,0} + s_{t-1,0} = C_{t-1}$. Shapiro [15] has given an interpretation of this sequence by adding diagonals in a certain Catalan triangle.

At first glance one might find it strange that the correction term differs from the bond case, since clearly all the non-nodal bond graphs that give rise to the first correction term have their counterparts as site graphs. In the following I shall always be talking only of non-nodal graphs consisting of two equal-length paths. The reason for the difference is quite simply that for some graphs the d-weight in (3.3) is 0 for the *site* graph but non-zero for the bond graph. A schematic representation of such a graph is shown in figure A1. A proof of this was given by Arrowsmith and Essam [16], who showed that $d(g)$ is non-zero

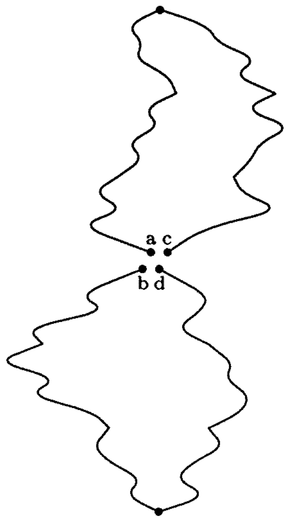

Figure A1. Schematic pictorial representation of a non-nodal graph which contributes to S^N in the bond problem but *not* in the site problem.

if and only if g is coverable by a set of directed paths *and has no circuit (or loop)*. From figure A1 we see that in the bond case the graph obtained by putting in the bonds a–b and c–d has no loops. However, in the site case there is a loop from the origin to point d and this graph does, therefore, not contribute in the site case. On the other hand it is clear that for any contributing site graph there is a corresponding contributing bond graph. So the contributing site graphs form a subset of the bond graphs.

In order to prove the formula for $s_{t,0}$ it is convenient to give another interpretation of the loop-free non-nodal graphs. Let us first characterize the graphs by the distance k between the paths. Since the graphs start and end with $k = 0$, and the distance zero appears nowhere else along the graph, these two 'steps' can be deleted. It is clear that in each step (increase of t by one) k changes by 0 or ±1. When k is unchanged there are two configurations corresponding to both paths moving either south-east or south-west, while for changes of ±1 there is just one configuration. The non-nodal graphs are thus in bijection with paths of length $t-1$ starting and ending at the ground level, which can take north-east, east and south-east steps, and where east steps come in two varieties or colours (such paths are known as *two-colour Motzkin paths*). It is one of the fundamental results of combinatorics that the number of two-colour Motzkin paths of length $n-1$ is C_n. It is easy to see that loop-free non-nodal graphs form the subset where the distance between paths is never 1 twice in a row, i.e. if $k_n = 1$ then $k_{n+1} = 2$. These graphs are in bijection with two-colour Motzkin paths with no east steps on the ground level.

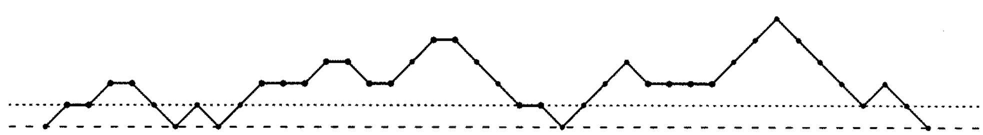

Figure A2. Typical two-colour Motzkin path with no east steps on the ground level.

Figure A2 shows an example of a two-colour Motzkin path with no east steps on the ground level. It is clear that all paths formed by taking the parts of the original path lying one level above the ground level (those above the dotted line), are ordinary unrestricted two-colour Motzkin paths, and these paths are therefore enumerated by the Catalan numbers. The number of no-loop non-nodal graphs can therefore be expressed in terms of Catalan numbers, by summing over the number of times m the associated restricted two-colour Motzkin path meets the ground level prior to the terminal point. Let D_n denote the number of two-colour Motzkin paths of length n with no east steps on the ground level. The number of such two-colour Motzkin paths, $D_{n,0}$, which does not hit the ground level prior to n is simply C_{n-1} because the path obtained by deleting the first and last step is an ordinary two-colour Motzkin path of length $n-2$. The number of restricted two-colour Motzkin paths $D_{n,1}$ which hit the ground level once is,

$$D_{n,1} = \sum_{k=0}^{n-4} C_{k+1} C_{n-4-k+1} = \sum_{i+j=n-2} C_i C_j \qquad i, j \geqslant 1.$$

