

# Portfolio Optimization under Tracking Error and Weights Constraints

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## **Abstract**

This article addresses the problem of an active portfolio manager whose performances are assessed against a benchmark and who must comply with a weights constraint. This situation is frequently encountered, in particular because the funds are often committed by their own prospectus to a minimum (or maximum) portfolio concentration. We characterize the optimal asset allocation which depends on the targeted ex ante Tracking Error and on the weights constraint. We analyze the implications of the weights constraint on the manager's performance and on the relevance of performance measures such as the Information Ratio. In particular, we obtain that, due to the weights constraint, at the optimum, the Information Ratio often decreases when the manager is free to deviate more from the benchmark.

Keywords: Tracking Error; Weights constraint; Portfolio Optimization; Information Ratio.

JEL classification: G11; D81

## 1 – Introduction

Active portfolio manager performances are commonly measured relative to a benchmark. This is most often done through a Tracking Error, defined as the standard deviation of the difference between the fund and the benchmark's returns. In such a situation, the fund manager sets a maximum value for the tracking error ex ante and maximizes an objective function such as the fund's expected return<sup>1</sup>. Portfolio selection under a tracking error constraint has been studied in the literature. In particular Roll (1992) and Jorion (2003) have examined the deformation of the efficient frontier due to these tracking error constraints. Their solution for the constrained optimal portfolio can be expressed as the sum of the optimal portfolio in absence of the tracking error constraint plus a "self-financing" portfolio as defined in Korkie and Turtle (2002).

Other restrictions on investment policies are commonly found in the contracts between investors and portfolio managers. Some of these restrictions state that the share of certain types of assets should be smaller, higher or equal to a given percentage. This type of restriction is called in this paper a portfolio weights constraint and writes mathematically:

$$\sum_{i \in \mathbb{L}} x_i \leq \bar{w}, \text{ or } \geq \bar{w}, \text{ or } = \bar{w}$$

where  $\mathbb{L}$  is a set of restricted securities,  $x_i$  is the weight in security  $i$  and  $\bar{w}$  is the given percentage.

These constraints are often inherent to the fund policy and are often specified in the fund's prospectus. For instance, an industry sector fund mainly invests in its corresponding sector; a fund dedicated to prudent investors may set an upper bound on its stock's holdings or a lower limit on its holdings of governmental bonds and bills; stock funds restrict the share of non

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<sup>1</sup> Note that this problem is quite general. Indeed, when the portfolio returns are compared to a benchmark, even when the tracking error constraint is not explicit, the optimization of a "risk averse" manager involves a tracking error constraint.

stock securities whereas bond and money market funds restrict the stock's share. For instance, US stock funds commonly include in their prospectus an obligation to hold less than 20% of non US stocks. Other funds restrict their investment in foreign securities. In fact, each "investment style" implies explicit or implicit constraints. These constraints appear worldwide and have been documented in the US context in Almazan et al (2004).

Weights constraints can also be set by regulators. Although some of these regulatory restrictions have been softened in recent years, they still apply in many countries. Regulations are made more explicit in "bank-based" financial markets like in Japan, Germany, France, or Italy where quantitative rules constraining stocks, foreign securities, real estate holdings, derivative securities, restricted stock or private equity as well as other classes of assets are imposed. Some of these regulatory constraints are specific to the fund profile and are often redundant with the prospectus commitments which can be more (but not less) stringent<sup>2</sup>. In addition, funds with tax benefits or tax deferred funds are often subject to weight restrictions. For instance, in most European countries, ceilings on non European security holdings are imposed<sup>3</sup>. Another regulatory restriction (important, in particular for mutual funds) is on short sales. Although short sales constraints are particular weights constraints, hence our framework applies; we do not directly address them in this paper, mainly because we focus our analysis on the case of a single weights constraint.

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<sup>2</sup> For instance, in Switzerland the "two thirds rule" applies: depending on the fund category at least two thirds of the assets must pertain to the relevant geographical sector, class of assets or maturity. In France, a strongly regulated country, bond and money market funds cannot hold more than 10% of stocks and stock funds must hold at least 60% of stocks of their relevant geographical sector (France, Euroland, EU, Asia...). Similar rules apply in most European countries. As other examples, stocks cannot exceed 65% of the assets of life insurance funds in France, 35% in Germany, 20% in Italy.

<sup>3</sup> For example, in France, tax advantaged stock funds (PEA) must be composed at least of 75% of European stocks and other tax advantaged insurance funds (DSK, NSK) are subject to a floor involving European stocks and small caps,...

Finally, portfolio managers and investors may be obligated to hold shares of a firm during a given period. These lock-in restrictions may be regulatory or non regulatory<sup>4</sup>.

The subject of this article, which is portfolio allocation under benchmarking and weights constraints, has not been studied so far, to the best of our knowledge. However, simultaneous tracking error and weights constraints are often encountered in practice. Indeed, many fund managers, in particular sector fund managers whose performance is compared to the corresponding sectorial benchmark, must satisfy weights constraints imposing a minimum concentration in the corresponding sector.

The first intuition could be that these two constraints are redundant, since the weights constraint imposes a concentration in a particular set of assets while the tracking error constraint imposes a maximum “distance” from an index representing this set. In fact, this first intuition is inaccurate as portfolio managers are often pointing out when complaining that the weights constraint limits their performance. Indeed, we show and explain in this paper that the weights constraint is usually different from the tracking error constraint and that both can be binding, even when the benchmark meets the weights constraint. On the one hand, when the tolerance of the fund manager towards deviation from the benchmark is high enough, by investing in a more diversified portfolio outside the sector, the performance can be enhanced. For instance, a high tech fund manager compelled to invest at least 90% in high-tech stocks whose performance is assessed towards a high-tech benchmark could eventually obtain a better mean-tracking error trade-off by investing only 70% of the portfolio in the high tech sector. On the other hand, the weights constraint can be met by investing in a portfolio that

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<sup>4</sup> For instance, SEC rule 144 forbids the incumbent shareholders to sell shares during the three months following an IPO. Such lock-in restrictions are also imposed to employees entering in an employee stock ownership plan, to firm managers who are granted restricted shares or stock-options.... Other “de facto” lock-in restrictions come from capital gain tax differentials between short term and long term investments, from the desire of individuals or families to keep control of companies...

could be 100% invested in the sector, but with very different weights than those of the index, therefore yielding a high tracking error.

In absence of a weights constraint, the maximum obtainable ex ante Information Ratio - defined as the ratio between the expected excess return and the tracking error (denoted  $TE$ ) is independent of the level of the  $TE$  (see for instance Roll (1992) or Jorion (2003)). Hence, the Information Ratio (denoted  $IR$ ) only depends on the ability of the fund manager to generate returns under a  $TE$  constraint (whatever the level of the constraint is), hence is a coherent performance measure. We show in this article that this is no longer true in presence of a weights constraint. In particular, an investor optimizing an expected return–tracking error tradeoff under a weights constraint does not necessarily choose the portfolio that maximizes the  $IR$ . Moreover, it often occurs that the weaker the  $TE$  constraint (the higher the  $TE$  allowed), the smaller the optimal  $IR$ . These results undermine the relevance of the Information Ratio as a performance measure.

Portfolio optimization, the analytics of the efficient frontier and its computational issues have been extensively studied in the Finance literature. Deviations from the standard framework such as general linear constraints (for instance Markowitz (1959), Sharpe (1970)), short sales constraints (for instance Ross (1977) and Dybvig (1984)), benchmarking (for instance Roll (1992) and Jorion (2003)), VaR constraints (for instance Alexander and Baptista (2004) in the case of a standard mean-variance framework or Alexander and Baptista (2007) in the case of a mean-tracking error framework), drawdown constraint (Alexander and Baptista (2006)) mean-variance dynamic rebalancing (for instance Richardson (1989) and Bajeux-Besnainou and Portait (1998)), have also been studied. Our paper differs from this literature in that it addresses portfolio selection under both tracking error and weights constraints.

Section 2 presents the general background and sets up an introductory example involving a weights constraint and the  $TE$  constraint.

Section 3 considers the case of an equality weights constraint and section 4 the case of an inequality constraint. In section 5, the loss and the reduction of the *IR* due to the weights constraint are evaluated, the coherence of the *IR* is questioned and an alternative performance measure is suggested. A numerical example that illustrates the main results of the paper is developed in section 6. Section 7 is a conclusion.

## 2. Background and introductory example

We present in 2.1 some known results about the efficient portfolios meeting a tracking error constraint. We introduce in 2.2 a weights constraint and analyze its implications through a simple numerical example.

### 2.1 Definitions, notations and background results

We consider a portfolio manager who can trade  $n$  risky assets ( $i = 1, \dots, n$ ) but no risk free asset. The expected returns of risky assets are represented by a  $n$ -dimensional vector  $\boldsymbol{\mu}$ , which  $i^{th}$  component is denoted  $\mu_i$ . The variance-covariance matrix ( $n \times n$ ) of the returns on risky assets is  $\mathbf{V}$ . A bold letter represents a vector or a matrix. ' indicates the transpose of a matrix or a vector;  $\mathbf{1}$  is the unit vector with  $n$  components equal to 1. We note  $\mathbf{x}$  a portfolio of risky assets and its corresponding vector of weights and  $\mu_x$  its expected return.

Recall the solutions of the optimization program when the performance is assessed against a benchmark (represented by a portfolio  $\mathbf{b}$ ). When the manager maximizes the expected return with a constraint on the tracking error *TE* (Roll's problem), or equivalently, when he/she optimizes a mean-tracking error trade-off, his/her optimization program is:

$$\text{Max}_{\mathbf{x}} \boldsymbol{\mu}'\mathbf{x}, \text{ subject to } (\mathbf{x} - \mathbf{b})' \mathbf{V} (\mathbf{x} - \mathbf{b}) \leq TE^2 \text{ and } \mathbf{1}'\mathbf{x} = 1.$$

It is convenient to consider the deviation from the benchmark  $\underline{y} \equiv \mathbf{x} - \mathbf{b}$  and write this program in the equivalent and alternative form:

$$(T) \quad \text{Max}_{\underline{y}} \quad \underline{\mu}' \underline{y} - \frac{\gamma}{2} \underline{y}' \mathbf{V} \underline{y}, \quad \text{with: } \mathbf{1}' \underline{y} = 0.$$

where  $\underline{y} \equiv \mathbf{x} - \mathbf{b}$  is a self-financing portfolio<sup>5</sup> (sum of the weights equal to zero), and the benchmark  $\mathbf{b}$  is the “host portfolio” as defined in Korkie and Turtle (2002). In the sequel, self financing portfolios are referred as SF, are underlined. In the case of weights summing up to one, the portfolio is called “fully invested” (for instance  $\mathbf{x}$  is the weights vector of a fully invested portfolio while  $\underline{y}$  is an SF weights vector).

The “multiplier”  $\gamma$  can be interpreted as a “risk aversion” parameter and the constraint states that the sum of the weights is equal to zero.

The self financing portfolio  $\underline{y}$  represents the “active part” of the portfolio management (a purely passive allocation replicates the benchmark  $\mathbf{b}$ , hence  $\underline{y} = \underline{0}$ ).

The solution of (T) is well known (see Roll (1992)) and writes:

$$(1) \quad \underline{y}_{\theta}^* = \theta \underline{\mathbf{u}}$$

where

$$(2) \quad \underline{\mathbf{u}} = \mathbf{t} - \mathbf{a},$$

$$(3) \quad \mathbf{a} = \frac{\mathbf{V}^{-1} \mathbf{1}}{\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}}$$

is the standard minimum variance portfolio (fully invested) in the Markowitz framework (with no TE constraint),

$$(4) \quad \mathbf{t} = \frac{\mathbf{V}^{-1} \underline{\mu}}{\mathbf{1}' \mathbf{V}^{-1} \underline{\mu}}$$

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<sup>5</sup> Also referred as “arbitrage”, “zero-weight” or “zero-investment” portfolios.



is the portfolio that maximizes the Sharpe ratio (with no risk-free asset), or “standard tangent portfolio” (in the Markowitz framework)<sup>6</sup> and:

$$(5) \quad \theta = \frac{\mathbf{1}' \mathbf{V}^{-1} \boldsymbol{\mu}}{\gamma} = \frac{\mu_a}{\sigma_a^2 \gamma}$$

Since  $\theta$  is inversely proportional to  $\gamma$ , the “multiplier”  $\theta$  (which lies between 0 and  $+\infty$  for risk averse investors) can be interpreted as a “risk tolerance” parameter. In the following,  $\mu_a$  is assumed positive, which is a standard assumption<sup>7</sup>.

In equations (1),  $\underline{\mathbf{u}}$  is self-financing (SF). The solutions  $\mathbf{b} + \underline{\mathbf{y}}_\theta^*$  for  $\theta$  positive are referred as  $T$ -portfolios and their representative points in the  $(\sigma, \mu)$  space span the upper branch  $T$  of an hyperbola. In the Excess-Return space (expected excess return as a function of the tracking error), we consider the SF portfolios  $\underline{\mathbf{y}}_\theta^*$  and  $T$  becomes linear as it follows from (1) and as shown in Jorion (2003).

