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A SURVEY OF ABSTRACT ALGEBRAIC LOGIC

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Introduction

Algebraic logic was born in the XIXth century with the work of Boole, De Morgan, Peirce, Schröder, etc. on classical logic, see [12, 16]. They took logical equivalence rather than truth as the primitive logical predicate, and, exploiting the similarity between logical equivalence and equality, they developed logical systems in which metalogical investigations take on a distinctly algebraic character. In particular, Boole's work evolved into the modern theory of Boolean algebras, and that of De Morgan, Peirce and Schröder into the theory of relation algebras. Algebraic logic in this sense developed more-or-less independently of the logical systems of Frege and Russell and Whitehead where truth and logical truth were the underlying logical predicates. Reinforced by Hilbert's idea of metamathematics, this trend in logic became focused around the formal notions of assertion (logical validity and theoremhood) and logical inference. Thus we have from the beginning of the contemporary era of logic two approaches to the subject, one centered on the notion of logical equivalence and the other centered on the notions of assertion and inference.

It was not until much later that logicians started to think about connections between these two ways of looking at logic. Tarski ([131]) gave the precise connection between Boolean algebra and the classical propositional calculus. His approach builds on Lindenbaum's idea of viewing the set of formulas as an algebra with operations induced by the logical connectives. Logical equivalence is a congruence relation on the formula algebra, and the associated quotient algebra turns out to be a free Boolean algebra. This is the so-called *Lindenbaum-Tarski method* (see below). The connection here between the two ways of looking at classical propositional logic is made by interpreting the logical equivalence of formulas φ and ψ as the theoremhood of a suitable formula $(\varphi \leftrightarrow \psi)$ in the assertional system. The connection between the predicate calculus and relation algebras is not so straightforward, and in fact, when the Lindenbaum-Tarski method is applied to the predicate calculus, it leads to cylindric and polyadic algebras rather than relation algebras.

Other logics not relaying on the (classical) notion of truth, like intuitionistic logic (centered on the notion of constructive mathematical proof) or multiple-valued logic, can also be approached from the two points of view, the equivalential and the assertional. And the connection between them, like in the predicate-logic case, can be complicated. For instance, when the Lindenbaum-Tarski method is applied to the infinite-valued logistic system of Łukasiewicz one gets not MV-algebras, but the so-called Wajsberg algebras. In contrast to Boolean, cylindric, polyadic, and Wajsberg algebras which were defined before the Lindenbaum-Tarski method was first applied to generate them from the appropriate assertional systems, Heyting algebras seem to be the first algebras of logic that were identified by applying the Lindenbaum-Tarski method to a known assertional system, namely the intuitionistic propositional calculus.

Traditionally algebraic logic has focused on the algebraic investigation of particular classes of algebras of logic, whether or not they could be connected to some known assertional system by means of the Lindenbaum-Tarski method. However, when such a connection could be established, there was interest in investigating the relationhip between various metalogical properties of the logistic system and the algebraic properties of the associated class of algebras (obtaining what are sometimes called "bridge theorems"). For example, it was discovered that there is a natural relation between the interpolation theorems of classical, intuitionistic, and intermediate propositional calculi, and the amalgamation properties of varieties of Heyting algebras. Similar connections were investigated between interpolation theorems in the predicate calculus and amalgamation results in varieties of cylindric and polyadic algebras.

Although interest in the traditional areas of algebraic logic has not diminished, the field has evolved considerably in other directions. The *ad hoc* methods by which a class of algebras is associated to a given logic have given way to a systematic investigation of broad classes of logics in an algebraic context. The focus has shifted to *the process by which a class of algebras is associated with an arbitrary logic* and away from the particular classes of algebras that are obtained when the process is applied to specific logics. The general theory of the algebraization of logical systems that has developed is called ***Abstract Algebraic Logic*** (***AAL*** from now on).

One of the goals of AAL is to discover general criteria for a class of algebras (or for a class of mathematical objects closely related to algebras such as, for instance, logical matrices or generalized matrices) to be *the algebraic counterpart* of a logic, and to develop the methods for obtaining this algebraic counterpart. In this connection an abstraction of the Lindenbaum-Tarski method plays a major role.

Bridge theorems relating metalogical properties of a logic to algebraic properties of its algebraic counterpart take on added interest in the context of AAL. For example, it was known for some time that there is a close connection between the deduction theorem and the property of a class of algebras that its members have uniformly equationally definable principal congruences, but it is only in the more general context of AAL that the connection can be made precise. Indeed, the desire to find the proper context in which this connection could be made precise partly motivated the development of AAL. There are other bridge theorems that relate metalogical properties such as (Beth) definability, the existence of sensible Gentzen calculi, etc. with algebraic properties such as the property that epimorphisms are surjective, congruence extension, etc.

Another important goal of AAL is a *classification* of logical systems based on the algebraic properties of their algebraic counterpart. Ideally, when it is known that a given logic belongs to a particular group in the classification, one hopes there will be general theorems that provide important information about its properties and behaviour.

In this survey we try to describe the present state of research in AAL after recalling its main building blocks, both historically and conceptually. We think the subject is young enough so that understanding some points in its early history is necessary in order to fully appreciate it. This is not however a work on the details of this history, and we are going to use mainly a unified terminology and notation in order to facilitate the task of reading the paper and as a guide to the current and forthcoming literature. Three papers of a historical or survey nature relating to AAL are [25, 29, 117]. The following five papers [22, 51, 71, 107, 127] deal with various aspects of the prehistory of AAL.

1 The First Steps

In this section we describe in more detail some of the milestones along the way from algebraic logic as it was perceived in middle of the twentieth century to the first truly general theory of the algebraization of logic as we see it today.

Following is some of the terminology and notation that will be used below. The expression "iff" is used as an abbreviation for "if and only if". The power set of a set A is denoted by $\mathcal{P}(A)$. For a function f with domain A and any $X \in \mathcal{P}(A)$, $f[X] = \{ f(x) : x \in X \}$. Algebraic structures, in particular algebras, will be denoted by boldface complexes of letters beginning with a capital Latin letter, e.g, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{Fm}, \ldots$, and their universes by the corresponding light-face letters, $A, B, Fm, \ldots$. The set of congruences of $\boldsymbol{A}$ is denoted by $\mathrm{Co}\,\boldsymbol{A}$. If K is a class of algebras and $\boldsymbol{A}$ an arbitrary algebra, then $\theta \in \mathrm{Co}\,\boldsymbol{A}$ is called a ***K-congruence of $\boldsymbol{A}$*** if $\boldsymbol{A}/\theta \in \mathsf{K}$; the set of K-congruences of $\boldsymbol{A}$ is denoted by $\mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$. When the class K is clear from context one speaks simply of the ***relative congruences of $\boldsymbol{A}$***. The set of homomorphisms between algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$.

In this survey a ***logical language*** will be simply a set of connectives (each with a fixed arity $n \geq 0$). Given a logical language $\mathcal{L}$ and a countably infinite set of propositional variables Var, the formulas are inductively defined in the usual way. The connectives can be considered as the operation symbols of an algebraic similarity type, and then the formulas are the terms of this similarity type in the now commonly used algebraic sense, over the set of propositional variables. Therefore we have at our disposal the algebra of terms, which is an absolutely free algebra of type $\mathcal{L}$ over a denumerable set of generators Var. We call it the ***algebra of formulas*** and denote it by $\boldsymbol{Fm}$. Thus, $\boldsymbol{Fm}$ consists in the set Fm of formulas together with the operations of forming complex formulas associated with each connective.

1.1 Consequence operations and logics

In 1930 Tarski defined what later on were called finitary consequence operations. In [130] he considered, for some countable set A, functions $C : \mathcal{P}(A) \to \mathcal{P}(A)$ that satisfy for every $X \subseteq A$,

(C1) $X \subseteq C(X)$,

(C2) $C\big(C(X)\big) = C(X)$,

(C3) $C(X) = \bigcup\{\, C(Y) : Y \subseteq X\,, Y \text{ finite} \,\}$ and

(C4) there is some $a \in A$ such that $C\big(\{a\}\big) = A$.

Condition (C3) is called the *finitarity* condition and implies the weaker property of *monotonicity* of C, i.e.,

(C5) if $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

These properties were intended to abstract the properties of the consequence operation of classical logic which do not depend on the meaning of the connectives. Nowadays a ***consequence operation*** on a (not necessarily countable) set A means any function $C : \mathcal{P}(A) \to \mathcal{P}(A)$ that satisfies conditions (C1), (C2) and (C5). If in addition the function satisfies the finitarity condition (C3), the consequence operation is said to be ***finitary***. A consequence operation C on a set A can be transformed into a *relation* $\vdash_C \subseteq \mathcal{P}(A) \times A$ between subsets of A and elements of A by postulating for every $X \subseteq A$ and every $a \in A$ that

$$X \vdash_C a \quad \text{iff} \quad a \in C(X). \tag{1.1}$$

The properties that $\vdash_C$ inherits from the conditions (C1), (C2) and (C5) on C define what is called a ***consequence relation***. These three conditions are usually translated into the following two conditions on $\vdash_C$.

(C1') if $a \in X$, then $X \vdash_C a$, and

(C2') if $Y \vdash_C a$ for all $a \in X$, and $X \vdash_C b$, then $Y \vdash_C b$,

which imply the monotonicity condition

(C5') if $X \vdash_C a$ and $X \subseteq Y$, then $Y \vdash_C a$.

Conversely, any consequence relation $\vdash$ defines a consequence operation $C_\vdash$ by stipulating that $a \in C_\vdash(X)$ iff $X \vdash_C a$, and the two processes are inverses to one another.

In 1958 Łoś and Suszko [101] added invariance under substitutions (which they called *structurality*) to the conditions of Tarski when the consequence operation acts on the set of formulas of a propositional logic; this was intended to express the *formal* character of the logical consequence. Given a logical language $\mathcal{L}$ a ***substitution*** is a function $\sigma : Var \to Fm$; it extends to a unique endomorphism of the formula algebra $\boldsymbol{Fm}$, which we denote also by σ. A consequence operation C on Fm is called ***substitution-invariant*** when for any substitution σ and any set of formulas $\varGamma$,

(C6) $C\big(\sigma[\varGamma]\big) = C\big(\sigma\big[C(\varGamma)\big]\big)$, or, equivalently, $\sigma\big[C(\varGamma)\big] \subseteq C\big(\sigma[\varGamma]\big)$.

The corresponding property for the associated consequence relation is

(C6') if $\varGamma \vdash_C \varphi$, then for every substitution σ, $\sigma[\varGamma] \vdash_C \sigma(\varphi)$,

for every $\varGamma \subseteq Fm$ and every $\varphi \in Fm$. The concept of formula algebra together with that of a substitution-invariant consequence relation seem to be the indispensable components of a general theory of the process of algebraizing different logical systems.

A formal concept of logic that considers only these two components has been extensively studied since Tarski's work mainly by logicians in Poland [46, 142] and has become one of the standard frameworks of contemporary AAL. Formally, given a logical language $\mathcal{L}$, a ***logic*** or ***deductive system*** in the language $\mathcal{L}$ is a pair $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ where $\boldsymbol{Fm}$ is the algebra of formulas of $\mathcal{L}$ and $\vdash_{\mathcal{S}}$ is a substitution-invariant consequence relation on Fm, that is, a relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(Fm) \times Fm$ satisfying (C1'), (C2') and (C6'), and hence also (C5'). A logic $\mathcal{S}$ is said to be ***finitary*** when its consequence relation satisfies the relational form of property (C3), that is, when for every $\Gamma \cup \{\varphi\} \subseteq Fm$,

(C3') if $\Gamma \vdash_{\mathcal{S}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathcal{S}} \varphi$.

Most of the research in algebraic logic and its applications have been limited to finitary logics, however the most general treatments benefit from the easing of this restriction. It is useful to recall that with every logic $\mathcal{S}$ a finitary logic $\mathcal{S}^f$ on the same formula algebra can be associated that for finite sets behaves like $\mathcal{S}$. It is called the ***finitary companion*** of $\mathcal{S}$ and its consequence relation is defined as follows.

$$\Gamma \vdash_{\mathcal{S}^f} \varphi \text{ iff there is a finite } \Delta \subseteq \Gamma \text{ such that } \Delta \vdash_{\mathcal{S}} \varphi. \tag{1.2}$$

The ***theorems*** of a logic $\mathcal{S}$ are the formulas φ such that $\emptyset \vdash_{\mathcal{S}} \varphi$; observe that in the general notion of logic considered here it is not required that this set be nonempty. A ***theory*** of $\mathcal{S}$, or simply an ***$\mathcal{S}$-theory***, is a set of formulas Σ closed under the consequence relation $\vdash_{\mathcal{S}}$, that is, such that if $\Sigma \vdash_{\mathcal{S}} \varphi$, then $\varphi \in \Sigma$; the set of all theories of $\mathcal{S}$ is denoted as $\mathcal{T}h\,\mathcal{S}$. A theory is ***consistent*** if it is not the set of all formulas. The following notational abbreviation will be very useful: If $\Gamma, \Delta \subseteq Fm$ then the expression $\Gamma \vdash_{\mathcal{S}} \Delta$ means that $\Gamma \vdash_{\mathcal{S}} \delta$ for each $\delta \in \Delta$.

Particular logics fall under this general concept in as much as they can be presented in a form that satisfies the definition, irrespective of the way they have been originally introduced. One of the most common ways of doing this in the case of finitary logics is by means of "Hilbert-style" calculi, which we now describe.

By a (***finitary***) ***inference rule***, or simply a ***rule***, over $\mathcal{L}$ we mean any pair $\langle \Gamma, \varphi \rangle$ where Γ is a finite set of formulas and φ is a single formula. An ***axiom*** is a rule of the form $\langle \emptyset, \varphi \rangle$, which is usually written as simply φ. The rule $\langle \Gamma, \varphi \rangle$ is often represented pictorially as $\dfrac{\Gamma}{\varphi}$. A formula ψ is ***directly derivable*** from a set Δ of formulas by the rule $\langle \Gamma, \varphi \rangle$ if there is a substitution σ such that $\sigma\varphi = \psi$ and $\sigma[\Gamma] \subseteq \Delta$. A logic $\mathcal{S}$ (over $\mathcal{L}$) is ***presented*** by a (possibly infinite) set of inference rules and axioms if $\Gamma \vdash_{\mathcal{S}} \varphi$ iff φ is contained in the smallest set of formulas that includes Γ together with all substitution instances of the axioms of $\mathcal{S}$, and is closed under direct derivability by the inference rules. The pair consisting of the sets of axioms and inference rules that present $\mathcal{S}$ is called a ***presentation*** of $\mathcal{S}$; every logic has of course many presentations. A presentation is ***finite*** if both the set of axioms and the set of inference rules are finite. A logic is ***finitely presented*** if it has a finite presentation. A logic $\mathcal{S}'$ is an ***extension*** of a logic $\mathcal{S}$ if they are both over the same language and $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{S}'}$; the extension is ***axiomatic*** if a presentation of $\mathcal{S}'$ can be obtained from a presentation of $\mathcal{S}$ by adjoining additional axioms, but no new rules of inference. $\mathcal{S}'$ is an ***expansion*** of a logic $\mathcal{S}$ and $\mathcal{S}$ is a ***fragment*** of $\mathcal{S}'$ if the language of $\mathcal{S}'$ includes the language of $\mathcal{S}$ and $\vdash_{\mathcal{S}'}$ coincides with $\vdash_{\mathcal{S}}$ when restricted to the language of $\mathcal{S}$.

There are various ways of defining a logic that can be naturally divided into "syntactical" and "semantical" methods. The "syntactical" methods use some combinatorial calculus, for example a Hilbert-style presentation (as discussed above), natural deduction, Gentzen calculus, resolution, tableaux, etc. to define the logic; the "semantical" methods use (mathematical) objects external to the set of formulas, for instance algebraic semantics, relational semantics, game-theoretic semantics, etc. Thus the above notion of a logic includes all the familiar sentential logics together with their various fragments and refinements—for example, the classical and the intuitionistic propositional calculi, the intermediate logics, the local and the global consequences associated with various modal logics, the multiple-valued logics of Łukasiewicz and Post, etc.

Another observation is in order here. At first sight the notion of logic considered in this survey does not seem to extend in scope beyond what is commonly considered to be "propositional" or "sentential" logic. In particular quantifier logics might seem to be beyond its scope. So do the so-called *substructural logics* [61, 113], such as BCK logic, relevance logic and linear logic, defined by means of Gentzen calculi that fail to satisfy the structural rules (see Section 4.2 below). But ordinary first-order logic can be reformulated in such a way that it falls within the scope of the present concept of logic; see for instance [23, Appendix C], and the substructural logics can also be accommodated if the notion of logic is generalized in a certain natural way; see Section 4.1.

With regard to the algebraization of quantifier logics, this is an appropriate place to mention an alternate approach to abstract algebraic logic that originated specifically from an attempt to abstract from the traditional algebraization of the first-order logic. It leads to a different conception of *logic* that is based on the conviction that there should be a semantical component in its definition. This approach is discussed briefly in Section 6.1; see [10] or [11] for a survey of the topic and more references.

1.2 Logical matrices

In the 1920's Łukasiewicz and Tarski [103] introduced the general concept of logical matrix, already implicit in the previous work of Łukasiewicz himself, Bernays, Post and others that used "truth tables" for several purposes. In current terminology a (***logical***) ***matrix*** for a logical language $\mathcal{L}$, or an ***$\mathcal{L}$-matrix***, is an ordered pair $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an algebra of type $\mathcal{L}$ with universe A, and $F \subseteq A$; this set F is called the set of ***designated values***. Logical matrices were first used as models of the theorems of a logic, in the study of particular logics (as for instance in the paper [105] by McKinsey and Tarski). The general theory was studied, mainly by Polish logicians, starting with Łoś [100] in 1949 and continuing with Łoś and Suszko's fundamental 1958 paper [101], where the general notion of a logic defined by a class of matrices is first given. Wójcicki's 1988 book [142] and Czelakowski's recent [46] are the best sources for the large body of research on this topic; Czelakowski's book is especially useful for understanding the role of logical matrices in AAL.

In any given $\mathcal{L}$-matrix $\langle \boldsymbol{A}, F\rangle$, each formula φ of $\mathcal{L}$ has a unique interpretation in $\boldsymbol{A}$ depending on the values in $\boldsymbol{A}$ that are assigned to its variables. Using the facts that $\boldsymbol{Fm}$ is absolutely freely generated by the set of variables and that $\boldsymbol{A}$ is an algebra over the same language, the interpretation of φ can be expressed algebraically as $h(\varphi)$, where h is a homomorphism from $\boldsymbol{Fm}$ to $\boldsymbol{A}$ that maps each variable of φ into its assigned value. (Since the set of variables of φ is a proper subset of the set of all variables, there are many homomorphisms with this property, but they all map φ into the same element of $\boldsymbol{A}$.) A homomorphism whose domain is the formula algebra is called an ***assignment***. We often write a formula φ in the form $\varphi(x_0, \dots, x_{n-1})$ to indicate that each of its variables occurs in the list $x_0, \dots, x_{n-1}$, and we write $\varphi^{\boldsymbol{A}}(a_0, \dots, a_{n-1})$ for $h(\varphi)$ where h is any assignment such that $h(x_i) = a_i$ for all $i < n$.

Given a logic $\mathcal{S}$ in a language $\mathcal{L}$, an $\mathcal{L}$-matrix $\langle \boldsymbol{A}, F\rangle$ is said to be a ***model of*** $\mathcal{S}$ if, for every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ and every $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\text{if } \ h[\Gamma] \subseteq F \ \text{ and } \ \Gamma \vdash_{\mathcal{S}} \varphi \ \text{ then } \ h(\varphi) \in F; \tag{1.3}$$

in this case it is also said that F is a *deductive filter* of $\mathcal{S}$ or, as is common now, an ***$\mathcal{S}$-filter*** of $\boldsymbol{A}$; this term was coined by Rasiowa in her 1974 book [120]. Given an algebra $\boldsymbol{A}$ of type $\mathcal{L}$, the set of all $\mathcal{S}$-filters of $\boldsymbol{A}$, which is denoted by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, is closed under intersections of arbitrary families and is thus a complete lattice. Therefore, given any set $X \subseteq A$, there is always the least $\mathcal{S}$-filter of $\boldsymbol{A}$ that contains X; it is called the ***$\mathcal{S}$-filter of $\boldsymbol{A}$ generated by*** X and is denoted by $Fi^{\boldsymbol{A}}_{\mathcal{S}}(X)$. The class of all matrix models of a logic $\mathcal{S}$ is denoted by $\mathbf{Mod}\,\mathcal{S}$.

It is easy to see that the $\mathcal{S}$-filters on the formula algebra are exactly the $\mathcal{S}$-theories, hence the $\mathcal{S}$-theory axiomatized, that is, generated by a set of formulas Γ, is exactly the $\mathcal{S}$-filter on the formula algebra $Fi^{\boldsymbol{Fm}}_{\mathcal{S}}(\Gamma)$ generated by Γ. The idea of considering $\mathcal{L}$-matrices on the formula algebra is due to Lindenbaum; for this reason the matrices of the form $\langle \boldsymbol{Fm}, \Sigma\rangle$ where Σ is an $\mathcal{S}$-theory are called the ***Lindenbaum matrices*** of $\mathcal{S}$; clearly they are models of $\mathcal{S}$.

A logic $\mathcal{S}$ in the language $\mathcal{L}$ is said to be ***complete relative to a class of $\mathcal{L}$-matrices*** $\mathbf{M}$ if

$\mathsf{M} \subseteq \mathsf{Mod}\,\mathcal{S}$ and for every $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \nvdash_{\mathcal{S}} \varphi$ there is a matrix $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ and an $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h[\Gamma] \subseteq F$ but $h(\varphi) \notin F$. When this holds it is also said that M is a ***matrix semantics for*** $\mathcal{S}$ or that M is ***strongly adequate*** or ***strongly characteristic for*** $\mathcal{S}$. Clearly every logic is complete relative to the class of all its Lindenbaum matrices; therefore each logic is complete relative to the class of all its matrix models too. In this sense every logic has a matrix semantics.

Logical matrices may also be used to *define* logics. Given a class of matrices M, two logics can be associated with it in a natural way: One is the logic $\mathcal{S}_{\mathsf{M}}$ that is defined by

$$\Gamma \vdash_{\mathcal{S}_{\mathsf{M}}} \varphi \quad \text{iff} \quad \big(\forall \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\big)\,\big(\forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\big),\ h[\Gamma] \subseteq F \Rightarrow h(\varphi) \in F\,; \tag{1.4}$$

the other is its finitary companion $\mathcal{S}^{f}_{\mathsf{M}}$. In general, the logic $\mathcal{S}_{\mathsf{M}}$ could be nonfinitary. Obviously the logic $\mathcal{S}_{\mathsf{M}}$ is complete relative to M, while $\mathcal{S}^{f}_{\mathsf{M}}$ is so only in case it coincides with $\mathcal{S}_{\mathsf{M}}$, that is, when the latter is finitary. Interesting problems for AAL are to find a natural (complete) matrix semantics for $\mathcal{S}^{f}_{\mathsf{M}}$ and natural conditions under which $\mathcal{S}_{\mathsf{M}}$ is finitary.

1.3 Lindenbaum-Tarski algebras

By the *classical Lindenbaum-Tarski method* we mean the process of forming the quotient of the formula algebra by the relation of logical equivalence. In the case of the classical and intuitionistic propositional logics, and some of their expansions, two formulas φ and ψ are logically equivalent, in symbols $\varphi \equiv \psi$, when $\varphi \leftrightarrow \psi$ is a theorem (or equivalently when $\varphi \to \psi$ and $\psi \to \varphi$ are both theorems). Tarski was the first to use this method [131, 132] to give the first precise formulation of the connection between the classical propositional calculus and Boolean algebra, but a number of different people became aware of essentially the same process about the same time; see [37, pp. 103–104]. This appears to be the first occurrence in the literature of what has become known as the *Lindenbaum* or *Lindenbaum-Tarski algebra*. See the remark in [107] in reference to the origin of these names. For another perspective see [71, 127] and the footnotes to Chapter VIII of [120]. The method can be applied whenever the relation $\equiv$ is a congruence, and has been generalized to theories as follows. Given a theory Σ of classical or intuitionistic logic, the relation $\equiv_{\Sigma}$ defined by

$$\varphi \equiv_{\Sigma} \psi \quad \text{iff} \quad \varphi \leftrightarrow \psi \in \Sigma \tag{1.5}$$

is also a congruence of the algebra of formulas, and the corresponding quotient algebra $\boldsymbol{Fm}/{\equiv_{\Sigma}}$ is known as *the Lindenbaum-Tarski algebra determined by* Σ. The Lindenbaum-Tarski algebras of classical propositional logic, obtained in this way, are (up to isomorphism) the countable Boolean algebras.

Around 1950 it was realized, by Henkin, Rasiowa, Sikorski and others that this method can be applied to other logics with a connective of implication satisfying some basic properties. This line of direct generalization culminated in Rasiowa's well-known 1974 monograph [120], where the class of (now) so-called ***implicative logics*** is studied. These are the logics $\mathcal{S}$ having a binary connective $\to$ satisfying the conditions $\varphi\,, \varphi \to \psi \vdash_{\mathcal{S}} \psi$ and $\varphi \vdash_{\mathcal{S}} \psi \to \varphi$ and such that, for each theory Σ, the relation $\leq_{\Sigma}$ (defined by: $\varphi \leq_{\Sigma} \psi$ iff $\Sigma \vdash_{\mathcal{S}} \varphi \to \psi$) is a quasi-order (reflexive and transitive) compatible with all connectives in $\boldsymbol{\mathcal{L}}$. Thus its symmetrization, that is the relation between formulas defined by

$$\varphi \equiv_{\Sigma} \psi \quad \text{iff} \quad \Sigma \vdash_{\mathcal{S}} \varphi \to \psi \text{ and } \Sigma \vdash_{\mathcal{S}} \psi \to \varphi, \tag{1.6}$$

is a congruence of the formula algebra that, in addition, has the following property.

$$\text{if } \varphi \equiv_{\Sigma} \psi \text{ and } \varphi \in \Sigma, \text{ then } \psi \in \Sigma. \tag{1.7}$$

To each theory Σ can be assigned the Lindenbaum-Tarski algebra $\boldsymbol{Fm}/{\equiv_{\Sigma}}$. Besides classical logic, other examples of implicative logics are intuitionistic logic, normal modal logics (with the rule of necessitation applicable to arbitrary sets of premises) and the many-valued logics of Post and Łukasiewicz. The corresponding Lindenbaum-Tarski algebras are the countable members of, respectively, the classes of Heyting algebras, several classes of Boolean algebras with operators (such as closure algebras, etc., depending on the specific modal logic), Post algebras, Wajsberg algebras, etc.

Rasiowa's book also contains what should be considered as the first general definition of the notion of *the algebraic counterpart of a logic*, one that had already been considered in her 1953 paper [121] with Sikorski. With each implicative logic $\mathcal{S}$ she associates the class of algebras, later denoted by $\mathsf{Alg}^{*}\mathcal{S}$, whose members she calls ***$\mathcal{S}$-algebras***: these are the algebras $\boldsymbol{A}$ of type $\mathcal{L}$ with an element $1 \in A$ such that the $\mathcal{L}$-matrix $\langle \boldsymbol{A}, \{1\} \rangle$ is a model of $\mathcal{S}$, and, for all $a, b \in A$, $a \to b = b \to a = 1$ implies $a = b$.

Although implicative logics were defined by conditions on implication, it is clear that it is the pair of formulas $\varphi \to \psi, \psi \to \varphi$ that together play the central role in the algebraization process. In view of (1.6) they act collectively as an equivalence, thus generalizing the construction (1.5) for classical logic. The next step in the generalization of the Lindenbaum-Tarski method was taken in 1974 by Prucnal and Wrónski, who in [118] defined the class of ***equivalential logics***. Its members are the logics that have a (possibly infinite) set E of formulas in at most two variables (x, y) that behaves like $\{x \to y, y \to x\}$, that is, for every theory Σ, the relation between formulas defined by

$$\varphi \equiv_{\Sigma} \psi \quad \text{iff} \quad \Sigma \vdash_{\mathcal{S}} \delta(\varphi, \psi) \text{ for every } \delta(x, y) \in E \tag{1.8}$$

is a congruence of the formula algebra satisfying (1.7). Using these congruence relations a Lindenbaum-Tarski-like process can also be applied to any equivalential logic. Equivalential logics were first systematically studied in [40] by Czelakowski. To some extent these papers can be seen as the starting point of the journey from traditional algebraic logic to abstract algebraic logic.

The final step in the full generalization of the Lindenbaum-Tarski method consists in disregarding the requirement that there are some formulas behaving collectively as equivalence, and instead finding a truly general definition of the congruence associated with a theory that can be applied to *every* logic. All this will be explained at length, but with less historical detail, in subsequent sections.