This formula is simply obtained by noting that the path to the left of the point where the restricted path meets the ground level for the first time can have a length k ranging from 0 to $n-4$ (the four steps connecting the ground level to the level above are discarded) while the length of the second path is $n-4-k$. Obviously the number of left and right paths are just C_{k+1} and $C_{n-4-k+1}$, independently, which leads to the formula above once we sum over the length of the left path. The generalization to $D_{n,m}$ is obvious

$$D_{n,m} = \sum_{i_1+i_2+\cdots+i_m=n-m-1} C_{i_1} C_{i_2} \cdots C_{i_m} \qquad i_1, \ldots, i_m \geqslant 1, m \leqslant \lfloor n/2 \rfloor - 1.$$

The sum $D_n = \sum_{m=0}^{\lfloor n/2 \rfloor - 1} D_{n,m}$ is exactly the same as that obtained by Shapiro [15] by adding diagonals in the Catalan triangle.

The higher-order correction terms are quite complicated though still expressible as linear functions of $s_{t,0}$,

$$2^r (r+1)! s_{t,r} = \sum_{k=1}^{n_a} a_{r,k} s_{t-r+k-1,0} + \sum_{k=1}^{r} \binom{t-r}{k} [b_{r,k}(s_{t-r-1,0} + 2 s_{t-r,0}) + c_{r,k} s_{t-r,0}] \tag{A.1}$$

where $n_a = r - 1 + \max(\lfloor r/2 \rfloor, 2)$. This representation leads to particularly simple coefficients $c_{r,k}$, since $c_{r,r-m} 2^4/(r+1)!$ are expressible as polynomials in r of order m for $r > m$.

The sequence determining the first correction term for X^N starts out as

$$x_{t,0} = 0, 0, 0, 2, 8, 34, 136, 538, 2112, 8264, \ldots.$$

In this case $x_{t,0} = u(t+1)$ is determined by the following recurrence relation

$$u(0) = 0 \qquad u(1) = 0 \qquad u(2) = 0 \qquad u(3) = 2 \qquad u(4) = 8$$

$$\begin{aligned} u(t+5) = {} & [(2+4t)u(t) + (10+13t)u(t+1) + (63/2 + 25/2t)u(t+2) \\ & + (4+2t)u(t+3) + (-53/2 - 11/2t)u(t+4)]/(t+6). \end{aligned}$$

The formulae for the higher-order correction terms are complicated though still expressible as functions of $x_{t,0}$,

$$\begin{aligned} 6^{r+1}(r+1)! x_{t,r} = {} & \sum_{k=0}^{2r} a_{r,k} x_{t-r+k-3,0} + \sum_{k=1}^{r} \binom{t-r}{k} [b_{r,k} x_{t-r-4,0} + c_{r,k} x_{t-r-3,0}] \\ & + (t-r)([d_{r,1} + (t-r-1) d_{r,2}/2] x_{t-r-2,0} \\ & + [d_{r,3} + (t-r-1) d_{r,4}/2] x_{t-r-1,0}). \end{aligned} \tag{A.2}$$

Table A1. New series terms for the directed square lattice site problem.