More precisely, it follows from (1) that, in the excess return space, the  $T$ -portfolios corresponding to positive values of  $\theta$  are represented by the semi-straight line stemming from the origin (representing portfolio  $\mathbf{b}$ ) with a slope equal to  $\mu_u/\sigma_u$ , where  $\mu_u$  and  $\sigma_u$  are respectively the expectation and the standard deviation of the return of  $\underline{\mathbf{u}}$ . Moreover, it is also convenient to represent the solutions of (T) corresponding to negative values of  $\theta$  (inefficient portfolios) by the semi-straight line  $T'$  stemming from the origin with a negative slope equal to  $-\mu_u/\sigma_u$ , symmetrical with respect the  $TE$  axis, as shown in Figure 1.

In the sequel, we deal mainly with SF portfolios and all the geometrical figures are represented in the excess return space.

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<sup>6</sup> In the absolute return space, this tangent is drawn from the origin since there is no risk-free security.

<sup>7</sup> In presence of a risk-free asset,  $\mu_a$  higher than the risk-free rate is an equilibrium condition. In absence of a risk-free rate,  $\mu_a > 0$  is the realistic assumption; as pointed out in Green (1986), a sufficient but much stronger condition is that there is no SF portfolio with a non-negative correlation with all assets.

## 2.2. The addition of a weight constraint: introductory numerical example.

We present in this section a simple numerical example that illustrates the impact of an additional linear constraint on the tracking error optimization program. This example shows the distortion of the solution and the reduction of the information ratio implied by the constraint. It also reveals a paradoxical consequence of the presence of a weights constraint: the Information Ratio may decrease when a higher tracking error is allowed.

Assume that three securities are traded, numbered 1, 2 and 3 where 1 and 2 are domestic and 3 is foreign. The benchmark is an equally weighted average of securities 1 and 2. The securities 1, 2 and 3 have expected returns respectively of 10%, 12% and 14%; standard deviations are all equal to 20% and the correlation between any two of them is equal to .5. A portfolio manager is compelled to a tracking error of 5% at most (program (T)). In addition, we also address the case in which he/she is also constrained to hold at least 90% of domestic assets (program Tracking-error with Inequality Constraint (TIC)). We can then write succinctly:

$$(T) \quad \text{Max } \mu_x, \text{ s.t. } TE_x \leq 5\% ;$$

$$(TIC) \quad \text{Max } \mu_x, \text{ s.t. } TE_x \leq 5\% \text{ and } x_1 + x_2 \geq 90\%$$

where  $x_1$  and  $x_2$  are the weights in securities 1 and 2;  $\mu_x$  is the expected return of the portfolio and  $TE_x$  is the Tracking error. The objective of maximizing the expected return (or equivalently expected excess return) subject to a  $TE$  constraint is naturally associated to the Information Ratio ( $IR = \text{expected excess return}/TE$ ) as a performance measure, which is analogous to the Sharpe ratio associated to the mean-variance optimization program.

The solutions of (T) and (TIC) are represented in table 1.

	$x_1$	$x_2$	$x_3$	$\mu_x - \mu_b$	$IR_x$
Solution of (T)	25%	50%	25%	1%	20%
Solution of (TIC)	21.55%	68.45%	10%	0.77%	15.4%

**Table 1**

The solution of (T) is obtained directly<sup>8</sup> from equation (1). This portfolio yields an expected return of 12%. Since the benchmark yields 11%, the Information Ratio is 20%.

This simple example shows that, although the benchmark satisfies the constraint, a rational manager would optimally invest in 25% of foreign security. When constrained to hold at most 10% of the later (as in (TIC)), this constraint is binding. In fact, the manager solving (TIC) chooses<sup>9</sup> a portfolio composed of 21.55%, 68.45% and 10% in securities 1, 2 and 3 respectively, yielding an expected return of 11.77%, which represents a loss of 23% in Information Ratio.

Note that the Information Ratio does not decrease necessarily when the constraint on the tracking error softens. Indeed, assume that the tracking error is constrained to be smaller than 8% (instead of 5%); the manager solving (TIC) with a constraint  $TE_x \leq 8\%$  chooses a portfolio composed of 5.95%, 84.05% and 10% in securities 1, 2 and 3 respectively, yielding an expected return of 12.08% and an Information Ratio of 13.51% (instead of 15.40% with a constraint  $TE_x \leq 5\%$ ).

This surprising result and its implication on the relevance of the Information Ratio as a measure of a constrained manager's ability to generate expected returns are thoroughly discussed in section 5. In particular, we prove there that optimizing  $\mu$  under an *inequality* constraint on  $TE$  is equivalent to maximizing  $IR$  under an *equality* constraint on  $TE$  (but not under an inequality constraint).

	$x_1$	$x_2$	$x_3$	$\mu_x - \mu_b$	$IR_x$
Solution of (TIC) with $TE_x \leq 5\%$	21.55%	68.45%	10%	0.77%	15.40%

<sup>8</sup> Note that there is a direct one-to-one relation between the risk aversion parameter  $\theta$  and the tracking error target:  $TE = \theta\sigma_u$ .

<sup>9</sup> In this simple case of only 3 securities, (TIC) is solved through its two binding constraints, forming a system of two equations and two unknown,  $x_1$  and  $x_2$ .

Solution of (TIC) with $TE_x \leq 8\%$	5.95%	84.05%	10%	1.08%	13.51%
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**Table 2**

In this paper, we generalize this simple example and analyze in a general framework the impact of weights constraints and the corresponding loss in expected returns and information ratios.

### 3. Tracking error with an equality weights constraint

Although the case of an equality weights constraint is not frequent in practice<sup>10</sup>, it is a necessary technical step to address the most frequent case of inequality studied in section 4. After presenting the framework (common to the cases of equality and inequality) we study the analytics and the geometrical representations of the optimal solutions.

#### 3.1 Framework and notations

We consider a subset  $\mathbb{L}$  of the  $n$  traded assets (for instance the first  $l$  securities of the list).  $\mathbb{L}$  is called the set of “limited” or “restricted assets” because, in some cases, portfolios are constrained to a limited weight in assets  $\mathbb{L}$ . Let  $\mathbf{1}_{\mathbb{L}}$  be the  $n$ -dimensional vector whose  $i^{th}$  component is either equal to 1, if asset  $i$  belongs to  $\mathbb{L}$ , or equal to 0 otherwise ( $\mathbf{1}_{\mathbb{L}}$  is the indicator vector of the subset  $\mathbb{L}$ ).

Generically,  $w_x$  represents the weight of a fully invested portfolio on the limited assets and  $w_y$  the weight of a SF portfolio on the limited assets; Therefore,  $w_x = \sum_{i \in \mathbb{L}} x_i = \mathbf{1}_{\mathbb{L}}' \mathbf{x}$  and  $w_y = \mathbf{1}_{\mathbb{L}}' \mathbf{y}$ .

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<sup>10</sup> Lock-in constraints, as mentioned in the introduction, would be an example of an equality constraint.

The weight constraint that may be imposed on the fully invested portfolio  $\mathbf{x}$  states that  $w_x$  is either equal to a constant  $\bar{w}$  or cannot exceed  $\bar{w}$ <sup>11</sup>; when  $\omega$  stands for  $\bar{w} - w_b$ , the constraints can be written for the SF active part,  $\underline{\mathbf{y}} = \mathbf{x} - \mathbf{b}$ , in function of  $\omega$ .

$$(ec) \text{ (equality constraint)} \quad w_x = \bar{w} \Leftrightarrow w_y = \bar{w} - w_b = \omega$$

$$(ic) \text{ (inequality constraint)} \quad w_x \leq \bar{w} \Leftrightarrow w_y \leq \bar{w} - w_b = \omega$$

Note first that  $\omega$  can be positive or negative, most likely between -1 and +1 (since  $\bar{w}$  and  $w_b$  are in most cases between 0 and +1).

Note also that (ic) while written as a “cap-constraint” is equivalent to a “floor-constraint” written as  $\sum_{i \in \mathbb{M}} x_i > 1 - \bar{w}$  on the subset  $\mathbb{M}$  of securities, the “complementary set” of  $\mathbb{L}$ . Hence, without any loss of generality, in the sequel, we consider cap constraints only<sup>12</sup>.

When an equality constraint (ec) is imposed on portfolio weights, which is the case considered in this section, the manager’s program writes, for a given value  $\omega$ :

$$(TEC-\omega) \quad \text{Max}_{\underline{\mathbf{y}}} \quad \underline{\boldsymbol{\mu}}' \underline{\mathbf{y}} - \frac{\gamma}{2} \underline{\mathbf{y}}' \mathbf{V} \underline{\mathbf{y}}, \text{ with: } \mathbf{1}' \underline{\mathbf{y}} = 0 ; \mathbf{1}'_{\mathbb{L}} \underline{\mathbf{y}} = \omega$$

The solution of program (TEC- $\omega$ ) for a given weight  $\omega$  is an SF portfolio representing the optimal “active part” of the asset allocation and is noted  $\underline{\mathbf{y}}_{\omega, \theta}^*$  (to be distinguished from  $\underline{\mathbf{y}}_{\theta}^*$  which solves (T)). When the risk aversion  $\gamma$  (related to  $\theta$  by (5)) is positive,  $\underline{\mathbf{y}}_{\omega, \theta}^*$  is referred as a TEC- $\omega$  portfolio. The solutions of program (TEC- $\omega$ ) for  $\gamma$  negative (“inefficient”

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<sup>11</sup> The parameter  $\bar{w}$  is not bounded in our analysis, although, in practise, it would most likely be between 0 and 1 as  $\bar{w} < 0$  would constrain the portfolio to hold globally short positions in the restricted securities; while  $\bar{w} > 1$  would constrain the “unrestricted” securities to be globally held short.

<sup>12</sup> Note also that constraints (ec) and (ic) are particular cases of general linear constraints where  $\mathbf{1}_{\mathbb{L}}$  would be substituted by any vector of  $\mathbb{R}^n$ . It would be a straightforward technical exercise to generalize the results of this paper to a general linear constraint. We choose not to do so, as we cannot think of any financial interpretation coming out of this generalization.

portfolios) are referred as  $TEC'\text{-}\omega$  portfolios. In the expected excess return– $TE$  space,  $TEC_\omega$  is defined as the set of points representing  $TEC\text{-}\omega$  portfolios (for a given  $\omega$  and for all positive  $\theta$ ) and  $TEC'_\omega$  the graphical representation of portfolios  $TEC'\text{-}\omega$  ( $\theta$  negative). We prove in 3.3 that  $TEC_\omega$  and  $TEC'_\omega$  are respectively the upper branch and the lower branch of an hyperbola.

### 3.2 Analytical solution of (TEC- $\omega$ ).

Before deriving the solution of program (TEC- $\omega$ ), we characterize in Lemma 1 a remarkable SF portfolio, the constrained minimum-tracking error portfolio satisfying the weights constraint. This portfolio, chosen by an investor with an infinite risk aversion  $\gamma$  and compelled to satisfy the constraint  $\mathbf{1}'_L \underline{\mathbf{y}} = \omega$  is denoted  $\underline{\mathbf{z}}_\omega$ .

#### Lemma 1

The weights vector  $\underline{\mathbf{z}}_\omega$  of the constrained minimum tracking error portfolio which solves:

$$\text{Min}_{\underline{\mathbf{y}}} \underline{\mathbf{y}}' \mathbf{V} \underline{\mathbf{y}}, \text{ with: } \mathbf{1}' \underline{\mathbf{y}} = 0; \mathbf{1}'_L \underline{\mathbf{y}} = \omega, \text{ writes:}$$

$$(6) \quad \underline{\mathbf{z}}_\omega = \omega \underline{\mathbf{s}}$$

$$\text{with } \underline{\mathbf{s}} = \frac{\mathbf{k} - \mathbf{a}}{w_k - w_a}^{13} \text{ and } \mathbf{k} = \frac{\mathbf{V}^{-1} \mathbf{1}_L}{\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}_L}$$

Besides, since  $\mu_a$  is assumed positive,  $\mu_s$  and  $w_u$  have the same sign, hence  $\mu_{z_\omega}$  and  $\omega w_u$  have the same sign.

Proof: see the Appendix.

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<sup>13</sup> It is easily shown that  $w_k$  cannot equal  $w_a$  except when all assets are restricted; in this case the problem is meaningless.

Note that portfolios  $\underline{\mathbf{s}}$  and  $\underline{\mathbf{k}}$  are both independent of  $\theta$  and of the constraint level  $\omega$  (but not on the list of restricted securities  $\mathbb{L}$  characterized by  $\mathbf{1}_L$ ),  $\underline{\mathbf{s}}$  is SF and  $\underline{\mathbf{k}}$  is fully invested.

Note that  $\underline{\mathbf{s}}$  is the constrained minimum tracking-error SF portfolio with a weight  $w_s$  on restricted assets equal to 1. Besides, it follows from (6) that the minimum- $TE$  portfolios  $\underline{\mathbf{z}}_\omega$  are geometrically represented by two semi- straight lines<sup>14</sup> stemming from the origin (representing the benchmark) and symmetrical around the  $TE$  axis, characterized in Proposition 3.