2 The Lindenbaum-Tarski Process Fully Generalized

The central concepts in metalogical studies are frequently *logical truth* or *theoremhood.* However, the concepts that have proven to be fundamental for the purpose of building a general theory of the algebraization of logic applicable to arbitrary systems are those of *logical equivalence* and of equivalence relative to a given theory. When the concepts of logical equivalence and of logical truth are reciprocally definable, a theory can be built using the second concept, but this is not the case in general. One of the reasons why classical logic has its distinctive algebraic character lies precisely in the fact that there the concepts of logical equivalence and logical truth are reciprocally definable.

2.1 Frege's principles

Real logics arise from the systematization of some inference phenomena. In the process intuitive notions of logical consequence, logical equivalence and logical truth are transformed by abstraction into precise mathematical formalizations of these notions. Logical equivalence can be expressed in terms of consequence by saying that two sentences are logically equivalent if each is a consequence of the other. Alternatively, it can be defined semantically: two sentences are logically equivalent if they have the same semantic value in every interpretation. But if consequence itself is expressed semantically, then these two ways of looking at logical equivalence usually amount to the same thing. This is the case for instance if the semantic values are partially ordered by their "degree of acceptance", and the semantic value of the conclusion is greater or equal to that of the premiss under this ordering. The expression 'semantic value' is used in the sense of Dummett's [62]. For Frege, the semantic value (denotation) of a sentence is its truth value. Thus, informally, two sentences of the language of classical logic are logically equivalent iff they have the same truth value in every possible interpretation. Furthermore, two interpreted sentences of classical logic have the same truth value iff, in every propositional context, if either one is replaced by the other, the resulting sentences have the same truth value. This second idea has been called *Frege's extensionality principle*, or simply *Frege's principle*; as a consequence one obtains what we call *Frege's weak principle*: two (uninterpreted) sentences of classical logic are logically equivalent iff, in every interpretation, they can be mutually substituted in a propositional context without altering its truth value. These principles reflect Frege's idea of *compositionality*, and when formalized along the following lines they can be generalized so as to be applicable to any logic.

When formalized, Frege's principle for classical propositional logic $\mathcal{CPL}$ takes the following form.

$$\text{For every theory } \Sigma \text{ of } \mathcal{CPL}, \text{ if } \ \Sigma \cup \{\varphi\} \vdash_{\mathcal{CPL}} \psi \ \text{ and } \ \Sigma \cup \{\psi\} \vdash_{\mathcal{CPL}} \varphi\,,$$
$$\text{then for every formula } \delta \text{ with the variable } x,$$
$$\Sigma \cup \big\{\delta(\varphi/x)\big\} \vdash_{\mathcal{CPL}} \delta(\psi/x) \ \text{ and } \ \Sigma \cup \big\{\delta(\psi/x)\big\} \vdash_{\mathcal{CPL}} \delta(\varphi/x),$$

while Frege's weak principle is equivalent to the following.

$$\text{If } \ \varphi \vdash_{\mathcal{CPL}} \psi \ \text{ and } \ \psi \vdash_{\mathcal{CPL}} \varphi\,, \text{ then for every formula } \delta \text{ with the variable } x,$$
$$\delta(\varphi/x) \vdash_{\mathcal{CPL}} \delta(\psi/x) \ \text{ and } \ \delta(\psi/x) \vdash_{\mathcal{CPL}} \delta(\varphi/x).$$

The only primitive logical notion involved in these formulations of Frege's principles is the consequence relation of classical logic; this makes them particularly well suited for application to the general notion of logic described in Section 1.1, and thus for incorporation into the general framework of AAL.

Let $\mathcal{S}$ be a logic in a logical language $\mathcal{L}$. By the ***Frege relation*** of $\mathcal{S}$, in symbols $\boldsymbol{\Lambda}\mathcal{S}$, we mean the relation of mutual implication between formulas, that is

$$\boldsymbol{\Lambda}\mathcal{S} = \big\{ \langle \varphi, \psi \rangle : \varphi \vdash_{\mathcal{S}} \psi \ \text{ and } \ \psi \vdash_{\mathcal{S}} \varphi \big\}. \tag{2.1}$$

The algebraic formulation of Frege's weak principle for $\mathcal{S}$ is then:

The Frege relation of $\mathcal{S}$ is a congruence of the formula algebra.

This principle will be called the *abstract weak Frege's principle* and the logics that satisfy it are called ***selfextensional logics*** following Wójcicki, see [142, Chapter 5].

An abstract counterpart of Frege's principle can we obtained in a similar way. Given an $\mathcal{S}$-theory Σ we say that two formulas are ***Σ-equivalent*** iff each is a consequence of the other when adjoined to Σ. The relation so defined is called the ***Frege relation of Σ relative to*** $\mathcal{S}$, and is denoted by $\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$. Hence,

$$\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma = \big\{ \langle \varphi, \psi \rangle : \Sigma, \varphi \vdash_{\mathcal{S}} \psi \ \text{ and } \ \Sigma, \psi \vdash_{\mathcal{S}} \varphi \big\}. \tag{2.2}$$

Notice that $\langle \varphi, \psi \rangle \in \boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$ iff φ and ψ belong to the same theories of $\mathcal{S}$ that include Σ. Now the formal general counterpart of Frege's principle for a logic $\mathcal{S}$ takes the following form.

For every theory Σ of $\mathcal{S}$, the Frege relation of Σ relative to $\mathcal{S}$
is a congruence of the formula algebra.

This principle may be called the *abstract Frege's principle*, and the logics satisfying it are called ***Fregean logics*** or ***extensional logics***; the logics that do not satisfy it may be called ***intensional logics***. The first *formalized* distinction between Fregean and non-Fregean logics appears to have been made by Suszko, see [128], and has been brought into AAL in [53, 54, 73, 115].

2.2 Algebras and matrices canonically associated with a logic

The Frege relation $\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$ for a theory Σ always satisfies the property analogous to (1.7), that is,

$$\text{if } \langle\varphi,\psi\rangle \in \boldsymbol{\Lambda}_{\mathcal{S}}\Sigma \text{ and } \varphi\in\Sigma \text{ then } \psi\in\Sigma. \tag{2.3}$$

Therefore, when a logic $\mathcal{S}$ is extensional, the Frege relation Σ gives rise to a Lindenbaum-Tarski-like construction for every $\mathcal{S}$-theory Σ: the quotient matrix $\langle \boldsymbol{Fm}/\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma, \Sigma/\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma\rangle$ can be defined, where $\boldsymbol{Fm}/\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$ is the quotient algebra and $\Sigma/\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$ is the set of equivalence classes of the elements of Σ. This construction really is a generalization of the Lindenbaum-Tarski method for Fregean logics. The classical logical equivalence relation, defined in terms of the biconditional as in (1.6), or in terms of the pair of implications $\varphi\to\psi$ and $\psi\to\varphi$ as mentioned above, coincides with the Fregean relation for classical propositional logic and for the other Fregean logics studied in the literature. Thus it makes sense to call the matrices obtained in this way the ***Lindenbaum-Tarski matrices*** of $\mathcal{S}$, and to call their algebra reducts (i.e., the algebras $\boldsymbol{Fm}/\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$) the ***Lindenbaum-Tarski algebras*** of $\mathcal{S}$.

For intensional logics the process just described does not work and must be modified in order to obtain a truly general version of the Lindenbaum-Tarski method. It is not difficult to see that, for any logic $\mathcal{S}$, there exists a largest congruence contained in the Frege relation of an arbitrary theory Σ of $\mathcal{S}$, which is called the ***Suszko congruence of Σ relative to $\mathcal{S}$***, and is denoted by $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma$. Obviously, $\mathcal{S}$ is Fregean iff $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma = \boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$ for all theories Σ of $\mathcal{S}$. The notion of the Suszko congruence was introduced in Suszko's 1977 talk [129] by the equivalent characterization

$$\langle\varphi,\psi\rangle \in \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma \quad\text{iff}\quad \Sigma\cup\big\{\delta(\varphi/x)\big\}\vdash_{\mathcal{S}}\delta(\psi/x) \text{ and } \Sigma\cup\big\{\delta(\psi/x)\big\}\vdash_{\mathcal{S}}\delta(\varphi/x) \tag{2.4}$$
$$\text{for every formula } \delta \text{ with the variable } x.$$

Finally, using the Suszko congruences, we get a version of the Lindenbaum-Tarski construction that can be applied to all logics. The ***Lindenbaum-Tarski matrices*** of an arbitrary logic $\mathcal{S}$ are then the matrices of the form $\big\langle \boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma\,, \Sigma/\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma\big\rangle$ for an $\mathcal{S}$-theory Σ, and their algebra reducts, that is, the algebras of the form $\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma$ with $\Sigma\in \mathcal{T}h\,\mathcal{S}$, are the ***Lindenbaum-Tarski algebras of*** $\mathcal{S}$. However, this class is too small (for instance, the cardinality of these algebras cannot exceed that of the formula algebra) and hence it cannot be taken as the canonical class of algebras to be associated with a given logic; but clearly it must be included in it.

The Lindenbaum-Tarski method just described encompasses the generalizations of the classical method discussed in Section 1.3. For equivalential logics, which can be non-Fregean, it can be proved that for each theory Σ, the relation $\equiv_\Sigma$ defined in (1.8) is precisely the Suszko congruence of Σ. The expression (1.8) shows that in this case the Suszko congruence of an $\mathcal{S}$-theory Σ is completely determined by Σ itself, as a set of formulas, independently of the underlying logic $\mathcal{S}$. But this is not true in general. This is clear from the definition (2.4) of the Suszko congruence, and becomes even clearer when (2.4) is reformulated in terms of $\mathcal{S}$-theories. We have then that $\langle\varphi\,,\psi\rangle \in \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma$ iff, for every formula δ with the variable x, $\delta(\varphi/x)$ and $\delta(\psi/x)$ belong to the same theories of $\mathcal{S}$ that include Σ. Thus all the $\mathcal{S}$-theories that include Σ must be taken into account in determining the Suszko congruence of Σ. For some logics, such as the equivalential logics, this global information is coded into the theory under consideration. This property establishes a sharp division within the class of all logics into the *protoalgebraic* and *nonprotoalgebraic logics* (see Section 3.2). In the most general version of the Lindenbaum-Tarski method just considered, the algebraic counterpart of a logic is not obtained from individual theories but rather from families of theories of the logic, that is, from the logic itself considered as an organic whole.

The class of algebras that is usually associated with a concrete logic $\mathcal{S}$ is obtained by extending the original Lindenbaum-Tarski method from theories to arbitrary matrix models of the logic. For instance, given a matrix $\langle \boldsymbol{A}, F\rangle$ that is a model of the classical propositional logic, a congruence relation $\equiv_F$ on $\boldsymbol{A}$ can be defined by relating two objects $a,b\in A$ iff $a\leftrightarrow b\in F$; this is the largest congruence on $\boldsymbol{A}$ that does not relate elements in F with elements outside F, and the algebra $\boldsymbol{A}/{\equiv_F}$ is a Boolean algebra; moreover, all Boolean algebras are obtainable in this way. More generally, if $\mathcal{S}$ is an implicative logic it can be proved that for any $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathcal{S}$, the relation $\equiv_F$ on $\boldsymbol{A}$ defined by $a \equiv_F b$ iff $a \to b\,, b \to a \in F$ is the largest congruence of $\boldsymbol{A}$ that does not relate elements in F with elements outside F, and that, moreover, F is the equivalence class of $a \to a$, for every $a \in A$. The class of the algebras $\boldsymbol{A}/{\equiv_F}$ obtained in this way coincides with Rasiowa's class of "$\mathcal{S}$-algebras". Similarly, for any logic $\mathcal{S}$, suitable analogues of the Frege relation and of the Suszko congruence (so far defined only for $\mathcal{S}$-theories) can be defined for the $\mathcal{S}$-filters of an arbitrary algebra. In this manner a class of algebras can be associated in a canonical way with any given logic. We now describe the process in detail.

Let $\mathcal{S}$ be a logic and let $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathcal{S}$. The ***Frege relation of*** $\langle \boldsymbol{A}, F\rangle$ ***relative to*** $\mathcal{S}$ is the relation $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}} F$ on A defined by:

$$\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}} F = \big\{ \langle a\,, b\rangle \in A \times A : \forall G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\,, F \subseteq G \implies (a \in G \iff b \in G)\big\} \tag{2.5}$$

That is, $\langle a, b\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}} F$ iff a and b belong to the same $\mathcal{S}$-filters of $\boldsymbol{A}$ that include F or, equivalently, iff $Fi^{\boldsymbol{A}}_{\mathcal{S}}\big(F \cup \{a\}\big) = Fi^{\boldsymbol{A}}_{\mathcal{S}}\big(F \cup \{b\}\big)$; this is clearly a generalization of (2.2). The ***Suszko congruence of*** $\langle \boldsymbol{A}, F\rangle$ ***relative to*** $\mathcal{S}$ is the largest congruence included in $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}} F$; it is denoted by $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} F$. We define *the canonical class of algebras* associated with the logic $\mathcal{S}$ by the extension of the Lindenbaum-Tarski method as the closure under isomorphisms of the class of quotient algebras $\{\,\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} F : F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\,, \boldsymbol{A}$ an $\mathcal{L}$-algebra$\,\}$. It will be denoted by $\mathbf{Alg}\,\mathcal{S}$ and its members will be called ***$\mathcal{S}$-algebras***.

In the general framework, however, the $\mathcal{S}$-algebras are less important than the matrices of the form $\langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} F\,, F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} F\rangle$, for arbitrary $\boldsymbol{A}$ and all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. These matrices have the important property of being Suszko-reduced in the sense that their Suszko congruence relative to $\mathcal{S}$ is the identity (the models of $\mathcal{S}$ whose Suszko congruence is the identity are called ***Suszko-reduced***). Every logic is complete with respect to the class of all its Suszko-reduced matrices. The elements of $\mathbf{Alg}\,\mathcal{S}$ are exactly the algebra reducts of the Suszko-reduced models of $\mathcal{S}$, but in general there is no canonical way of associating a single filter $F_{\boldsymbol{A}}$ with each $\boldsymbol{A} \in \mathbf{Alg}\,\mathcal{S}$ so that the logic is complete relative to the class $\big\{\langle \boldsymbol{A}\,, F_{\boldsymbol{A}}\rangle : \boldsymbol{A} \in \mathbf{Alg}\,\mathcal{S}\big\}$. This is possible in the best behaved cases, such as classical logic or intuitionistic logic, where one can take $F_{\boldsymbol{A}} = \{1\}$, so that in this sense the Boolean algebras and the Heyting algebras are, respectively, a complete algebraic semantics for them. By contrast, there are many cases of a pair of logics that have the same class of $\mathcal{S}$-algebras and which can be distinguished through their Suszko-reduced matrices. A typical example of this phenomenon are the pairs of the local and the global consequences associated with a normal modal logic such as $\mathcal{S}4$ or $\mathcal{S}5$; some aspects of this situation have been explored in general in [74, 97].

Further research on Suszko congruences and Suszko-reduced matrices can be found in Czelakowski's paper [47] included in the present volume.

3 The Core Theory of Abstract Algebraic Logic

The concepts that constitute the core of AAL were introduced in the 1980's and the basic results were obtained. This development was anticipated in the considerable work on the algebraic theory of logical matrices that had been done during the previous decade. In the 1990's progress was made in systematizing the core and generalizing many of its concepts and results. In this section we present a systematic but non-historical description of the core theory of AAL including the algebraic theory of logical matrices that preceded it and is now considered an integral part of ALL. Czelakowski's recent monograph [46] is a comprehensive exposition of a substantial part of AAL, the only one presently available. It also includes previously unpublished material and detailed historical notes for each chapter. A comprehensive review [66] of it is included in the present volume.

3.1 Elements of the general theory of matrices

The basic concepts that turn out to be central to the development of the algebraic theory of logical matrices and, more generally, to the development of AAL, are those of a congruence of a matrix, in particular, the largest congruence of a matrix, and the corresponding notion of a reduced matrix. Closely related to the notion of matrix congruence is that of strict homomorphism.

A (***matrix***) ***congruence*** of a matrix $\langle \boldsymbol{A}, F \rangle$ is a binary relation θ on A that is a congruence of $\boldsymbol{A}$ and is ***compatible*** with F in the sense that,

$$\text{for all } a, b \in A, \text{ if } \langle a\,, b\rangle \in \theta \text{ and } a \in F, \text{ then } b \in F.$$

It is easy to see that every matrix $\langle \boldsymbol{A}, F \rangle$ has a largest matrix congruence; it is called the ***Leibniz congruence*** of $\langle \boldsymbol{A}, F \rangle$, or the *Leibniz congruence of F in $\boldsymbol{A}$*, and is denoted by $\boldsymbol{\Omega}_{\boldsymbol{A}} F$. The notion, introduced for Lindenbaum matrices by Łoś in [100] and in general by Wójcicki in [141], was given the name Leibniz congruence by Blok and Pigozzi in [23] because of the following characterization of $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ that can be viewed as the first-order analogue of Leibniz's second-order definition of identity.

$$\begin{aligned} \langle a, b \rangle \in \boldsymbol{\Omega}_{\boldsymbol{A}} F \quad \text{iff} \quad & \text{for every } \varphi(x, x_0, \dots, x_{n-1}) \in \boldsymbol{Fm} \text{ and all } c_0, \dots, c_{n-1} \in A, \\ & \varphi^{\boldsymbol{A}}(a, c_0, \dots, c_{n-1}) \in F \quad \text{iff} \quad \varphi^{\boldsymbol{A}}(b, c_0, \dots, c_{n-1}) \in F. \end{aligned} \tag{3.1}$$

Observe that the definition of the Leibniz congruence is completely independent of any logic: it is intrinsic to $\boldsymbol{A}$ and F. When applied to models of a logic $\mathcal{S}$, one finds the following relation with the Suszko congruences relative to $\mathcal{S}$. For every $\mathcal{S}$-filter F on an algebra $\boldsymbol{A}$,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} F = \bigcap \big\{ \boldsymbol{\Omega}_{\boldsymbol{A}} G : G \text{ is an } \mathcal{S}\text{-filter of } \boldsymbol{A} \text{ and } F \subseteq G \big\}. \tag{3.2}$$

The following useful characterization of $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} F$ is easily obtained from (3.1) and (3.2).

$$\begin{aligned} \langle a, b \rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} F \quad \text{iff} \quad & \text{for every } \varphi(x, x_0, \dots, x_{n-1}) \in \boldsymbol{Fm} \text{ and all } c_0, \dots, c_{n-1} \in A, \\ & Fi^{\boldsymbol{A}}_{\mathcal{S}}\big(F \cup \{\varphi^{\boldsymbol{A}}(a, c_0, \dots, c_{n-1})\}\big) = Fi^{\boldsymbol{A}}_{\mathcal{S}}\big(F \cup \{\varphi^{\boldsymbol{A}}(b, c_0, \dots, c_{n-1})\}\big) \end{aligned} \tag{3.3}$$

Observe that this is not intrinsic to $\boldsymbol{A}$ and F but depends on $\mathcal{S}$ through the operator $Fi^{\boldsymbol{A}}_{\mathcal{S}}$ of $\mathcal{S}$-filter generation on $\boldsymbol{A}$.

For the Leibniz and the Suszko congruences of $\mathcal{S}$-filters on the formula algebra $\boldsymbol{Fm}$, that is, of $\mathcal{S}$-theories, we write simply $\boldsymbol{\Omega}$ and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$ in place of $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$ and $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}$, respectively; similarly, for the Frege relation, $\boldsymbol{\Lambda}_{\mathcal{S}}$ is written in place of $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}$.

We define ***quotient matrices*** by matrix congruences. Given a matrix $\langle \boldsymbol{A}, F \rangle$ and a matrix congruence θ of $\langle \boldsymbol{A}, F \rangle$, the quotient of $\langle \boldsymbol{A}, F \rangle$ by θ is the matrix $\langle \boldsymbol{A}/\theta, F/\theta \rangle$, where $\boldsymbol{A}/\theta$ is the quotient algebra and F/θ the set of equivalence classes of the elements of F. There is only one matrix congruence on the quotient of a matrix by its Leibniz congruence; this is the identity relation. A logical matrix $\langle \boldsymbol{A}, F \rangle$ is said to be ***reduced*** (or ***Leibniz-reduced***) if its Leibniz congruence is the identity. Thus to each matrix $\langle \boldsymbol{A}, F \rangle$ corresponds the reduced matrix $\langle \boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}} F, F/\boldsymbol{\Omega}_{\boldsymbol{A}} F \rangle$, which is called its ***reduction***

The class of reduced matrix models of a logic $\mathcal{S}$ is denoted by $\mathbf{Mod}^*\mathcal{S}$. The class of algebras that by tradition is associated with a logic $\mathcal{S}$ is the class of algebraic reducts of the reduced models of $\mathcal{S}$; it is denoted by $\mathbf{Alg}^*\mathcal{S}$. Hence

$$\mathbf{Alg}^*\mathcal{S} = \big\{ \boldsymbol{A} : \exists\, F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \boldsymbol{\Omega}_{\boldsymbol{A}} F \text{ is the identity} \big\}.$$

Since clearly $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} F$, it is true in general that $\mathbf{Alg}^*\mathcal{S} \subseteq \mathbf{Alg}\,\mathcal{S}$. These classes can be different, but they always generate the same quasivariety; in many cases they coincide (see Theorem 3.4 below).

A ***strict*** or (***matrix***) ***homomorphism*** from $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{B}, G\rangle$ is an $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ such that $F = h^{-1}[G]$. The relation between strict homomorphisms and matrix congruences parallels the well-known one in universal algebra between homomorphisms and congruences: for every strict homomorphism from $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{B}, G\rangle$, its kernel is a matrix congruence of $\langle \boldsymbol{A}, F\rangle$, and every matrix congruence θ of $\langle \boldsymbol{A}, F\rangle$ is obtainable in this way: it is the kernel of the projection of $\langle \boldsymbol{A}, F\rangle$ onto $\langle \boldsymbol{A}/\theta, F/\theta\rangle$.

The general theory of matrices has strong links with the model theory of first-order logic, more precisely, with the model theory of equality-free languages. In 1975 Bloom [31] made the observation that a finitary logic in a language $\mathcal{L}$ can be treated as a universal strict Horn theory over the first-order language $\mathcal{L}^P$ without equality that is obtained by adjoining the single, unary relation symbol P to the set of operation (function) symbols of $\mathcal{S}$. The terms of $\mathcal{L}^P$ are thus the formulas of $\mathcal{L}$, and the atomic formulas of $\mathcal{L}^P$ have the form $P\varphi$ for φ an $\mathcal{L}$-formula. If the logic is not finitary it can be treated as a universal strict Horn theory in a suitable infinitary language. The correspondence between finitary logics over $\mathcal{L}$ and universal strict Horn theories over $\mathcal{L}^P$ goes as follows.

Each formula φ of $\mathcal{L}$ can be translated into the atomic formula $P\varphi$ of $\mathcal{L}^P$, and each rule $\langle \Gamma, \varphi\rangle$ with a finite Γ can be translated into the equality-free universal strict basic Horn sentence $\mathbf{tr}\big(\langle \Gamma, \varphi\rangle\big)$ over $\mathcal{L}^P$ that is defined as follows.

$$\mathbf{tr}\big(\langle \emptyset\,, \varphi\rangle\big) = \overrightarrow{\forall x} P\varphi,$$
$$\mathbf{tr}\big(\langle \{\varphi_0, \dots, \varphi_{n-1}\}\,, \varphi\rangle\big) = \overrightarrow{\forall x}\big(P\varphi_0 \wedge \dots \wedge P\varphi_{n-1} \to P\varphi\big),$$

where $\overrightarrow{\forall x}$ represents the sequence of universal quantifiers for all variables appearing in the appropriate formula, and $\wedge$ and $\to$ are the classical propositional connectives of the first-order language $\mathcal{L}^P$. Then $\mathbf{tr}(\mathcal{S})$ is defined to be the equality-free universal strict Horn theory axiomatized by the translations of all the pairs $\langle \Gamma, \varphi\rangle$ such that $\Gamma \vdash^{fin}_{\mathcal{S}} \varphi$. If $\mathcal{S}$ is finitary, it is clear that the $\mathcal{L}^P$-structures coincide with the $\mathcal{L}$-matrices, and that the models of $\mathbf{tr}(\mathcal{S})$ are exactly the matrix models of $\mathcal{S}$.

Conversely, for every equality-free universal strict Horn theory Σ over a language $\mathcal{L} \cup \{P\}$ with an arbitrary set of function symbols and a single, unary relation symbol P, there is a unique finitary logic $\mathcal{S}$ over $\mathcal{L}$ whose translation $\mathbf{tr}(\mathcal{S})$ coincides with Σ. This correspondence shows that the theory of logical matrices for finitary logics can be identified with the model theory of the equality-free universal strict Horn theories of the first-order languages whose language contains, besides function symbols, exactly one unary relation symbol. Thus results from the theory of *implicative classes* (sometimes also called *quasivarieties*, extending the usual universal algebraic term) can be applied to the theory of logical matrices. For example, an easy adaptation of the ultraproduct proof of the compactness theorem for first-order logic gives:

Theorem 3.1 *A logic $\mathcal{S}$ is finitary iff the class of all its matrix models $\mathbf{Mod}\,\mathcal{S}$ is closed under ultraproducts.*

This result, due to Bloom [31], is an example of a so-called bridge theorem, a theorem that relates a metalogical property with an "algebraic" one. In this case the correspondence is between a metalogical property of $\mathcal{S}$ and a property of its model class $\mathbf{Mod}\,\mathcal{S}$. The bridge theorems that are often of most interest for AAL relate metalogical properties of $\mathcal{S}$ to algebraic properties of its reduced model class $\mathbf{Mod}^{*}\mathcal{S}$ or, of even more interest, to algebraic properties of $\mathbf{Alg}\,\mathcal{S}$ or $\mathbf{Alg}^{*}\mathcal{S}$. We discuss several results of this kind below.

A natural question in AAL is to find an algebraic characterization of the classes of matrices $\mathbf{Mod}\,\mathcal{S}_{\mathbf{M}}$ and $\mathbf{Mod}\,\mathcal{S}^{f}_{\mathbf{M}}$, where $\mathcal{S}_{\mathbf{M}}$ is the logic defined by an arbitrary class of matrices $\mathbf{M}$, and $\mathcal{S}^{f}_{\mathbf{M}}$ is its finitary companion. By "algebraic" here one usually has in mind a characterization in terms of the natural algebraic "operators" that act on classes of matrices, in this case on $\mathbf{M}$, analogous to Birkhoff's and Mal'cev's algebraic characterizations respectively of the variety and quasivariety generated by a class of algebras. In the following result from [39] we use the operations $\mathbb{H}_S$ and

$\mathbb{H}_S^{-1}$ of taking images and inverse images respectively under strict surjective homomorphisms, the operation $\mathbb{S}$ of taking submatrices, and the operations $\mathbb{P}$, $\mathbb{P}_R$, $\mathbb{P}_{R\omega_1}$ and $\mathbb{P}_U$ of taking respectively, direct products, reduced products, reduced products by an ω_1-complete filter, and ultraproducts; all these constructions are assumed to include closure under isomorphisms. A ***trivial matrix*** is a matrix whose algebra is a trivial (one-element) algebra and whose designated element is the unique element of the algebra.

Theorem 3.2 *Let* $\mathbf{M}$ *be a class of matrices in a countable language containing a trivial matrix. Then:*

1. $\mathbf{Mod}\,\mathcal{S}^f_{\mathbf{M}} = \mathbb{H}_S^{-1}\,\mathbb{H}_S\,\mathbb{S}\,\mathbb{P}_R(\mathbf{M}) = \mathbb{H}_S^{-1}\,\mathbb{H}_S\,\mathbb{S}\,\mathbb{P}\,\mathbb{P}_U(\mathbf{M})$.
2. $\mathbf{Mod}\,\mathcal{S}_{\mathbf{M}} = \mathbb{H}_S^{-1}\,\mathbb{H}_S\,\mathbb{S}\,\mathbb{P}_{R\omega_1}(\mathbf{M})$.

The analogous problem of characterizing the class of reduced matrix models of the logics defined by a given class of reduced matrices is a more natural one for AAL, and was solved in [26] using related but less familiar class operators. For each operator $\mathbb{O} \in \{\mathbb{S}, \mathbb{P}, \mathbb{P}_R, \mathbb{P}_{R\omega_1}, \mathbb{P}_U\}$, one considers the operator $\mathbb{O}^*$ that, when applied to a class of matrices $\mathbf{M}$, gives the class of matrices isomorphic to the reduction of some member of the class $\mathbb{O}(\mathbf{M})$, i.e., the composition of $\mathbb{O}$ with the "reduction" operator plus closure under isomorphisms. Recall that the class of reduced matrix models of a logic $\mathcal{S}$ is denoted by $\mathbf{Mod}^*\mathcal{S}$. Then:

Theorem 3.3 *Let* $\mathbf{M}$ *be a class of reduced matrices in a countable language containing a trivial matrix. Then:*

1. $\mathbf{Mod}^*\mathcal{S}^f_{\mathbf{M}} = \mathbb{S}^*\mathbb{P}^*_R(\mathbf{M}) = \mathbb{S}^*\mathbb{P}^*\mathbb{P}^*_U(\mathbf{M})$.
2. $\mathbf{Mod}^*\mathcal{S}_{\mathbf{M}} = \mathbb{S}^*\mathbb{P}^*_{R\omega_1}(\mathbf{M})$.