n	$S(p)$	$\mu_{0,1}(p)$	$\mu_{0,2}(p)$	$\mu_{2,0}(p)$
49	−3 989 867 712 261	−89 398 788 718 610	−1 997 686 213 754 238	−33 424 766 465 974
50	8 580 315 717 912	196 529 687 645 088	4 518 812 046 618 304	74 612 004 326 120
51	−18 450 741 974 659	−430 495 003 001 046	−10 037 202 119 113 882	−162 365 675 913 426
52	39 714 443 919 946	945 547 880 790 844	22 567 072 966 501 772	359 569 023 440 940
53	−85 497 506 155 974	−2 073 414 529 248 340	−50 294 843 206 169 288	−786 814 538 064 912
54	184 179 126 806 512	4 552 510 986 519 760	112 723 362 298 382 724	1 735 941 752 380 332
55	−396 886 210 803 357	−9 988 878 134 103 694	−251 571 214 569 296 384	−3 808 660 144 091 956
56	855 734 249 183 509	21 932 260 509 927 964	562 892 405 947 427 952	8 388 677 741 860 840
57	−1 845 825 000 749 297	−48 144 989 472 917 392	−1 256 868 058 330 309 144	−18 428 858 054 805 456
58	3 983 384 787 346 219	105 726 756 215 995 148	2 809 559 318 045 916 036	40 560 117 237 064 492
59	−8 599 535 636 965 444	−232 161 203 280 983 216	−6 273 640 218 845 134 136	−89 163 845 530 458 120
60	18 572 320 618 806 137	509 899 256 031 942 796	14 014 639 196 725 078 868	196 179 493 197 815 148
61	−40 124 529 644 388 180	−1 119 917 082 819 437 502	−31 288 707 556 698 610 790	−431 410 135 430 855 566
62	86 720 156 055 109 481	2 460 124 423 504 262 496	69 862 837 357 022 993 636	949 102 253 844 386 388
63	−187 496 448 569 247 473	−5 404 595 046 584 365 902	−155 940 077 601 170 801 964	−2 087 589 817 858 270 048
64	405 534 728 909 684 450	11 874 858 693 213 744 120	348 067 893 959 380 971 188	4 592 790 116 926 019 372
65	−877 426 805 166 151 991	−26 093 306 262 441 552 830	−776 726 385 163 036 787 130	−10 103 619 566 512 091 258
66	1 899 030 133 479 082 710	57 341 753 939 024 263 200	1 733 139 806 171 397 955 248	22 229 577 257 948 816 928
67	−4 111 343 988 945 455 470	−126 021 332 759 023 525 888	−3 866 501 287 920 863 924 260	−48 907 979 301 629 134 116
68	8 903 649 049 790 473 757	276 984 891 538 277 258 264	8 624 938 634 269 764 601 752	107 612 749 966 418 186 776
69	−19 287 966 769 691 177 647	−608 845 025 562 943 897 104	−19 236 751 196 703 556 849 764	−236 787 215 239 641 195 332
70	41 796 381 922 966 439 485	1 388 441 468 813 888 582 992	42 900 449 394 862 448 744 324	521 053 631 653 640 266 324
71	−90 598 011 496 116 317 009	−2 942 588 348 045 572 798 924	−95 661 510 876 264 852 436 682	−1 146 629 680 458 206 972 302
72	196 434 657 717 734 195 605	6 469 843 385 401 094 502 448	213 286 445 757 191 496 518 268	2 523 405 614 475 087 992 884
73	−426 020 319 924 814 111 080	−14 226 181 709 514 806 962 484	−475 483 808 303 867 302 659 350	−5 553 505 976 059 430 032 826
74	924 172 481 642 000 231 190	31 283 381 554 808 983 424 564	1 059 887 209 300 551 335 726 068	12 222 712 333 836 676 378 204
75	−2 005 335 678 304 212 930 642	−68 797 304 028 652 689 337 280	−2 362 307 477 572 924 432 972 394	−26 902 131 263 080 690 712 678
76	4 352 442 737 004 987 018 437	151 308 488 974 126 286 057 844	5 264 658 663 276 025 029 764 160	59 214 386 284 087 515 891 824
77	−9 449 058 961 219 086 599 817	−332 803 963 349 353 164 835 862	−11 731 710 535 855 830 921 986 156	−130 343 092 179 640 177 080 016
78	20 518 626 196 069 403 747 527	732 056 224 712 682 938 741 308	26 140 275 838 901 762 860 484 152	286 925 229 989 719 714 345 800
79	−44 566 294 136 459 273 950 057	−1 610 379 469 736 015 604 828 142	−58 239 279 737 889 211 493 881 184	−631 637 066 943 482 168 917 668
80	96 818 818 012 187 977 898 273	3 542 734 638 940 687 117 182 184	129 741 790 947 720 603 074 722 100	1 390 543 017 557 046 212 795 804
81	−210 381 380 688 137 675 218 788	−7 794 283 613 503 270 838 084 276	−289 003 883 877 478 452 390 760 910	−3 061 394 805 973 924 320 554 690
82	457 245 573 160 144 114 585 903	17 149 135 831 082 379 952 735 720	643 709 356 712 958 070 363 747 636	6 740 206 974 126 201 296 139 132