Proposition 1 (proved in the appendix) characterizes the solution  $\underline{\mathbf{y}}_{\omega,\theta}^*$  of program (TEC- $\omega$ ).

#### Proposition 1

For any values of  $\theta$  and  $\omega$ , the solutions of (TEC- $\omega$ ) are combinations of two self-financing portfolios  $\underline{\mathbf{u}}$  and  $\underline{\mathbf{s}}$  (both independent of  $\omega$  and  $\theta$ ) and write:

$$(7) \quad \underline{\mathbf{y}}_{\omega,\theta}^* = (\omega - \theta w_u) \underline{\mathbf{s}} + \theta \underline{\mathbf{u}}$$

Equivalently:

$$(8) \quad \underline{\mathbf{y}}_{\omega,\theta}^* = \underline{\mathbf{y}}_\theta^* + (\omega - \theta w_u) \underline{\mathbf{s}}$$

where  $\underline{\mathbf{y}}_\theta^* = \theta \underline{\mathbf{u}}$  is solution of program (T) as given in (1).

Equations (7) and (8) lead to four different interpretations:

First, (7) writes equivalently as  $\underline{\mathbf{y}}_{\omega,\theta}^* = \omega \underline{\mathbf{s}} + \theta (\underline{\mathbf{u}} - w_u \underline{\mathbf{s}})$  which expresses the solution  $\underline{\mathbf{y}}_{\omega,\theta}^*$  as a combination of two portfolios: the first one  $\omega \underline{\mathbf{s}}$ , which is the constrained minimum tracking error portfolio  $\underline{\mathbf{z}}_\omega$  with a weight  $\omega$  in the restricted assets, is chosen by an investor with a zero risk tolerance. The second portfolio,  $\theta (\underline{\mathbf{u}} - w_u \underline{\mathbf{s}})$ , has a zero weight on the restricted assets

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<sup>14</sup> Most likely two segments since  $-1 \leq \omega \leq +1$ .

and provides for additional expected return at the expense of increasing the tracking error without changing the weight on restricted assets.

Second, the previous decomposition shows a linear and separable impact of any change in the two parameters  $\omega$  and  $\theta$  on the solution  $\underline{\mathbf{y}}_{\omega,\theta}^*$ .

Third, (7) implies a two fund separation, the two separating funds being portfolios  $\underline{\mathbf{u}}$  and  $\underline{\mathbf{s}}$ , which are independent of  $\theta$  and  $\omega$  (however,  $\underline{\mathbf{s}}$  depends on the list  $\mathbb{L}$  of restricted assets). Besides, the weights allocated to the two separating funds do not sum up to one, which at first may not seem to be a problem since only self financing portfolios are involved, but actually raises some issues discussed in the interpretation of Proposition 2 of section 3.3. Moreover, the *three* funds  $\mathbf{b}$ ,  $\underline{\mathbf{s}}$  and  $\underline{\mathbf{u}}$  are necessary to span all the *fully invested* portfolios  $\mathbf{b} + \underline{\mathbf{y}}_{\omega,\theta}^*$  solutions of (TEC), for all values of  $\theta$  and  $\omega$ .

Fourth, equation (8) implies that any TEC-portfolio is obtained by adding to the unconstrained optimum portfolio  $\underline{\mathbf{y}}_{\theta}^* = \theta \underline{\mathbf{u}}$  solution of (T) a “fraction”  $(\omega - \theta w_u)$  of the self financing portfolio  $\underline{\mathbf{s}}$ . The addition of portfolio  $\underline{\mathbf{s}}$  is used to obtain the required weight  $\omega$ .

### 3.3. Geometrical representation of the solutions of (TEC- $\omega$ )

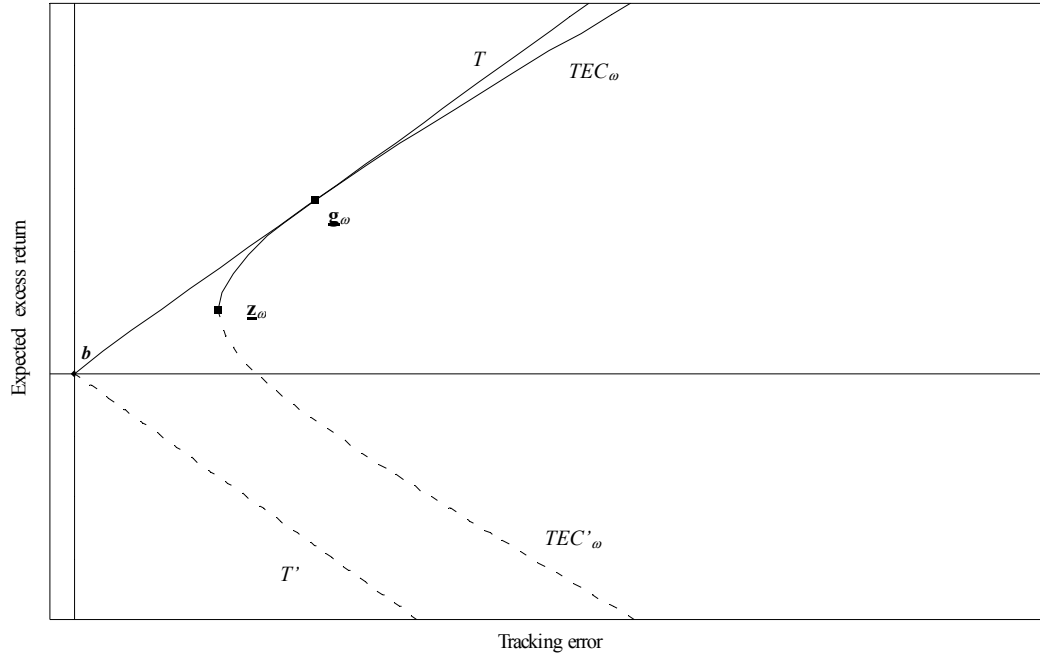
Recall that the solutions  $\underline{\mathbf{y}}_{\theta}^* = \theta \underline{\mathbf{u}}$  of the unconstrained program (T) are represented by the two symmetrical semi-straight lines  $T'$  and  $T$  (as in Figure 1). Each  $T'-T$  portfolio corresponds to a particular value of  $\theta$  and has a weight  $w_{y_{\theta}^*}$  on the restricted assets.  $T'-T$  is spanned clockwise by increasing  $\theta$  (the weight in  $\underline{\mathbf{u}}$  increases). When  $\theta$  increases from  $-\infty$  to  $+\infty$ ,  $w_{y_{\theta}^*}$  increases from  $-\infty$  to  $+\infty$  if  $w_u > 0$ , or decreases from  $+\infty$  to  $-\infty$  if  $w_u < 0$ .

Since  $w_u$  is the difference  $w_t - w_a$  between the weights of restricted assets in the tangent portfolio and in the minimum variance portfolio, it may be positive or negative: intuitively, it is positive when the returns of the restricted assets have a high expected value and volatility

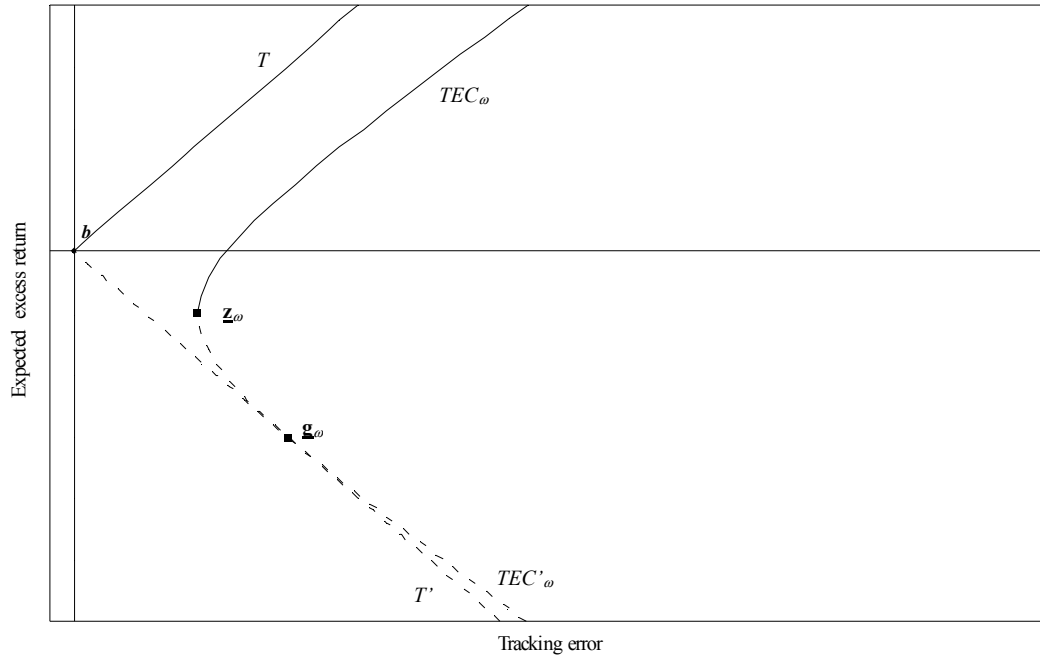


(more aggressive portfolios contain more restricted assets), and is negative otherwise. Since it is more common to restrict aggressive assets,  $w_u$  is positive in most circumstances.

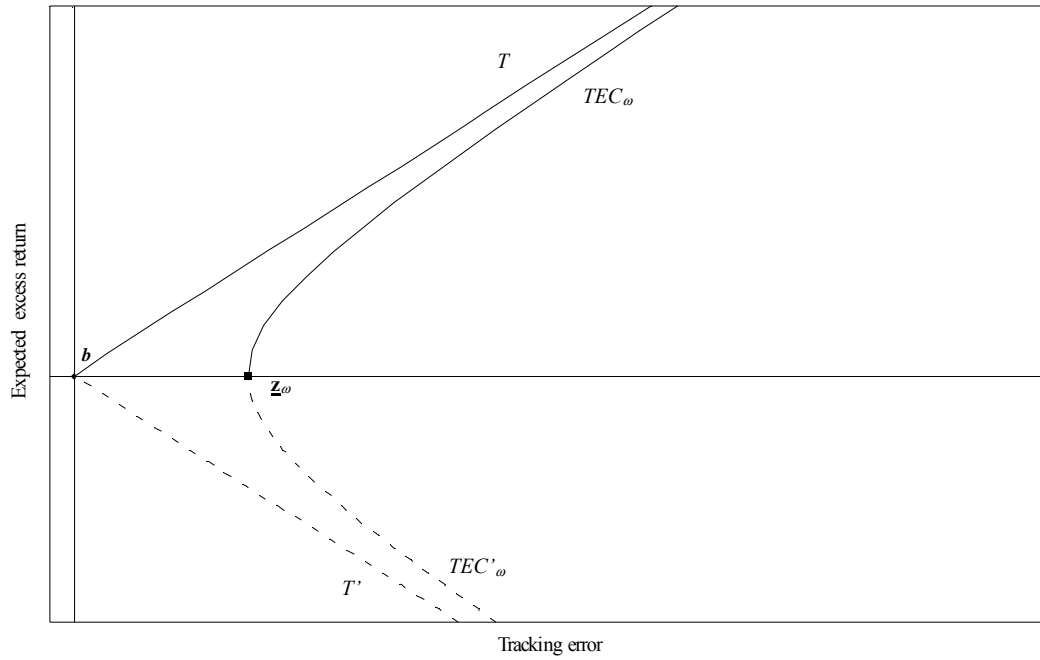
Figures 1, 2 and 3 represent  $TEC_\omega, TEC'_\omega$  (forming a hyperbola as shown later),  $T$  and  $T'$  when  $w_u > 0$ ,  $w_u < 0$  and  $w_u = 0$  respectively.



**Figure 1:  $T$  and  $TEC_\omega$  tangent frontiers in excess return space ( $w_u > 0$ )**



**Figure 2:  $T'$  and  $TEC'_\omega$  tangent frontiers in excess return space ( $w_u < 0$ )**



**Figure 3:  $T$  and  $TEC_\omega$  frontiers in excess return space ( $w_u = 0$ )**

When  $w_u > 0$  or  $< 0$  (Figures 1 and 2 respectively), the two frontiers  $TEC_\omega$  and  $T$  (or  $TEC'_\omega$  and  $T'$ ) are tangent at point  $\underline{\mathbf{g}}_\omega$  which represents the unique solution of program (T) with a weight  $\omega$  on the restricted assets. Indeed, as pointed previously, considering the different portfolios of  $T'-T$  when moving clockwise, the corresponding weight  $w_{y^*}$  increases (when  $w_u > 0$ ), or decreases (when  $w_u < 0$ ), taking all values between  $-\infty$  and  $+\infty$ . Hence, there is a unique  $T'-T$  portfolio with  $w_{y^*} = \omega$ . Since this portfolio solves program (T) and satisfies the weight constraint  $\omega$ , it also solves (TEC- $\omega$ ) and therefore is common to  $TEC_\omega$  and  $T$  (or  $TEC'_\omega$  and  $T'$ ). This portfolio called the tangent portfolio and denoted  $\underline{\mathbf{g}}_\omega$  can be characterized using equation (7):

$$(9) \quad \underline{\mathbf{g}}_\omega = \theta_\omega \underline{\mathbf{u}},$$

with:

$$(10) \quad \theta_\omega = \omega / w_u$$

Note that  $\theta_\omega$  is the risk tolerance parameter of the unconstrained investor choosing the self-financing portfolio  $\underline{\mathbf{g}}_\omega$  (which implies a weight  $\omega$  on restricted assets). Since portfolio  $\underline{\mathbf{g}}_\omega$  is chosen by the unconstrained investor with a risk tolerance parameter  $\theta = \theta_\omega$ , it is efficient if  $\theta_\omega \geq 0$  and inefficient (on the lower part of frontier  $T$ ) otherwise<sup>15</sup>.