Other typical problems addressed in this area have to do with isolating natural subclasses of $\mathbf{M}$ that define the same (finitary) logic as $\mathbf{M}$. For example, by the matrix analogue of the well-known Birkhoff Subdirect Representation Theorem of universal algebra we have that the class of all subdirectly irreducible factors of the members of $\mathbf{M}$ defines the same finitary logic as $\mathbf{M}$. Another problem is the existence of a finite or recursive presentation of the logic $\mathcal{S}^f_{\mathbf{M}}$, in a suitable proof system either of the so-called Hilbert style or in the Gentzen style.

The close connection between the matrix model semantics of logics and the model theory of equality-free universal strict Horn theories also has ramifications in the other direction. In Section 4.3 we survey some work done in the model theory of equality-free logic that was directly inspired by recent work in AAL.

3.2 Protoalgebraic logics

For almost all logics considered in traditional algebraic logic the following property holds for each theory Σ and each pair of formulas φ and ψ.

$$\begin{gathered}\text{If for every formula } \delta \text{ with the variable } x,\ \delta(\varphi/x) \in \Sigma \text{ iff } \delta(\psi/x) \in \Sigma\,,\\ \text{then}\ \ \Sigma \cup \{\varphi\} \vdash_{\mathcal{S}} \psi \ \text{ and }\ \Sigma \cup \{\psi\} \vdash_{\mathcal{S}} \varphi.\end{gathered} \tag{3.4}$$

In general, two formulas φ and ψ that satisfy the premiss of (3.4) are called *Σ-indiscernible* [46]; in certain contexts and for the particular case where Σ is the set of theorems of $\mathcal{S}$ this relation has been called *synonymity* [126]. Condition (3.4) says that Σ-indiscernible formulas are also Σ-interderivable. The logics that satisfy it for all theories were called ***protoalgebraic*** by Blok and Pigozzi in their 1986 paper [21] and have been extensively studied. In [43] Czelakowski defined a class of logics by syntactical means that he called *non-pathological.* They were shown to be all protoalgebraic in [21], and, apart from some trivial examples, they included all known protoalgebraic logics. Later it was shown that these trivial examples are in fact the only protoalgebraic logics that did not meet Czelakowski's criterion. The protoalgebraic logics include almost all those mentioned in Section 1.1, and they constitute the main class of logics for which the advanced methods of universal algebra can be applied to their matrices to give strong and interesting results.

By comparing the premiss of (3.4) with the expression (3.1) we see that the relation of Σ-indiscernibility coincides with the Leibniz congruence relation $\boldsymbol{\Omega}\Sigma$; hence a logic is protoalgebraic iff $\boldsymbol{\Omega}\Sigma \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$ for each of its theories Σ. Since $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma$ is the largest congruence below $\boldsymbol{\Lambda}_{\mathcal{S}}\Sigma$, this inclusion holds iff $\boldsymbol{\Omega}\Sigma \subseteq \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma$; but by (3.2) it is always true that $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma \subseteq \boldsymbol{\Omega}\Sigma$. Therefore $\mathcal{S}$ is protoalgebraic iff $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\Sigma = \boldsymbol{\Omega}\Sigma$ for all $\Sigma \in \mathcal{T}h\,\mathcal{S}$. Hence *for protoalgebraic logics the Leibniz and the Suszko congruences coincide*, and each theory of a protoalgebraic logic contains all the information needed to determine its Suszko congruence, which is therefore an intrinsic property of the theory. As a consequence, the Lindenbaum-Tarski matrices are intrinsically characterized; this turns out to be a fundamental property of protoalgebraic logics. As an immediate consequence of the fact that Suszko and Leibniz congruences coincide we get

Theorem 3.4 *If $\mathcal{S}$ is a protoalgebraic logic then $\mathsf{Alg}\,\mathcal{S} = \mathsf{Alg}^*\mathcal{S}$.*

Before the notion of protoalgebraicity was isolated and for sometime after while it was thought that all interesting logics were protoalgebraic, it was generally agreed that the proper generalization of the Lindenbaum-Tarski process was reduction by the Leibniz congruence, and all attention was focused on the class of Leibniz-reduced models of a logic $\mathcal{S}$ and the corresponding class of algebra reducts $\mathsf{Alg}^*\mathcal{S}$. Since almost all logics then under study are actually protoalgebraic, in hindsight one understands that these were the proper notions to consider. However, beginning in 1991 a number of interesting nonprotoalgebraic logics have been identified and studied algebraically, and it has become clear that the right generalization of the Lindenbaum-Tarski process is reduction by what we now call the Suszko congruence (or, as we see in Section 5, by the Tarski congruence applied to generalized matrix models); moreover, the proper semantic classes to consider are the Suszko-reduced models of $\mathcal{S}$ and $\mathsf{Alg}\,\mathcal{S}$.

Among the nonprotoalgebraic logics that have been recently studied from the AAL perspective we find the conjunction-disjunction fragment of classical logic [80], the implication-less fragment of intuitionistic logic [23], positive modal logic [96], Belnap's four-valued logic [70], the weak version of system $\mathcal{R}$ of relevance logic [78], and some subintuitionistic logics [36]. In some of these cases (but not all: observe that Theorem 3.4 is not an equivalence) the classes $\mathsf{Alg}\,\mathcal{S}$ and $\mathsf{Alg}^*\mathcal{S}$ are different, and it is the larger one $\mathsf{Alg}\,\mathcal{S}$ that is most naturally associated with the logic. This confirms the choice of the Suszko congruence as the proper generalization of the Tarski-Lindenbaum congruence. Research on the general theory of nonprotoalgebraic logics has only recently begun. [47] and [73] are to our knowledge the only published works with interesting general results on logics of this type. Nevertheless there are several works in progress dealing with nonprotoalgebraic logics, and any truly general theory of AAL has to encompass them.

The feature of protoalgebraic logics that seems most accountable for the richness of their algebraic theory is the way so many metalogical properties, as reflected in the algebraic properties of the Lindenbaum matrices, transfer to arbitrary matrix models, in particular to Leibniz-reduced matrix models and their algebra reducts. The following result by Czelakowski and Pigozzi, see [46], is one *transfer theorem* of this kind.

Theorem 3.5 *Let $\mathcal{S}$ be a finitary protoalgebraic logic. Any property expressible by a universal sentence of elementary lattice theory holds in the lattice $\mathcal{T}h\,\mathcal{S}$ of all $\mathcal{S}$-theories iff it holds in all lattices $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ of $\mathcal{S}$-filters on arbitrary algebras $\boldsymbol{A}$.*

One can obtain as particular cases of this theorem a number of classical transfer theorems. For example, if the lattice of theories of a logic $\mathcal{S}$ is distributive, then so is the lattice $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$ [42].

Another important feature of protoalgebraicity is its distinctive protean nature; it can be characterized in many different and often surprising ways. We consider one of the more interesting ones now that takes the form of a generalization of the familiar *deduction theorem* of classical logic. In the following definition Fm^ω denotes the set of assignments of formulas over $\mathcal{L}$ to all the variables.

A logic $\mathcal{S}$ has the ***parameterized local deduction-detachment theorem*** (***DDT***) if there is a family of sets of formulas $\mathcal{E} \subseteq \mathcal{P}(Fm)$ such that for all $\Gamma \subseteq Fm$ and all $\varphi, \psi \in Fm$,

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \big(\exists\, \Delta(x, y, \vec{z}) \in \mathcal{E}\big)\ \big(\exists\, \vec{\gamma} \in Fm^\omega\big)\ \big(\forall\, \delta \in \Delta(\varphi, \psi, \vec{\gamma})\big)\ \Gamma \vdash_{\mathcal{S}} \delta.$$

The following result can be found in [48].

Theorem 3.6 *A logic $\mathcal{S}$ is protoalgebraic iff it has the parameterized local DDT.*

Thus all protoalgebraic logics possess this very weak form of the familiar deduction theorem.

Transfer theorems become bridge theorems when the property of the class of matrix models that the given metalogical property transfers to turns out to be equivalent to a natural algebraic property in its own right. The natural abstraction of the classical deduction theorem, and its localized version, given in the following two definitions are two important metalogical properties that have bridges to properties of the matrix models of this kind.

$\mathcal{S}$ has the ***local deduction-detachment theorem*** if it has the parameterized local DDT with an empty set of parameters, more precisely, if there is a family of sets of formulas $\mathcal{E} \subseteq \mathcal{P}(Fm)$ in two variables such that for all $\Gamma \subseteq Fm$ and all $\varphi, \psi \in Fm$,

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \big(\exists\, \Delta(x, y) \in \mathcal{E}\big)\ \big(\forall\, \delta \in \Delta(\varphi, \psi)\big)\ \Gamma \vdash_{\mathcal{S}} \delta.$$

Finally, $\mathcal{S}$ has the ***deduction-detachment theorem***, or for emphasis the ***global deduc-tion-detachment theorem***, if it has the local DDT such that the set $\mathcal{E}$ consists of a single finite set of formulas, i.e., if there is a finite set $\Delta(x, y)$ of formulas in two variables such that for all $\Gamma \subseteq Fm$ and all $\varphi, \psi \in Fm$,

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \big(\forall\, \delta \in \Delta(\varphi, \psi)\big)\ \Gamma \vdash_{\mathcal{S}} \delta.$$

If Δ can be taken to be unitary, it is said that the logic has the ***uniterm DDT***; the general case is referred to simply as the ***DDT*** or, for emphasis if needed, as the ***multiterm DDT***.

In the presence of protoalgebraicity equivalences can be established respectively between the (non-parameterized) local and global deduction-detachment theorems and analogues for matrices of the congruence-extension and equationally definable principal congruence properties of universal algebra.

An $\mathcal{L}$-matrix $\langle \boldsymbol{A}, F\rangle$ is a ***submatrix*** of an $\mathcal{L}$-matrix $\langle \boldsymbol{B}, G\rangle$, in symbols $\langle \boldsymbol{A}, F\rangle \subseteq \langle \boldsymbol{B}, G\rangle$, if $\boldsymbol{A} \subseteq \boldsymbol{B}$ (i.e., $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$) and $G \cap A = F$. A class M of matrix models of a deductive system $\mathcal{S}$ is said to have the ***$\mathcal{S}$-filter-extension-property*** if, for all $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{B}, G\rangle \in \mathsf{M}$ such that $\langle \boldsymbol{A}, F\rangle \subseteq \langle \boldsymbol{B}, G\rangle$, and every $F' \in Fi_{\mathcal{S}}\, \boldsymbol{A}$ such that $F \subseteq F'$ and $\langle \boldsymbol{A}, F'\rangle \in \mathsf{M}$, there exists a $G' \in Fi_{\mathcal{S}}\, \boldsymbol{B}$ such that $G \subseteq G'$, $\langle \boldsymbol{B}, G'\rangle \in \mathsf{M}$, and $G' \cap A = F'$.

Theorem 3.7 ([24, 44]) *Let $\mathcal{S}$ be a finitary protoalgebraic logic. Then the following statements are equivalent.*

1. *$\mathcal{S}$ has the local deduction-detachment theorem.*
2. *The class $\mathbf{Mod}\,\mathcal{S}$ has the $\mathcal{S}$-filter-extension property.*
3. *The class $\mathbf{Mod}^*\mathcal{S}$ has the $\mathcal{S}$-filter-extension property.*

A class M of matrix models of a deductive system $\mathcal{S}$ is said to have ***formula-definable principal $\mathcal{S}$-filters*** if there is a finite set $\Delta(x, y) = \{\, \delta_i(x, y) : i < n \,\}$ of formulas in two variables such that, for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ and every $a \in A$,

$$Fi_{\mathcal{S}}^{\boldsymbol{A}}\big(F \cup \{a\}\big) = \{\, b \in A : \forall \delta \in \Delta,\ \delta^{\boldsymbol{A}}(a, b) \in F \,\}.$$

Theorem 3.8 ([24]) *Let $\mathcal{S}$ be a finitary protoalgebraic logic. Then the following statements are equivalent.*

1. *$\mathcal{S}$ has the deduction-detachment theorem.*
2. *The class $\mathsf{Mod}\,\mathcal{S}$ has formula-definable principal $\mathcal{S}$-filters.*
3. *The class $\mathsf{Mod}^*\mathcal{S}$ has formula-definable principal $\mathcal{S}$-filters.*

The property of having formula-definable principal $\mathcal{S}$-filters, and hence by extension the DDT, has a purely lattice-theoretic characterization, which we now describe.

A (***dual***) ***Brouwerian semilattice*** is an algebra $\boldsymbol{A} = \langle A, *^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ such that $\langle A, \vee^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a bounded (join-) semilattice, and, for $a, b \in A$, $a *^{\boldsymbol{A}} b$ is the smallest element c (with respect to the semilattice order) such that $a \leq b \vee^{\boldsymbol{A}} c$. Thus $*^{\boldsymbol{A}}$ is a binary operation with the property that, for all $a, b, c \in A$,

$$a *^{\boldsymbol{A}} b \leq c \quad \text{iff} \quad a \leq b \vee^{\boldsymbol{A}} c.$$

The operation $*^{\boldsymbol{A}}$ is called (***dual***) ***relative pseudo-complementation***. Although it is not immediately obvious, the class of Brouwerian algebras can be defined by identities alone and thus forms a variety.

For a finitary protoalgebraic logic $\mathcal{S}$, $\mathsf{Mod}\,\mathcal{S}$ has formula-definable principal $\mathcal{S}$-filters iff the join-semilattice of the finitely axiomatizable theories of $\mathcal{S}$ is dually Brouwerian iff, for every $\boldsymbol{A}$, the join-semilattice of the finitely generated $\mathcal{S}$-filters of $\boldsymbol{A}$ is dually Brouwerian.

Combining this result with Theorem 3.8 we get:

Theorem 3.9 ([43]) *Let $\mathcal{S}$ be a finitary protoalgebraic logic. Then the following statements are equivalent.*

1. *$\mathcal{S}$ has the deduction-detachment theorem.*
2. *The join-semilattice of the finitely axiomatizable theories of $\mathcal{S}$ is dually Brouwerian.*
3. *For every $\boldsymbol{A}$, the join-semilattice of the finitely generated $\mathcal{S}$-filters of $\boldsymbol{A}$ is dually Brouwerian.*

This theorem combines a bridge theorem (the equivalence between 1 and 3) and a transfer theorem (the equivalence between 2 and 3). A brief discussion of some elements of this theorem can be found in [29].

3.3 Algebraizable logics

In the 1989 monograph by Blok and Pigozzi *Algebraizable Logics* [23] the concept of algebraizable logic was given a mathematically precise definition for the first time. The idea underlying the definition is the following: a logic is algebraizable if there exists a class of algebras related to the logic in the same way as the class of Boolean algebras is related to classical propositional logic. This relation can be expressed in more than one form. Blok and Pigozzi chose the following one.

Given a class of algebras K of algebraic similarity type $\mathcal{L}$, the ***equational consequence*** associated with K is the relation $\models_{\mathsf{K}}$ between a set of equations $\Gamma \approx \Delta = \{\gamma_i \approx \delta_i : i \in I\}$ and a single equation equation $\varphi \approx \psi$ of type $\mathcal{L}$ defined by:

$$\begin{aligned} \Gamma \approx \Delta \models_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad & \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}), \\ & \text{if } \ h(\gamma_i) = h(\delta_i) \text{ for every } i \in I, \text{ then } h(\varphi) = h(\psi). \end{aligned}$$

$\models_{\mathsf{K}}$ is a substitution-invariant consequence relation in the sense of conditions (C1'), (C2'), and (C6') of Section 1.1, but with obvious changes in the later due to the fact that it is on the set of $\mathcal{L}$-equations rather than $\mathcal{L}$-formulas (see Section 4.1 below).

Let $\mathcal{S}$ be a logic over the language $\mathcal{L}$. Let K be a class of algebras of type $\mathcal{L}$. The formulas of type $\mathcal{L}$ and the terms of the algebraic language of type $\mathcal{L}$ are the same objects (or, if one prefers, can be identified). This is crucial in connecting the consequence relation of the logic $\mathcal{S}$ with the equational consequence associated with K, and, more generally, in establishing the connection between the properties of $\mathcal{S}$ and K. A set of $\mathcal{L}$-equations $K(x){\approx}\Lambda(x) = \{\kappa_j(x) \approx \lambda_j(x) : j \in J\}$ in at most one variable is said to be a ***faithful interpretation of*** $\mathcal{S}$ ***in*** $\models_{\mathsf{K}}$ if for every $\Gamma \subseteq Fm$ and every $\varphi \in Fm$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad K(\Gamma){\approx}\Lambda(\Gamma) \models_{\mathsf{K}} K(\varphi){\approx}\Lambda(\varphi), \tag{3.5}$$

where $K(\Gamma){\approx}\Lambda(\Gamma) = \{\kappa_j(\psi) \approx \lambda_j(\psi) : j \in J, \psi \in \Gamma\}$ and $K(\varphi){\approx}\Lambda(\varphi) = \{\kappa_j(\varphi) \approx \lambda_j(\varphi) : j \in J\}$. When there is a faithful interpretation of a logic $\mathcal{S}$ in the equational consequence $\models_{\mathsf{K}}$ associated with a class of algebras K, the class is called an ***algebraic semantics*** for the logic $\mathcal{S}$.

A set of formulas $E(x, y) = \{\varepsilon_i(x, y) : i \in I\}$ in at most two variables is said to be a ***faithful interpretation of*** $\models_{\mathsf{K}}$ ***in*** $\mathcal{S}$ if for every set of equations $\Gamma{\approx}\Delta$ and every equation $\varphi \approx \psi$ we have

$$\Gamma{\approx}\Delta \models_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad E(\Gamma, \Delta) \vdash_{\mathcal{S}} E(\varphi, \psi), \tag{3.6}$$

where $E(\Gamma, \Delta) = \{\varepsilon_i(\gamma, \delta) : \gamma \approx \delta \in \Gamma{\approx}\Delta, i \in I\}$ and $E(\varphi, \psi) = \{\varepsilon_i(\varphi, \psi) : i \in I\}$. The two interpretations are said to be *mutually inverse* if

$$x \dashv\vdash_{\mathcal{S}} E\big(K(x), \Lambda(x)\big) \qquad \text{and} \qquad x \approx y \mathrel{=\!\mid\!\models_{\mathsf{K}}} K\big(E(x, y)\big){\approx}\Lambda\big(E(x, y)\big), \tag{3.7}$$

where $x \dashv\vdash_{\mathcal{S}} E\big(K(x), \Lambda(x)\big)$ is the conjunction of the entailments $x \vdash_{\mathcal{S}} E\big(K(x), \Lambda(x)\big)$ and $E\big(K(x), \Lambda(x)\big) \vdash_{\mathcal{S}} x$, and similarly for the other part of (3.7). A logic $\mathcal{S}$ over a language $\mathcal{L}$ is ***algebraizable*** if there is a class of algebras K of type $\mathcal{L}$ and a pair of faithful interpretations $E(x, y)$ and $K(x){\approx}\Lambda(x)$, respectively, of $\models_{\mathsf{K}}$ in $\mathcal{S}$ and conversely that are mutually inverse.

If the interpretations $E(x, y)$ and $K(x){\approx}\Lambda(x)$ are finite sets it is said that the logic is ***finitely algebraizable***. The notion of algebraizable logic introduced by Blok and Pigozzi in [23] is actually what is now called finitely algebraizable logic; moreover they considered exclusively finitary logics. They did not consider the wider notion. It and its extension to possibly nonfinitary consequences were considered by Herrmann [89, 90, 91] and by Czelakowski [45]. Examples of algebraizable logics that are not finitely algebraizable can be found in [91, 99].

If $E(x, y)$ is a faithful interpretation of the equational consequence relation of a class K of algebras in $\mathcal{S}$, then it is not difficult to see that the condition (1.8) defines, for each $\mathcal{S}$-theory Σ a congruence relation $\equiv_\Sigma$ on the formula algebra satisfying (1.7). So every algebraizable logic $\mathcal{S}$ is equivalential and hence protoalgebraic, and thus $\mathsf{Alg}\,\mathcal{S} = \mathsf{Alg}^*\mathcal{S}$.

If $\mathcal{S}$ is an algebraizable logic, then $\mathsf{Alg}\,\mathcal{S}$ is largest class K of algebras such that there exists mutually inverse faithful interpretations between $\mathcal{S}$ and $\models_{\mathsf{K}}$. It is called the ***equivalent algebraic semantics*** of $\mathcal{S}$. If $\mathcal{S}$ is finitely algebraizable then $\mathsf{Alg}\,\mathcal{S}$ is always a *quasivariety*, and in fact this property characterizes finitely algebraizable logics.

It turns out that to verify that $\mathcal{S}$ is algebraizable with equivalent algebraic semantics K it suffices only to show that there exists a faithful interpretation $K(x){\approx}\Lambda(x)$ of $\mathcal{S}$ in the equational consequence relation of K, and a set of formulas $E(x, y)$ in two variables such that $x \dashv\vdash_{\mathcal{S}} E\big(K(x), \Lambda(x)\big)$, i.e., it suffices to verify only (3.5) and the first equivalence of (3.7), because (3.6) and the second equivalence of (3.7) are easily shown to be a consequence of these two conditions. Symmetrically, it also suffices only to verify (3.6) and the second equivalence of (3.7).

If $\mathsf{Alg}\,\mathcal{S}$ is a *variety* then one says that $\mathcal{S}$ is ***strongly*** (***finitely***) ***algebraizable***. Most of the best known logics are finitely algebraizable, and most of these are strongly algebraizable. Rasiowa's implicative logics, described in Section 1.3, are all finitely algebraizable. The interpretations are defined by the set of formulas $E(x, y) = \{x \to y\,, y \to x\}$ and the set of equations $K(x){\approx}\Lambda(x) = \{x \approx x \to x\}$. Since $(x \to x)^{\boldsymbol{A}} = 1$ in all $\boldsymbol{A} \in \mathsf{Alg}\,\mathcal{S}$, condition (3.5) becomes, informally,

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathsf{Alg}\,\mathcal{S}} \varphi \approx 1, \tag{3.8}$$

which is the familiar completeness/soundness theorem for $\mathcal{S}$ relative to the class of matrices $\big\{\langle \boldsymbol{A}\,, \{1\}\rangle : \boldsymbol{A} \in \mathsf{Alg}\,\mathcal{S}\big\}$. Thus condition (3.5) in the definition of algebraizability is a generalization of a common kind of completeness/soundness condition (3.8), but the notion of algebraizability requires more, namely, the reverse "completeness/soundness" condition (3.6), and moreover, that the two interpretations be mutually inverse in the sense of (3.7).

The implicative logics include almost all the familiar logics of classical algebraic logic: the classical propositional calculus $\mathcal{CPC}$ and the intuitionistic propositional calculus $\mathcal{IPC}$, together with their various implication fragments, are all strongly, finitely algebraizable. The equivalent algebraic semantics of $\mathcal{CPC}$ and $\mathcal{IPC}$ are of course the varieties of Boolean and Heyting algebras, respectively. The equivalent algebraic semantics of each fragment of $\mathcal{CPC}$ or $\mathcal{IPC}$ that contains either $\to$ or $\leftrightarrow$ is the class of all subalgebras of the appropriate reducts of Boolean or Heyting algebras, respectively. In particular, the equivalent algebraic semantics of the $\{\wedge, \to\}$, the $\{\wedge, \leftrightarrow\}$, the $\{\to\}$, and the $\{\leftrightarrow\}$ fragments are called the varieties of *Brouwerian semilattices*, *Skolem semilattices*, *Hilbert algebras*, and *equivalential algebras*.

The two relevance logics $\mathcal{R}$ and $\mathcal{RM}$ are also algebraizable; in both cases the faithful interpretation of the logic in the equational logic of the equivalent algebraic semantics is defined by the same set of formulas $E(x\,,y)$ as for implicative logics, and the interpretation in the other direction is given by the set of equations $\big\{x \wedge (x \to x) \approx (x \to x)\big\}$. In the case of $\mathcal{R}$, the theory of algebraizability facilitated the identification of the class of $\mathcal{R}$-algebras $\mathbf{Alg}\,\mathcal{R}$ in [78]; this turns out to lie strictly between the class of De Morgan semigroups and the class of De Morgan monoids (the latter was formerly believed to be the proper algebraic counterpart of $\mathcal{R}$). All algebraizable logics are protoalgebraic, but there are many non-algebraizable protoalgebraic logics, such as the local consequences associated with normal modal logics (those which restrict the application of the necessitation rule to theorems).

One important line of investigation in traditional algebraic logic has been the connection between a particular metalogical property of a specific logic—which is almost always algebraizable—and an algebraic property of its associated class of algebras; these are the most typical bridge results. Once a connection is established results from one domain can be translated to the other. Historically, the first results along this line had to do with the equivalence between metalogical "interpolation" properties on one hand and algebraic "amalgamation" properties on the other. The specific logics considered were first-order predicate logic, the logics intermediate between classical and intuitionistic propositional logic, and modal logics, and the corresponding classes of algebras were respectively cylindric (or polyadic) algebras, subvarieties of Heyting algebras, and varieties of modal algebras. Later the connection between properties of definability, related to the Beth definability theorem, and the property that epimorphisms (in the category-theoretic sense) are surjective were studied. These investigations were for the most part ad hoc and not part of a general theory of connections of this kind, although historically their scope did become progressively wider. It was not until the concept of an algebraizable logic became available that the stage was set for the development of a general theory of the bridge theorems. A new connection of this kind that AAL has brought to light is that between the deduction theorem and the algebraic property of an algebra having its principal congruences equationally definable. Indeed, the main motivation underlying the theory of algebraizable logics developed by Blok and Pigozzi was the need to establish a general context in which this connection can be stated in a sharp, mathematically precise way. The goal, besides clarifying the concept of algebraization, was to apply the large body of results in universal algebra on the definability of principal congruences to the problem of the existence, and more importantly the nonexistence, of a deduction theorem for a wide class of logics. The first significant bridge theorem establishing the connection between a metalogical and algebraic property that was obtained in the context of algebraizable logics (due to Blok and Pigozzi [29]) is explained in the following.

Recall that a logic $\mathcal{S}$ has the (*multiterm global*) *deduction-detachment theorem* (the *DDT*) if there is a finite set $\Delta(x,y)$ of formulas in two variables such that, for every $\Gamma \subseteq Fm$ and all $\varphi, \psi \in Fm$,

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Sigma \vdash_{\mathcal{S}} \Delta(\varphi, \psi), \tag{3.9}$$

where $\Delta(\varphi,\psi) = \{\delta(\varphi,\psi) : \delta(x,y) \in \Delta(x,y)\}$. The set Δ is called a *deduction-detachment set* for $\mathcal{S}$.

Recall also that Theorem 3.8 provides a bridge between the DDT for a protoalgebraic logic and a "quasi-algebraic" property of its classes of matrix models and reduced matrix models (the filter-extension property). For algebraizable logics the bridge can be extended to the relative congruence-extension property of the equivalent algebraic semantics.

A quasivariety K has ***equationally definable principal relative congruences*** (***EDPRC***) if there is a finite set of equations in at most four variables $\{\varepsilon_i(x_0, x_1, y_0, y_1) \approx \delta_i(x_0, x_1, y_0, y_1) : i \leq n\}$ such that for every algebra $\boldsymbol{A} \in \mathsf{K}$ and all $a, b, c, d \in A$,

$$\langle c, d\rangle \in \Theta^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \quad \text{iff} \quad \forall i \leq n \quad \varepsilon^{\boldsymbol{A}}_i(a, b, c, d) = \delta^{\boldsymbol{A}}_i(a, b, c, d),$$

where $\Theta^{\boldsymbol{A}}_{\mathsf{K}}(a, b)$ is the smallest congruence θ of $\boldsymbol{A}$ such that $\langle a, b\rangle \in \theta$ and $\boldsymbol{A}/\theta \in \mathsf{K}$.

Theorem 3.10 ([29]) *If $\mathcal{S}$ is a finitary and finitely algebraizable logic, then $\mathcal{S}$ has a DDT iff its equivalent algebraic semantics, i.e., the quasivariety $\mathsf{Alg}\,\mathcal{S}$, has* EDPRC*.*

We now present a sampling of the large number of bridge results that exist dealing with definability and interpolation results in AAL.