83	−993 998 356 903 718 319 199 641	−37 734 364 348 756 216 753 502 994	−1 433 638 977 288 645 999 020 939 660	−14 840 423 239 587 571 243 043 080
84	2 161 289 069 292 339 668 165 416	83 034 651 548 600 236 802 398 900	3 192 670 465 583 319 427 635 760 444	32 676 690 454 094 483 385 862 812
85	−4 700 314 114 038 311 031 841 115	−182 728 855 195 529 461 160 585 126	−7 109 392 867 826 626 914 837 449 716	−71 952 773 516 188 632 888 545 144
86	10 224 077 808 938 757 568 364 783	402 141 349 920 870 671 599 037 548	15 829 794 487 407 914 130 651 664 444	158 443 333 160 753 123 241 082 372
87	−22 243 488 016 690 782 862 992 086	−885 062 275 453 463 475 531 759 414	−35 243 856 214 887 254 433 554 861 726	−348 913 039 391 110 399 836 823 030
88	48 401 950 300 344 256 475 637 712	1 948 018 474 475 088 201 326 334 892	78 461 861 674 533 680 323 863 667 932	768 382 759 013 781 010 299 412 124
89	−105 342 556 574 383 686 701 245 163	−4 287 824 718 991 072 217 933 002 402	−174 663 572 419 357 214 397 960 394 054	−1 692 214 996 408 139 015 463 119 278
90	229 310 760 086 509 442 832 526 722	9 438 551 172 124 661 276 621 472 080	388 790 296 374 975 505 848 281 680 740	3 726 926 745 592 918 193 624 099 284
91	−499 254 332 826 485 830 405 089 496	−20 777 664 596 947 896 696 430 266 604	−865 362 328 805 313 041 911 227 036 936	−8 208 483 558 388 276 019 386 972 720
92	1 087 158 906 295 203 197 314 525 443	45 741 434 128 774 338 756 337 752 228	1 925 972 519 952 860 729 220 428 345 684	18 079 692 468 497 167 015 871 015 788
93	−2 367 749 947 910 536 462 457 691 310	−100 703 295 399 263 069 696 732 444 520	−4 286 197 764 440 865 108 983 542 970 368	−39 823 072 996 281 556 640 972 600 352
94	5 157 620 492 591 221 745 845 382 303	221 716 756 474 626 351 542 666 349 488	9 538 177 217 550 360 368 042 356 768 560	87 719 100 925 863 061 037 547 551 360
95	−11 236 560 697 354 134 632 594 642 616	−488 174 068 949 070 149 281 128 899 830	−21 224 177 013 741 918 069 997 560 647 638	−193 227 743 472 334 785 915 666 225 854
96	24 484 272 295 246 490 240 996 603 369	1 074 910 533 745 861 510 816 316 084 564	47 224 735 893 260 940 697 619 756 232 312	425 657 949 575 680 334 538 486 829 888
97	−53 359 176 119 558 192 258 725 059 781	−2 366 958 912 206 990 532 708 241 937 752	−105 070 747 584 428 334 692 618 935 926 926	−937 708 039 740 162 031 390 681 348 930
98	116 304 588 119 008 605 380 082 758 269	5 212 293 324 137 570 131 956 081 910 068	233 758 716 381 324 649 345 956 203 550 124	2 065 806 255 709 766 914 529 923 370 436
99	−253 541 064 664 534 333 448 574 654 272	−11 478 517 921 790 669 543 324 316 788 398	−520 029 205 468 351 028 057 499 552 325 942	−4 551 202 015 946 486 108 841 527 592 910
100	552 792 621 363 735 927 069 722 202 340	25 279 082 608 208 297 840 376 986 116 580	1 156 810 413 899 443 238 201 095 537 631 608	10 027 139 072 529 312 666 003 007 391 872
101	−1 205 419 189 592 079 128 067 122 848 352	−55 674 391 944 146 677 972 360 325 076 636	−2 573 191 283 063 154 192 367 322 432 822 154	−22 092 376 889 614 817 985 940 202 648 678
102	2 628 908 431 680 539 275 382 640 619 768	122 622 030 951 814 229 187 110 993 853 380	5 723 454 485 424 354 892 796 024 259 804 260	48 676 843 932 534 358 770 102 134 948 212
103	−5 734 206 699 826 900 169 571 754 951 593	−270 084 799 228 725 528 259 525 901 837 952	−12 729 789 261 108 652 289 128 945 512 185 508	−107 254 816 832 237 290 531 615 089 289 476
104	12 509 218 040 472 363 621 751 154 085 913	594 907 882 100 104 114 064 827 637 816 852	28 311 392 794 520 974 801 663 824 530 397 816	
105	−27 292 552 060 674 090 069 382 730 103 133	−1 310 437 975 264 638 135 266 056 264 966 662	−62 961 989 896 370 296 367 186 662 669 460 974	
106	59 554 409 261 647 295 405 476 354 898 380	2 886 688 264 100 702 091 406 743 320 354 752	140 014 554 490 416 449 591 789 656 684 308 532	