Note also that the unconstrained SF optimal portfolio  $\underline{\mathbf{y}}_\theta^*$  can be generated by any single fund belonging to the frontier  $T$ . Selecting  $\underline{\mathbf{g}}_\omega$  as the generating fund of  $T$  and since (from (9))  $\underline{\mathbf{u}} =$

$\frac{1}{\theta_\omega} \underline{\mathbf{g}}_\omega$ , equation (1) yields:

$$(11) \quad \underline{\mathbf{y}}_\theta^* = \frac{\theta}{\theta_\omega} \underline{\mathbf{g}}_\omega.$$

---

<sup>15</sup> Note that the terms “efficient” and “inefficient” refer here to the mean-tracking error space.

When  $w_u = 0$  (Figure 3), the two branches of the hyperbola generated by the solutions of (TEC- $\omega$ ) are symmetrical around the  $TE$  axis and its two asymptotes are  $T^*$  and  $T$ .

We can now state Proposition 2 which follows directly from equation (8), the definition of  $\theta_\omega$  and the composition of  $\underline{\mathbf{g}}_\omega$ :

Proposition 2

When  $w_u \neq 0$ , the solution of (TEC- $\omega$ ) can be written as:

$$(12) \quad \underline{\mathbf{y}}_{\omega,\theta}^* = \left(1 - \frac{\theta}{\theta_\omega}\right) \underline{\mathbf{z}}_\omega + \frac{\theta}{\theta_\omega} \underline{\mathbf{g}}_\omega$$

Equation (12) provides a « standard » two-fund separation. It expresses the solution  $\underline{\mathbf{y}}_{\omega,\theta}^*$  of (TEC- $\omega$ ), for any value of  $\theta$ , as a combination of two funds ( $\underline{\mathbf{z}}_\omega$  and  $\underline{\mathbf{g}}_\omega$ ), the sum of the weights allocated to these two funds being equal to 1; these combinations generate an hyperbola. Note that equations (7) and (12) provide two different forms of separation. In a way, (7) is a more general form of separation than (12) since it involves two SF separating funds  $\underline{\mathbf{s}}$  and  $\underline{\mathbf{u}}$  which are independent of both parameters  $\theta$  and  $\omega$ . The two separating funds are therefore common to all investors restricted on the same set  $\mathbb{L}$  of assets, for any value of their risk tolerance  $\theta$  and of the level  $\omega$  of their weights constraint. However, as pointed previously, the sum of the two weights in  $\underline{\mathbf{s}}$  and  $\underline{\mathbf{u}}$  is generally different from 1 and such non convex combinations do not necessarily generate an hyperbola.

There are three possible situations as far as the sign of  $\omega$  is concerned:

- $\omega = 0$ : the benchmark satisfies the equality constraint;

- $\omega > 0$ : the benchmark satisfies the inequality weights constraint but not the equality;
- $\omega < 0$ : the benchmark does not satisfy the inequality constraint, which is unlikely<sup>16</sup>.

For realistic values ( $\omega \geq 0$ ), assumed in the rest of the paper, we characterize the set of hyperbolas  $TEC_\omega$  in Proposition 3 (proved in Appendix).

**Proposition 3**

- For  $\omega > 0$  (case represented in Figure 4), the set of constrained frontiers  $TEC'_\omega$   $TEC_\omega$ , (when  $\omega$  varies) is a network of hyperbolas characterized by:
  - Parallel asymptotes which slopes  $h$  and  $-h$  are given by:

$$(13) \quad h = \sqrt{\frac{\mu_u \mu_a}{\sigma_a^2} - \frac{\mu_s^2}{\sigma_s^2}}$$

- The slopes of the two semi-straight lines stemming from the origin and symmetrical around the  $TE$  axis representing the minimum  $TE$  self financing portfolios  $\underline{z}_\omega$  are equal to:

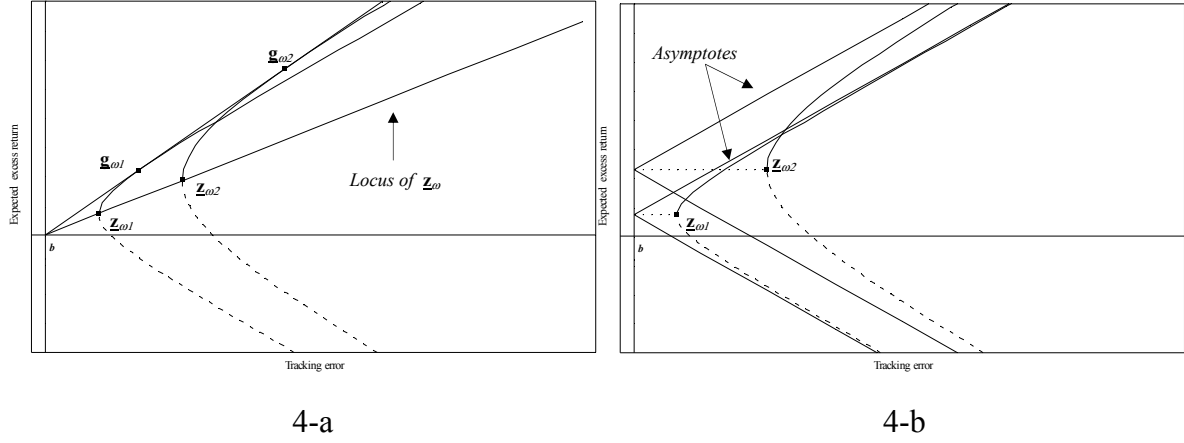
$$\pm \frac{\mu_s}{\sigma_s}.$$

- For  $\omega = 0$  (the benchmark satisfies the equality constraint), the constrained efficient frontier  $TEC_0$  is a semi-straight line stemming from the origin, which slope is the same as the slope  $h$  of the hyperbolas upper asymptote.

Note that the slope  $h$  of the hyperbolas upper asymptotes is lower than the slope of the unconstrained frontier  $T(\sqrt{\frac{\mu_u \mu_a}{\sigma_a^2}})$ , which implies that  $T$  intercepts the hyperbolas  $TEC_\omega$  (for all  $\omega > 0$ ).

---

<sup>16</sup> Such a situation may prevail for instance when meeting the “constraint” is a condition for tax benefits which is not necessarily satisfied by the benchmark.



**Figure 4: The locus of minimum tracking error portfolios  $\underline{z}_\omega$  (4-a) and the network of hyperbolas  $TEC_\omega$  when  $\omega > 0$  (and  $w_u > 0$ ) with their parallel asymptotes (4-b)**

#### 4. Tracking error with an inequality weights constraint

In this section, we consider an inequality constraint (ic- $\omega$ ) ( $\mathbf{1}'_L \underline{\mathbf{y}} \leq \omega$ ), most frequently encountered in practice. Recall that we assume  $\omega \geq 0$  (equivalent to  $w_b \leq \bar{w}$ ), which means that the benchmark satisfies the inequality weights constraint (the most realistic case).

The optimization program then writes:

$$(TIC-\omega) \quad \text{Max}_{\underline{\mathbf{y}}} \quad \underline{\boldsymbol{\mu}}' \underline{\mathbf{y}} - \frac{\gamma}{2} \underline{\mathbf{y}}' \mathbf{V} \underline{\mathbf{y}}, \quad \text{with: } \mathbf{1}' \underline{\mathbf{y}} = 0; \mathbf{1}'_L \underline{\mathbf{y}} \leq \omega$$

For a given value of  $\omega$  and any  $\theta \geq 0$ , the solutions are referred as TIC- $\omega$  portfolios and their geometrical representation in the expected excess return-TE space are referred as  $TIC_\omega$ .

Note that the weights constraint (ic- $\omega$ ) is either binding (in this case, it satisfies the equality constraint and the solution  $\underline{\mathbf{y}}_{\omega,\theta}^*$  derived in the Proposition 1 holds), or not binding and the unconstrained optimum  $\underline{\mathbf{y}}_{\theta}^*$  given in (1) prevails.

More precisely, the constraint (ic) is not binding if and only if the unconstrained optimum satisfies the weights constraint:  $\theta \mathbf{1}_L' \underline{\mathbf{u}} \leq \omega$ , or equivalently:

$$(14) \quad \omega \geq \theta w_u$$

When  $w_u \neq 0$ , inequality (14) can also be written, depending on the sign of  $w_u$ :

$$(15) \quad \theta < \frac{\omega}{w_u} \quad (\text{or } \theta < \theta_{\omega}) \text{ if } w_u > 0$$

( $w$  increases with  $\theta$  when moving clockwise along  $T$ ); or:

$$(16) \quad \theta \geq \frac{\omega}{w_u} \quad (\text{or } \theta > \theta_{\omega}) \text{ if } w_u < 0$$

( $w$  decreases with  $\theta$  when moving clockwise along  $T$ ).

Recall that  $w_u$  is the difference  $w_t - w_a$  between the weights of restricted assets in the tangent portfolio and in the minimum variance portfolio and that its sign is probably more often positive than negative although we cannot discard a negative sign.

When  $w_u < 0$ , since  $\omega \geq 0$  and  $\theta \geq 0$ , the condition of equation (16) is always satisfied which means that the weights constraint is always unbinding and that  $TIC_{\omega}$  is the same as  $T$ .

We characterize the TIC- $\omega$  portfolios (for  $\theta > 0$  and  $\omega > 0$ ) in proposition 4:

#### Proposition 4

Assume  $\omega \geq 0$  and  $\theta \geq 0$ .

- If  $w_u > 0$  the solution of (TIC- $\omega$ ) writes:

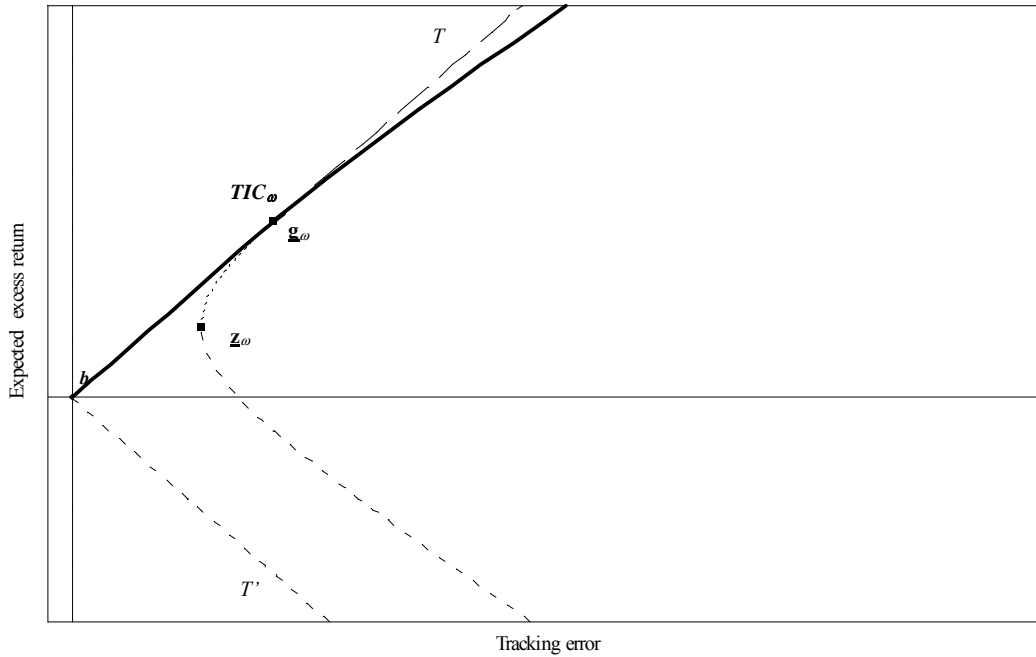
$$(17) \quad \begin{cases} \underline{\mathbf{y}}_{-\omega, \theta}^* = \theta \underline{\mathbf{u}} & \text{for } \theta < \frac{\omega}{w_u} \\ \underline{\mathbf{y}}_{-\omega, \theta}^* = \theta \underline{\mathbf{u}} + (\omega - \theta w_u) \underline{\mathbf{s}} & \text{for } \theta \geq \frac{\omega}{w_u} \end{cases}$$

These TIC- $\omega$  portfolios are represented, in the excess return –TE space, by the segment  $[0, \underline{\mathbf{g}}_\omega]$  of  $T$  and the right branch of  $TEC_\omega$ , tangent at point  $\underline{\mathbf{g}}_\omega$  (see Figure 5).