Let $\mathcal{S}$ be a logic and let P and R be disjoint sets of variables. Let $\Gamma = \Gamma(\overline{p}, \overline{r})$ be a set of formulas with its variables either in P (those in the sequence $\overline{p}$) or in R (those in the sequence $\overline{r}$). We say that $\Gamma(\overline{p}, \overline{r})$ ***defines*** R ***explicitly*** in terms of P if for every $r \in R$ there is a $\varphi_r \in Fm$ with variables in P such that $\langle r, \varphi_r\rangle \in \boldsymbol{\Omega}\big(Fi^{(P\cup R)}_{\mathcal{S}}(\Gamma)\big)$, where $Fi^{(P\cup R)}_{\mathcal{S}}$ denotes $\mathcal{S}$-filter generation in the subalgebra of $\boldsymbol{Fm}$ generated by $P \cup R$. We say that $\Gamma(\overline{p}, \overline{r})$ ***defines*** R ***implicitly*** in terms of P if for every set of variables R' disjoint from R and P of the same cardinality as R, it is the case that, for every bijection f between R and R', if $r' = f(r)$ for $r \in R$, then for every $r \in R$, $\langle r, r'\rangle \in \boldsymbol{\Omega}\big(Fi^{(P\cup R\cup R')}_{\mathcal{S}}(\Gamma(\overline{p}, \overline{r}) \cup \Gamma(\overline{p}, \overline{r'}))\big)$, where $Fi^{(P\cup R\cup R')}_{\mathcal{S}}$ denotes $\mathcal{S}$-filter generation in the subalgebra of $\boldsymbol{Fm}$ generated by $P \cup R \cup R'$. A logic $\mathcal{S}$ has the ***Beth property*** if for all disjoint sets of variables P and R, each set of formulas $\Gamma(\overline{p}, \overline{r})$ that defines R implicitly in terms of P, defines also R explicitly in terms of P.

A class of algebras K of the same similarity type has the property that the epimorphisms are surjective, the ***property ES*** for short, if for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, every epimorphism $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ is surjective. Recall that an algebraic homomorphism $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ is an ***epimorphism*** (in the category-theoretic sense) in K when for every $\boldsymbol{C} \in \mathsf{K}$ and all $g, g' \in \mathrm{Hom}(\boldsymbol{B}, \boldsymbol{C})$, if $g \circ h = g' \circ h$ then $g = g'$.

Theorem 3.11 *If $\mathcal{S}$ is an algebraizable logic, then $\mathcal{S}$ has the Beth property iff $\mathsf{Alg}\,\mathcal{S}$ has the property ES.*

This theorem was first obtained by Hoogland [92] for the semantics-based framework of Section 6.1. Its present form and its extension to equivalential logics are joint work with Blok, see [93]. The study of algebraic forms of Beth-like properties goes back to 1982 results by Németi in the context of cylindric algebras, see [88, Theorem 5.6.10].

A logic $\mathcal{S}$ has the ***Craig interpolation property for consequence*** if for every $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$, there is $\Gamma' \subseteq Fm$ with variables in $\mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi)$ such that $\Gamma \vdash_{\mathcal{S}} \Gamma'$ and $\Gamma' \vdash_{\mathcal{S}} \varphi$. A class of algebras K of the same similarity type has ***the amalgamation property*** if for all $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathsf{K}$ and all (isomorphic) embeddings $f : \boldsymbol{C} \to \boldsymbol{A}$ and $g : \boldsymbol{C} \to \boldsymbol{B}$ there is a $\boldsymbol{D} \in \mathsf{K}$ and (isomorphic) embeddings $h : \boldsymbol{A} \to \boldsymbol{D}$ and $t : \boldsymbol{B} \to \boldsymbol{D}$ such that $h \circ f = t \circ g$.

Theorem 3.12 *If $\mathcal{S}$ is an algebraizable logic with the* DDT*, then $\mathcal{S}$ has the Craig interpolation property for consequence iff $\mathsf{Alg}\,\mathcal{S}$ has the amalgamation property.*

This particular theorem is due essentially to Czelakowski [41]. In [52] there are many other versions for equivalential and algebraizable logics due to Czelakowski and Pigozzi, using matrices instead of algebras; this paper also contains many references to the extensive literature on the relationship between interpolation properties (for logics) and amalgamation properties (for classes of algebras or of matrices).

3.4 The Leibniz hierarchy

Neither the protoalgebraic nor the algebraizable logics were defined explicitly in terms of the Leibniz operator, but it turns out that they can be characterized this way. More precisely, they can be characterized in terms of the way in which the Leibniz congruence of a filter of the logic can be explicitly defined in terms of the filter and vice versa. This gives a informative way of classifying logics $\mathcal{S}$ by the degree to which they can be faithfully represented by the equational logic of the $\mathcal{S}$-algebras.

For an algebraizable logic $\mathcal{S}$ the following holds. Let $E(x, y)$ and $K(x)\approx\Lambda(x)$ be respectively the sets of formulas and of equations that define the mutually inverse faithful interpretations involved in the definition of algebraizability. Let $\langle \boldsymbol{A}, F\rangle$ be a model of $\mathcal{S}$ and let $\equiv_F$ be the relation on A defined by the condition that $a \equiv_F b$ iff, for every $\varepsilon(x, y) \in E(x, y)$, $\varepsilon^{\boldsymbol{A}}(a, b) \in F$. It is not difficult to see that $\equiv_F$ is a congruence relation on $\boldsymbol{A}$ that is compatible with F (the special case for Lindenbaum models was observed earlier). Thus $\equiv_F \subseteq \boldsymbol{\Omega_A} F$ by the definition of the Leibniz congruence. For the reverse inclusion, assume $\langle a, b\rangle \in \boldsymbol{\Omega_A} F$. Then, for each $\varepsilon(x, y) \in E(x, y)$, $\left\langle \varepsilon^{\boldsymbol{A}}(a, a)\,, \varepsilon^{\boldsymbol{A}}(a, b)\right\rangle \in \boldsymbol{\Omega_A} F$, and hence $\varepsilon^{\boldsymbol{A}}(a, b) \in \boldsymbol{\Omega_A} F$ by the compatibility of $\boldsymbol{\Omega_A} F$ with F and the fact that $\varepsilon^{\boldsymbol{A}}(a, a) \in F$ since $a \equiv_F a$. Thus, for every model $\langle \boldsymbol{A}, F\rangle$ of $\mathcal{S}$ and all $a, b \in A$,

$$\langle a, b\rangle \in \boldsymbol{\Omega_A} F \quad \text{iff} \quad E^{\boldsymbol{A}}(a, b) \subseteq F, \tag{3.10}$$

where $E^{\boldsymbol{A}}(a, b) = \{\varepsilon^{\boldsymbol{A}}(a, b) : \varepsilon(x, y) \in E(x, y)\}$. That is, the Leibniz congruence is defined by the set $E(x, y)$. Moreover, if $\langle \boldsymbol{A}, F\rangle$ is a reduced model of $\mathcal{S}$, then

$$F = \left\{a \in A : \kappa^{\boldsymbol{A}}(a) = \lambda^{\boldsymbol{A}}(a)\,,\ \forall\, \kappa(x) \approx \lambda(x) \in K(x)\approx\Lambda(x)\right\}; \tag{3.11}$$

that is, the filter is defined by the equations in $K(x)\approx\Lambda(x)$.

In general, we say that a set of formulas $E(x, y)$ in at most two variables ***defines the Leibniz congruence*** of a matrix $\langle \boldsymbol{A}, F\rangle$ when (3.10) holds for all $a, b \in A$, and we say that a set of equations $K(x)\approx\Lambda(x)$ in at most one variable ***defines the filter*** of a matrix $\langle \boldsymbol{A}, F\rangle$ when (3.11) holds. The algebraizable logics can be characterized as the logics for which there is a set of formulas $E(x, y)$ and a set of equations $K(x)\approx\Lambda(x)$ that define, respectively, the Leibniz congruence of each model of the logic and the filter of each reduced model. If the set $E(x, y)$ is finite we have the finitely algebraizable logics.

Each of these two definability conditions, taken separately, characterizes an interesting type of protoalgebraic logic. The ***equivalential logics*** (already mentioned in Section 1.3) are the logics which have a set of formulas $E(x, y)$ that defines the Leibniz congruences of their models. The ***weakly algebraizable logics*** are the protoalgebraic logics that have a set of equations $K(x)\approx\Lambda(x)$ that defines the filters of their reduced models; these logics have been studied in [50, 73]. Among the equivalential logics there are the ***finitely equivalential logics***. They are the equivalential logics with a finite set $E(x, y)$ of equivalence formulas. These three classes of logics together with the algebraizable logics constitute the principal levels of the so-called *Leibniz* or *algebraic hierarchy* of protoalgebraic logics.

The Leibniz congruences of models of arbitrary protoalgebraic logics are also definable, albeit in a much weaker sense than for the logics higher up in the Leibniz hierarchy. But, as in the case of the higher level logics, the condition does serve to characterize protoalgebraic logics. Let $E(x, y, \overline{u})$ be a set of formulas where we fix two variables x and y and we consider the others, represented in the sequence $\overline{u}$, as parameters. It is said that $E(x, y, \overline{u})$ ***parametrically defines the Leibniz congruence*** of a matrix $\langle \boldsymbol{A}, F\rangle$ if, for every pair a, b of elements of A,

$$\langle a, b\rangle \in \boldsymbol{\Omega_A} F \quad \text{iff} \quad \forall\, \overline{c} \in A\,,\ E^{\boldsymbol{A}}(a, b, \overline{c}) \subseteq F.$$

The protoalgebraic logics are exactly those logics that have a set of formulas $E(x, y, \overline{u})$ that parametrically defines the Leibniz congruences of their models [26]; it is interesting to compare this with the characterization in terms of the parameterized local deduction theorem given in Theorem 3.6.

Another way of characterizing the various levels of the Leibniz hierarchy is by the degree to which the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ preserves the order structure of the lattice of $\mathcal{S}$-filters of an arbitrary algebra $\boldsymbol{A}$. For proofs of the various results collected in Theorems 3.13 to 3.16, and historical information about them, see [46].

Theorem 3.13 *Let $\mathcal{S}$ be a logic. Then:*

1. *$\mathcal{S}$ is protoalgebraic iff for every algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotone on the set of $\mathcal{S}$-filters of $\boldsymbol{A}$, i.e. if $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $F \subseteq G$, then $\boldsymbol{\Omega}_{\boldsymbol{A}}F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}G$.*
2. *$\mathcal{S}$ is equivalential iff $\mathcal{S}$ is protoalgebraic and $\boldsymbol{\Omega}_{\boldsymbol{A}}$ commutes with inverse images by homomorphisms, that is, if for all algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, every $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$, and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}h^{-1}[F] = h^{-1}[\boldsymbol{\Omega}_{\boldsymbol{B}}F]$.*
3. *$\mathcal{S}$ is finitely equivalential iff $\mathcal{S}$ is protoalgebraic and $\boldsymbol{\Omega}_{\boldsymbol{A}}$ commutes with unions of directed families of $\mathcal{S}$-filters whose union is an $\mathcal{S}$-filter, that is, if for every algebra $\boldsymbol{A}$ and every family $\mathcal{F} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ upwards directed by inclusion such that $\bigcup \mathcal{F} \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}\big(\bigcup \mathcal{F}\big) = \bigcup\{\boldsymbol{\Omega}_{\boldsymbol{A}}F : F \in \mathcal{F}\}$.*
4. *$\mathcal{S}$ is weakly algebraizable iff $\mathcal{S}$ is protoalgebraic and for every algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is injective on the family of the $\mathcal{S}$-filters of $\boldsymbol{A}$.*
5. *$\mathcal{S}$ is algebraizable iff $\mathcal{S}$ is equivalential and for every algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is injective on the family of the $\mathcal{S}$-filters of $\boldsymbol{A}$.*
6. *$\mathcal{S}$ is finitely algebraizable iff $\mathcal{S}$ is finitely equivalential and for every algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is injective on the family of the $\mathcal{S}$-filters of $\boldsymbol{A}$.*

As a corollary we have the following lattice isomorphism characterizations of algebraizable and weakly algebraizable logics. Recall that for an arbitrary algebra $\boldsymbol{A}$, an $(\mathsf{Alg}\,\mathcal{S})$-congruence of $\boldsymbol{A}$ is any $\theta \in \mathrm{Co}\,\boldsymbol{A}$ such that $\boldsymbol{A}/\theta \in \mathsf{Alg}\,\mathcal{S}$.

Corollary 3.14 *Let $\mathcal{S}$ be a logic. Then:*

1. *$\mathcal{S}$ is weakly algebraizable iff for every algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is an isomorphism between the lattice of $\mathcal{S}$-filters of $\boldsymbol{A}$ and the lattice of $(\mathsf{Alg}\,\mathcal{S})$-congruences of $\boldsymbol{A}$.*
2. *$\mathcal{S}$ is algebraizable iff for every algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is an isomorphism between the lattice of $\mathcal{S}$-filters of $\boldsymbol{A}$ and the lattice of $(\mathsf{Alg}\,\mathcal{S})$-congruences of $\boldsymbol{A}$ that commutes with inverse images by homomorphisms.*
3. *$\mathcal{S}$ is finitely algebraizable iff for every algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is an isomorphism between the lattice of $\mathcal{S}$-filters of $\boldsymbol{A}$ and the lattice of $(\mathsf{Alg}\,\mathcal{S})$-congruences of $\boldsymbol{A}$ that commutes with unions of directed families of $\mathcal{S}$-filters whose union is an $\mathcal{S}$-filter.*

Theorem 3.13 and its corollary are particularly useful in applications to specific logics or classes of algebras, especially for showing that a logic does *not* belong at a certain level of the hierarchy. For example, an effective way to show that a logic $\mathcal{S}$ is not weakly algebraizable is by finding an algebra and two distinct $\mathcal{S}$-filters with the same Leibniz congruence; the Leibniz congruences are easy to construct if the algebra is of small (finite) cardinality. This is done for several examples in Section 5.2 of [23]. Theorem 3.13 and its corollary can also be used to show that a given class of algebras cannot be the class of $\mathcal{S}$-algebras for any logic $\mathcal{S}$ at a specific level of the hierarchy by looking at the structure of the congruence lattice of selected members of the class. This is done for the class of distributive lattices in [80], for the class of pseudo-complemented lattices in [124], for the class of De Morgan algebras in [70], and for the class of positive modal algebras in [96]. Finally, the theorem and corollary constitute a starting point for a further abstraction of the notion of algebraizability that focuses on its *purely lattice theoretical* nature, such as [18]; see Section 4.1 and Theorem 6.3 in Section 6.2 below.

A third, quite different and more "model-theoretic", way of characterizing the different classes of logics in the Leibniz hierarchy uses closure properties of their classes of reduced models under the matrix-class operators considered in Section 3.1.

Theorem 3.15 *Let $\mathcal{S}$ be a logic. Then:*

1. *$\mathcal{S}$ is protoalgebraic iff $\mathsf{Mod}^*\,\mathcal{S}$ is closed under subdirect products.*
2. *$\mathcal{S}$ is equivalential iff $\mathsf{Mod}^*\,\mathcal{S}$ is closed under submatrices and direct products.*
3. *$\mathcal{S}$ is finitely equivalential iff $\mathsf{Mod}^*\,\mathcal{S}$ is closed under submatrices, direct products and ultraproducts, that is, is a quasivariety in the sense of Mal'cev.*
4. *$\mathcal{S}$ is weakly algebraizable iff it is protoalgebraic and for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\,\mathcal{S}$, F is the least $\mathcal{S}$-filter of $\boldsymbol{A}$.*
5. *$\mathcal{S}$ is algebraizable iff it is equivalential and for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\,\mathcal{S}$, F is the least $\mathcal{S}$-filter of $\boldsymbol{A}$.*
6. *$\mathcal{S}$ is finitely algebraizable iff it is finitely equivalential and for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\,\mathcal{S}$, F is the least $\mathcal{S}$-filter of $\boldsymbol{A}$.*

The given definition of algebraizable logic fails to be *intrinsic* in the sense that it requires a priori knowledge of a class K of algebra such that the logic and the consequence $\models_{\mathsf{K}}$ are mutually interpretable. But, as seen in Section 3.3, K can be taken to be the class of $\mathcal{S}$-algebras for the logic $\mathcal{S}$, and the $\mathcal{S}$-algebras can be defined strictly in terms of $\mathcal{S}$ via the Leibniz operator, so intrinsic charaterizations of algebraizability do exist. Theorem 3.13 provides a more direct intrinsic characterization of algebraizability in terms of the Leibniz operator, and similarly for the other classes of the Leibniz hierarchy.

Of special interest among the intrinsic characterizations are those that are "syntactical" in the sense that they refer only to intrinsic properties of the consequence relation $\vdash_{\mathcal{S}}$; these characterizations are the most useful for verifying where in the Leibniz hierarchy a specific logic actually lies. We consider a number of characterizations of this kind.

Let $E(x,y)$ be a set of formulas with at most two variables. It is said that $E(x,y)$ is a ***set of implication formulas*** for a logic $\mathcal{S}$ if

$$\vdash_{\mathcal{S}} E(x,x) \qquad \text{and} \qquad x, E(x,y) \vdash_{\mathcal{S}} y \quad (E\text{-Modus Ponens}).$$

An implication set for $\mathcal{S}$ is said to be a ***set of equivalence formulas*** if moreover

$$E(x,y) \vdash_{\mathcal{S}} E(y,x) \qquad , \qquad E(x,y) \cup E(y,z) \vdash_{\mathcal{S}} E(x,z) \qquad \text{and}$$

$$E(x_0,y_0) \cup \ldots \cup E(x_{n-1},y_{n-1}) \vdash_{\mathcal{S}} E(\lambda x_0 \ldots x_{n-1}, \lambda y_0 \ldots y_{n-1})$$

for every connective λ of the language of $\mathcal{S}$, where n is its arity. It turns out that a set of formulas $E(x,y)$ is a set of equivalence formulas for a logic $\mathcal{S}$ iff it defines the Leibniz congruences of the models of $\mathcal{S}$.

Theorem 3.16 *Let $\mathcal{S}$ be a logic. Then:*

1. *$\mathcal{S}$ is protoalgebraic iff it has a set of implication formulas.*
2. *$\mathcal{S}$ is equivalential iff it has a set of equivalence formulas.*
3. *$\mathcal{S}$ is finitely equivalential iff it has a finite set of equivalence formulas.*
4. *$\mathcal{S}$ is algebraizable iff it has a set of equivalence formulas $E(x,y)$ and a set of equations in one variable $K(x)\approx\varLambda(x)$ such that $x \dashv\vdash_{\mathcal{S}} E\big(K(x),\varLambda(x)\big)$.*
5. *$\mathcal{S}$ is finitely algebraizable iff it has a finite set of equivalence formulas $E(x,y)$ and a set of equations in one variable $K(x)\approx\varLambda(x)$ such that $x \dashv\vdash_{\mathcal{S}} E\big(K(x),\varLambda(x)\big)$.*

The verification of each of these equivalences is straightforward in view of the preceding characterizations. For example, the idea behind the proof of part 4 is easy to explain: Recall that $\mathcal{S}$ is algebraizable iff there exists a class K of algebras and a faithful interpretation $E(x,y)$ of the equational consequence relation of K in $\mathcal{S}$, and a set $K(x)\approx\varLambda(x)$ of equations in one variable such that $x \dashv\vdash_{\mathcal{S}} E\big(K(x), \Lambda(x)\big)$. But if $E(x, y)$ is a set of equivalence formulas, then as previously observed $E(x, y)$ defines Leibniz congruences, and it follows from the definition of $\mathsf{Alg}\,\mathcal{S}$ that $E(x, y)$ is a faithful interpretation of the equational consequence relation of $\mathsf{Alg}\,\mathcal{S}$ in $\mathcal{S}$. So taking K to be $\mathsf{Alg}\,\mathcal{S}$ we get one-half of the equivalence of 4; the other is straightforward.

Other relevant classes of logics have been considered in the hierarchy. Each of the classes of weakly algebraizable, algebraizable and finitely algebraizable logics contains a subclass of logics for which all the reduced models have a unitary designated set, i.e., such that, if $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{*}\mathcal{S}$, then $F = \{1\}$ for some element $1 \in A$. For these subclasses of logics the qualifying term ***regularly*** is added to the name denoting the particular class in the Leibniz hierarchy. In the algebraizable and finitely algebraizable cases this property can be easily expressed syntactically by the so-called Gödel's rule or ***G-rule***: the set of rules

$$\{x\,, y\} \ \vdash\ \varepsilon(x, y) \qquad \text{for all } \varepsilon \in E(x, y)$$

where $E(x, y)$ is any set of equivalence formulas for the logic. All implicative logics (see Section 1.3) are in fact regularly, finitely algebraizable, with $1 = (x \to x)^{\boldsymbol{A}}$ in all $\boldsymbol{A} \in \mathsf{Alg}\,\mathcal{S}$. See Section 4.4 for a discussion of the algebraic investigations motivated by this kind of algebraizability. In a similar manner the name of a class of logics is qualified by the term ***strongly*** to indicate that, for those logics $\mathcal{S}$ in the class, $\mathsf{Alg}\,\mathcal{S}$ is a variety. Finally, one can consider the interaction between the Leibniz hierarchy and the classification of logics according to Frege's principles, as discussed in Section 2.1, that is, with the classes of ***selfextensional*** and of ***Fregean*** logics. From what can be discerned from the known examples, it seems selfextensionality is a property that is independent of the different classes of the hierarchy, however this is not so in the case of the Fregean property. This is evident from the next result, obtained independently by Font and Jansana [73] and by Czelakowski and Pigozzi [46, 53].

Theorem 3.17

1. *Every protoalgebraic Fregean logic with theorems is regularly algebraizable.*
2. *Every finitary and protoalgebraic Fregean logic with theorems is regularly, finitely algebraizable.*
3. *Every finitary and protoalgebraic Fregean logic satisfying the uniterm DDT or having a conjunction and theorems is strongly and regularly, finitely algebraizable.*

One should not confuse the term "finitely" when referring to equivalential or algebraizable logics with the notion of a "finitary" logic. In fact, there are finitary logics that are regularly algebraizable but not even finitely equivalential [55]; obviously their equivalent algebraic semantics are not quasivarieties. Protoalgebraic Fregean logics are studied in more detail in [46, 53, 54].

Figure 1 shows the relative positions of some of the classes in the Leibniz hierarchy.

4 Extensions of the Core Theory

The core theory expounded in the previous section has been extended in several directions. Some of these extensions, e.g., k-deductive systems and Gentzen systems, are more-or-less straightforward, mathematically, but they represent a conceptual advancement; in particular the extension to Gentzen systems opened new perspectives on the study of the algebraization of logic in a broad sense. Others, like the extension of the methods of AAL to the study of equality-free logic, are, strictly speaking, beyond the scope of algebraic logic, but they show the power of some of its methods and concepts. We also discuss some recent research of a more purely algebraic character that focuses on properties that distinguish those classes of algebras that arise as the algebraic counterpart of a logic.

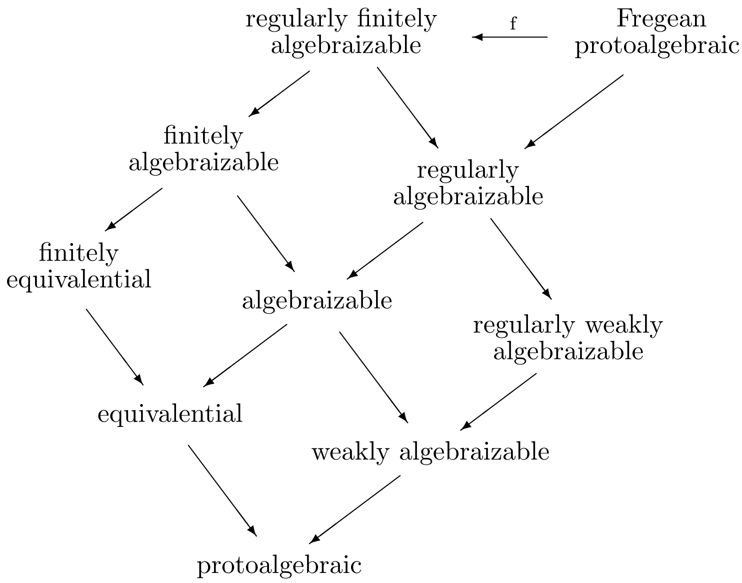

Figure 1: The main classes of logics in the Leibniz hierarchy. $\rightarrow$ means $\subseteq$, and the “f” superscript means “for finitary logics”.

4.1 k-deductive systems and universal algebra

The motivation behind the study of deductive systems of higher dimension is the desire to have a common framework for studying the logic of logical equivalence, and more generally equational logic, and the logic of assertion. Recall that a ***quasi-equation*** of an algebraic similarity type $\mathcal{L}$ is a formula of the form

$$t_0 \approx s_0 \wedge \ldots \wedge t_{n-1} \approx s_{n-1} \rightarrow t \approx s$$

where $t_0, s_0, \ldots, t_{n-1}, s_{n-1}, t, s$ are terms of $\mathcal{L}$, and $\wedge, \rightarrow$ are the logical connectives of first-order logic. A quasi-equation is valid in a class of algebras K, i.e., a ***quasi-identity*** of K, if its universal closure is true in every member of K. This is equivalent to saying that $\{t_0 \approx s_0, \ldots, t_{n-1} \approx s_{n-1}\} \models_{\mathsf{K}} t \approx s$, where $\models_{\mathsf{K}}$ is the relation of equational consequence associated with K defined in Section 3.3. The equations valid in a class of algebras K, i.e., the identities of K, are the consequences of the empty set of equations under $\models_{\mathsf{K}}$. Thus both the theories of identities and quasi-identities of a class of algebras K are encompassed in the study of the equational consequence relation $\models_{\mathsf{K}}$.

The equational consequence $\models_{\mathsf{K}}$ determined by K has the following properties, where Π and Φ are arbitrary sets of equations:

(E1) If $\varphi \approx \psi \in \Pi$, then $\Pi \models_{\mathsf{K}} \varphi \approx \psi$.

(E2) If for every $\varphi \approx \psi \in \Phi$, $\Pi \models_{\mathsf{K}} \varphi \approx \psi$ and $\Phi \models_{\mathsf{K}} \delta \approx \varepsilon$, then $\Pi \models_{\mathsf{K}} \delta \approx \varepsilon$.

(E3) If $\Pi \models_{\mathsf{K}} \varphi \approx \psi$, then for every substitution σ, $\sigma[\Pi] \models_{\mathsf{K}} \sigma(\varphi) \approx \sigma(\psi)$.

(E4) $\models_{\mathsf{K}} \varphi \approx \varphi$.

(E5) $\varphi \approx \psi \models_{\mathsf{K}} \psi \approx \varphi$.

(E6) $\varphi \approx \psi\, ,, \psi \approx \delta \models_{\mathsf{K}} \varphi \approx \delta$.

(E7) $\varphi_1 \approx \psi_1\, , \ldots, \varphi_n \approx \psi_n \models_{\mathsf{K}} f\varphi_1 \ldots \varphi_n \approx f\psi_1 \ldots \psi_n$, for every n-ary operation symbol f.

Moreover, every relation between sets of equations and equations which satisfies properties (E1)–(E7) is the equational consequence of some class of algebras. Observe that the first three conditions are analogous to the conditions (C1'), (C2') and (C6') in the definition of a logic. This suggests generalizing the definition of logic by replacing the formulas by ordered pairs of formulas, which can be identified with equations. A 2-***dimensional deductive system*** (or 2-***deductive system*** for short) $\mathcal{S}$ in a language $\mathcal{L}$ is a pair $\langle \boldsymbol{Fm}, \vdash^2_{\mathcal{S}} \rangle$ where $\boldsymbol{Fm}$ is the algebra of formulas of $\mathcal{L}$ and $\vdash^2_{\mathcal{S}}$ is a substitution-invariant consequence relation on the set Fm^2, that is, a relation between sets of pairs of formulas and pairs of formulas such that

(C^{2}1) $\langle \varphi, \psi \rangle \in \Pi$, then $\Pi \vdash^2_{\mathcal{S}} \langle \varphi, \psi \rangle$,

(C^{2}2) if for every $\langle \varphi, \psi \rangle \in \Phi$, $\Pi \vdash^2_{\mathcal{S}} \langle \varphi, \psi \rangle$ and $\Phi \vdash^2_{\mathcal{S}} \langle \delta, \varepsilon \rangle$, then $\Pi \vdash^2_{\mathcal{S}} \langle \delta, \varepsilon \rangle$, and

(C^{2}6) if $\Pi \vdash^2_{\mathcal{S}} \langle \varphi, \psi \rangle$, then for every substitution σ, $\sigma[\Pi] \vdash^2_{\mathcal{S}} \langle \sigma(\varphi), \sigma(\psi) \rangle$.

If in addition the following condition holds,

(C^{2}3) if $\Pi \vdash^2_{\mathcal{S}} \langle \varphi, \psi \rangle$, then there is a finite $\Pi' \subseteq \Pi$ such that $\Pi' \vdash^2_{\mathcal{S}} \langle \varphi, \psi \rangle$,

then the 2-deductive system is said to be ***finitary***.