Table A2. New series terms for the directed triangular lattice bond problem.

n	$S(p)$	$\mu_{0,1}(p)$	$\mu_{0,2}(p)$	$\mu_{2,0}(p)$
26	5 337 497 418	209 994 728 977	28 038 948 604 228	357 799 862 456
27	11 678 931 098	487 411 142 729	68 883 587 787 794	841 629 097 226
28	25 513 719 388	1 127 362 924 089	168 327 542 017 154	1 972 059 110 234
29	55 663 119 018	2 599 086 582 635	409 289 987 873 146	4 604 235 247 626
30	121 272 163 372	5 973 768 053 766	990 554 419 328 610	10 713 215 525 118
31	263 864 408 629	13 690 809 855 903	2 386 824 242 808 628	24 848 543 707 616
32	573 556 848 773	31 292 824 198 260	5 727 568 988 920 190	57 462 309 456 098
33	1 245 063 650 267	71 342 703 947 141	13 690 818 307 565 964	132 505 664 249 544
34	2 700 144 659 216	162 261 360 324 560	32 605 625 326 065 898	304 737 782 904 598
35	5 851 221 147 909	368 214 693 911 431	77 383 096 278 813 208	699 075 297 747 540
36	12 660 942 847 609	833 758 529 144 166	183 049 343 643 929 384	1 599 836 631 974 088
37	27 392 697 005 550	1 884 144 109 110 908	431 652 603 971 595 032	3 652 954 620 022 208
38	59 166 631 983 818	4 249 400 422 872 492	1 014 868 412 269 977 442	8 322 867 118 585 614
39	127 777 294 036 668	9 566 581 135 474 702	2 379 355 385 563 105 336	18 923 690 215 681 768
40	275 696 162 276 153	21 499 276 492 272 919	5 563 403 530 205 036 262	42 943 367 206 142 286
41	594 048 482 357 433	48 233 388 196 399 900	12 974 964 525 963 569 978	97 265 602 603 253 438
42	1 281 000 979 206 493	108 047 966 744 458 962	30 186 354 559 080 349 712	219 921 104 676 935 224
43	2 755 074 940 142 431	241 645 525 989 717 809	70 064 113 568 387 529 280	496 383 864 923 234 468
44	5 932 229 201 093 542	539 692 019 601 879 166	162 259 519 144 323 831 762	1 118 569 140 266 192 598
45	12 754 620 464 996 577	1 203 634 572 376 367 923	374 966 937 946 540 768 796	2 516 752 401 957 810 240
46	27 393 502 356 280 237	2 680 685 119 486 373 279	864 732 112 976 429 729 296	5 653 852 976 905 997 716
47	58 904 482 286 533 364	5 963 270 787 963 481 223	1 990 292 162 650 597 920 198	12 683 846 242 039 392 030
48	126 300 979 513 067 199	13 247 560 344 786 965 319	4 572 211 932 174 265 999 574	28 413 833 808 390 157 526
49	271 153 388 225 432 487	29 397 708 611 765 878 122	10 484 509 048 736 986 795 242	63 570 493 940 799 673 654
50	581 799 707 017 985 602	65 162 373 599 194 694 838	23 999 926 816 621 820 432 406	142 041 285 657 057 320 738
51	1 245 200 040 883 711 881	144 265 291 339 186 480 170	54 845 072 436 992 120 826 262	316 981 854 770 124 968 722
52	2 672 296 117 689 586 731	319 107 834 898 349 284 317	125 131 020 334 445 948 974 496	706 573 223 473 121 970 044
53	5 721 610 946 798 161 890	705 067 186 518 337 735 671	285 043 213 836 022 414 418 910	1 573 161 190 417 955 836 862
54	12 219 537 226 294 787 605	1 556 202 374 122 366 410 976	648 336 112 166 418 027 074 000	3 498 618 026 159 745 044 592
55	26 278 769 763 797 603 705	3 432 580 531 634 699 049 051	1 472 529 893 791 471 135 605 612	7 773 224 302 066 420 178 488
56	55 868 130 245 151 778 098	7 561 145 873 732 448 408 790	3 339 705 956 678 263 537 822 184	17 250 739 435 533 913 221 856
57	120 005 563 753 505 676 014	16 647 643 650 693 934 045 389	7 564 345 024 108 961 163 420 714	