- If  $w_u \leq 0$ , the constraint (ic- $\omega$ ) is not binding, the unconstrained optimum satisfies the constraint and the solution of (TIC- $\omega$ ) writes:

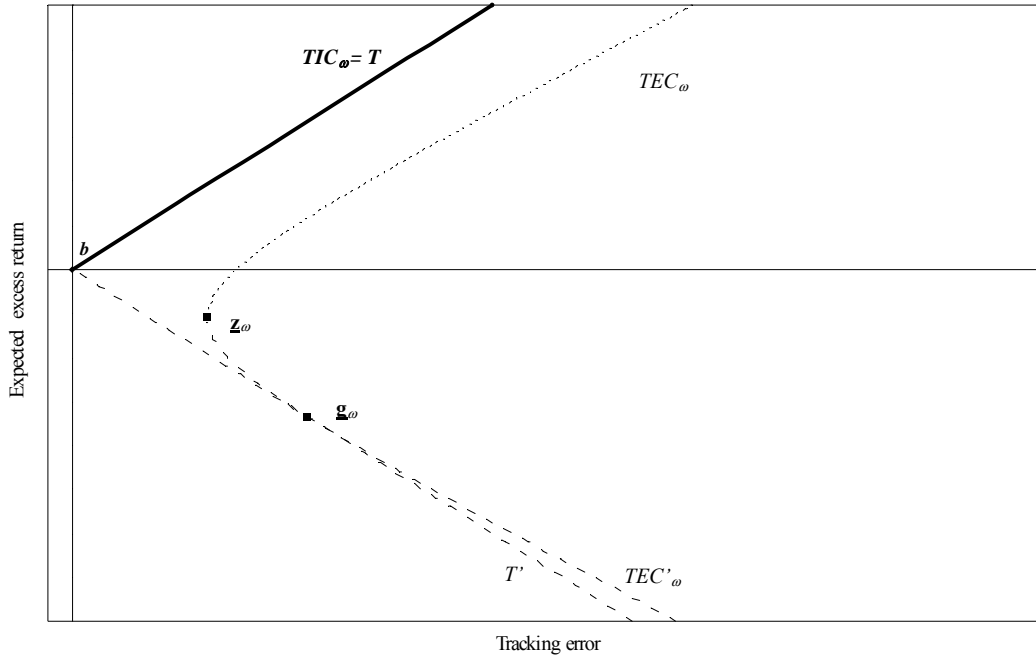
$$(18) \quad \underline{\mathbf{y}}_{-\omega, \theta}^* = \theta \underline{\mathbf{u}} \quad \text{for } \theta \geq 0$$

The TIC- $\omega$  portfolios are represented, in the excess return- $TE$  space, by the frontier  $T$  (see Figure 6).



**Figure 5: Efficient portfolios in the case of an inequality constraint ( $w_u > 0$ )**





**Figure 6: Efficient portfolios in the case of an inequality constraint ( $w_u < 0$ )**

Note that, when  $w_{y^*}$  is higher than the ceiling  $\omega$ , portfolio  $\underline{z}$  is added in order to lower this weight and meet the inequality constraint. When  $w_{y^*}$  is smaller than  $\omega$ , the inequality constraint is not binding and the unconstrained optimum  $\underline{y}_\theta^*$  solves (TIC- $\omega$ ).

## 5. Loss assessment, Information Ratios and graphical interpretations

In this section, we evaluate the loss due to the weights constraint and its impact on the Information Ratio and question its legitimacy as a performance measure in presence of a weights constraint.

Two different approaches are possible depending on the considered optimization program:

- In the first approach (followed in the previous paragraphs), we consider the optimal trade-off between the expected excess return and the  $TE$  for a given individual, characterized by his/her

risk aversion parameter  $\gamma$  and an objective function  $\underline{\mu}' \underline{y} - \frac{\gamma}{2} \underline{y}' \mathbf{V} \underline{y}$ . This is the approach followed in 5.1 where the loss due to the weights constraint is defined as the decrease in the value function due to the constraint.

- In the second approach, which is more relevant for many asset managers, the level  $TE$  of Tracking Error is set in the fund's policy and the manager maximizes the expected return under a Tracking Error constraint (in the presence or the absence of a weights constraint). These two approaches are mathematically equivalent. In 5.2, we follow the second approach and compute the information ratios with and without the weights constraint.

The relevance of the Information Ratio as a performance is questioned in 5.3. In 5.4., we consider an alternative definition of the Information Ratio and of the Tracking Error which implies similar results. Finally, in 5.5, we introduce a new definition of a performance measure under tracking error and weights constraints

### 5.1. Optimal trade-off between expected excess return and tracking error: loss analysis

The fund manager maximizes the objective function  $\Phi(\underline{y}) = \underline{\mu}' \underline{y} - \frac{\gamma}{2} \underline{y}' \mathbf{V} \underline{y}$  that can be interpreted as a certainty equivalent excess rate of return.

The loss  $L(\omega)$  due to the equality weights constraint (ec- $\omega$ ) is measured by the decrease in the certainty equivalent excess rate of return. It depends on the risk aversion parameter  $\gamma$  or, equivalently, on the risk tolerance  $\theta$  and writes:  $L(\omega) = \Phi(\underline{y}_{\theta}^*) - \Phi(\underline{y}_{\omega, \theta}^*)$ .

In the case of an inequality weights constraint, the constraint is either binding and the loss is  $L(\omega)$ , or not binding, and there is no loss. Therefore, we only address the case of an equality.

Proposition 5 (proved in the Appendix) provides a simple expression of this loss:

Proposition 5:

The loss  $L(\omega)$  in certainty equivalent return is given by equation (19) and is proportional to the square of the difference between the required weight  $\omega$  and the unconstrained optimal weight  $w_{y^*_\theta}$ :

$$(19) \quad L(\omega) = \frac{\mu_a}{2\theta} \frac{\sigma_s^2}{\sigma_a^2} \left( \omega - w_{y^*_\theta} \right)^2$$

$$\text{where } w_{y^*_\theta} = \theta w_u$$

## 5.2. Impact of a weights constraint on the information ratio ( $IR$ )

The optimization program of the manager considered in this section writes:

Mean- $TE$  tradeoff:  $\text{Max}_{\underline{y}} \underline{\mu}' \underline{y}$ , with:  $\underline{y}' \mathbf{V} \underline{y} \leq TE^2$ , with or without (ec- $\omega$ ) or (ic- $\omega$ ).

Since the constraint on the tracking error is binding at the optimum we can write this Mean- $TE$  tradeoff as:

$$\text{Max}_{\underline{y}} \underline{\mu}' \underline{y}, \text{ with: } \underline{y}' \mathbf{V} \underline{y} = TE^2, \text{ with or without (ec-}\omega\text{) or (ic-}\omega\text{)}.$$

We assume that  $TE \geq \omega \sigma_s$  ( $= \sigma_{z_\omega}$  the minimum tracking error obtainable), which is a necessary and sufficient condition for the existence of a solution. The performance of an active asset manager who tries to outperform a benchmark under a  $TE$  constraint is often assessed through an Information Ratio  $IR$  defined as the Expected excess return divided by the Tracking Error  $TE$ . An alternative definition of the Information Ratio is considered later on.

Since the Information Ratio  $IR_y$  is equal to  $\underline{\mu}' \underline{y} / TE$ , for any given value of  $TE$  the Mean- $TE$  program defined above is equivalent to the following  $IR$ - $TE$  maximization program:

$$(IR_{TE}) \quad \text{Max}_{\underline{y}} IR_y, \text{ with: } \underline{y}' \mathbf{V} \underline{y} = TE^2, \text{ constrained or not by (ec-}\omega\text{) or (ic-}\omega\text{)}.$$

This program must be carefully distinguished from the  $IR-TE$  trade-off:  $\text{Max}_{\underline{y}} IR_{\underline{y}}$ , with:

$\underline{y}' \mathbf{V} \underline{y} \leq TE^2$ , constrained by (ec- $\omega$ ) or (ic- $\omega$ ). Indeed, as pointed out in the introductory example and shown in section 5.4., this last program may yield one or an infinity of interior solutions which do not optimize the Mean- $TE$  tradeoff; in this respect, the Mean- $TE$  and the  $IR-TE$  tradeoffs are not equivalent. We consider thus  $(IR_{TE})$  whose solution will be qualified as the “optimal  $IR$ ” (from the viewpoint of the mean- $TE$  trade-off).

A binding weights constraint lowers the Information Ratio. The impact of the weights constraint on the Information Ratio is characterized in Proposition 6 (proved in the Appendix). We consider only the most realistic cases when the benchmark satisfies the inequality constraint but not the equality.

#### Proposition 6

- In absence of a weights constraint, the optimal  $IR$  is independent of  $TE$  and writes:

$$(20) \quad IR^* = \frac{\mu_{y^*}}{\sigma_{y^*}} = \sqrt{\frac{\mu_u \mu_a}{\sigma_a^2}}$$

- In presence of an equality weights constraint, the optimal  $IR$  depends on  $\omega$  and  $TE$ :

$$(21) \quad IR_{\omega}^*(TE) = \frac{\omega \mu_s + h \sqrt{TE^2 - \omega^2 \sigma_s^2}}{TE}$$

When  $w_u > 0$ ,  $IR_{\omega}^*(TE)$  first increases with  $TE$  to its maximum value  $IR^*$  and then decreases to its asymptotic value  $h$ ; when  $w_u \leq 0$ , it increases asymptotically to  $h$ .

- In presence of an inequality constraint, the optimal  $IR$  writes:

$$IR^* \text{ for } TE \leq TE(\underline{\mathbf{g}}_{\omega})$$

$$IR_{\omega}^*(TE) \text{ for } TE > TE(\underline{\mathbf{g}}_{\omega}) \text{ (the tracking error of } \underline{\mathbf{g}}_{\omega} \text{); in this case, } IR_{\omega}^*(TE) < IR^*.$$

This optimal  $IR$  always increases with  $\omega$ .

Recall that  $h = \sqrt{\frac{\mu_u \mu_a}{\sigma_a^2} - \frac{\mu_s}{\sigma_s^2}}$  is the slope of the upper asymptotes of the hyperbolas  $TEC-\omega$ .

Recall also that, as stated in Lemma 1,  $\underline{z}_\omega = \omega \underline{s}$ ,  $\mu_s$  and  $w_u$  have the same sign and that we assume  $\omega > 0$ . The expressions of  $\mu_s$  and  $\sigma_s$  are given in the Appendix.

Proposition 6 has several theoretical and practical implications, in particular on the relevance of the Information Ratio as a performance measure; these implications are studied in 5.3.

### 5.3 Limits of the Information Ratio as a performance measure

Proposition 6 characterizes the relation between the Information Ratio and the Tracking Error.

In absence of a weights constraint, according to equation (20), the maximum Information Ratio  $IR^*$  is independent of the level of the tracking error  $TE$  set in the fund's policy. This result is also a direct consequence of Roll (1992) and Jorion (2003) analysis, since  $T$  (Figure 1) is a semi-straight line starting at the origin and with a slope equal to  $IR^*$ . It means that the Information ratio  $IR^*$  of the manager:

- only depends on his/her technical ability;
- cannot be improved with more flexibility by allowing a higher tracking error;
- is a theoretically meaningful tool for comparing the performances of two funds tracking the same benchmark, even if the funds operate under different tracking errors.

In presence of a weights constraint (equality or inequality), according to Proposition 6, the  $IR$  chosen by the manager depends on the tracking error  $TE$  as well as on  $\omega$  and  $\mathbb{L}$ . In particular, an individual, managing two different funds constrained by different tracking errors operates under two different Information Ratios. This “duality” prevails even if these two funds track

the same benchmark and face the same weights constraint (characterized by the same restricted assets  $\mathbb{L}$  and the same  $\omega$ ). Therefore, the Information Ratio becomes a spurious ex-ante measure of the technical ability of the fund's manager. It is thus important to stress that, in presence of a weights constraint, it may be theoretically flawed to compare on the basis of their Information Ratios<sup>17</sup>, the performances of two funds operating under different values of  $\omega$  or  $TE$ .

As stated in Proposition 6, the optimal  $IR$  increases with  $\omega$  in the case of an inequality constraint. This intuitive result is easily obtained from program  $(IR_{TE})$ : when relaxing the weights constraint (increasing  $\omega$ ) the value function (optimal  $IR$ ) increases. This result questions the legitimacy of using  $IR$  for comparing the performance of two funds with different constraint levels (and the same  $TE$ ). However, we focus on the impact of  $TE$  on the optimal  $IR$ , as it not only challenges the relevance of the  $IR$  but also implies a result that may be considered counter-intuitive: the optimal  $IR$  may decrease with a less stringent (higher) Tracking Error. Therefore, we study the optimal  $IR$  and its graphical representation as a function of  $TE$  (for given  $\mathbb{L}$  and  $\omega$ ), first in the case of an equality constraint and then in the case of an inequality.