Logical-matrix semantics can be generalized to 2-deductive systems in a natural way. A 2-***matrix*** is pair $\langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an algebra and $F \subseteq A \times A$. A 2-matrix is a ***model*** of a 2-deductive system $\mathcal{S}$ if, for every assignment $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, every $\Pi \subseteq Fm^2$ and every $\langle \varphi, \psi \rangle \in Fm^2$ such that $\Pi \vdash^2_{\mathcal{S}} \langle \varphi, \psi \rangle$ and $h[\Pi] \subseteq F$, it follows that $\big\langle h(\varphi), h(\psi) \big\rangle \in F$. In this situation we say that F is an $\mathcal{S}$-2-***filter*** of $\boldsymbol{A}$. The notion of Leibniz congruence of a matrix generalizes easily to 2-matrices and 2-filters: the ***Leibniz congruence*** of a 2-matrix $\langle \boldsymbol{A}, F \rangle$ is the largest congruence θ of $\boldsymbol{A}$ such that

$$\text{if } \langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \in \theta \text{ and } \langle a_0, a_1 \rangle \in F, \text{ then } \langle b_0, b_1 \rangle \in F.$$

When this condition is fulfilled it is said that θ is *compatible with* F. Using this notion the concepts of a *protoalgebraic, equivalential, weakly equivalential* and *algebraizable* 2-deductive system can be defined and the theory of 2-deductive systems can be developed in parallel with the theory of logics.

The equational consequence relations associated with classes of algebras can be identified with extensions of the 2-deductive system presented by the axioms and rules corresponding to conditions (E4)–(E7), that is the conditions

(E4$'$) $\vdash^2_{\mathcal{S}} \langle \varphi, \varphi \rangle$.

(E5$'$) $\langle \varphi, \psi \rangle \vdash^2_{\mathcal{S}} \langle \psi, \varphi \rangle$.

(E6$'$) $\langle \varphi, \psi \rangle, \langle \psi, \delta \rangle \vdash^2_{\mathcal{S}} \langle \varphi, \delta \rangle$.

(E7$'$) $\{\langle \varphi_0, \psi_0 \rangle, \dots, \langle \varphi_{n-1}, \psi_{n-1} \rangle\} \vdash^2_{\mathcal{S}} \langle f\varphi_0 \dots \varphi_{n-1}, f\psi_0 \dots \psi_{n-1} \rangle$, for every n-ary connective f.

Let us denote by $\mathcal{EQ}_{\mathcal{L}}$ the least 2-deductive system in language $\mathcal{L}$ which satisfies conditions (E4$'$)–(E7$'$); its models are exactly the 2-matrices $\langle \boldsymbol{A}, \theta \rangle$ such that θ is a congruence of $\boldsymbol{A}$. Thus, the $\mathcal{EQ}_{\mathcal{L}}$-2-filters of an algebra $\boldsymbol{A}$ are the congruences of $\boldsymbol{A}$. It is easy to see that the Leibniz congruence of a model $\langle \boldsymbol{A}, \theta \rangle$ of $\mathcal{EQ}_{\mathcal{L}}$ is precisely θ (i.e., $\boldsymbol{\Omega}_{\boldsymbol{A}}\theta = \theta$). Therefore, the reduced models of $\mathcal{EQ}_{\mathcal{L}}$ are the 2-matrices of the form $\langle \boldsymbol{A}, Id_A \rangle$ where Id_A is the identity relation on A, and, accordingly, they can be identified with their algebra reducts. This implies that the equational consequence relations determined by arbitrary classes of algebras correspond to the 2-deductive systems that extend $\mathcal{EQ}_{\mathcal{L}}$. Furthermore, the equational consequence relations determined by quasivarieties are the finitary 2-deductive systems that extend $\mathcal{EQ}_{\mathcal{L}}$, and this establishes a correspondence between these systems and quasivarieties that can be stated as follows.

1) A class of algebras K is a quasivariety iff there is a finitary 2-deductive system extending $\mathcal{EQ}_{\mathcal{L}}$ such that K is the class of algebra reducts of the reduced models of $\mathcal{S}$.

Conversely,

2) a 2-deductive system $\mathcal{S}$ is a finitary deductive system that extends $\mathcal{EQ}_{\mathcal{L}}$ iff there is a quasivariety K such that the class of the reduced models of $\mathcal{S}$ is the class of 2-matrices $\{\langle \boldsymbol{A}, Id_A\rangle : \boldsymbol{A} \in \mathsf{K}\}$.

There is also a correspondence between varieties and finitary 2-deductive systems which are axiomatic extensions of $\mathcal{EQ}_{\mathcal{L}}$. Indeed, if a variety K is axiomatized by a set of equations Π, and $\mathcal{S}$ is the extension of $\mathcal{EQ}_{\mathcal{L}}$ by the axioms $\{\langle\varphi,\psi\rangle : \varphi \approx \psi \in \Pi\}$, then $\mathcal{S}$ is finitary and its reduced models are the 2-matrices $\langle \boldsymbol{A}, Id_A\rangle$ where $\boldsymbol{A}$ belongs to K. Conversely, given an axiomatic extension of $\mathcal{EQ}_{\mathcal{L}}$ by a set Γ of pairs of formulas, the class of algebraic reducts of its reduced models is the variety axiomatized by the equations $\varphi \approx \psi$ with $\langle\varphi,\psi\rangle \in \Gamma$. Thus, the study of quasivarieties and varieties correspond to the study of the finitary 2-deductive systems that are extensions of $\mathcal{EQ}_{\mathcal{L}}$ and of the axiomatic ones, respectively.

From a mathematical point of view, the next step is obvious: for each natural number $k > 0$ the ***k*-deductive systems**, or ***k*-dimensional deductive systems**, are defined by replacing in the definition of 2-deductive systems the pairs of formulas by sequences of formulas of length k (k-sequences) and making the other obvious changes. Then logics in the sense of Section 1.1 can be identified with the 1-deductive systems. Matrix semantics generalizes in a natural way to k-deductive systems, and the notion of a congruence of an algebra being compatible with a set of k-sequences of elements of its domain is also defined naturally. This leads to a generalization of the Leibniz operator that operates on sets of k-formulas of an algebra. A theory of k-deductive systems within the framework of AAL can be developed along the lines described above; the basic elements of such a theory were worked out by Blok and Pigozzi in [26].

The generalization to arbitrary finite dimensions provides a framework for a common development of logics in the sense of the so-called *assertional* logics of Section 1.1 ($k = 1$) and equational logic ($k = 2$). This in turn provides a context in which the notion of algebraizability can be viewed as a special case of a more general, symmetric relation between deductive systems of different dimensions. More precisely, in [26, 29] algebraizability appears as a particular case of *the more general notion of equivalence between a k-deductive system and an m-deductive system* that is given by the appropriate generalizations of (3.5), (3.6) and (3.7). Since by definition the equivalence is effected by syntactic transformations, one would expect that metalogical properties that are characterized syntactically would automatically transfer from a deductive system to each of its equivalent systems. The (multiterm global) deduction-detachment theorem (DDT) for k-deductive systems, which is a straightforward generalization of the DDT for logics, is an example of a property of this kind.

Theorem 4.1 ([29]) *The* DDT *is preserved under equivalence between k- and m-deductive systems* (*in the case where both deductive systems are finitary and the equivalence and interpretations are finite*).

Many familiar metalogical properties of assertional logics (i.e., 1-deductive systems) when applied to the extensions of the 2-deductive system $\mathcal{EQ}_{\mathcal{L}}$ take the form of a well-known algebraic property, and the DDT is of this kind. In fact:

Theorem 4.2 *A quasivariety K has* EDPRC *iff its associated 2-deductive system $\models_{\mathsf{K}}$ has the* DDT (*in its 2-dimensional form*).

Thus Theorem 3.10 turns out to be a corollary of these two more abstract results, once algebraizability is defined as equivalence with a 2-deductive system of the form $\models_{\mathsf{K}}$ for some K.

Combining Theorem 4.2 with the 2-dimensional version of Theorem 3.9 (and using the fact that filters of the 2-deductive system $\models_{\mathsf{K}}$ are K-congruences) we have that K has EDPRC iff, for every $\boldsymbol{A} \in \mathsf{K}$, the join-semilattice of finitely generated K-congruences of $\boldsymbol{A}$ is dually Brouwerian. Since it is well known that any algebraic lattice whose join sub-semilattice of compact elements is dually Brouwerian is distributive, we get:

Theorem 4.3 *Every quasivariety of algebras with* EDPRC *is relatively congruence-distri-butive.*

The special case of this result for varieties was proved in [98]; it solved a then open problem in universal algebra, and was one of the first examples of the kind of cross fertilization between logic and algebra that AAL stimulates. It is explained in [116] how, thanks to the introduction of the technical notion of algebraizability, this purely algebraic theorem was actually developed with a logical intuition, now formalized as Theorem 3.10, in mind. The relationship between the DDT and the EDPRC motivated a series of papers [20, 19, 27, 28, 98] on varieties with equationally definable principal congruences with important universal-algebraic content.

The much more abstract point of view has paved the way to further extensions of the key idea of algebraizability and of the Leibniz hierarchy to other domains, such as those described in Sections 4.2 and 6.2 below, and to a lattice-theoretical abstraction [18] of the behaviour of the Leibniz operator for algebraizable logics.

Finally, we mention that the more general notion of equivalence can also be applied to two logics of the same dimension, in particular to assertional (i.e., 1-dimensional) logics. It should not be confused with the more common notion of *definitional equivalence*, where a change in language is involved. If it is necessary to emphasize the distinction the AAL notion, which applies only to logics in the same language, is referred to as *deductive equivalence*. See [29] for more details and examples, and [87] for an integrated treatment of the two notions in the context of AAL.

The theory of k-deductive systems applies ideas and techniques borrowed from its two original sources, the theory of matrices of 1-dimensional logics and universal algebra. As in the case $k = 1$, the protoalgebraic k-deductive systems have the richest theory. At the same time several theorems of universal algebra turn out to be particular cases of more general theorems for k-deductive systems. One of them is Mal'cev's theorem characterizing the quasivariety generated by a class of algebras K as the class $\mathbb{S}\mathbb{P}_R(\mathsf{K})$; it is a consequence of the version for 2-deductive systems of Theorem 3.3. Another example is Birkhoff's theorem on subdirect representation. Other theorems of universal algebra can be obtained by simple arguments by specializing more general results for protoalgebraic k-deductive systems to $\mathcal{EQ}_{\mathcal{L}}$, since the latter 2-deductive system is clearly protoalgebraic. One result from universal algebra that has motivated a particularly large amount of research in AAL is Baker's well-known *finite basis theorem.* It says that every congruence-distributive variety of algebras (in a finite language) generated by a finite set of finite algebras is finitely axiomatizable (finitely based). This is generalized in [114] to quasivarieties: every finitely-generated quasivariety (in a finite language) that is relatively congruence-distributive is finitely based. The matrix counterpart of congruence-distributivity is ***filter-distributivity*** (the property that for each algebra $\boldsymbol{A}$ the lattice $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is distributive), a property shared by a wide variety of logics, for instance, all those having a disjunction or the DDT. After several progressive generalizations by Czelakowski, Blok and Pigozzi, Pałasińska [110] obtained the version below for protoalgebraic k-deductive systems from which the original form follows as a particular case; the result and some refinements and extensions is included in her paper [112] in this volume.

Theorem 4.4 *Every filter-distributive and protoalgebraic finitary k-deductive system over a finite language that is defined by a finite set of finite matrices is finitely presented.*

In Section 4.3 similar generalizations of the well-known *Jónsson's lemma* are expounded. According to the previous remarks, parts of universal algebra can be considered as a chapter of the theory of k-deductive systems. Nevertheless, as usually happens, the historical process has been the other way around: results of universal algebra have been generalized first to protoalgebraic 1-deductive systems and subsequently, or sometimes simultaneously, to k-deductive systems. Universal algebra has been a constant source of inspiration for the development of AAL, but, with some justification, it can be claimed that a large part of universal algebra is encompassed by AAL.

4.2 Gentzen systems and their generalizations

The notion of Gentzen system was introduced into AAL by Torrens [134], and by Rebagliato and Verdú in [123]. Its original purpose was to deal with logics that are not algebraizable in the sense of Section 3.3, and possibly not even protoalgebraic, but which nevertheless have a clear algebraic character. This applies for example to certain fragments of some algebraizable logics. But although algebraizability in the strict sense of Section 3.3 is lost in passing to the fragment, in some important cases it can be recovered in a somewhat weakened sense by means of a Gentzen calculus.

A Gentzen system is the abstract notion that arises from considering the relation between sets of sequents and sequents that can be defined by the rules of a given Gentzen calculus. Roughly speaking Gentzen systems are in the same relation to Gentzen calculi as logics, as defined in Section 1.1, are in relation to their Hilbert-style presentations. They can also be viewed as a generalization of the notion of a k-deductive system.

A ***sequent*** over a logical language $\mathcal{L}$ is a pair $\big\langle\langle\varphi_0,\dots,\varphi_{n-1}\rangle,\langle\psi_0,\dots,\psi_{m-1}\rangle\big\rangle$ of finite sequences of formulas of $\mathcal{L}$ that is frequently written in one of the two forms $\varphi_0,\dots,\varphi_{n-1}\vdash\psi_0,\dots,\psi_{m-1}$ and $\varphi_0,\dots,\varphi_{n-1}\Rightarrow\psi_0,\dots,\psi_{m-1}$, but which we prefer to write as

$$\varphi_0,\dots,\varphi_{n-1}\rhd\psi_0,\dots,\psi_{m-1}$$

in order to avoid misunderstandings. We will say that a sequent of this form is ***of type*** (n,m). Some calculi apply only to sequents of certain types, for example Gentzen's calculus LJ for intuitionistic logic applies only to sequents of types (n,m) with $n\geq 0$ and $m\leq 1$, but his calculus LK for classical logic applies to arbitrary sequents.

We denote finite sequences of formulas by overlined lowercase Greek letters ($\overline{\varphi},\overline{\psi}$, etc.). We identify the sequences of length one with their unique component, and denote concatenation of sequences by juxtaposition (e.g., $\overline{\xi},\varphi,\psi,\overline{\xi}'$). Sequents are written in the form $\overline{\varphi}\rhd\overline{\psi}$. Finally, if $\overline{\varphi}=\langle\varphi_0,\dots,\varphi_{n-1}\rangle$ and σ is a substitution, then we put $\sigma\overline{\varphi}:=\langle\sigma\varphi_0,\dots,\sigma\varphi_{n-1}\rangle$. Consider a language $\mathcal{L}$ and a set Seq of sequents of $\mathcal{L}$ that is closed under sequent types, that is, if it contains a sequent of type (n,m) then it contains all the sequents of type (n,m). A ***Gentzen system*** on Seq is a pair $\mathcal{G}=\langle\boldsymbol{Fm},\vdash_{\mathcal{G}}\rangle$, where $\vdash_{\mathcal{G}}$ is a substitution-invariant consequence relation on the set Seq, that is, $\vdash_{\mathcal{G}}\subseteq\mathcal{P}(\mathrm{Seq})\times\mathrm{Seq}$ and the following conditions corresponding to (C^{2}1), (C^{2}2), and (C^{2}6) are satisfied for all subsets Π and Φ of Seq.

(C$^{\mathrm{G}}$1) If $\overline{\varphi}\rhd\overline{\psi}\in\Pi$, then $\Pi\vdash_{\mathcal{G}}\overline{\varphi}\rhd\overline{\psi}$.

(C$^{\mathrm{G}}$2) If for every $\overline{\varphi}\rhd\overline{\psi}\in\Phi$, $\Pi\vdash_{\mathcal{G}}\overline{\varphi}\rhd\overline{\psi}$ and $\Phi\vdash\overline{\delta}\rhd\overline{\varepsilon}$, then $\Pi\vdash_{\mathcal{G}}\overline{\delta}\rhd\overline{\varepsilon}$.

(C$^{\mathrm{G}}$6) If $\Pi\vdash_{\mathcal{G}}\overline{\varphi}\rhd\overline{\psi}$, then for every substitution σ, $\sigma[\Pi]\vdash_{\mathcal{G}}\sigma\overline{\varphi}\rhd\sigma\overline{\psi}$.

The Gentzen system is said to be ***finitary*** if

(C$^{\mathrm{G}}$3) if $\Pi\vdash_{\mathcal{G}}\overline{\varphi}\rhd\overline{\psi}$, then there is a finite $\Pi'\subseteq\Pi$ such that $\Pi'\vdash_{\mathcal{B}}\overline{\varphi}\rhd\overline{\psi}$.

The set of pairs (n,m) such that Seq has an element of type (n,m) will be called the ***type*** of the Gentzen system and will be denoted by $\mathrm{tp}(\mathcal{G})$.

In addition to these conditions, certain so-called ***structural rules*** are often imposed on a Gentzen system. We formulate just one, the *rule of weakening on the left*, as an example of the form they take:

$$\{\overline{\xi},\overline{\xi'}\rhd\overline{\psi}\}\ \vdash_{\mathcal{G}}\ \overline{\xi},\varphi,\overline{\xi'}\rhd\overline{\psi}.$$

This condition is assumed to hold for all finite sequences of formulas $\overline{\xi},\overline{\xi'},\overline{\psi}$ and every formula φ, such that the sequents $\overline{\xi},\overline{\xi'}\rhd\overline{\psi}$ and $\overline{\xi},\varphi,\overline{\xi'}\rhd\overline{\psi}$ are of admissible types. A Gentzen system that satisfies all the structural rules is called ***structural***; otherwise, ***substructural***.

The effect of all the structural rules, taken together, holding in a Gentzen system $\mathcal{G}$ (of suitable type) is that the relation $\vdash\subseteq\mathcal{P}_\omega(Fm)\times Fm$ defined by the condition

$$\{\varphi_0,\dots,\varphi_{n-1}\}\vdash\psi\quad\text{iff}\quad\vdash_{\mathcal{G}}\varphi_0,\dots,\varphi_{n-1}\rhd\psi\tag{4.1}$$

coincides with the finite part $\vdash^{fin}_{\mathcal{S}}$ of a (unique) finitary logic $\mathcal{S}$. Well-known finitary logics are often defined by a structural Gentzen system in this way as a useful alternative to a Hilbert-style presentation. The prime examples of this phenomena are Gentzen calculi LK and LJ for classical and intuitionistic propositional logic. Substructural Gentzen systems are also used to define logics, but a different kind of logic from the deductive systems considered in Section 1.1. Here $\mathcal{S}$ is viewed simply as a set of formulas that is invariant under substitution, i.e., there is no primitive notion of consequence and thus $\mathcal{S}$ is entirely determined by its theorems. However the language of $\mathcal{S}$ normally has a connective $\to$ that can be viewed as taking on some of the function of the missing consequence relation. The definition of a logic $\mathcal{S}$ of this kind by a Gentzen system $\mathcal{G}$ takes the following form.

$$\varphi_0 \to (\cdots \to (\varphi_{n-1} \to \psi)\cdots) \in \mathcal{S} \quad \text{iff} \quad \vdash_{\mathcal{G}} \varphi_0, \ldots, \varphi_{n-1} \rhd \psi \tag{4.2}$$

Logics $\mathcal{S}$ defined in this way are called ***substructural*** if $\mathcal{G}$ is substructural. Examples of substructural logics are BCK logic, relevance logic and linear logic. In each of these cases it turns out that there is an algebraizable deductive system whose set of theorems coincides with the given logic; this is shown in [23] for BCK and relevance logic, and it is not difficult to extend the method used there to linear logic. Normally, the sole purpose of such a deductive system is to generate the theorems of the substructural logic $\mathcal{S}$, and no special proof-theoretic meaning is attributed to its consequence relation; for instance, in regard to the implicative connective $\to$ of $\mathcal{S}$, this is reflected in the fact that $\to$ can never constitute a singleton deduction-detachment system for the deductive system if $\mathcal{S}$ is truly substructural. Indeed substructurality is not an intrinsic property of $\mathcal{S}$ but rather an atribute of the Gentzen system that defines it. Hence any algebraic study of substructural logics that is intended to take this property into account should properly focus on the algebraization of the defining Gentzen systems.

In analogy with the case for deductive systems, Gentzen systems are normally defined by specifying a subset of Seq as axioms and a set of pairs $\langle \mathit{\Pi}, \overline{\varphi} \rhd \overline{\psi}\rangle \in \mathcal{P}_\omega(\mathrm{Seq}) \times \mathrm{Seq}$ as rules of inference. The consequence relation $\vdash_{\mathcal{G}}$ is defined in terms of the axioms and rules of inference in the standard way. The axioms and inference rules together is called a ***presentation*** of $\mathcal{G}$; a Gentzen system together with a presentation is called a ***Gentzen calculus***.

The notion of logical matrix can be extended to obtain a semantics for Gentzen systems. Let $\mathcal{L}$ be a language and let $\mathcal{G}$ be a Gentzen system on the language $\mathcal{L}$. Given an arbitrary set A let us consider the finite sequences of elements of A and denote them by overlined small Latin letters ($\overline{a}, \overline{b}$, etc.). We denote the pairs of these finite sequences by $\overline{a} \rhd \overline{b}$. The ***type*** of a pair $\overline{a} \rhd \overline{b}$ is the pair of natural numbers (n, m) where n is the length of $\overline{a}$ and m the length of $\overline{b}$. Given an algebra $\boldsymbol{A}$ of type $\mathcal{L}$, an assignment $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ and a sequence $\overline{\varphi} = \langle \varphi_0, \ldots, \varphi_{n-1}\rangle$, we put $h\overline{\varphi} := \big\langle h(\varphi_0), \ldots, h(\varphi_{n-1})\big\rangle$; this sequence is the ***interpretation of $\overline{\varphi}$ in $\boldsymbol{A}$ by*** h. Given a sequent $\overline{\varphi} \rhd \overline{\psi}$, its ***interpretation in $\boldsymbol{A}$ by*** h is the pair $h\overline{\varphi} \rhd h\overline{\psi}$. A ***G-matrix for the type of*** $\mathcal{G}$ (the "G" here stands for "Gentzen") is a pair $\langle \boldsymbol{A}, \mathrm{R}\rangle$ where R is a set of pairs of finite sequences of elements of A whose type belongs $\mathrm{tp}(\mathcal{G})$. A G-matrix $\langle \boldsymbol{A}, \mathrm{R}\rangle$ of the type of $\mathcal{G}$ is a ***model of*** $\mathcal{G}$ if, for every assignment $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, every set $\mathit{\Pi}$ of sequents of $\mathcal{G}$ such that, for all $\overline{\varepsilon} \rhd \overline{\delta} \in \mathit{\Pi}$, $h\overline{\varepsilon} \rhd h\overline{\delta} \in \mathrm{R}$, and every sequent $\overline{\varphi} \rhd \overline{\psi}$ of $\mathcal{G}$ such that $\mathit{\Pi} \vdash_G \overline{\varphi} \rhd \overline{\psi}$, we have $h\overline{\varphi} \rhd h\overline{\psi} \in \mathrm{R}$. The notion of a congruence relation on an algebra $\boldsymbol{A}$ being compatible with a set R of pairs of finite sequences of element of $\boldsymbol{A}$ is defined just as in the case of sets of k-sequences. The Leibniz congruence of a G-matrix $\langle \boldsymbol{A}, \mathrm{R}\rangle$ is the largest congruence on $\boldsymbol{A}$ compatible with R. Using the Leibniz congruence the abstract algebraic theory of logics and of k-deductive systems can be generalized in a natural way to Gentzen systems. In particular the various classes of the Leibniz hierarchy can be defined using, for instance, the characterizations found in 3.13; see [119, 122, 123]. Moreover, ordinary logics can be viewed as Gentzen systems of type $\big\{(0,1)\big\}$; more generally k-deductive systems can viewed as Gentzen systems of type $\big\{(0,k)\big\}$. So this formalism provides a unified view of the three notions of consequence involved in the algebraic treatment of logical systems.

The general notion of the equivalence of deductive systems of different dimension that was discussed in Section 4.1 can be extended to Gentzen systems of different types in a straightforward way. The interpretations are based on *translations* of sequents into finite sets of equations and of equations into finite sets of sequents. This gives a third way of expressing the connection between a Gentzen system and a logic that can be an alternative to (4.1) and (4.2), although in many cases the interpretation of the logic $\mathcal{S}$ in a Gentzen system $\mathcal{G}$ given by (4.1) or (4.2) turns out to be one half of an equivalence. Indeed, in the case of BCK logic, the interpretation described in (4.2) is in part an equivalence between BCK logic and a certain substructural Gentzen system with a very natural presentation; this is shown in [2].

It is natural to take a Gentzen system to be ***algebraizable*** if it is equivalent to some extension of $\mathcal{EQ}_{\mathcal{L}}$, that is to a logic of the form $\langle \boldsymbol{Fm}, \models_{\mathsf{K}} \rangle$ for some class K of algebras of type $\mathcal{L}$. As mentioned above, part of the motivation for considering Gentzen systems in the abstract algebraic theory of logics is to try to classify the algebraic character of a logic $\mathcal{S}$ that is not algebraizable or maybe not even protoalgebraic. One way to do this is to find an algebraizable Gentzen system in which $\mathcal{S}$ can be faithfully interpreted. The implication-less fragment $\mathcal{IPL}^*$ of the intuitionistic propositional logic is a classic example of this kind. It was shown in [23] to be nonprotoalgebraic, but a Gentzen calculus of type $\omega \smallsetminus \{0\} \times \{1\}$ that faithfully interprets it was presented in [123] whose associated Gentzen system was shown to be algebraizable with equivalent algebraic semantics the class of pseudo-complemented distributive lattices. The relevant interpretations are:

$$\varphi_0, \ldots, \varphi_{n-1} \rhd \psi \quad \longrightarrow \quad (\varphi_0 \wedge \ldots \wedge \varphi_{n-1} \wedge \psi) \approx (\varphi_0 \wedge \ldots \wedge \varphi_{n-1}) \tag{4.3}$$

$$\{\delta \rhd \varepsilon \,,\, \varepsilon \rhd \delta\} \quad \longleftarrow \quad \delta \approx \varepsilon \tag{4.4}$$

The interpretation (4.4) is the one most commonly used for algebraizing Gentzen systems that define selfextensional logics, as is the case here. If we compare it with the one used for implicative logics (Section 3.3) we see why Gentzen systems are useful in capturing the algebraic nature of logics without an implication connective; the role of the missing implication is played by the "entailment symbol" $\rhd$.

Different Gentzen systems can have the same sequents as theorems, and hence define the same logic, but have different behaviour regarding algebraizability. Similarly, different Gentzen calculi defining the same logic can have different proof-theoretic properties. Gil and Rebagliato [82] have investigated the relationships between proof-theoretic properties of a Gentzen calculus and the algebraic character of the Gentzen system it defines. One of the things they show is that any Gentzen system defined by a so-called *regular* Gentzen calculus that has the exchange rule (one of the structural rules) is protoalgebraic iff it satisfies the cut rule (another well known structural rule). The regularity assumption is a technical requirement that guarantees the logical rules of the calculus have a certain "natural" form. Actually, the systems that Gil and Rebagliato work with in [82] are slightly more general than Gentzen systems. They work with the so-called ***m-sided Gentzen systems*** ($m \geq 2$). These are based on a generalization of the notion of sequent to ***m-sided sequents***, which are sequences of m finite sequences of formulas, these being of varying length subject to possible restrictions; they are denoted similarly to the following.

$$\varphi_0^0, \ldots, \varphi_{k_0}^0 \mid \varphi_0^1, \ldots, \varphi_{k_1}^1 \mid \ldots \mid \varphi_0^{m-1}, \ldots, \varphi_{k_{m-1}}^{m-1} .$$

Ordinary sequents are identified with 2-sided sequents. m-sided Gentzen systems were introduced by Rousseau [125] in order to build proof systems of Gentzen style for Łukasiewicz's finitely-valued logics, and have recently re-appeared in connection with automated deduction issues, see [14].

More recent, but in the same vein, is the introduction of ***hypersequents***. They are finite sequences (of arbitrary length) of ordinary (i.e., 2-sided) sequents, and have been used to obtain proof systems for several kinds of logics: intermediate, many-valued, fuzzy, substructural, computational. Proof systems based on hypersequents seem to show more flexibility and better proof-theoretic behaviour than either the 2-sided or arbitrary m-sided Gentzen calculi. See [13] for a survey and further references. It is clear that these generalizations can also be treated algebraically in a similar (but notationally more complicated) way as 2-sided Gentzen calculi. Both the idea of algebraizability via mutually inverse interpretations to and from equational logics, and the Leibniz hierarchy based on the notion of matrix, can be generalized in a straightforward way so that sequents, m-sided sequents or hypersequents play the role of formulas. Some works where this is done, at varying levels of generality, are [1, 81, 83, 84].