Table A3. New series terms for the directed triangular lattice site problem.

n	$S(p)$	$\mu_{0,1}(p)$	$\mu_{0,2}(p)$	$\mu_{2,0}(p)$
27	31 086 416	2 537 201 920	180 162 619 784	3 493 604 968
28	54 484 239	4 696 226 432	351 465 799 212	6 578 499 844
29	95 220 744	8 662 963 994	682 372 429 474	12 255 365 130
30	166 451 010	15 938 662 652	1 319 072 709 540	22 945 871 212
31	290 209 573	29 236 920 460	2 539 112 346 126	42 418 505 522
32	506 071 134	53 506 963 048	4 868 795 865 052	79 065 895 100
33	880 465 145	97 662 175 022	9 301 026 350 316	145 071 334 272
34	1 532 283 109	177 894 354 832	17 707 215 868 596	269 543 696 068
35	2 660 274 891	323 249 218 548	33 597 579 475 250	490 798 690 662
36	4 621 898 737	586 336 769 144	63 552 411 513 904	910 306 336 312
37	8 009 846 706	1 061 171 804 692	119 850 074 633 534	1 644 056 437 386
38	13 891 471 400	1 917 510 976 440	225 393 528 150 372	3 049 141 333 676
39	24 041 215 812	3 457 940 539 676	422 719 590 219 566	5 456 382 479 138
40	41 625 532 064	6 226 878 220 792	790 809 981 499 104	10 141 493 117 240
41	71 931 529 791	11 192 318 698 210	1 475 724 176 635 586	17 948 875 370 594
42	124 411 612 350	20 092 269 205 896	2 747 568 614 463 200	33 532 113 165 512
43	214 621 391 390	36 004 956 808 838	5 103 796 857 539 224	58 529 997 237 324
44	370 839 553 549	64 452 114 092 524	9 460 996 104 306 040	110 351 718 228 800
45	639 024 696 294	115 182 948 294 020	17 501 002 169 903 066	189 161 996 834 038
46	1 102 419 174 084	205 638 719 322 044	32 311 701 334 358 584	361 978 973 535 312
47	1 898 477 439 658	366 587 483 305 266	59 540 588 349 689 460	605 431 024 385 712
48	3 271 434 676 999	652 904 591 166 608	109 522 752 581 367 792	1 185 609 582 832 608
49	5 624 820 363 027	1 161 134 164 194 872	201 098 347 347 198 582	1 916 175 057 214 282
50	9 693 710 116 271	2 063 632 450 148 240	368 654 569 738 994 916	3 885 789 400 216 356
51	16 634 472 160 666	3 661 795 173 290 544	674 667 552 855 892 942	5 981 962 784 372 730
52	28 649 053 574 116	6 494 555 752 892 524	1 232 887 441 544 215 856	12 779 152 925 915 688
53	49 158 925 607 599	11 502 147 999 885 690	2 249 412 773 359 085 386	18 336 104 911 125 754
54	84 477 695 445 892	20 358 932 047 872 636	4 098 441 587 758 882 072	42 326 707 460 800 448
55	144 947 819 272 120	35 990 408 059 294 200	7 456 350 674 610 337 790	54 742 323 913 847 946
56	249 148 051 950 911	63 598 870 606 450 408	13 548 513 117 372 733 000	

From the polynomials for $S^N(t_{\max})$ and $X^N(t_{\max})$ with $t_{\max} = 47$, and using the extrapolation formulae, I extended the series for $S(p)$, $\mu_{0,1}(p)$ and $\mu_{0,2}(p)$ to order 106 and the series for $\mu_{2,0}(p)$ to order 103. The new series terms are listed in table A1.