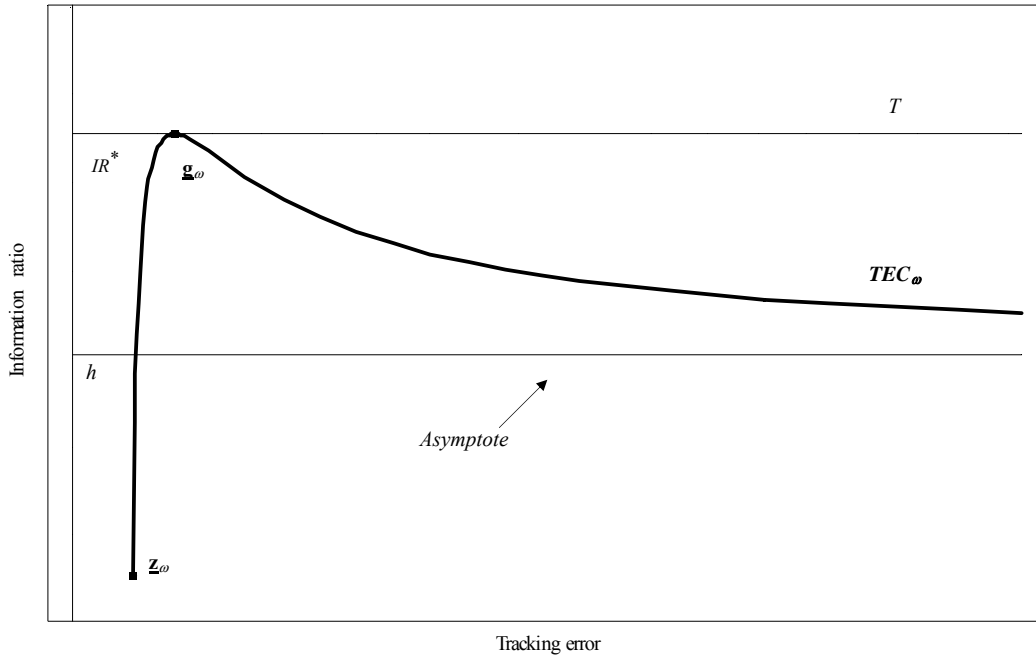
- In the case of an equality weights constraint (ec- $\omega$ ), the dependence between the optimal  $IR$  and the tracking error goal is characterized by the function  $IR_{\omega}^*(TE)$ , given by (21). Its graphical representation can be interpreted as the efficient frontier in the  $TE$ - $IR$  space and depends on  $\mathbb{L}$  and  $\omega$ . The shape of this efficient frontier depends on the sign of  $w_u$ , or

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<sup>17</sup> In fact, the performance measure is an “ex post” or “empirical” Tracking Error  $\hat{IR}$ , which is an estimator of the “true” but unobservable parameter  $IR$ . In addition to the theoretical problem raised in this paper (the estimated parameter  $IR$  is not necessarily relevant), well known statistical problems come from the volatility of the estimator ( $\hat{IR}$ ), which implies that many years are necessary to estimate  $IR$  with some confidence. These important statistical problems are beyond the scope of this paper.

equivalently on the sign of the return  $\mu_{z_\omega}$  of  $\underline{z}_\omega$  (the minimum  $TE$  portfolio satisfying the weights constraint), as stated in proposition 6 and shown in Figures 7 and 8.

- If  $w_u > 0$  (roughly, if restricted assets are more volatile), the information ratio increases with  $TE$  until it reaches the unconstrained information ratio  $IR^*$  and then decreases towards its asymptotical value  $h$ , as shown in Figure 7. This result can be obtained from simple geometric considerations. Indeed, assume  $w_u > 0$  and consider any point  $\underline{y}$  of the frontier  $TEC_\omega$  represented in Figure 1. The slope of the segment  $[0, \underline{y}]$  is equal to  $IR(\underline{y})$ . From Figure 1, this slope first increases with  $TE$  until it reaches the slope of the segment  $[0, \underline{g}_\omega]$  (which is equal to  $IR^*$ ) and then decreases (asymptotically towards  $h$  as shown in Figure 4). This result contrasts with the constant  $IR^*$  (due to the linearity of  $T$ ) in absence of weights constraint; it is a direct consequence of the concavity of  $TEC_\omega$ . In this case, the manager chooses the maximum  $IR$  attainable only if her/his  $TE$  goal corresponds to the tracking error of the tangent portfolio  $\underline{g}_\omega$  (characterized in 3.3). Note that a given manager does not always choose a higher Information Ratio when allowed a higher  $TE$ : beyond  $TE(\underline{g}_\omega)$ , even though a higher  $TE$  softens the constraint and increases the expected return  $\underline{\mu}'\underline{y}$ , it decreases the value function  $\underline{\mu}'\underline{y}/TE$  of program  $(IR_{TE})$ . Since the optimal ex-ante (theoretical) Information Ratio depends on  $TE$ , it is very questionable to assess assets managers' performances on the basis of their ex-post (empirical) Information Ratios.



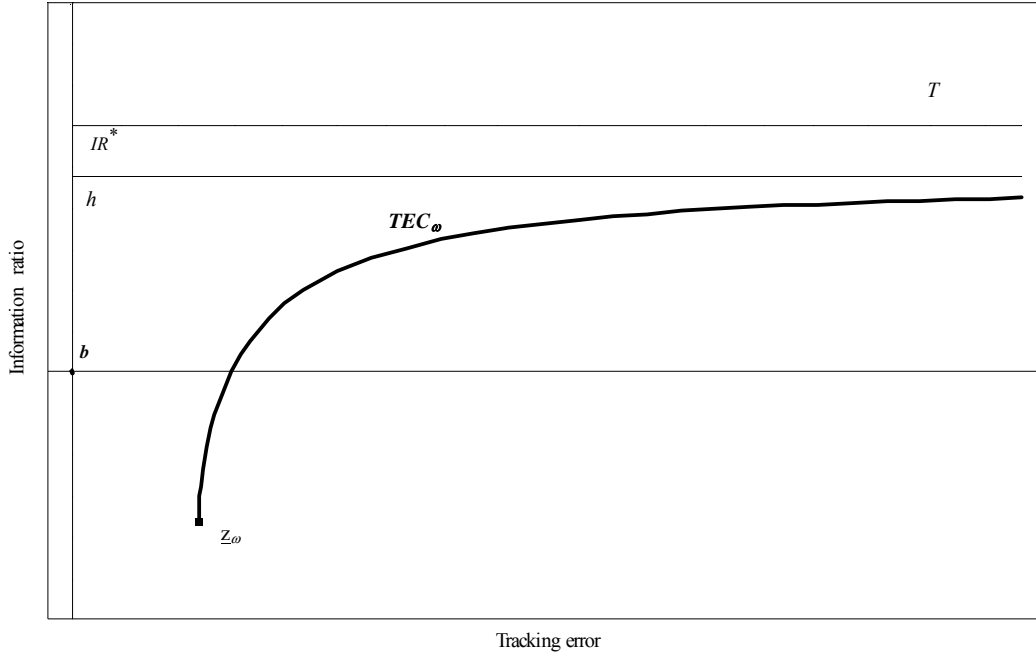
**Figure 7: Information ratios in the case of an equality constraint ( $w_u > 0$ )**

- If  $w_u < 0$  (roughly, if restricted assets are less volatile), the information ratio increases steadily with  $TE$  towards its asymptotical value  $h$  as shown in Figure 8. In this case, we get the intuitive result that the optimal Information Ratio increases when a higher tracking error is allowed. This result can be obtained geometrically from Figure 2. However, it is still questionable to rank different constrained funds on the basis of their Information Ratios, since a higher  $IR$  may simply mean that the manager operates under a higher  $TE$ .

- If  $w_u = 0$ , the unconstrained information ratio  $IR^*$  is equal to the asymptotical value  $h$ ,  $\mu_{z_\omega} = 0$  and the information ratio increases with  $TE$  from 0 to  $IR^*$ . The graphical representation is analogous to that of Figure 8, where  $h = IR^*$  and  $\mu_{z_\omega} = 0$ .

Note that the solution of  $\text{Max}_{\underline{y}} IR_{\underline{y}}$ , with:  $\underline{y}' \mathbf{V} \underline{y} \leq TE^2$ , constrained by  $(ec-\omega)$  yields the interior solution  $\underline{g}_\omega$ .





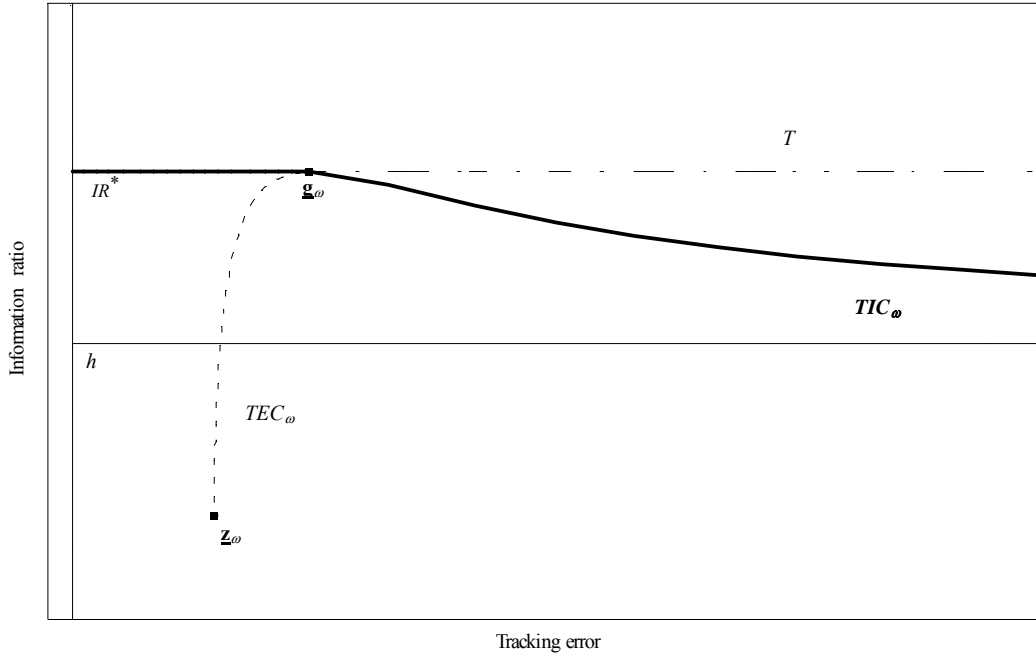
**Figure 8: Information ratios in the case of an equality constraint ( $w_u < 0$ )**

- The case of an inequality constraint (ic- $\omega$ ) is the most frequently encountered and we only consider a benchmark satisfying this constraint.

- When  $w_u > 0$  (Figure 9), the optimal Information Ratio is constant and equal to  $IR^*$  for values of  $TE$  lower than  $TE(\underline{\mathbf{g}}_\omega)$  and decreases towards its asymptotical value  $h$  when  $TE$  increases beyond  $TE(\underline{\mathbf{g}}_\omega)$ .

Note again that, in the range of  $TE$  values where the Information Ratio decreases with  $TE$ , it is fallacious to use Information Ratios to assess and compare the performance of different funds if they operate under different values of  $TE$ . Besides, we obtain again that the higher the tolerance to deviations from the benchmark, the smaller the optimal  $IR$ . We may give the following intuitive explanation of this result: when the weights constraint is not binding the optimal  $IR$  is constant and a binding weights constraint decreases its value; but the

weights constraint is relatively more restrictive when the TE bound is larger; hence the optimal  $IR$  decreases with  $TE$ .



**Figure 9: Information ratios in the case of an inequality constraint ( $w_u > 0$ )**

- When  $w_u < 0$ , according to Figure 6, the Information Ratio is constant and equal to  $IR^*$ . Note that in this case, the Information Ratio remains an adequate measure of portfolio performance.
- The case  $w_u = 0$  is analogous to the case  $w_u < 0$  but with  $IR^* = h$ .

Note that the solution of  $\text{Max}_{\underline{y}} IR_{\underline{y}}$ , with:  $\underline{y}' \mathbf{V} \underline{y} \leq TE^2$ , constrained by (ic- $\omega$ ) is indeterminate (as shown in Figure 9, any (TIC- $\omega$ ) portfolio with  $TE \leq \text{Min}(TE(\underline{g}_\omega), TE)$  solves the program).

#### 5.4. An alternative definition of the Information Ratio

Note that two alternative definitions of the Information Ratio and of the Tracking Error are commonly used (see for instance Goodwin (1998)). In this paper, it is the ratio between the

Expected Excess Return and the Tracking error  $TE$  being defined as the standard deviation of the excess return. These definitions (see for example Roll (1992) and Jorion (2003)), are used respectively as a measure of “global” performance of the portfolio and as a measure of the “distance” to the benchmark.  $IR$  measures the global performance (stock picking and market timing due to a beta<sup>18</sup> different from 1) and the associated  $TE$  is the “total distance” of the portfolio return to the benchmark return  $R_b$ . This definition, that does not rely on any factor model is the one used in our paper for different reasons specified hereafter.