It may seem that in each of the generalizations of the notion of algebraizability described in this section the theory has to be developed in extenso, since the syntactical objects to which the extended notion of deductive system is applied are different each time. However, there are more general and comprehensive frameworks that cover all these cases as particular ones, and hence can be used as the theoretical background on which to base a particular application. One is based on the expression of all the preceding formalisms in terms of (ordinary) first-order languages without equality, as is explained in detail in the next section. The other is less dependent on grammatical issues and concentrates on the lattice-theoretical aspect of algebraizability as a special isomorphism between two lattices of theories (Corollary 3.14). It has been developed by Blok and Jónsson [18] and it seems it may have a wide range of applications.

4.3 Equality-free universal Horn logic

Bloom's observation mentioned in Section 3.1 showing how to correlate finitary logics with equality-free strict universal Horn theories can be extended to the finitary k-deductive systems and also to Gentzen systems in a natural way.

The finitary k-deductive systems over a language $\mathcal{L}$ are correlated with the equality-free strict universal Horn theories over a first-order language that contains, apart from the function symbols corresponding to the logical connectives of $\mathcal{L}$, exactly one k-ary relation symbol P. A k-sequence of formulas $\langle \varphi_0, \dots, \varphi_{k-1} \rangle$ translates into the atomic formula $P\varphi_0 \dots \varphi_{k-1}$, and a rule $\big\langle \{\overline{\varphi}_0, \dots, \overline{\varphi}_{n-1}\}, \overline{\varphi} \big\rangle$ translates into the universal Horn formula $\overrightarrow{\forall x}\big(P\overline{\varphi}_0 \wedge \dots \wedge P\overline{\varphi}_{n-1} \rightarrow P\overline{\varphi}\big)$. k-matrices $\langle \boldsymbol{A}, F \rangle$ where $F \subseteq A^k$ can be identified with first-order structures.

The relation between finitary Gentzen systems and equality-free strict universal Horn theories is established as follows. Let $\mathcal{G}$ be a finitary Gentzen system over the language $\mathcal{L}$. For every (n, m) in the type of $\mathcal{G}$ one introduces an $(n+m)$-ary relation symbol $R_{(n,m)}$ and translates a sequent $\langle \varphi_0, \dots, \varphi_{n-1} \rangle \rhd \langle \psi_0, \dots, \psi_{m-1} \rangle$ of type (n, m) into the atomic formula $R_{(n,m)}\varphi_0 \dots \varphi_{n-1}\psi_0 \dots \psi_{m-1}$; note that the use of ordered pairs to index the relation symbols makes it possible to parse the atomic formulas into sequents in a unambiguous way. Given a sequent $\overline{\varphi} \rhd \overline{\psi}$ we denote by $\mathbf{tr}(\overline{\varphi} \rhd \overline{\psi})$ its translation. A sequent rule $\langle \mathit{\Pi}, \overline{\varphi} \rhd \overline{\psi} \rangle$ where $\mathit{\Pi}$ is a finite set of sequents and $\overline{\varphi} \rhd \overline{\psi}$ is a sequent is translated into the universal closure of the formula

$$\Big(\bigwedge_{\overline{\varepsilon} \rhd \overline{\delta} \in \mathit{\Pi}} \mathbf{tr}(\overline{\varepsilon} \rhd \overline{\delta}) \Big) \rightarrow \mathbf{tr}(\overline{\varphi} \rhd \overline{\psi})$$

when $\mathit{\Pi} \neq \emptyset$, and into the universal closure of $\mathbf{tr}(\overline{\varphi} \rhd \overline{\psi})$ when $\mathit{\Pi} = \emptyset$. The formula so obtained is an equality-free strict universal Horn formula in the similarity type $\mathcal{L}_{\mathcal{G}} = \mathcal{L} \cup \big\{ R_{(n,m)} : (n, m) \in \mathrm{tp}(\mathcal{G}) \big\}$. Thus, to each finitary Gentzen system $\mathcal{G}$ corresponds an equality-free strict universal Horn theory.

The G-matrices for a Gentzen system $\mathcal{G}$ have to be modified in order to obtain structures suitable as models—in the usual first-order logic sense—for the equality-free strict universal Horn theory of $\mathcal{G}$. The modification of a G-matrix $\langle \boldsymbol{A}, \mathrm{R} \rangle$ for $\mathcal{G}$ consists, loosely speaking, in breaking up the set R into its 'type-homogeneous' parts. One obtains the structure $\mathfrak{A} = \big\langle \boldsymbol{A}, \langle R^{\mathfrak{A}}_{(n,m)} : (n, m) \in \mathrm{tp}(\mathcal{G}) \rangle \big\rangle$ of similarity type $\mathcal{L}_{\mathcal{G}}$, where $R^{\mathfrak{A}}_{(n,m)}$ is the set of sequences $\langle a_0, \dots, a_{n-1}, a_n, \dots, a_{n+m-1} \rangle$ such that $\big\langle \langle a_0, \dots, a_{n-1} \rangle, \langle a_n, \dots, a_{n+m-1} \rangle \big\rangle \in \mathrm{R}$. Obviously $\langle \boldsymbol{A}, \mathrm{R} \rangle$ and $\mathfrak{A} = \big\langle \boldsymbol{A}, \langle R^{\mathfrak{A}}_{(n,m)} : (n, m) \in \mathrm{tp}(\mathcal{G}) \rangle \big\rangle$ are essentially the same entity and can be identified. Under this identification a Gentzen system and its associated equality-free strict universal Horn theory have exactly the same models. This correspondence is expounded in detail in [75].

The correspondence just described can be extended to the more complicated extensions of the notion of Gentzen system described in the preceding section. The basic "syntactic unit" of each of these extensions, finite sequences of finite sequences of formulas, finite sequences of finite sequences of finite sequences of formulas, etc. translates into an atomic formula with a relation symbol whose index reflects the logical type of the given syntactic unit. Then each "rule" translates into a strict universal Horn formula in the obvious way. The translation of the matrices appropriate for these generalized Gentzen systems is based on the idea that an ordinary assignment $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$

can be extended componentwise in a routinely way to finite sequences of formulas, sequences of sequences of formulas, etc. The interpretations of each of these linguistic constructs has the same structure over the domain A of the algebra $\boldsymbol{A}$, i.e., finite sequences of finite sequences of formulas, finite sequences of finite sequences of finite sequences of formulas, etc. Hence the matrices for more complicated systems can be presented as structures for a first-order language whose algebraic part corresponds to the logical language, and whose relation symbols are indexed according to the structure of the "syntactic units" of the system (sequents, m-sided sequents, hypersequents, etc.). Since the basic component of each of the generalized Gentzen systems under consideration is a substitution-invariant (in a generalized sense) consequence relation on some set, the associated first-order theory will always be an equality-free strict universal Horn theory.

Consideration of these correspondences awakened a renewed interest in *the model theory of equality-free languages* beyond universal Horn theories without equality. The basic tools used here turn out to be the same ones used in the theory of k-deductive systems and Gentzen systems, namely the notions of Leibniz congruence, reduced structures and strict homomorphisms between first-order structures. A ***congruence of a first-order structure*** $\mathfrak{A}$ is a congruence relation θ on its algebraic part that is ***compatible*** with each fundamental relation of $\mathfrak{A}$, i.e., for every n-ary relation symbol R,

$$\text{if for every } i < n\,,\ \langle a_i\,, b_i\rangle \in \theta \ \text{ and }\ \langle a_0, \dots, a_{n-1}\rangle \in R^{\mathfrak{A}},\ \text{ then }\ \langle b_0, \dots, b_{n-1}\rangle \in R^{\mathfrak{A}}.$$

The ***Leibniz congruence*** $\boldsymbol{\Omega}\mathfrak{A}$ ***of a first-order structure*** $\mathfrak{A}$ is the largest congruence of $\mathfrak{A}$; $\mathfrak{A}$ is ***reduced*** when $\boldsymbol{\Omega}\mathfrak{A} = Id_A$, and the ***reduction*** of a structure is its quotient by its Leibniz congruence. Finally, a ***strict homomorphism*** from a structure $\mathfrak{A}$ for a first-order language into a structure $\mathfrak{B}$ for the same language is a function $h : A \to B$ that is an homomorphism relative to the algebraic parts of $\mathfrak{A}$ and $\mathfrak{B}$ that in addition satisfies the condition that, for every n-ary relation symbol R and all $a_0, \dots, a_{n-1} \in A$,

$$\langle a_0, \dots, a_{n-1}\rangle \in R^{\mathfrak{A}} \quad \text{iff} \quad \big\langle h(a_0), \dots, h(a_{n-1})\big\rangle \in R^{\mathfrak{B}}.$$

Using these definitions one obtains the operators $\mathbb{H}_S$ and $\mathbb{H}_S^{-1}$ of forming images and inverse images of structures by strict homomorphisms.

The matrix-model theory outlined above has motivated two distinct lines of research in the model theory of equality-free first-order logic that have gone considerably beyond strict universal Horn theories. One line, pursued in the doctoral dissertation of Dellunde [56] and the related papers [35, 57, 58], has investigated a number of the standard topics of first-order model theory in an equality-free context. As examples of the kind of results obtained (those that do not correspond to a result for Gentzen systems) we mention an analogue of the Keisler-Shelah theorem and a characterization result from [35].

Theorem 4.5 *Two structures over the same language satisfy the same equality-free first-order sentences iff they have ultrapowers with isomorphic reductions.*

Theorem 4.6 *A first-order sentence of an arbitrary language is logically equivalent to an equality-free sentence iff it is preserved under* $\mathbb{H}_S$ *and* $\mathbb{H}_S^{-1}$.

The other line, taken up in the doctoral dissertation of Elgueta [63] and the related papers [49, 64, 65, 67] addresses questions of a more universal-algebraic character for classes of structures definable in equality-free languages, such as the the existence of free structures and subdirect representation. The following theorem from [64] is typical of the type of results obtained in this work. By a ***quasivariety*** in this context we mean the model class of a strict universal Horn theory without equality; it is ***protoalgebraic*** if the corresponding Horn theory is protoalgebraic in the sense that naturally generalizes the condition characterizing the protoalgebraic logics given in Theorem 3.13, part 1. L is a ***relative subvariety*** of a quasivariety K if it is the model class of an axiomatic extension of the Horn theory of K, i.e., if it is the set of all structures in K that satisfy a fixed set of universal closures of atomic formulas. For any $\mathsf{M} \subseteq \mathsf{K}$, $\mathbb{V}_{\mathsf{K}}(\mathsf{M})$ denotes the relative subvariety of K generated by M.

Theorem 4.7 *Let $\mathcal{L}$ be a first-order language without equality and let K be a protoalgebraic quasi-variety of $\mathcal{L}$-structures. Let $\mathsf{M} \subseteq \mathsf{K}$ be such that $\mathbb{V}_{\mathsf{K}}(\mathsf{M})$ is relatively filter-distributive. Then the relatively finitely subdirectly irreducible reduced members of $\mathbb{V}_{\mathsf{K}}(\mathsf{M})$ belong to the class $\mathbb{F}^*_{\mathsf{K}}\, \mathbb{S}^*\, \mathbb{P}^*_U(\mathsf{M}^*)$.*

$\mathbb{S}^*$ and $\mathbb{P}^*_U$ denote respectively the closure under the operations of forming substructures and ultraproducts and then reducing by the Leibniz congruence. $\mathbb{F}^*_{\mathsf{K}}$ represents closure under the operation of forming "K-filter extensions" of a given structure and then reducing. See [64] for details. This result is a generalization of the well-known *Jónsson's lemma* saying that if K is a class of algebras and the variety $\mathbb{V}(\mathsf{K})$ it generates is congruence-distributive, then the subdirectly irreducible elements of $\mathbb{V}(\mathsf{K})$ belong to $\mathbb{H}\,\mathbb{S}\,\mathbb{P}_U(\mathsf{K})$. The classic result becomes a special case of the preceding theorem, which is the culmination of a series of generalizations by Blok, Pigozzi, Czelakowski and Dziobiak.

4.4 Algebras of Logic

In traditional algebraic logic much effort has been spent investigating the properties of algebras of a logical character without much regard for the process by which they are identified or the exact nature of their connection with a specific logic. Although the algebras can usually be associated in some way with a specific assertional logic, i.e., a 1-deductive system in the sense of Section 1.1, there are many situations in which the algebras under investigation, while having a clear metalogical nature, are only loosely tied if at all to a specific assertional logic. This is manifestly the case when logical equivalence is taken as the primitive logical predicate, as in the early history of algebraic logic discussed in the Introduction. But even in the case where the perceived underlying logical content is clearly assertional in nature, the connection with an actual assertional logic may not be clear. The various kinds of lattices endowed with additional operations that have been considered in the literature certainly fall in this category. As just one important example of the situation just described we mention the work of Monteiro and his students and collaborators ([38, 108, 109] is a small sample of the work of this school). An example of a somewhat different kind can be found in the work of the Barcelona group prior to the late 1980's, such as [79, 137], that centered around algebraic structures that can be characterized by the existence of a closure system with a certain list of properties having some logical "form".

For the purposes of studying, in the widest possible context, the precise connection that exists between assertional logic and the various algebras that have been considered under the rubric of algebraic logic, AAL provides the appropriate venue. The work in AAL has also stimulated interest within universal algebra in quasivarieties, which historically have taken a back seat to varieties. As we have seen, the natural equivalent algebraic semantics for finitary, finitely algebraizable assertional logics are in general quasivarieties. The consequence relation of a finitary, finitely algebraizable logic is reflected in the quasi-identities of its equivalent algebraic semantics, whereas the identities reflect only the theorems of the logic. One aspect of this renewed interest is the search for generalizations to quasivarieties of some important theorems of universal algebra obtained originally for varieties, for example the generalizations of Baker's finite basis theorem and Jónsson's lemma mentioned in Sections 4.1 and 4.3.

A question that can now receive a precise technical answer is: *what is an algebra of logic*? If one interprets algebra of logic to be the equivalent algebraic semantics $\mathsf{K} = \mathbf{Alg}\,\mathcal{S}$ of an algebraizable logic $\mathcal{S}$, then the answer is easy in view of the symmetry of the definition of the equivalence between $\mathcal{S}$ and the equational consequence relation of K. Taking the dual of the proof of part 4 of Theorem 3.16 (see the remarks following the statement of the theorem) we have that K is the equivalent algebraic semantics of some algebraizable logic $\mathcal{S}$ iff there is a faithful interpretation $K(x)\approx\Lambda(x)$ of $\mathcal{S}$ in the equational consequence relation of K, and a set of formulas $E(x, y)$ in two variables such that the second equivalence of (3.7) holds. But it is not difficult to see that, for any set of equations $K(x)\approx\Lambda(x)$, the formula (3.5) that expresses the fact that $K(x)\approx\Lambda(x)$ is a faithful interpretation of $\mathcal{S}$ in the equational consequence relation of K can be used to define a logic $\mathcal{S}$ with this property (this only uses basic properties of equality). So the second equivalence of (3.7) is the only condition that K must satisfy to be a class of algebras of logic in the present sense. In the most interesting case, where K is a quasivariety, the characterization that we have just outlined takes the following form; see [46] for a detailed proof.

Theorem 4.8 *A quasivariety K is the equivalent algebraic semantics of some finitary and finitely algebraizable logic iff there exist a finite set $E(x,y)$ of formulas and a finite set $K(x){\approx}\Lambda(x)$ of equations such that:*

$$\models_{\mathsf{K}} K\big(E(x,x)\big){\approx}\Lambda\big(E(x,x)\big) \quad \textit{and} \quad K\big(E(x,y)\big){\approx}\Lambda\big(E(x,y)\big) \models_{\mathsf{K}} x \approx y$$

In this case, the logic is given by the expression (3.5)*.*

This theorem is not particularly useful however in showing that a given quasivariety is not a quasivariety of logic, i.e., it is not of the form $\mathbf{Alg}\,\mathcal{S}$ for any finitary, finitely algebraizable logic $\mathcal{S}$. For this purpose Corollary 3.14, part 2 is better suited. According to this result, if we can verify that for some algebra $\boldsymbol{A}$ the structure of the lattice $\mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$ of K-congruences of $\boldsymbol{A}$ precludes it from being isomorphic (via the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$) to any lattice of the form $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ for some $\mathcal{S}$ (and this computation can be feasible if the algebra is small), then we know K is not of the form $\mathbf{Alg}\,\mathcal{S}$ for any finitary, finitely algebraizable logic $\mathcal{S}$. The problem is that, although every such logic has a unique equivalent quasivariety, a quasivariety of logic can have many different equivalent logics.

As we have previously observed, for every quasivariety K and every finite system of equations $K(x) \approx \Lambda(x)$ in one variable, (3.5) defines a finitary logic, denoted by $\mathcal{S}(K{\approx}\Lambda, \mathsf{K})$, that is faithfully interpreted in $\models_{\mathsf{K}}$ by $K(x){\approx}\Lambda(x)$; it is called the $(K{\approx}\Lambda)$***-assertional logic*** of K (a 1-dimensional logic, as opposed to the equational logic of K, which is 2-dimensional). Among all the assertional logics of K are those for which K is the equivalent algebraic semantics, i.e., the second equivalence of (3.7) holds for some $E(x,y)$ and $K(x){\approx}\Lambda(x)$. There may be several different assertional logics with this property; for example, it turns out that a certain 3-valued paraconsistent logic first considered in [60] and the $(\to, \neg)$-fragment of Łukasiewicz's 3-valued logic are both finitely algebraizable with the same equivalent quasivariety (see [29]).

The notion of $\mathcal{S}(K{\approx}\Lambda, \mathsf{K})$-filter of an algebra can be viewed as a natural common generalization of normal subgroups, ring ideals, and other general notions of ideal in algebra. Indeed, taking K to be respectively the varieties of groups and rings, the $\mathcal{S}\big(\{x{\approx}0\}, \mathsf{K}\big)$-filters of a group or ring $\boldsymbol{A}$ coincide respectively with the normal subgroups or ideals of $\boldsymbol{A}$. The basic elements of a general theory of ideals based on the filters of the assertional logics of a quasivariety are developed in [30].

Normal subgroups and ring ideals coincide with the 0-equivalence classes of congruences. For a quasivariety K the connection between $\mathcal{S}\big(K{\approx}\Lambda, \mathsf{K}\big)$-filters and K-congruences is somewhat weaker. By the $(K{\approx}\Lambda)$***-class*** of a K-congruence θ of $\boldsymbol{A}$ we mean the set

$$\big\{a \in A : \big\langle \kappa^{\boldsymbol{A}}(a)\,, \lambda^{\boldsymbol{A}}(a)\big\rangle \in \theta\,,\ \forall\, \kappa(x) \approx \lambda(x) \in K(x){\approx}\Lambda(x)\big\}.$$

Note that the $\big(\{x{\approx}0\}\big)$-class of a congruence θ on a group or ring is $0/\theta$. In general, the $(K{\approx}\Lambda)$-class of a K-congruence is a $\mathcal{S}(K{\approx}\Lambda, \mathsf{K})$-filter but not conversely. The converse does hold however if K is a variety or if K is $(K{\approx}\Lambda)$***-regular*** in the sense that each K-congruence is determined by its $(K{\approx}\Lambda)$-class.

Theorem 4.9 ([30]) *Let K be a quasivariety and $K(x){\approx}\Lambda(x)$ a finite set of equations in one variable. Then K is $(K{\approx}\Lambda)$-regular iff $\mathcal{S}(K{\approx}\Lambda, \mathsf{K})$ is finitely algebraizable with K as its equivalent algebraic semantics.*

In a Fregean (extensional) logic all theorems are logically equivalent since they have the same truth value in any interpretation, and the same is true of the familiar non-Fregean (intensional) logics, which are expansions of Fregean logics. Each logic $\mathcal{S}$ of this kind that is protoalgebraic is finitely algebraizable, and its equivalent quasivariety K is ***pointed*** in the sense that there is a constant term that singles out a "distinguished" element in every algebra of K. This distinguished element is the common value that all the theorems of $\mathcal{S}$ take in the algebra. It can thus be represented by any chosen theorem of $\mathcal{S}$, which can then be taken to be the constant term of K. This term and its interpretation are usually denoted by 1 (or sometimes 0). In all the familiar cases $\mathcal{S}$ turns out to be the ($x\approx 1$)-assertional logic of K and is ($x\approx 1$)-regular. These logics are paradigmatic for the regularly, finitely algebraizable logics discussed in the remarks following Theorem 3.16.

In a more general context the ***assertional logic*** (without qualification) of an arbitrary pointed quasivariety K is its ($x\approx 1$)-assertional logic, and it (or rather its consequence relation) is denoted by $\vdash^{\text{AL}}_{\mathsf{K}}$. Hence it is defined by the condition:

$$\Gamma \vdash^{\text{AL}}_{\mathsf{K}} \varphi \quad \text{iff} \quad \{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathsf{K}} \varphi \approx 1. \tag{4.5}$$

When compared with (3.5), this tells us that $x\approx 1$ is a faithful interpretation, and K is an algebraic semantics for its own assertional logic $\vdash^{\text{AL}}_{\mathsf{K}}$. This logic will be algebraizable (and K will be its equivalent algebraic semantics) if (3.7) holds for some $E(x, y)$.

A pointed quasivariety K is ***relatively point-regular*** if it is ($x\approx 1$)-regular, that is, if each K-congruence θ is uniquely determined by its ($x\approx 1$)-class $1/\theta$. By Theorem 4.9, then, K is the equivalent algebraic semantics of its own assertional logic, which is finitary and finitely algebraizable, and the mapping

$$\theta \longmapsto 1/\theta\,, \tag{4.6}$$

is the inverse mapping to the isomorphism $\boldsymbol{\Omega}_{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$, if $\boldsymbol{A} \in \mathsf{K}$.

It turns out that for each algebraic similarity type with a distinguished constant there is a one-to-one correspondence between finitary, regularly, finitely algebraizable logics and relatively point-regular quasivarieties. Moreover, expression (4.6) can be viewed as an algebraic characterization of the filters of the logic, and shows that the correspondence given by the Leibniz operator and its inverse can be characterized independently of its logical origin or its logical significance.

An alternative general theory of ideals for pointed varieties, devoid of the logical considerations that motivated the one described above, was initiated in 1970's by Ursini [135] and developed later in [5, 86]. Although Ursini and his collaborators were concerned mainly with varieties, their theory extends to quasivarieties without difficulty. It turns out that the theory is particularly well-behaved when the pointed quasivariety is ***subtractive*** [6, 7, 8, 136]; this means that there is a term $s(x, y)$ in two variables such that $s(x, x) \approx 1$ and $s(1, x) \approx x$ are identities of the quasivariety.

Relatively point-regular quasivarieties and subtractive pointed quasivarieties are incomparable in the sense that neither notion encompasses the other. Blok and Raftery [30] consider a natural generalization of both, the protoregular quasivarieties. A pointed quasivariety is ***protoregular*** if its assertional logic is protoalgebraic. Protoregularity clearly generalizes both relative point-regularity and subtractivity ($\big\{s(x, y)\big\}$ is an implication set for the assertional logic). The following are established in [30]. If K is a pointed variety, then K is subtractive iff it is protoregular and, for each algebra in K, the filters of the assertional logic and the ideals in the sense of Ursini are the same. From the results in [50] it follows that the assertional logics of protoregular quasivarieties coincide with regularly, weakly algebraizable and finitary logics.

We have seen that the well-known universal algebraic notion of point-regularity has a natural metalogical content. A natural question to ask is if there is an analogous content to the better known notion of ***full regularity***, that is, the property that each K-congruence is determined by any one of its equivalence classes. The paper [17] by Barbour and Raftery in this volume explores one possible answer, which gives rise to the new notion of *parameterized algebraizability*. In the process they generalize the notion of assertional logic in order to encompass a wider class of logics intrinsically associated with a quasivariety, such as the so-called *membership logic*, which is never protoalgebraic. They also study parameterized versions of protoalgebraicity, of algebraic semantics, etc. These generalizations are by no means straightforward, and constitute a good example of the kind of interplay between logic and algebra that AAL has given rise to.

As previously observed, the protoalgebraic Fregean logics, and certain of their expansions, are the paradigms for regularly, finitely algebraizable logics. In this sense the equivalent quasivarieties of protoalgebraic Fregean logics are paradigmatic for relatively point-regular quasivarieties, but not every relatively point-regular quasivariety is obtained this way; those that are are called Fregean. Thus a relatively point-regular quasivariety is ***Fregean*** if its assertional logic is Fregean. This notion, here defined solely in logical terms, can be given a purely algebraic characterization as follows, see [53]:

Theorem 4.10 *A relatively point-regular quasivariety* K *is Fregean iff it has the following property. For every* $\boldsymbol{A} \in \mathsf{K}$ *and all* $a, b \in A$*, if* $\Theta_{\mathsf{K}}(a, 1) = \Theta_{\mathsf{K}}(b, 1)$ *then* $a = b$*, where* $\Theta_{\mathsf{K}}(c, d)$ *is the* K*-congruence generated by the pair* $\langle c, d\rangle$*.*

An arbitrary pointed (but not necessarily point-regular) quasivariety that satisfies the condition of the theorem is said to be ***congruence-orderable*** (the term comes from the fact the quasi-ordering defined by $a \leq b$ iff $\Theta_{\mathsf{K}}(a, 1) \subseteq \Theta_{\mathsf{K}}(b, 1)$ is actually a partial ordering in this case).

Fregean relatively point-regular quasivarieties and, more generally, congruence-orderable pointed quasivarieties have been studied by a number of different authors, although most attention has been on varieties. Fregean point-regular varieties were introduced in [115] and studied further, along with Fregean relatively point-regular quasivarieties, in [46, 53, 54]

A relatively point-regular quasivariety K is ***strongly relatively point-regular*** if every finitely generated relative congruence is principal and, in fact, of the form $\Theta_{\mathsf{K}}(a, 1)$. Building on [34], Pigozzi [115] proves that a strongly point-regular variety is Fregean iff it is (equationally definitionally equivalent to) a variety of Brouwerian semilattices with possible additional operations that preserve the congruences of the Brouwerian semilattice in a natural way. This was the first of several "representation" results of this kind for Fregean point-regular varieties. Idziak, Słomczyńska and Wronski [94] prove that a congruence-permutable point-regular variety is Fregean iff it is a variety of equivalential algebras (see Section 3.3) possibly with additional congruence-preserving operations. They also showed that every Fregean strongly point-regular variety is arithmetical. Agliano [3] shows that every Fregean point-regular variety that has definable ($\{x{\approx}1\}$)-classes (i.e., sets of the form $1/\theta$ for every congruence θ) is a variety of Hilbert algebras (see Section 3.3) with congruence-preserving operations. (However Agliano formulates his result within the theory of ideals in the sense of Ursini). See also [4].

Metalogical representation theorems closely related to the above algebraic representation theorems have also been obtained. It is proved in [53] that every Fregean logic with the uniterm deduction-detachment theorem (a single deduction-detachment formula) is definitionally equivalent to an axiomatic extension $\mathcal{S}$ of an expansion of the $\to$ fragment of $\mathcal{CPL}$ by new *extensional* connectives; for a new binary connective $*$ to be extensional, for example, $(x_1 \to y_1) \to (y_1 \to x_1) \to (x_2 \to y_2) \to (y_2 \to x_2) \to (x_1 * y_1) \to (x_2 * y_2)$ must be a theorem of $\mathcal{S}$ (association of outer arrows is assumed to the right). This is a metalogical analogue of the result of Agliano mentioned above. Conversely, it is easy to see that every $\mathcal{S}$ of this kind is Fregean with the uniterm deduction-detachment theorem. It is also proved in [53] that every protoalgebraic Fregean deductive system with conjunction is definitionally equivalent to an axiomatic extension of an expansion of the $\to, \wedge$ fragment of $\mathcal{IPC}$ by new extensional connectives, and conversely. This is the analogue of the result of [94]. See [54] for similar results involving the multiterm DDT.

A systematic investigation of both Fregean and self-extensional nonprotoalgebraic logics can be found in [73]. A survey of some of these results and earlier ones can be found in [69]. Analogous, but necessary weaker, versions of two of the representation theorems mentioned above for arbitrary self-extensional deductive systems with the uniterm DDT and with conjunction are obtained in [73].

Of considerable interest in AAL is the problem of isolating the abstract properties of the familiar algebraizable logics that account for their almost universal strong (finite) algebraizability, i.e., the fact that the algebraic counterparts of most of them are varieties. As a sample of the kind of results that have been obtained along this line we mention the following (see [46, Chapter 6] for proofs).

Theorem 4.11

1. *Every strongly point-regular Fregean quasivariety is in fact a variety and has* EDPC (*see Section* 3.3).
2. *A quasivariety of Skolem lattices has the relative congruence-extension property iff it is a variety.*

A novel viewpoint in examining the role of conjunction in Fregean varieties is adopted in [111]. Here the relation between the quasivariety of BCK algebras, which is relatively point-regular but not Fregean, and the sequent calculus $LBCK$ defining BCK logic is studied. By combining algebraic and proof-theoretical techniques it is shown that the fact that $LBCK$ is obtained by deleting the rule of contraction from the usual sequent calculus LJ for intuitionistic logic (a Fregean logic whose equivalent algebraic semantics is the Fregean variety of Heyting algebras) is directly related to the loss of the Fregean property. In this situation the metalogical and algebraic properties of conjunction in intuitionistic logic and in Heyting algebras are split between the lattice operation of conjunction and a new operation, called *fusion*, which is residuated with respect to the implication.