A.2. The triangular bond problem

The correction terms for the triangular bond problem are very simple. The first correction term for S^N is just a constant $s_{t,0} = 2$, while the first correction term for X^N alternates between 0 and 2. The non-nodal graphs responsible for these correction terms are almost trivial. It is clear (see figure 1) that the non-nodal graphs terminating at level $t+1$ having the smallest possible number of bonds are those composed of two paths meeting on the centre line (t odd) or on the site next to the centre-line (t even), with each path having as few south-east and south-west steps as possible. These sites can be reached by a non-nodal graph with $t+1$ bonds. For t odd the only two such graphs are those consisting of a path with $\lfloor t/2 \rfloor + 1$ south steps and a path starting with a south-east (south-west) step followed by $\lfloor t/2 \rfloor$ south steps, while ending with a south-west (south-east) step. For t even, the two graphs are those consisting of a path with $\lfloor t/2 \rfloor$ south steps terminating with a south-east (south-west) step and a path starting with a south-east (south-west) step followed by $\lfloor t/2 \rfloor$

south steps. It is easy to check that any other non-nodal graph contains more bonds. So $s_{t,0} = 2$ while $x_{t,0}$ alternate between 0 and 2 since for t odd the non-nodal graphs terminate on the centre-line and therefore do not contribute to X^N.

The sequence determining the second correction terms for S^N is

$$1, 2, 5, 10, 17, 26, 37, 50, 65, \ldots$$

from which it is clear that $s_{t,1}$ grows as a polynomial in t, $s_{t,1} = t^2 - 2t + 2$. In general the correction terms can be represented as a polynomial in t of order $2r$. The alternation between odd and even values of t seen in $x_{t,0}$ eventually also manifests itself in the correction terms for S^N. The general formulae for the correction term is,

$$s_{t,r} = \frac{1}{r!(r+1)!}\sum_{j=0}^{2r} a_{r,j}(t-1)^j + \frac{t \bmod 2}{r!(r+1)!}\sum_{j=0}^{\lfloor (r-3)/2 \rfloor} b_{r,j}(t-1)^j \qquad t \geqslant 2r-2. \quad \text{(A.3)}$$

The prefactors and the expression of the polynomials in terms of $t - 1$ has been chosen to make the leading coefficients particularly simple. Once again it should be noted that the leading coefficients $a_{r,2r-m}$ are polynomials in r of order $m + \lfloor m/2 \rfloor$ (this is valid for $m \leqslant 5$), which again was used to obtain a few additional correction formulae.

The extrapolation formulae for X^N are very similar to the ones above,

$$x_{t,r} = \frac{1}{r!(r+1)!}\sum_{j=0}^{2r} a_{r,j}(t-1)^j + \frac{t \bmod 2}{r!(r+1)!}\sum_{j=0}^{r} b_{r,j}(t-1)^j \qquad t \geqslant 2r-2. \quad \text{(A.4)}$$

In this case the leading coefficients of both $a_{r,2r-m}$ and $b_{r,r-m}$ can be predicted. For $r > m$ I find that $a_{r,2r-m}$ can be expressed as a polynomial in r of order $\leqslant m+2$. Likewise $(-1)^r b_{r,r-m}/(r+1)!$ is a polynomial in r of order $2m$ for $r > 2m$.

As stated earlier, the non-nodal contribution to the series for the triangular bond case were calculated up to $t_{\max} = 45$. With the extrapolation formulae I derived the series for $S(p)$, $\mu_{0,1}(p)$ and $\mu_{0,2}(p)$ to order 57 and the series for $\mu_{2,0}(p)$ to order 56. The resulting new series terms are listed in table A2.

A.3. The triangular site problem

In this case the first correction term for S^N alternates between 0 and 1 while the first correction term for X^N is 0. The graphs giving rise to these correction terms are very simple. First note that the graphs giving rise to the bond correction terms all have loops when viewed as site graphs. The non-nodal site graphs with fewest elements for t odd consist of the two paths starting with a south-east (south-west) step followed by $\lfloor t/2 \rfloor$ south steps and ending with a south-west (south-east) step. These graphs have $t + 2$ random elements (remember that the origin is not a random element). For t even one can easily see that there are no loop-free non-nodal graphs with $t + 2$ or fewer elements. This fully accounts for the first correction terms.

The other extrapolation formulae for the triangular site problem are very similar to those for the bond case. The only difference is that the order of the polynomials correcting the odd-t values is somewhat higher. Once again the leading coefficients are low-order polynomials in r. With the help of the extrapolation formulae I extended the series for $S(p)$, $\mu_{0,1}(p)$ and $\mu_{0,2}(p)$ to order 56 and the series for $\mu_{2,0}(p)$ to order 55. The new series terms are listed in table A3.

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