An alternative definition of the Information Ratio and Tracking error is based on a regression of the return of the portfolio on the return of the benchmark.  $IR'$  is then the ratio of the alpha of the portfolio by the non-systematic risk of the portfolio and  $TE'$  is a measure of specific risk:  $IR' = \alpha/\sigma(\varepsilon)$  and  $TE' = \sigma(\varepsilon)$ <sup>19</sup>. These definitions are broadly used by practitioners and academics, in particular for performance analysis.  $IR'$ , also commonly called Appraisal Ratio measures the stock picking performance. It is associated with the partial distance to the benchmark  $TE' = \sigma(\varepsilon)$ . Therefore, it does not account for the excess returns due to leverage ( $\beta \neq 1$ ) that may be interpreted as a consequence of market timing. These two alternative definitions only coincide when  $\beta=1$ .

In our framework, we could have used one definition or another. We chose the first one mainly because it fits better the basic problem at hand: a manager is committed to a maximum “total distance”  $TE$  to the benchmark and tries to maximize the expected return by all means (stock picking and betas different from 1). However, substituting  $R_b$  by  $\beta R_b$ , and considering the maximization of  $\alpha$  subject to a constrained  $TE'$ , all of our results hold ( $\beta R_b$  can be considered as a “pseudo” benchmark).

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<sup>18</sup> Defined as the regression coefficient of the portfolio return against the return of the benchmark.

<sup>19</sup> When using an index model,  $\varepsilon$  is the residual noise and  $\alpha$  is the “abnormal” return (à la Jensen).

### 5.5. The Adjusted Information Ratio: An alternative performance measure under tracking error and weights constraints

We look for an alternative and unambiguous performance measure in order to overcome the problem of an optimal  $IR$  that varies with the tracking error  $TE$  and the constraint level  $\omega$ .

From equation (7), subtracting the minimum tracking error portfolio  $\underline{z}_\omega$  to the optimal solution  $\underline{y}_{\omega,\theta}^*$  of (TEC- $\omega$ ), we get:

$$(22) \quad \underline{y}_{\omega,\theta}^* - \underline{z}_\omega = \theta(\underline{u} - w_u \underline{s})$$

Consider the Information Ratio of portfolio  $\underline{y}_{\omega,\theta}^* - \underline{z}_\omega$ ; according to equation (22), this ratio is equal to  $\frac{\mu_{u-w_u s}}{\sigma_{u-w_u s}}$  and is independent of the risk tolerance parameter  $\theta$ , or equivalently of the

Tracking Error goal. It is also independent of the level of the constraint  $\omega$ , but depends on the set  $\mathbb{L}$  of restricted assets, since portfolio  $\underline{s}$  depends on these particular assets through  $\mathbf{1}_L$ .

We call “Adjusted Information Ratio” or “Asymptotic Information Ratio” (for reasons detailed below), of a portfolio  $\underline{y}$  subject to a weights constraint, the Information Ratio of the portfolio  $\underline{y}_{\omega,\theta}^* - \underline{z}_\omega$ . The “Asymptotic  $IR$ ” can be interpreted by considering an “adjusted benchmark”:  $\mathbf{b} + \underline{z}_\omega$ . Among the fully invested portfolios satisfying the equality weights constraint,  $\mathbf{b} + \underline{z}_\omega$  is the “closest” to the benchmark (it minimizes the tracking error, satisfies the equality constraint, and thus the corresponding inequality constraint). The active part of the portfolio management process may be defined with respect to this adjusted benchmark. The asymptotic  $IR$  is the standard Information Ratio computed against the adjusted benchmark.

Let  $AIR = \frac{\mu_{u-w_u, S}}{\sigma_{u-w_u, S}}$  be the maximum adjusted  $IR$  attainable under any weights constraint

involving the set of restricted assets  $\mathbb{L}^{20}$ . Since  $AIR$  is independent of  $\theta$  as well as of  $\omega$ , it is theoretically valid for comparing the performances of different funds subject to different constraint and  $TE$  levels.

The value and the geometrical interpretation of  $AIR$  are given in Proposition 7 (proved in the Appendix).

#### Proposition 7

The Adjusted Information Ratio  $AIR$  is equal to the slope of the asymptotes of  $TEC_\omega$  and equal to the asymptotical constrained information ratio:

$$(23) \quad AIR = h$$

This result can be related to Figure 4: the hyperbolas  $TEC_\omega$  have a common slope, independent of  $\omega$ , which is the  $IR$  of the portfolio with an infinite tracking error as well as the  $AIR$ . Hence, the asymptotical  $IR$  must be equal to the common slope of these asymptotes.

Although this approach is theoretically appealing, the computation of the empirical  $AIR$  (which is the suggested “ex post” performance measure) raises a practical problem. Indeed, contrary to the benchmark which is an observable portfolio, the adjusted benchmark is not observable, since the composition of  $\underline{z}_\omega$  depends on non observable parameters. However, the only unobservable input necessary to construct  $\underline{z}_\omega$  is the matrix of variance-covariance  $\mathbf{V}$  (as opposed, for example to portfolio  $\underline{u}$  for which expected returns are also needed) and  $\mathbf{V}$  can be estimated from historical returns using standard methods.

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<sup>20</sup> The dependence of the  $AIR$  on  $\mathbb{L}$  is not a major problem since, in most cases, the restricted assets are unambiguously related to the benchmark (when the benchmark is a domestic index the restricted assets are the foreign securities, the stock funds are constrained on non stock assets ...).

## 6. Numerical example

We consider 7 traded securities (or asset classes), corresponding to 5 domestic ( $x_1^d, \dots, x_5^d$ ) and 2 foreign assets ( $x_1^f, x_2^f$ ). Their expected returns  $\mu$  and standard deviations  $\sigma$  are:

	$x_1^d$	$x_2^d$	$x_3^d$	$x_4^d$	$x_5^d$	$x_1^f$	$x_2^f$
$\mu$	12%	11%	12%	12%	14%	16%	17%
$\sigma$	22%	35%	25%	20%	35%	20%	28%

**Table 3**

The correlations between any two domestic assets as well as the correlation between the two foreign assets are equal to .3. The correlation between a domestic and a foreign asset is equal to .2. The benchmark is an equally weighted average of domestic assets. It follows from standard computation that  $w_u = w_t - w_a = 15.44\%$ , which is positive.

We consider a portfolio manager in charge of two different funds, with the same benchmark but bound by two different tracking errors respectively equal to 5% and 10%. Program (T) implies an unconstrained optimal *IR* ( $IR^*$ ), equal to 23.14%, for any values of  $\theta$  or  $TE$ . Assume now that in both funds, the sum of the weights on foreign assets is constrained to be smaller than 20%. The optimal allocation for both tracking errors is presented in Table 4.

	$x_1^d$	$x_2^d$	$x_3^d$	$x_4^d$	$x_5^d$	$x_1^f$	$x_1^f$	$\mu$	$\sigma(\mathbf{x}-\mathbf{b})$	$IR$
$IR^* = 23.14\%$ ; $AIR = 8.25\%$										
<b>b</b>	20.00%	20.00%	20.00%	20.00%	20.00%	0.00%	0.00%	12.20%	0.00%	0.00%
<b><math>\mathbf{z}_\omega + \mathbf{b}</math></b>	14.28%	18.54%	15.93%	12.71%	18.54%	13.63%	6.37%	13.05%	3.93%	21.62%
<b><math>\mathbf{g}_\omega + \mathbf{b}</math></b>	13.80%	16.09%	15.49%	12.18%	22.44%	11.52%	8.48%	13.17%	4.20%	23.14%
$TE = 5\%$										
T	12.62%	15.35%	14.64%	10.70%	22.91%	13.70%	10.08%	13.36%	5.00%	23.14%
TIC	13.27%	13.48%	15.02%	11.62%	26.61%	9.28%	10.72%	13.30%	5.00%	22.09%
$TE = 10\%$										
T	5.24%	10.70%	9.28%	1.40%	25.81%	27.41%	20.16%	14.51%	10.00%	23.14%
TIC	11.28%	3.52%	13.23%	9.47%	42.51%	0.70%	19.29%	13.81%	10.00%	16.08%

**Table 4**

Following first the approach of 5.1 (certainty equivalent excess returns), we consider a given risk tolerance parameter  $\theta$ . The tracking error of 5% is chosen by a manager with a risk tolerance parameter  $\theta$  equal to 1.54 (see footnote 8). The decrease in the value function implied by the constraint is given in Proposition 5 (equation (19)). The solution of the unconstrained program (T) yields a value for the objective function equal to .578% (which is a certainty equivalent excess return). When the same portfolio manager is constrained to hold 20% of foreign assets at most, the value function of (TIC- $\omega$ ) is equal to .566%. It roughly represents a loss of 1.2 basis points in return, which is very small in this example.

Following now the approach of 5.2 (Information Ratios), we consider a portfolio manager solving (T) and (TIC- $\omega$ ) with a tracking error constraint at a given level  $TE$  and assess the impact of the weights constraint on the Information Ratio. Since  $w_u$  is positive, we are in the

situation of Figure 9: due to the weights constraint, when  $TE$  is greater than 4.20% (which is  $TE(\underline{\mathbf{g}}_\omega)$ ) and increases, the Information Ratio decreases towards its asymptotical value 8.25%. When the portfolio manager is restricted to a tracking error of 5% at most, the solution of (T) does not satisfy the inequality constraint, which is thus binding for program (TIC). Then, the solution of (TIC) yields an expected return of 13.30%, which implies a decrease of the Information Ratio from 23.14% to 22.09%. When the tracking error constraint is weakened from 5% to 10%, the optimal Information Ratio decreases about six times more, from 22.09% to 16.08%. As pointed in paragraph 5.3, the Information Ratio is an inadequate measure of performance. Indeed, the same manager ends up with very different values of  $IR$  when managing, with the same skills and efforts, different funds constrained by the same restricted assets, at the same level  $\omega$ .

This example shows also that, even when the impact of the weights constraint on the value function (optimal certainty equivalent excess return) is very small, its impact on the optimal Information Ratio can be very significant. Moreover, the better the return- $TE$  tradeoff (the higher the certainty-equivalent return), the smaller is the  $IR$ .

The alternative performance measure, the  $AIR$ , is equal to 8.25% for all values of  $\theta$  (or  $TE$ ) and  $\omega$ , it only depends on the manager's skills, not on the constraints levels.

## 7. Concluding remarks and extensions

In many situations a minimum concentration in a subset of securities is required and the portfolio performance is assessed against a benchmark often composed of these securities. This article analyzes the consequences of a weights constraint on benchmarked asset allocation. It presents three different sets of results:



- Separation results: in particular, the active part of all optimal portfolios is a combination of the same two funds  $\underline{u}$  and  $\underline{s}$  for any value of the risk tolerance parameter  $\theta$  and any level  $\omega$  of the weights constraint, for a given set of restricted assets  $\mathbb{L}$ .
- The analytics and the geometry of the efficient frontier in the excess return-Tracking Error space: when the weights constraint is not binding, the efficient frontier is linear and hyperbolic otherwise. In case of an inequality constraint, it is thus generally composed of successive linear and hyperbolic segments.
- The influence of the weights constraint on the optimal Information Ratio: in the inequality constraint case, the optimal  $IR$  increases when the weights constraint softens (higher  $\omega$ ) and decreases, in most cases, when  $TE$  increases beyond a threshold. Moreover, even when the weights constraint has a very small impact on the certainty equivalent excess return, it can change the  $IR$  dramatically. This undermines the appropriateness of the Information Ratio as a performance measure. We suggest an alternative performance indicator, the Adjusted Information Ratio, which avoids most of the theoretical drawbacks of the standard  $IR$  at the cost of a more complicated estimation of this performance measure.

The results obtained in this article can be extended in different directions. A first possible extension is the inclusion of a risk-free security among the traded assets. The presence of a risk free asset does not change the results substantially. The unconstrained efficient frontier is again a straight line, but with a higher slope and the constrained frontier is still an hyperbola (in the case of an equality constraint), or a combination of linear or/and hyperbolic segments (in the case of an inequality constraint). A second possible extension of our analysis is the introduction of multiple weights constraints. Consider for instance the case of two compatible equality constraints:  $1_{L_1}' \underline{y} = \omega_1$  and  $1_{L_2}' \underline{y} = \omega_2$ . From the previous results, the solutions of the two optimization programs span two hyperbolas  $TEC_{\omega_1}$  and  $TEC_{\omega_2}$ . The curve

representing the solutions of the optimization program with the two binding constraints  $TEC_{\omega_1\omega_2}$  is an hyperbola tangent to  $TEC_{\omega_1}$  and  $TEC_{\omega_2}$ . With more constraints, a network of such tangent hyperbolas is obtained.