5 Generalized Matrix Semantics and Full Models

As we mentioned in Section 3.1 the class of algebras $\mathbf{Alg}^*\mathcal{S}$, that is obtained from the classical matrix semantics for a logic $\mathcal{S}$ by reducing by the Leibniz congruence, does not necessarily coincide with the canonical class $\mathbf{Alg}\,\mathcal{S}$. The paradigm here because of its simplicity is the conjunction-disjunction fragment of classical logic (or of intuitionistic logic: they coincide). In this case $\mathbf{Alg}\,\mathcal{S}$ is the class of distributive lattices while $\mathbf{Alg}^*\mathcal{S}$ is a proper subclass of the class of distributive lattices, characterized in [80]; other examples are found in [70, 78, 96].

The tool we used to define the canonical class $\mathbf{Alg}\,\mathcal{S}$ of algebras of a logic is the Suszko congruence. According to expression (3.2), the Suszko congruence relative to a logic $\mathcal{S}$ of a matrix $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathcal{S}$ depends not only on F but also on the family of all $\mathcal{S}$-filters of $\boldsymbol{A}$ that include F:

$$\left[F\right)_{\mathcal{S}} = \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : F \subseteq G\} \tag{5.1}$$

This collection of $\mathcal{S}$-filters is a ***closure system*** (or ***closed-set system***) on A, that is, a family of subsets of A containing A and closed under intersections of arbitrary nonempty subfamilies. A closure system is ***inductive*** if it is closed under unions of subfamilies that are upwards directed by inclusion. For most purposes it is useful to think of the Suszko congruence as a function of the family of matrices $\{\langle \boldsymbol{A}, G\rangle : G \in \left[F\right)_{\mathcal{S}}\}$, or equivalently of the pair $\langle \boldsymbol{A}\,, \left[F\right)_{\mathcal{S}}\rangle$, rather than as a function of the single matrix $\langle \boldsymbol{A}, F\rangle$. In general, objects of the form $\mathcal{A} = \langle \boldsymbol{A}\,, \mathcal{C}\rangle$, where $\boldsymbol{A}$ is an algebra and $\mathcal{C}$ is an inductive closure system on the domain of $\boldsymbol{A}$ are called ***generalized matrices***, or ***g-matrices*** for short. For every finitary logic $\mathcal{S}$, the structure $\langle \boldsymbol{Fm}, \mathcal{T}h\,\mathcal{S}\rangle$ is, for instance, a generalized matrix, as are all structures of the form $\langle \boldsymbol{A}\,, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$. Generalized matrices appeared under this name for the first time in [140], and are called ***abstract logics*** in [33] and related papers, and in many papers of the algebraic logic group in Barcelona published before the mid-nineties, including [73].

In [73] a general theory of the algebraization of propositional logics is built around the central idea of using generalized matrices both as models for finitary logics and as models of Gentzen systems, thus establishing a bridge between the algebraic studies of these two notions. In particular, this includes another "canonical" way to associate a class of algebras with a given finitary logic $\mathcal{S}$, which uses the notion of the Tarski congruence of a g-matrix (to be defined later). The class of algebras obtained is exactly $\mathbf{Alg}\,\mathcal{S}$, the same as the class obtained by using the Suszko congruence.

Generalized matrices can also be presented in terms of consequence operations (also called closure operators). Given a closure system $\mathcal{C}$ on a set A, its associated consequence operation $\mathrm{Clo}_{\mathcal{C}}$ on A is defined by

$$\mathrm{Clo}_{\mathcal{C}}(X) = \bigcap\{F \in \mathcal{C} : X \subseteq F\},$$

for each $X \subseteq A$. This operation satisfies Tarski's axioms of Section 1.1, in particular it is finitary (or algebraic). Moreover, given a finitary consequence operation C on A, the family $\mathcal{C}_C$ of the ***C-closed*** subsets of A (i.e., those $X \subseteq A$ such that $C(X) = X$) is an inductive closure system on A, and hence the pair $\langle \boldsymbol{A}, \mathcal{C}_C \rangle$ is a g-matrix. These two correspondences are inverse to one another. The consequence operation of the g-matrix $\langle \boldsymbol{Fm}, \mathcal{T}h\,\mathcal{S} \rangle$ of a finitary logic $\mathcal{S}$ is clearly the consequence operation determined by the consequence relation of $\mathcal{S}$.

A g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is said to be a ***generalized model***, ***g-model*** for short, of a finitary logic $\mathcal{S}$ if every element of $\mathcal{C}$ is an $\mathcal{S}$-filter of $\boldsymbol{A}$. Equivalently, $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is a g-model of $\mathcal{S}$ iff its associated consequence operation is a ***model of the consequence operation of*** $\mathcal{S}$ in the following sense. For all $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\text{if } \Gamma \vdash_{\mathcal{S}} \varphi \text{ then } h(\varphi) \in \mathrm{Clo}_{\mathcal{C}}\big(h[\Gamma]\big) \text{ for every } h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}). \tag{5.2}$$

The g-matrix $\langle \boldsymbol{Fm}, \mathcal{T}h\,\mathcal{S} \rangle$ is obviously one of the g-models of $\mathcal{S}$. The class of g-models of $\mathcal{S}$ will be denoted by $\mathbf{GMod}\,\mathcal{S}$. For g-matrices we have the notion of completeness of a logic relative to a class of g-matrices analogous to the notion of completeness of a logic relative to a class of matrices. In this sense every logic is complete relative to $\mathbf{GMod}\,\mathcal{S}$.

Many concepts and constructions of the general theory of matrices have a natural counterpart in the theory of g-matrices. Given a g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$, its ***Tarski congruence***, which is denoted by $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\mathcal{C}$ or by $\widetilde{\boldsymbol{\Omega}}\mathcal{A}$, is the largest congruence of $\boldsymbol{A}$ compatible with every element of $\mathcal{C}$; it is easy to see that

$$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\mathcal{C} = \bigcap_{F \in \mathcal{C}} \boldsymbol{\Omega}_{\boldsymbol{A}}(F). \tag{5.3}$$

The function $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is the ***Tarski operator*** of $\boldsymbol{A}$. Observe that, like the Leibniz congruence $\boldsymbol{\Omega}_{\boldsymbol{A}}F$ of a matrix $\langle \boldsymbol{A}, F \rangle$, the Tarski congruence $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\mathcal{C}$ of a g-matrix is intrinsic to the g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$, while its Suszko congruence $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}F$ is not. The precise relationship between the Tarski and the Suszko congruences is the following. If $\langle \boldsymbol{A}, F \rangle$ is a matrix model of $\mathcal{S}$ then $\big\langle \boldsymbol{A}, \big[F\big)_{\mathcal{S}} \big\rangle$ is a g-model of $\mathcal{S}$ and

$$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big[F\big)_{\mathcal{S}} = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}F. \tag{5.4}$$

This justifies the choice of notation. The Suszko congruences appear as particular cases of the Tarski congruence, namely those of the closure systems of the form (5.1) for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

A g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is said to be ***reduced*** if its Tarski congruence is the identity. The class of reduced g-models of a logic $\mathcal{S}$ will be denoted by $\mathbf{GMod}^{*}\mathcal{S}$. It turns out that the class of algebra reducts of the members of $\mathbf{GMod}^{*}\mathcal{S}$ is $\mathbf{Alg}\,\mathcal{S}$ and that $\mathcal{S}$ is complete relative to the class $\mathbf{GMod}^{*}\mathcal{S}$.

A ***strict homomorphism*** from a g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ to a g-matrix $\mathcal{B} = \langle \boldsymbol{B}, \mathcal{D} \rangle$ is an $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ such that $\mathcal{C} = \big\{h^{-1}[F] : F \in \mathcal{D}\big\}$. The bijective strict homomorphisms are called ***isomorphisms*** and the surjective strict homomorphisms are called ***bilogical morphisms*** after [33]. Two g-matrices $\mathcal{A}$ and $\mathcal{B}$ are said to be ***bilogically morphic*** if there are bilogical morphisms from both $\mathcal{A}$ and $\mathcal{B}$ to the same g-matrix; in particular, $\mathcal{A}$ and $\mathcal{B}$ are bilogically morphic if there is a bilogical morphism from either one to the other. We see later that this is an equivalence relation. In AAL, and in logic without equality in general, the relation of being bilogically morphic plays much the same role that the isomorphism relation plays in logic with equality.

Given a g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$, its ***reduction*** is the g-matrix $\mathcal{A}^{*} = \big\langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}(\mathcal{A}), \mathcal{C}/\widetilde{\boldsymbol{\Omega}}\mathcal{A} \big\rangle$, where $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{A}$ is the quotient algebra and $\mathcal{C}/\widetilde{\boldsymbol{\Omega}}\mathcal{A} = \big\{F/\widetilde{\boldsymbol{\Omega}}\mathcal{A} : F \in \mathcal{C}\big\}$. A reduced g-matrix is isomorphic to its reduction, so every reduction is reduced. The projection homomorphism $\pi : \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{A}$ is a bilogical morphism from $\mathcal{A}$ onto $\mathcal{A}^{*}$. It follows that two g-matrices are bilogically morphic iff their reductions are isomorphic, and hence that being bilogically morphic is an equivalence relation.

The fundamental notion introduced in [73] is that of *full* g-model of a finitary logic. The class of full g-models of a logic is a natural subclass of the class of all g-models of a logic that has been useful in revealing interesting connections between the algebraic theory of logics and that of the Gentzen systems defining them. It has proved to be especially useful in the study of the algebraic theory of nonprotoalgebraic logics; among other things it has led to a new notion of algebraizability that is applicable to logics of this kind. During the 1980's, before the concept of full g-model emerged and its theory was developed, the algebraic logic group in Barcelona studied in detail the algebraic theory of several specific logics using g-matrices; [79] is a representative example of this work. The class of g-matrices that played the key role in each of these investigations turned out to be exactly the class of full g-models. This early work was the source for the concept of full g-model and an inspiration to the later development of the concept (see [73] for historical and bibliographical references).

A g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is a ***basic full g-model*** of a finitary logic $\mathcal{S}$ when $\mathcal{C} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and it is a ***full g-model*** of $\mathcal{S}$ if it is bilogically morphic to a basic full g-model of $\mathcal{S}$. In [73] it is proved that $\mathcal{A}$ is a full g-model iff there is a bilogical morphism from $\mathcal{A}$ onto a basic full g-model. Moreover, the reduction of every full g-model is a basic full g-model. In the metatheory of a logic $\mathcal{S}$ the basic full g-models clearly play an important role and by extension so do all full g-models. The class of all full g-models of a logic $\mathcal{S}$ is denoted by $\mathsf{FGMod}\,\mathcal{S}$. Every logic is complete relative to the class $\mathsf{FGMod}\,\mathcal{S}$, and also relative to the class $\mathsf{FGMod}^{*}\mathcal{S}$ of the reduced full g-models of $\mathcal{S}$. The class of algebra reducts of $\mathsf{FGMod}^{*}\mathcal{S}$ is $\mathsf{Alg}\,\mathcal{S}$, so the latter can be characterized by:

$$\boldsymbol{A} \in \mathsf{Alg}\,\mathcal{S} \quad \text{iff} \quad \langle \boldsymbol{A}\,, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle \in \mathsf{FGMod}^{*}\mathcal{S}. \tag{5.5}$$

An important property of the full g-models of a logic is given in the following theorem generalizing the lattice isomorphism characterizations of weakly algebraizable and algebraizable logics (Corollary 3.14); a noteworthy fact is the absence of any restricting assumptions on the logic, other than that it be finitary. The full g-models $\langle \boldsymbol{A}\,, \mathcal{C}\rangle$ of a logic $\mathcal{S}$ on the same algebra $\boldsymbol{A}$ can be identified with their closure system reduct $\mathcal{C}$. The family of these closure systems ordered by inclusion forms a lattice: the lattice of full g-models of $\mathcal{S}$ with the same underlying algebra $\boldsymbol{A}$.

Theorem 5.1 ([73]) *For every finitary logic $\mathcal{S}$ and every algebra $\boldsymbol{A}$, the Tarski operator $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is a dual isomorphism between the lattice of full g-models of $\mathcal{S}$ on $\boldsymbol{A}$ and the lattice of the $(\mathsf{Alg}\,\mathcal{S})$-congruences of $\boldsymbol{A}$.*

The relation between the semantics of full g-models and the logical matrix semantics in the case of finitary protoalgebraic logics is as follows. If $\mathcal{S}$ is a finitary protoalgebraic logic and $\boldsymbol{A}$ is any algebra, an $\mathcal{S}$-filter F of $\boldsymbol{A}$ is said to be a ***Leibniz filter*** if it is the smallest among all $\mathcal{S}$-filters of $\boldsymbol{A}$ whose Leibniz congruence coincides with $\boldsymbol{\Omega}_{\boldsymbol{A}}F$ (such a smallest filter always exists by protoalgebraicity). Weakly algebraizable logics can be characterized as the protoalgebraic logics $\mathcal{S}$ with the property that for every algebra all its $\mathcal{S}$-filters are Leibniz filters. Leibniz filters are studied in [74, 97] and also appear in [73] although not under this name. They can be used to characterize the full g-models of finitary protoalgebraic logics in a way that also yields a characterization of finitary protoalgebraic logics.

Theorem 5.2 *If $\mathcal{S}$ is a finitary logic, then $\mathcal{S}$ is protoalgebraic iff the full g-models of $\mathcal{S}$ are the g-models of $\mathcal{S}$ of the form $\langle \boldsymbol{A}\,, [F)_{\mathcal{S}}\rangle$ where F is Leibniz.*

Moreover, if $\mathcal{S}$ is protoalgebraic, for each full g-model $\langle \boldsymbol{A}, [F)_{\mathcal{S}}\rangle$ of $\mathcal{S}$, (5.4) becomes $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}[F)_{\mathcal{S}} = \boldsymbol{\Omega}_{\boldsymbol{A}}F$. Clearly, if $\mathcal{S}$ is in addition weakly algebraizable then the full g-models of $\mathcal{S}$ are exactly all the g-models $\langle \boldsymbol{A}\,, [F)_{\mathcal{S}}\rangle$ where F is an arbitrary $\mathcal{S}$-filter. Thus, the isomorphism theorem (Corollary 3.14) that characterizes weakly algebraizable and algebraizable logics can be obtained from Theorem 5.1.

One of the most interesting aspects of g-matrices is how they can be used in completely natural ways both as models of finitary logics and as models of Gentzen systems. This double nature allows one to tie the algebraic theory of logics to that of Gentzen systems. We proceed to explain the situation. The Gentzen systems that define logics in the most natural way are those that satisfy all the structural Gentzen rules; as in Section 4.2 we call them *structural Gentzen systems.* In fact we will be interested only in the structural Gentzen systems whose type is either the set $\{(n,1) : n \in \omega\}$ or the set $\{(n,1) : n \geq 1\}$ (see Section 4.2). From now on the expression ***Gentzen system*** will be used to refer to these kind of Gentzen systems without further qualification. Given a Gentzen system $\mathcal{G}$, the logic $\mathcal{S}_\mathcal{G}$ defined by $\mathcal{G}$ is the finitary logic determined by the derivable sequents of $\mathcal{G}$ as in (4.1): $\Gamma \vdash_{\mathcal{S}_\mathcal{G}} \varphi$ iff $\emptyset \rhd \varphi$ is derivable or there are $\varphi_0, \ldots, \varphi_{n-1} \in \Gamma$ such that $\varphi_0, \ldots, \varphi_{n-1} \rhd \varphi$ is derivable. It is not difficult to see that the same finitary logic can be the logic of more than one Gentzen system. An interesting issue then is to find natural criteria for selecting one of these Gentzen systems among all others. The full g-models lead to one such criterion of an algebraic character.

A g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is a ***model of a Gentzen system*** $\mathcal{G}$ if, for every assignment $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, every set $\{\overline{\varphi}_i \rhd \psi_i : i \in I\}$ of sequents and every sequent $\overline{\varphi} \rhd \psi$ such that $\{\overline{\varphi}_i \rhd \psi_i : i \in I\} \vdash_\mathcal{G} \overline{\varphi} \rhd \psi$,

$$\text{if } \; h(\psi_i) \in \mathrm{Clo}_\mathcal{C}\big(h(\overline{\varphi}_i)\big) \text{ for all } i \in I \,, \;\text{ then }\; h(\varphi) \in \mathrm{Clo}_\mathcal{C}\big(h(\overline{\varphi})\big).$$

Comparing this with (5.2) we see almost immediately that, if a g-matrix is a model of a Gentzen system $\mathcal{G}$, then it is a g-model of the associated finitary logic $\mathcal{S}_\mathcal{G}$.

Generalized matrices admit a relational presentation. For each g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ the relation $\mathrm{R}_\mathcal{A}$ between finite sequences and individual elements of A is defined as follows.

$$\big\langle \langle a_0, \ldots, a_{n-1} \rangle, a_n \big\rangle \in \mathrm{R}_\mathcal{A} \quad \text{iff} \quad a_n \in \mathrm{Clo}_\mathcal{C}\big(\{a_0, \ldots, a_{n-1}\}\big).$$

Note that $\langle \boldsymbol{A}, \mathrm{R}_\mathcal{A} \rangle$ is a G-matrix of type $\{(n,1) : n \geq 1\}$ in the sense of Section 4.2. $\mathcal{A}$ is a model of a Gentzen system $\mathcal{G}$ iff the G-matrix $\langle \boldsymbol{A}, \mathrm{R}_\mathcal{A} \rangle$ is a model of $\mathcal{G}$ (see Section 4.2). If we apply to $\langle \boldsymbol{A}, \mathrm{R}_\mathcal{A} \rangle$ the process described in Section 4.3 we obtain a structure of a suitable first-order language without equality that can be viewed as a first-order representation of the models of the Gentzen system $\mathcal{G}$. This duality in the presentation of g-matrices has been exploited in [75, 76]; some results are mentioned below.

In order to connect the algebraic theory of logics with that of Gentzen systems through the full g-models we say that a finitary logic $\mathcal{S}$ has a ***fully adequate Gentzen system*** if the class of its full g-models is the class of models of some Gentzen system. (In case $\mathcal{S}$ does not have theorems, a technical adjustment in the definition is required, but this can be disregarded for the purpose of this survey.) If a fully adequate Gentzen system $\mathcal{G}$ for $\mathcal{S}$ exists, it is clearly unique, and $\mathcal{S}$ can be recovered from it in the sense that $\{\varphi_0, \ldots, \varphi_{n-1}\} \vdash_\mathcal{S} \psi$ iff the sequent $\vdash_\mathcal{G} \varphi_0, \ldots, \varphi_{n-1} \rhd \psi$ is derivable in $\mathcal{G}$. The property of having a fully adequate Gentzen system seems to be related to a number of different issues in AAL. For instance, [75] gives:

Theorem 5.3 *A finitary logic has a fully adequate Gentzen system iff the class of first-order structures corresponding to its full g-models is closed under* $\mathbb{S}$ *and* $\mathbb{P}_R$.

A different characterization, of more restricted application, is obtained in [76]:

Theorem 5.4 *A finitary weakly algebraizable logic has a fully adequate Gentzen system iff it satisfies the multiterm* DDT.

The existence of a fully adequate Gentzen system leads to a new notion of algebraizability for logics. A finitary logic is ***G-algebraizable*** if it has a fully adequate Gentzen system that is algebraizable (in the sense of Section 4.2). If $\mathcal{S}$ is a finitary logic that is G-algebraizable, then the associated fully adequate and algebraizable Gentzen system is unique and its equivalent algebraic semantics coincides with $\mathsf{Alg}\,\mathcal{S}$. This new concept is not an extension of the notion of algebraizable logic, as there are algebraizable logics that are not G-algebraizable and vice versa, but it allows one to establish a very strong link between the G-algebraizable logics and their associated class of algebras $\mathsf{Alg}\,\mathcal{S}$. Observe that if $\mathcal{G}$ is the fully adequate Gentzen system for a logic $\mathcal{S}$ then (5.5) becomes:

$$\boldsymbol{A} \in \mathsf{Alg}\,\mathcal{S} \quad \text{iff} \quad \langle \boldsymbol{A}, \mathcal{F}i_\mathcal{S}\boldsymbol{A} \rangle \text{ is a reduced model of } \mathcal{G}. \tag{5.6}$$

Being a model of a Gentzen system amounts to being a model of any Gentzen calculus that defines it, and $\mathcal{S}$ will satisfy all the rules of this calculus. Hence the interesting problem is to find Gentzen calculi for $\mathcal{G}$ that give some insight into the structure of the algebras in $\mathsf{Alg}\,\mathcal{S}$ via the properties of the closure system of the $\mathcal{S}$-filters on them. The best results have been obtained for logics $\mathcal{S}$ such that the $\mathcal{S}$-filters on the $\mathcal{S}$-algebras form a closure system intrinsically associated with the algebraic structure. For instance, in many cases where the algebras in $\mathsf{Alg}\,\mathcal{S}$ have a lattice reduct, the set of $\mathcal{S}$-filters in these algebras is the set of their lattice filters.

In a sense that we hope further research will make more precise, one can say that G-algebraizable logics are those logics whose class of $\mathcal{S}$-algebras is determined by the properties of the logic that can be formulated by Gentzen-style rules, and the Gentzen system defined by these rules is algebraizable. There is considerable empirical evidence supporting this view in the analysis of examples, and in some partial results of [73] of a more general nature concerning selfextensional logics, a class that includes protoalgebraic logics as well as nonprotoalgebraic logics.

The first papers, prior to the the present conceptual framework developed in [73], in which Gentzen-style rules and g-matrices are exploited for this purpose are [80, 123]. Among other logics, the disjunction-conjunction fragment of $\mathcal{CPL}$ (or of $\mathcal{IPL}$) is treated; let us denote it by $\mathcal{S}_{\vee\wedge}$. It is the simplest natural example of a G-algebraizable logic. The following results are obtained.

1. $\mathsf{Alg}\,\mathcal{S}_{\vee\wedge}$ is the class of distributive lattices, and the $\mathcal{S}_{\vee\wedge}$-filters on a distributive lattice are its ordinary (lattice) filters.
2. $\mathcal{S}_{\vee\wedge}$ is G-algebraizable, and its fully adequate Gentzen system is defined by the calculus consisting of all structural rules plus the usual Gentzen rules for disjunction and conjunction in LJ.
3. The equivalent algebraic semantics of this Gentzen system is the class of distributive lattices, and the translations that establish the equivalence of the Gentzen system and the equational logic of distributive lattices are those given in (4.3) and (4.4).
4. An algebra $\boldsymbol{A}$ is a distributive lattice (that is, it belongs to $\mathsf{Alg}\,\mathcal{S}_{\vee\wedge}$) iff there is an inductive closure system $\mathcal{C}$ on A satisfying, for all $a\,,b \in A$ and all $X \subseteq A$:

$$\mathrm{Clo}_{\mathcal{C}}\big(\{a \wedge b\}\big) = \mathrm{Clo}_{\mathcal{C}}\big(\{a\,,b\}\big). \tag{5.7}$$

$$\mathrm{Clo}_{\mathcal{C}}\big(X \cup \{a \vee b\}\big) = \mathrm{Clo}_{\mathcal{C}}\big(X \cup \{a\}\big) \cap \mathrm{Clo}_{\mathcal{C}}\big(X \cup \{b\}\big). \tag{5.8}$$

$$\mathrm{Clo}_{\mathcal{C}}\big(\{a\}\big) = \mathrm{Clo}_{\mathcal{C}}\big(\{b\}\big) \text{ implies } a = b. \tag{5.9}$$

In particular, (5.7)–(5.9) hold in every reduced full g-model $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ of $\mathcal{S}_{\vee\wedge}$.

Conditions similar to these are called *Tarski-style conditions* by Wójcicki [142] and are the direct translation of the property of being a model of the Gentzen-style rules mentioned in item 2 above. In other cases this translation can be more complicated and involve a number of different aspects of the closure operator or the closure system, such as having a basis with certain properties, etc. These results have motivated considerable interest in trying to find characterizations of the full g-models of various logics that are interesting for either metalogical, algebraic or topological reasons; see for example [70, 77, 78, 95, 96]. A summary of the earliest contributions in the area can be found in Chapter 5 of [73].

Condition (5.9) is another manifestation of the Frege principle. The notion of the Frege relation of a logic can be naturally generalized to g-matrices. Given a g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C}\rangle$, its ***Frege relation***, denoted by $\boldsymbol{\Lambda}\mathcal{A}$, is defined by:

$$\langle a, b\rangle \in \boldsymbol{\Lambda}\mathcal{A} \quad \text{iff} \quad \mathrm{Clo}_{\mathcal{C}}\big(\{a\}\big) = \mathrm{Clo}_{\mathcal{C}}\big(\{b\}\big).$$

It is easy to check that $\widetilde{\boldsymbol{\Omega}}\mathcal{A}$ is the largest congruence of $\boldsymbol{A}$ included in $\boldsymbol{\Lambda}\mathcal{A}$. A g-matrix $\mathcal{A}$ has ***the congruence property*** when its Frege relation is a congruence, or equivalently when $\widetilde{\boldsymbol{\Omega}}\mathcal{A} = \boldsymbol{\Lambda}\mathcal{A}$; thus a logic $\mathcal{S}$ is selfextensional iff, when presented as the g-matrix $\langle \boldsymbol{Fm}\,, \mathcal{T}h\,\mathcal{S}\rangle$, it has the congruence property. It is easy to see that a g-matrix $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ satisfies condition (5.9) iff it has the congruence property and is reduced. Thus, since all reduced full g-models of the logic $\mathcal{S}_{\vee\wedge}$ satisfy condition (5.9) (by item 4 above), they all have the congruence property. Thus all full g-models of $\mathcal{S}_{\vee\wedge}$ have the congruence property since the latter is preserved under bilogical morphisms. So $\mathcal{S}_{\vee\wedge}$ is selfextentional and the congruence property transfers from the logic to its full g-models. From (5.7) and (5.8) we also have that the basic properties of "conjunction" and "disjunction" also transfer in this way. Thus for the properties of congruence, conjunction and disjunction, the so-called transfer problem has a positive solution for $\mathcal{S}_{\vee\wedge}$. The *transfer problem* for a given metalogical property is to find general conditions under which the property is guaranteed to transfer from the logic to its full g-models; this has turned out to be an important problem in AAL. A binary connective $\wedge$ is said to be a ***conjunction*** for a logic or a g-matrix when condition (5.7) is satisfied by the associated consequence operation. A partial solution to the transfer problem for the congruence property is contained in the following result.

Theorem 5.5 ([73]) *Let $\mathcal{S}$ be a selfextensional finitary logic $\mathcal{S}$ with either a conjunction or the uniterm DDT. Then:*

1. *The Frege relation of its full g-models is a congruence, i.e., the congruence property transfers to full g-models.*
2. *$\mathcal{S}$ is G-algebraizable, i.e., $\mathcal{S}$ has an algebraizable fully adequate Gentzen system whose equivalent algebraic semantics is $\mathsf{Alg}\,\mathcal{S}$.*
3. *$\mathsf{Alg}\,\mathcal{S}$ is the variety generated by the class of the Lindenbaum-Tarski algebras of $\mathcal{S}$.*

Part 3 of this theorem, when applied to finitary and finitely algebraizable logics, explains why so many algebraizable logics have a variety as their equivalent algebraic semantics when the general theory only guarantees that it is a quasivariety: it is in large part a consequence of the Frege Principle, which is reflected in the fact that most of the logics of classical algebraic logic are selfextentional. But some additional property, like the existence of a conjunction or the uniterm DDT, is also needed. It can be shown that a variety is of the form $\mathsf{Alg}\,\mathcal{S}$ for an $\mathcal{S}$ satisfying the assumptions in the previous theorem iff it has a semilattice reduct or a Hilbert-algebra reduct. A similar description of the varieties appearing when 'selfextensional' is replaced by 'Fregean' in the above theorem is obtained in [53, 54]. Several related problems left open in [73] have been recently solved by Babyonyshev [15] and independently by Bou [32]. Among other things it has been proved that neither the congruence property nor that of being Fregean transfer in general.

6 Extensions to More Complex Notions of Logic

All the work in AAL discussed so far builds on the formalization of the intuitive notion of a *logic* as a *deductive system*, that is as a pair $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ where only the language ($\boldsymbol{Fm}$) and the consequence relation ($\vdash_{\mathcal{S}}$) play a role. Thus, the process of algebraizing a logic $\mathcal{S}$ depends entirely on its consequence relation $\vdash_{\mathcal{S}}$ regardless of the particular way the latter is defined. In particular, semantical considerations influence the process only indirectly. For some purposes however this "consequence-based" conception of logic is oversimplified as there are important aspects of the algebraization process that cannot be captured by consideration of the consequence relation only. For these the underlying semantics of the logic must also be taken directly into account by including an abstract notion of model in the formalization. By taking logic to be a pair $\langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ we bind ourself to a single, fixed language, and there are many who would view this as another source of oversimplification. From this point of view any proper notion of logic should be able to deal with different presentations of essentially the same logic in different languages.