## Appendix

### Notations

$A = \mathbf{1}' \mathbf{V}^{-1} \boldsymbol{\mu}$      $B = \boldsymbol{\mu}' \mathbf{V}^{-1} \boldsymbol{\mu}$      $C = \mathbf{1}' \mathbf{V}^{-1} \mathbf{1}$      $D = \mathbf{1}_L' \mathbf{V}^{-1} \mathbf{1}$      $E = \mathbf{1}_L' \mathbf{V}^{-1} \mathbf{1}_L$      $F = \mathbf{1}_L' \mathbf{V}^{-1} \boldsymbol{\mu}$   
We define the following scalars:

and the (fully invested) portfolio structures:

$$\mathbf{t} = \frac{\mathbf{V}^{-1} \boldsymbol{\mu}}{\mathbf{1}' \mathbf{V}^{-1} \boldsymbol{\mu}} \quad ; \quad \mathbf{a} = \frac{\mathbf{V}^{-1} \mathbf{1}}{\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}} \quad ; \quad \mathbf{k} = \frac{\mathbf{V}^{-1} \mathbf{1}_L}{\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}_L}$$

Expected returns  $\mu$ , return variances  $\sigma^2$  and covariances  $V$  write:

$$\mu_u = \mu_t - \mu_a = \frac{BC - A^2}{AC} \quad ; \quad \mu_k - \mu_a = \frac{CF - AD}{CD} \quad ; \quad \sigma_t^2 - \sigma_a^2 = \frac{BC - A^2}{A^2 C} \quad ; \quad \sigma_k^2 - \sigma_a^2 = \frac{CE - D^2}{CD^2}$$

$$V_{kt} = \frac{F}{AD} \quad \text{and for any portfolio } x \quad V_{ax} = \sigma_a^2 \quad V_{ak} = V_{at} = \sigma_a^2 = \frac{1}{C}$$

Shares in the restricted assets write:

$$w_u = w_t - w_a = \frac{CF - AD}{AC} \quad ; \quad w_k - w_a = \frac{CE - D^2}{CD} \quad ; \quad D = \frac{w_a}{\sigma_a^2} \quad ; \quad A = \frac{\mu_a}{\sigma_a^2}$$

Define:  $\underline{\mathbf{u}} = \mathbf{t} - \mathbf{a} \quad ; \quad \underline{\mathbf{s}} = \frac{\mathbf{k} - \mathbf{a}}{w_k - w_a} \quad ; \quad \sigma_u^2 = \sigma_a^2 \frac{\mu_u}{\mu_a} \quad ; \quad \mu_{y_\theta}^* = \theta \mu_u \quad ; \quad \sigma_{y_\theta}^2 = \theta^2 \frac{\mu_u}{A}$

$$\mu_s = \frac{CF - AD}{CE - D^2} \quad ; \quad \sigma_s^2 = \frac{C}{CE - D^2} = \frac{1}{D(w_k - w_a)} = \quad ; \quad V_{us} = \frac{\mu_s}{\mu_a} \sigma_a^2 \quad ; \quad \sigma_s^2 = \frac{\mu_s}{A w_u} = \sigma_a^2 \frac{\mu_s}{\mu_a w_u}$$

### Proof of Lemma1:

The minimum tracking error portfolio  $\underline{\mathbf{z}}_\omega$  solves:

$$\text{Min}_{\underline{\mathbf{y}}} \quad \underline{\mathbf{y}}' \mathbf{V} \underline{\mathbf{y}} \quad \text{s.t.} \quad \mathbf{1}' \underline{\mathbf{y}} = 0 \quad \text{and} \quad \mathbf{1}_L' \underline{\mathbf{y}} = \omega$$

Using Lagrange multipliers, the program can be rewritten as

$$\text{Min}_{\underline{y}} \underline{y}' \underline{V} \underline{y} - \lambda \mathbf{1}' \underline{y} - \nu \mathbf{1}_L' \underline{y}$$

where  $\lambda$  and  $\nu$  are the multipliers. The standard first-order conditions are:

$$(A1) \quad \underline{z}_\omega = (\mathbf{1}/2)\lambda \underline{V}^{-1} \mathbf{1} + (\mathbf{1}/2)\nu \underline{V}^{-1} = \alpha \mathbf{a} + \beta \mathbf{k}$$

$$(A2) \quad \mathbf{1}' \underline{z}_\omega = 0$$

$$(A3) \quad \mathbf{1}_L' \underline{z}_\omega = \omega$$

From (A2) and (A3) and using (A1) we obtain:

$$(A4) \quad \begin{aligned} \alpha &= -\beta \\ \beta &= \frac{\omega}{(w_k - w_a)} \end{aligned}$$

Substituting for  $\alpha$  and  $\beta$  from (A4) into (A1), we obtain:

$$(A5) \quad \underline{z}_\omega = \frac{\omega}{w_k - w_a} (\mathbf{k} - \mathbf{a}) = \omega \underline{s}$$

From (A5), we obtain:

$$(A6) \quad \mu_{z_\omega} = \omega \mu_s = \omega A w_u \sigma_s^2$$

Since  $\mu_a$  is assumed positive  $A$  is positive and  $w_u$  and  $\mu_s$  have the same sign. When the benchmark satisfies the inequality constraint ( $\omega > 0$ ),  $w_u$  and  $\mu_{z_\omega}$  have the same sign.

### Proof of Proposition 1:

Using Lagrange multipliers, TEC- $\omega$  can be rewritten as

$$\text{Max}_{\underline{y}} \underline{\mu}' \underline{y} - \frac{A}{2\theta} \underline{y}' \underline{V} \underline{y} - \lambda \mathbf{1}' \underline{y} - \nu \mathbf{1}_L' \underline{y}$$

where  $\lambda$  and  $\nu$  are the multipliers. The standard first-order conditions are

$$(A7) \quad \underline{\mu} - \frac{A}{\theta} \underline{V} \underline{y}_{\omega, \theta}^* - \lambda \mathbf{1} - \nu \mathbf{1}_L = 0$$

$$(A8) \quad \mathbf{1}' \underline{y}_{\omega, \theta}^* = 0$$

$$(A9) \quad \mathbf{1}'_L \mathbf{y}_{\omega, \theta}^* = \omega$$

It follows from (A7) that:

$$(A10) \quad \mathbf{y}_{\omega}^* = \frac{\theta}{A} (\mathbf{V}^{-1} \boldsymbol{\mu} - \lambda \mathbf{V}^{-1} \mathbf{1} - \nu \mathbf{V}^{-1} \mathbf{1}_L) = \theta \left( \mathbf{t} - \frac{\lambda C}{A} \mathbf{a} - \frac{\nu D}{A} \mathbf{k} \right)$$

Substituting for  $\mathbf{y}_{\omega, \theta}^*$  from (A10) into (A8) and (A9), we obtain:

$$(A11) \quad \begin{aligned} \lambda C + \nu D &= A \\ \lambda D + \nu E &= F - \frac{A}{\theta} \omega \end{aligned}$$

and by substituting for  $\lambda$  and  $\nu$  from (A11) into (A10) and rearranging:

$$(A12) \quad \mathbf{y}_{\omega, \theta}^* = \theta(\mathbf{t} - \mathbf{a}) + (\omega - \theta w_u) \mathbf{s} = \mathbf{y}_{\theta}^* + (\omega - \theta w_u) \mathbf{s}$$

### Proof of Proposition 3:

- It follows from (A12) that:

$$(A13) \quad \mu_{y_{\omega, \theta}^*} = \theta \mu_u + (\omega - \theta w_u) \mu_s = \omega \mu_s + \theta (\mu_u - w_u \mu_s)$$

$$(A14) \quad \sigma_{y_{\omega, \theta}^*}^2 = \theta^2 \sigma_u^2 + \sigma_s^2 (\omega^2 - \theta^2 w_u^2) = \omega^2 \sigma_s^2 + \theta^2 \frac{\mu_u}{A} - w_u^2 \sigma_s^2 = \omega^2 \sigma_s^2 + \frac{\theta^2}{A} (\mu_u - w_u \mu_s)$$

Note that (A14) applied to  $\omega = 0$  (when the benchmark satisfies the equality constraint)

implies that  $\frac{\theta^2}{A} (\mu_u - w_u \mu_s)$  is the variance of  $\mathbf{y}_{0, \theta}^*$  and is therefore positive.

Then,  $\sigma_{y_{\omega, \theta}^*}^2 - \sigma_s^2 \omega^2$  is positive, we can take its square root and (A13) and (A14) lead to the

equation of the efficient frontier TEC- $\omega$ .

$$(A15) \quad \mu_{y_{\omega, \theta}^*} - \omega \mu_s = \sqrt{\sigma_{y_{\omega, \theta}^*}^2 - \omega^2 \sigma_s^2} \sqrt{A(\mu_u - w_u \mu_s)} = h \sqrt{\sigma_{y_{\omega, \theta}^*}^2 - \omega^2 \sigma_s^2}$$

where

$$(A16) \quad h = \sqrt{A(\mu_u - w_u \mu_s)} = \sqrt{\frac{\mu_u \mu_a}{\sigma_a^2} - \frac{\mu_s}{\sigma_s^2}}$$

For  $\omega \neq 0$  (A15) is the equation of the upper branch of an hyperbola. For  $\omega = 0$  (in this case the benchmark satisfies the equality constraint) the efficient frontier is a straight line. Asymptotically, we can write:

$$(A17) \quad \mu_{y_{\omega,\theta}^*} \approx h \sigma_{y_{\omega,\theta}^*}$$

- From (A5), we obtain:

$$(A18) \quad \mu_{\underline{z}_\omega} = \sigma_{\underline{z}_\omega} \frac{\mu_s}{\sigma_s}$$

The minimum tracking error SF portfolios are thus located on a straight line.

Proof of Proposition 5:

$$\Phi(\underline{y}_\theta^*) = \mu_{y_\theta^*} - \frac{A}{2\theta} \sigma_{y_\theta^*}^2 = \frac{\theta}{2} \mu_u$$

(A13) and (A14) lead to:

$$(A19) \quad \Phi(\underline{y}_{\omega,\theta}^*) = \mu_{y_{\omega,\theta}^*} - \frac{A}{2\theta} \sigma_{y_{\omega,\theta}^*}^2 = \Phi(\underline{y}_\theta^*) - L$$

where

$$(A20) \quad L = \frac{A}{2\theta} \sigma_s^2 \left( \omega - \theta w_u \right)^2 = \frac{\mu_a}{2\theta} \frac{\sigma_s^2}{\sigma_a^2} \left( \omega - w_{y_\theta^*} \right)^2$$

Proof of Proposition 6:

From (A15), we obtain:

$$(A21) \quad \mu_{y_{\omega,\theta}^*} = \omega \mu_s + h \sqrt{TE^2 - \omega^2 \sigma_s^2}$$

and then

$$(A22) \quad IR_{\omega}^*(TE) = \frac{\mu_{y_{\omega,\theta}}^*}{TE} = \frac{\omega\mu_s + h\sqrt{TE^2 - \omega^2\sigma_s^2}}{TE}$$

Standard calculus shows that, when  $w_u > 0$ ,  $IR_{\omega}^*(TE)$  is maximum and equal to  $IR^*$  for

$TE = \frac{\omega}{w_u} \sqrt{\frac{\mu_u}{A}} = TE(\underline{\mathbf{g}}_{\omega})$  and that its limit at positive infinity is equal to  $h$ . These results are

also easily obtained geometrically: Figure 1 shows that the slope of the semi-straight line  $[0, \underline{\mathbf{y}}_{\omega,\theta}^*]$  (which is equal to  $IR_{\omega}^*$ ) increases with  $TE$  until it reaches the slope of the semi-straight

line  $[0, \underline{\mathbf{g}}_{\omega}]$  (which is equal to  $IR^*$ ) and then decreases. The results for  $w_u \leq 0$  are obtained

geometrically from Figure 2 which shows that the slope of the semi-straight line  $[0, \underline{\mathbf{y}}_{\omega,\theta}^*]$

increases with  $TE$  when the latter increases from  $TE(\underline{\mathbf{z}}_{\omega})$  to infinity.

Finally, in the case of an inequality constraint, the optimal  $IR$  increases with  $\omega$  since it is the solution of program  $(IR_{TE})$  which itself increases when the tracking error constraint is softened (i.e. when  $TE$  increases).

#### Proof of Proposition 7:

From (A12), we obtain:

$$(A23) \quad \mu_{y_{\omega,\theta}^* - z_{\omega}}^* = \theta(\mu_u - w_u \mu_s) = \frac{\theta}{A} h^2$$

$$(A24) \quad \sigma_{y_{\omega,\theta}^* - z_{\omega}}^2 = \theta^2(\sigma_u^2 + w_u^2 \sigma_s^2 - 2w_u V_{us}) = \theta^2 \left( \sigma_a^2 \frac{\mu_u}{\mu_a} + w_u^2 \sigma_a^2 \frac{\mu_s}{\mu_a w_u} - 2w_u \frac{\mu_s}{\mu_a} \sigma_a^2 \right) = \frac{\theta^2}{A^2} h^2$$

From (A23) and (A24), we have:

$$(A25) \quad AIR = \frac{\mu_{y_{\omega,\theta}^* - z_{\omega}}^*}{\sigma_{y_{\omega,\theta}^* - z_{\omega}}^*} = h$$

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