In this section we discuss two approaches to AAL designed to overcome such oversimplifications. The origin of the first can be traced back to the theory of relations developed by De Morgan, Peirce, Schroeder, and Tarski (mentioned briefly in the Introduction); a closer ancestor is the algebraization of the first-order predicate logic that leads to cylindric and polyadic algebras. The second approach focuses on arguably the most important aspect of a logic from the standpoint of AAL, its substitutional invariance. It is embodied here in the notion of a logic as a category, more precisely in the category-theoretic notion of a institution. It also incorporates the viewpoint that a logic should be invariant under certain linguistic transformations, such as the selection of the primitive logical connectives. These two approaches can be described respectively as "semantics-based" and "categorical" and are briefly described in Sections 6.1 and 6.2 below, where due references are given. An ambitious program that attempts to incorporate both of them, together with the consequence-based approach into one coherent theory of AAL, has been initiated by Diskin [59]. Here again category theory provides the basic framework and some methods of abstract model theory are used. We will not discuss this program further.

In both the semantics-based and categorical approaches a substitution-invariant consequence relation is naturally associated with the logic, and hence the Lindenbaum-Tarski process is an integral part of the algebraization. Consequently, most of the notions of the previous sections (e.g. protoalgebraizability, algebraizability, etc.) also apply in the more complex environments of the present section.

6.1 The semantics-based approach

This approach has been under continuous development since Andréka's 1973 Ph. D. dissertation [9] on the algebraization of first-order logic, and has its roots in the even earlier work on cylindric and relation algebras previously mentioned. Special cylindric algebras arise directly from first-order logic in two ways. One is by means of the familiar Lindenbaum-Tarski process. Since the sentential part of the logic is classical, the relation $\equiv_{\Sigma}$ associated with a first-order theory Σ as in (1.6) is a congruence on the formula algebra (of first-order formulas) and the quotient algebra is a cylindric algebra of (sets of) formulas specifically associated with the theory Σ; it is the free algebra of the equational theory of cylindric algebras that is naturally associated with Σ. It is a Boolean algebra that has been enriched by unary operations corresponding to the quantifications $\exists v_i$ and by constants corresponding to the atomic formulas $v_i \approx v_j$. A cylindric algebra can also be obtained from every model (in the first-order sense) of Σ. Let $\mathfrak{A}$ be such a model. To each first-order formula φ is associated the set $\varphi^{\mathfrak{A}}$ of all ω-sequences of elements of A (the domain of $\mathfrak{A}$) that satisfy φ in $\mathfrak{A}$, that is, $\varphi^{\mathfrak{A}} = \{\overline{a} \in A^{\omega} : \mathfrak{A} \models \varphi\ [\overline{a}]\}$. The set $\{\varphi^{\mathfrak{A}} : \varphi \text{ a formula}\}$ has the structure of a cylindric algebra of sets of a special kind. It is a model of the equational theory associated with Σ.

AAL in this context had its origins in the investigation of the relation between logical properties of first-order theories and algebraic properties of the corresponding varieties of cylindric algebras. The algebraic form of certain versions of the Beth definability property for a first-order theory Σ can only be formulated in terms of the special cylindric-set algebras in the variety associated with Σ that are obtained from the models of Σ in the way indicated above. It is thus essential to incorporate an abstract notion of model in the formalization of the notion of logic that underlies AAL in this context. The article [11] is an excellent and detailed exposition of this approach; we survey only the essential features and the connections with the rest of this survey of AAL.

By a ***semantics-based logic*** we mean a four-tuple $L = \langle \boldsymbol{Fm}, \mathsf{M}, mng, \models\rangle$ where $\boldsymbol{Fm}$ is a formula algebra of some logical language $\mathcal{L}$ over some set of variables; M is a class called the class of ***models***; mng is a function that assigns to each model $\mathfrak{A} \in \mathsf{M}$ a function $mng_{\mathfrak{A}}$ with domain $\boldsymbol{Fm}$ and unspecified range, called the ***meaning function*** of $\mathfrak{A}$; and $\models\ \subseteq \mathsf{M} \times \boldsymbol{Fm}$ is a binary relation called the ***validity relation***; moreover, the following conditions hold.

(1) L satisfies Frege's *principle of compositionality* in the sense that the meaning of a formula does not change if a subformula is replaced by one with the same meaning; i.e., $mng_{\mathfrak{A}}$ is a homomorphism, and hence its range $mng_{\mathfrak{A}}[\boldsymbol{Fm}]$ is an $\mathcal{L}$-algebra, called the ***meaning algebra*** of $\mathfrak{A}$.

(2) L has the *semantical substitution property*: For every substitution σ in $\boldsymbol{Fm}$ and every $\mathfrak{A} \in \mathsf{M}$ there is a $\mathfrak{B} \in \mathsf{M}$ such that $mng_{\mathfrak{B}} = mng_{\mathfrak{A}} \circ \sigma$. This is equivalent to saying that for every $\mathfrak{A} \in \mathsf{M}$ and every $h \in \mathrm{Hom}(\boldsymbol{Fm}, mng_{\mathfrak{A}}[\boldsymbol{Fm}])$ there is some $\mathfrak{B} \in \mathsf{M}$ such that $h = mng_{\mathfrak{B}}$. For some purposes one needs the *enhanced semantical substitution property*, which requires $\mathfrak{B}$ to satisfy the additional condition: $\mathfrak{A} \models \sigma(\varphi)$ iff $\mathfrak{B} \models \varphi$ for all $\varphi \in \boldsymbol{Fm}$, see [72].

(3) *The validity of a formula in a model depends only on its meaning in that model*, that is, if $mng_{\mathfrak{A}}(\varphi) = mng_{\mathfrak{A}}(\psi)$ then $\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \psi$.

In some quarters the meaning function is interpreted as the *intensional* aspect of the semantics, and the validity relation as its *extensional* side. The imposition of connections of different kinds between these two elements gives rise to various interesting classes of logics.

For some purposes an independent consequence relation (denoted as $\vdash$) is adjoined to the data that constitutes the logic; this is intended to represent a proof system for the logic. In most cases however this coincides with the consequence relation defined from the validity relation in the natural way.

Two classes of algebras are associated with each logic L in this context. The first class reflects the special role of the models and consists of their meaning algebras:

$$\mathsf{Alg}_m(L) = \big\{ mng_{\mathfrak{A}}[\boldsymbol{Fm}] : \mathfrak{A} \in \mathsf{M} \big\} \tag{6.1}$$

Notice that this class need not be closed under isomorphisms. These are the "concrete" algebras that arise directly from the models of L.

The more abstract algebras of L are obtained by a semantical variant of Lindenbaum-Tarski process in which the formula algebra is factored by an equivalence relation associated with an arbitrary class of models. More precisely, two formulas are taken to be equivalent modulo a subclass K of M if they have the same meaning in each member of K. We take:

$$\mathsf{Alg}(L) = \mathbb{I}\big(\{\boldsymbol{Fm}/{\sim_K} : K \subseteq \mathsf{M}\}\big) \tag{6.2}$$

where $\varphi \sim_K \psi$ iff $mng_{\mathfrak{A}}(\varphi) = mng_{\mathfrak{A}}(\psi)$ for all $\mathfrak{A} \in K$; note that $\sim_K$ is a congruence on $\boldsymbol{Fm}$ because the meaning function is a homomorphism. The algebras of the form $\boldsymbol{Fm}/{\sim_K}$ are called in this framework the "Lindenbaum-Tarski algebras" of L.

Referring again to first-order logic, the paradigm for a semantics-based logics, we see that the logical language $\mathcal{L}$ of a semantics-based logic is intended to abstract the logical connectives of first-order logic, that is the sentential connectives, quantifications, and equalities between variables. The extralogical predicates of a first-order language on the other hand are abstracted by the variables (which in this context should be thought of as sentential variables rather than individual variables); more properly, the variables are intended to abstract the atomic formulas from which the extralogical predicates are indirectly extracted. (The details of this process in the special case of the semantics-based logic associated with cylindric algebras are given in Appendix C of [23].)

The concept of first-order logic as a collection of logics, one for each choice of extralogical predicate symbols, has its abstract analogue in the notion of a ***general semantics-based logic***. This is a function $\mathbf{L} = \langle L^P : P \in \mathbb{C}\rangle$ where $\mathbb{C}$ is some class of sets, containing at least one of each cardinality. For each $P \in \mathbb{C}$, L^P is a semantics-based logic, in the preceding sense, and $\boldsymbol{Fm}^P$ is the formula algebra of a fixed, common language $\mathcal{L}$ over the set of variables P (the abstraction of the atomic formulas of a first-order language). In addition the various component logics are related by some natural compatibility conditions that abstract the transformations between first-order languages over different sets of predicate symbols that are induced by defining the predicates of one as formulas in the other: Each mapping from P to $\boldsymbol{Fm}^Q$, besides extending to a unique homomorphism from $\boldsymbol{Fm}^P$ to $\boldsymbol{Fm}^Q$, also induces a mapping between the models of L^Q and those of L^P that commutes with the meaning functions in a natural way. In particular, if $P, Q \in \mathbb{C}$ have the same cardinality, there is an isomorphism between $\boldsymbol{Fm}^P$ and $\boldsymbol{Fm}^Q$ that commutes with mng and with $\models$, and if $P \subseteq Q$, then every model of L^P extends to a model of L^Q and every model of L^Q restricts to a model of L^P in natural ways. The associated algebras of a general semantics-based logic L are defined as follows.

$$\mathsf{Alg}_m \mathbf{L} = \bigcup_{P \in \mathbb{C}} \mathsf{Alg}_m L^P \qquad \text{and} \qquad \mathsf{Alg}\, \mathbf{L} = \bigcup_{P \in \mathbb{C}} \mathsf{Alg}\, L^P.$$

In [10, 11] several examples of well-known sentential and first-order logics are presented as general semantics-based logics, and it is checked that $\mathsf{Alg}\,\mathbf{L}$ always is the expected class of algebras.

Moreover the following fundamental theorem is proved.

Theorem 6.1 *For each general semantics-based logic* $\mathbf{L}$*,* $\mathsf{Alg}\,\mathbf{L} = \mathbb{S}\,\mathbb{P}\,\mathsf{Alg}_m\,\mathbf{L}$*.*

This is important in that it gives a uniform and purely algebraic way of obtaining the class $\mathsf{Alg}\,\mathbf{L}$, the abstract or general algebraic counterpart of $\mathbf{L}$, from the class $\mathsf{Alg}_m\,\mathbf{L}$ of the concrete algebras arising from the specific semantical presentation of $\mathbf{L}$; it also confirms that in turn $\mathsf{Alg}\,\mathbf{L}$ is independent of such presentation.

Some connections with the consequence-based approach to AAL can be readily established, even in the case of a specific (non-general) semantics-based logic L, if the semantical substitution property is taken in its enhanced form. First, the validity relation naturally engenders a consequence relation: $\Gamma \models_L \varphi$ iff φ is valid in all models in M in which all formulas of Γ are valid. Then $\mathcal{S}_L = \langle \boldsymbol{Fm}\,, \models_L \rangle$ is a consequence-based logic (deductive system) in the sense of Section 1.1 (substitution-invariance is assured by the enhanced semantical substitution property), and we can apply to it the theory developed in the preceding sections. Accordingly one can say that L is *protoalgebraic, equivalential, algebraizable, etc.*, when $\mathcal{S}_L$ has the property. Secondly, each model of L gives rise to a logical matrix in a natural way: if $\mathfrak{A} \in \mathsf{M}$, then the *theory of* $\mathfrak{A}$ is defined as the set $Th(\mathfrak{A}) = \{\varphi \in Fm : \mathfrak{A} \models \varphi\}$, and the *meaning matrix of* $\mathfrak{A}$ is the matrix $\mathcal{M}_{\mathfrak{A}} = \big\langle mng_{\mathfrak{A}}[\boldsymbol{Fm}]\,, mng_{\mathfrak{A}}\big[Th(\mathfrak{A})\big]\big\rangle$. If one considers its Leibniz reduction $\mathcal{M}^{*}_{\mathfrak{A}}$ then one can prove results like the following.

Theorem 6.2 *Let L be a semantics-based logic and $\mathfrak{A}\,,\mathfrak{B}$ two of its models. Then $\mathfrak{A}$ and $\mathfrak{B}$ are* ***elementarily equivalent*** (*i.e., they have the same theory*) *iff there is an isomorphism h between the reduced matrices $\mathcal{M}^{*}_{\mathfrak{A}}$ and $\mathcal{M}^{*}_{\mathfrak{B}}$ that commutes with the meaning functions, i.e., such that* $mng_{\mathfrak{B}} = h \circ mng_{\mathfrak{A}}$*.*

If L is one of the presentations of first-order logic (with a specific set of predicate symbols) that fits the schema of a semantics-based logic, then this theorem provides a way of investigating elementary equivalence algebraically.

A similar process can be followed in dealing with a general semantics-based logic $\mathbf{L}$. For most purposes it is sufficient to work with the logic L^P for a a fixed denumerable P; for instance one defines the associated deductive system $\mathcal{S}_{\mathbf{L}}$ to be $\mathcal{S}_{L^P}$ for such a P, and a general semantics-based logic is called *protoalgebraic, equivalential, algebraizable, etc.* when $\mathcal{S}_{L^P}$ has the property; it can be shown that this is independent of the choice of P.

Starting from a consequence-based logic $\mathcal{S}$ in the sense of Section 1.1, one can associate with it a general semantics-based logic using its matrix semantics in the following way. For each cardinal $\kappa \geq 2$, we put $L^{\kappa}_{\mathcal{S}} = \langle \boldsymbol{Fm}^{\kappa}\,, \mathsf{M}^{\kappa}\,, mng^{\kappa}\,, \models^{\kappa} \rangle$, where

> $\boldsymbol{Fm}^{\kappa}$ is the formula algebra of type $\mathcal{L}$ over a set of variables of cardinality κ,
>
> $\mathsf{M}^{\kappa} := \big\{ \langle \boldsymbol{A}\,, F\,, h\rangle : \langle \boldsymbol{A}\,, F\rangle \in \mathsf{Mod}^{*}\mathcal{S} \text{ and } h \in \mathrm{Hom}(\boldsymbol{Fm}^{\kappa}\,, \boldsymbol{A}) \big\}$,
>
> $mng^{\kappa}_{\langle \boldsymbol{A}\,,F\,,h\rangle}(\varphi) := h(\varphi)$, and
>
> $\langle \boldsymbol{A}\,, F\,, h\rangle \models^{\kappa} \varphi \quad \text{iff} \quad h(\varphi) \in F$.

In [72] it is shown that $\mathbf{L}_{\mathcal{S}} = \langle L^{\kappa}_{\mathcal{S}} : \kappa \geq 2\rangle$ is a general semantics-based logic, that its associated consequence-based logic $\mathcal{S}_{\mathbf{L}_{\mathcal{S}}}$ coincides with $\mathcal{S}$, and that the class $\mathsf{Alg}\,\mathbf{L}_{\mathcal{S}}$ coincides with $\mathsf{Alg}\,\mathcal{S}$. It is clear that the reverse process cannot yield parallel results by its nature: if we start from a particular general semantics-based logic $\mathbf{L}$, then we may lose information in passing to $\mathcal{S}_{\mathbf{L}}$ since the reduced matrix semantics $\mathbf{L}_{\mathcal{S}_{\mathbf{L}}}$ of the latter need bear no relation to the original semantics of the former.

The kind of general properties that have been studied for these semantics-based logics include soundness and completeness, compactness relative to consequence or to satisfiability, finite axiomatizability, decidability, the relation between interpolation and amalgamation, and the relation between Beth-like properties and conditions on surjectivity of epimorphisms in several senses, etc. Since every consequence-based logic can be interpreted in a natural way as a semantics-based logic, essentially everything that can be done with the former can be transported to the latter. Conversely, a large part of the theory of a semantics-based logic depends only on its associated consequence relation, and hence can be formulated entirely in terms of the associated consequence-based logic. Generally speaking, most of the examples that were originally studied within the semantics-based framework correspond to algebraizable consequence-based logics. See [11, 72] for details.

6.2 The categorial approach

All the logics we have considered so far have had as their basic syntactical elements formulas (or finite sequences of formulas) that are constructed recursively from a fixed set of logical connectives and (sentential) variables, or atomic formulas, in the familiar way. A minor generalization allows for the set of atomic formulas to be varied, as for example in the case of general semantics-based logics considered in Section 6.1 (a similar generalization was made in [52]). These generalizations still keep the logical language, i.e., the set of logical connectives, fixed. However, the informal notion of logic is not bound so rigidly to the language in which it is formalized: when considering the different formalizations of, say, classical logic, by taking different sets of primitive connectives, one actually thinks of them as *different presentations of the same logic*. This viewpoint calls for a notion of logic that incorporates a variety of languages together with mappings between them induced by certain translations of one language into another. Another problem with restricting the languages to essentially sentential languages is the awkwardness this introduces in the algebraization of quantifier logics. The first-order predicate logic, in its standard formalization, cannot be algebraized by the methods discussed so far because it fails to be substitution-invariant. To be so, it must be first reformulated as a substitution-invariant logic in the sense of Section 1.1, and this entails its radical transformation into what is essentially a sentential logic (see Appendix C of [23] for more details). Finally, recent developments in the formal languages of computation have led to even greater levels of generalization and to the consideration of category theory as the appropriate framework for the kind of AAL that seems to emerge from these considerations.

One proposal in this direction is the theory of institutions, which was created by Goguen and Burstall [85] to formalize the relationship between language and semantics at a more abstract level than abstract model theory. An ***institution*** is a four-tuple $\langle \mathbf{SIGN}, sen, Mod, \models \rangle$ where $\mathbf{SIGN}$ is a category whose objects are called ***signatures***, $sen : \mathbf{SIGN} \to \mathbf{SET}$ is a functor associating with each signature $\mathcal{L}$ a corresponding set $sen(\mathcal{L})$ of ***$\mathcal{L}$-sentences***, $Mod : \mathbf{SIGN} \to \mathbf{CAT}^{op}$ is a contravariant functor assigning to each signature $\mathcal{L}$ a category whose objects are called ***$\mathcal{L}$-models***, and $\models$ is a function, called ***$\mathcal{L}$-satisfaction***, that assigns to each signature $\mathcal{L}$ in $\mathbf{SIGN}$ a binary relation $\models_{\mathcal{L}}$ between the objects of $Mod(\mathcal{L})$ and the members of $sen(\mathcal{L})$; $\mathcal{L}$-satisfaction is required only to satisfy the condition that, for each arrow $f : \mathcal{L} \to \mathcal{L}'$ in $\mathbf{SIGN}$, each object M' of $Mod(\mathcal{L}')$ and each $\varphi \in sen(\mathcal{L})$, $M' \models_{\mathcal{L}'} sen(f)(\varphi)$ iff $Mod(f)(M') \models_{\mathcal{L}} \varphi$.

The reader can check that this is very similar to the notion of a general semantics-based logic, without the intensional component represented by the meaning functions, and with the formula algebras $\boldsymbol{Fm}^P$, with $P \in \mathbb{C}$, together with the homomorphisms between them replaced by an abstract category.

The closely related notion of a ***π-institution*** has been proposed by Fiadeiro and Sernadas [68]. It is a triple $\mathcal{I} = \langle \mathbf{SIGN}, sen, \vdash \rangle$ where $\mathbf{SIGN}$ and sen are the same as for institutions, and $\vdash$ is a function associating with each signature $\mathcal{L}$ a relation $\vdash_{\mathcal{L}} \subseteq \mathcal{P}\big(sen(\mathcal{L})\big) \times sen(\mathcal{L})$, called ***$\mathcal{L}$-closure*** or ***$\mathcal{L}$-entailment***, such that:

(1) $\vdash_{\mathcal{L}}$ is a consequence relation on $sen(\mathcal{L})$ in the sense of Section 1.1.

(2) If $\Gamma \vdash_{\mathcal{L}} \varphi$ then for each arrow $f : \mathcal{L} \to \mathcal{L}'$ in $\mathbf{SIGN}$, $sen(f)[\Gamma] \vdash_{\mathcal{L}'} sen(f)(\varphi)$.

This second condition is an abstract version of substitution-invariance: the entailment is required to be "invariant" not only under substitutions within the same language, but also under "changes of primitive connectives".

Observe that in this definition nothing is assumed about the nature of the "sentences" of a signature. In this way substructural consequence-based logics can be handled in a natural way by taking the $\mathcal{L}$-sentences to be $\mathcal{L}$-sequents rather than the ordinary formulas, and then taking the $\mathcal{L}$-entailment relation to be the consequence relation of a Gentzen system adequate for the logic that fails to satisfy all the structural rules. The π-institutions were independently introduced by Meseguer [106] who called them *entailment systems*; by gluing together an institution and an entailment system he obtained a more general notion of logic.

However, only the simplest part of the formalism is needed in order to develop an adequate *categorical abstract algebraic logic.*

Voutsadakis, in the paper [139] in this volume, generalizes the notion of equivalence between deductive systems of different dimension (discussed briefly in Section 4.1) to institutions, and in the process obtains a generalization of the notion of algebraizability in terms of an abstract version of the Leibniz operator. We only summarize the main ideas here.

As in the case of semantics-based logics, each institution defines in a natural way, via the satisfaction relation, a consequence relation satisfying the conditions of a π-institution. Voutsadakis develops his theory within the framework of π-institutions and then extends it to general institutions using this construction. He characterizes equivalence between π-institutions by abstracting the following lattice-theoretic characterization of the finite algebraizability of consequence-based logics found in [23, Theorem 3.7]:

Theorem 6.3 *Let $\mathcal{S}$ be a finitary logic and K a quasivariety. Then $\mathcal{S}$ is algebraizable and has K as its equivalent algebraic semantics iff there exists an isomorphism between the lattice $\mathcal{T}h\,\mathcal{S}$ and the lattice $\mathrm{Co}_{\mathsf{K}}\boldsymbol{Fm}$ that commutes with substitutions (in an appropriate sense).*

The isomorphism, if it exists, is unique and coincides of course with the Leibniz operator $\boldsymbol{\Omega}$ on the formula algebra. From the point of view of Section 4.1, the K-congruences of the formula algebra coincide with the theories of the equational logic associated with K. This is the manner in which the equivalence of deductive systems of different dimension is characterized in [29], and it seems suitable for categorical abstraction. Thus Voutsadakis is led to study the *category of theories* of a π-institution (which are signature-dependent) and the corresponding *theory functor* that assigns to each signature its family of theories and transforms each arrow of **SIGN** into an arrow of theories. Most of the hard work in [139] is done in studying the functors between categories of theories that play the role of the Leibniz operator and its inverse. Several notions of *deductive equivalence* between π-institutions are introduced, in terms of functors between signatures and natural transformations of the theory functors satisfying certain conditions that generalize (3.5), (3.6) and (3.7) in categorical terms; the natural transformations play the role of the faithful interpretations used to define algebraizability in Section 3.3, while the functors between signatures incorporate the new feature of change of logical language.

In the main theorem of [139], Voutsadakis proves that two so-called term institutions are deductively equivalent iff there exists a signature-respecting adjoint equivalence between their respective categories of theories that commutes with substitutions. Roughly speaking, for an institution, the *term* property is a categorical analogue of the absolutely free nature of ordinary formula algebras; this condition seems to be required in order to generalize the notion of algebraizability given in Section 3.3. At this point the reader should have at least some idea what the notions of *signature-respecting* and *substitution* mean; the precise definitions can be found in [139].

The research is continued in [138], where the restricted notion of an *algebraic institution* is developed; it is more abstract than the institution normally associated with the 2-deductive system corresponding to the equational consequence of a class of algebras; the greater abstraction is needed in order to algebraize institutions associated with more exotic logics. Then, algebraizable institutions are naturally defined as those deductively equivalent to an algebraic institution. Voutsadakis works out in detail the algebraization of several kinds of logics, from ordinary sentential logics to equational logic and first-order logics, without having to transform the latter into sentential logics in an artificial way. As a proof of the wider applicability of this framework, an institution is described which is not so-to-speak "linguistically based", but diagram-based: its signature objects are graphs, and the sentences of a given signature are the arrows of the category freely generated by the graph corresponding to the signature; this institution is shown to be algebraizable.

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References

[1] Adillon, R. *Contribució a l'estudi de les lògiques proposicionals intuïcionistes i de Gödel sense contracció, i dels seus fragments.* Ph. D. Dissertation, University of Barcelona, 2001.

[2] Adillon, R., and Verdú, V. A Gentzen system equivalent to the BCK-logic. *Bulletin of the Section of Logic 25*, 2 (1996), 73–79.

[3] Agliano, P. Fregean subtractive varieties with definable congruences. *J. Austral. Math. Soc. Ser. A 71* (2001), 353–366.

[4] Agliano, P. Congruence quasi-orderability in subtractive varieties. *J. Austral. Math. Soc. Ser. A 71* (2001), 421–455.

[5] Agliano, P., and Ursini, A. Ideals and other generalizations of congruence classes. *J. Austral. Math. Soc. Ser. A 53* (1992), 103–115.

[6] Agliano, P., and Ursini, A. On subtractive varieties II: general properties. *Algebra Universalis 36* (1996), 222–259.

[7] Agliano, P., and Ursini, A. On subtractive varieties III: from ideals to congruences. *Algebra Universalis 37* (1997), 296–333.

[8] Agliano, P., and Ursini, A. On subtractive varieties IV: definability of principal ideals. *Algebra Universalis 38* (1997), 355–389.

[9] Andréka, H. *Algebraic investigation of first-order logic.* Ph. D. Dissertation, Eötvös Lorand university, Budapest, 1973.

[10] Andréka, H., and Németi, I. General algebraic logic: A perspective on 'What is logic'. In *What is a logical system ?*, D. Gabbay, Ed., vol. 4 of *Studies in Logic and Computation.* Oxford University Press, 1994, pp. 393–443.

[11] Andréka, H., Németi, I., and Sain, I. Algebraic logic. In *Handbook of philosophical logic, second edition*, D. Gabbay and F. Guenthner, Eds., vol. 2. Kluwer, 2001, pp. 133–247.

[12] Anellis, I. H., and Houser, N. Nineteenth century roots of algebraic logic and universal algebra. In *Algebraic logic*, H. Andréka, J. D. Monk, and I. Németi, Eds., vol. 54 of *Colloquia Math. Soc. Janos Bolyai.* North-Holland, Amsterdam, 1991, pp. 1–36.

[13] Avron, A. The method of hypersequents in the proof theory of propositional nonclassical logics. In *Logic: from foundations to applications*, W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, Eds. Oxford Science Publications, Oxford, 1996, pp. 1–32.

[14] Baaz, M., Fermueller, C. G., and Zach, R. Elimination of cuts in first-order finite-valued logics. *Journal of Information Processing and Cybernetics EIK 29*, 6 (1994), 333–355.

[15] Babyonyshev, S. Strongly Fregean logics. Manuscript, 2001.

[16] Badesa, C. *The origins of model theory.* Princeton University Press. To appear.

[17] Barbour, G., and Raftery, J. Quasivarieties of logic, regularity conditions and parameterized algebraization. In this volume.

[18] Blok, W., and Jónsson, B. Algebraic structures for logic. A course given at the *23rd Holiday Mathematics Symposium*, New Mexico State University, January 1999. Available at `http://math.nmsu.edu/~holysymp/`.

[19] Blok, W., Köhler, P., and Pigozzi, D. On the structure of varieties with equationally definable principal congruences II. *Algebra Universalis 18* (1984), 334–379.

[20] Blok, W., and Pigozzi, D. On the structure of varieties with equationally definable principal congruences I. *Algebra Universalis 15* (1982), 195–227.

[21] Blok, W., and Pigozzi, D. Protoalgebraic logics. *Studia Logica 45* (1986), 337–369.

[22] Blok, W., and Pigozzi, D. Alfred Tarski's work on general metamathematics. *The Journal of Symbolic Logic 53* (1988), 36–50.

[23] Blok, W., and Pigozzi, D. *Algebraizable logics*, vol. 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[24] Blok, W., and Pigozzi, D. Local deduction theorems in algebraic logic. In *Algebraic Logic*, H. Andréka, J. D. Monk, and I. Németi, Eds., vol. 54 of *Colloq. Math. Soc. János Bolyai.* North-Holland, Amsterdam, 1991, pp. 75–109.

[25] Blok, W., and Pigozzi, D. Foreword to Special Issue on Algebraic Logic. *Studia Logica 50* (1991).

[26] Blok, W., and Pigozzi, D. Algebraic semantics for universal Horn logic without equality. In *Universal Algebra and Quasigroup Theory*, A. Romanowska and J. D. H. Smith, Eds. Heldermann, Berlin, 1992, pp. 1–56.

[27] Blok, W., and Pigozzi, D. On the structure of varieties with equationally definable principal congruences III. *Algebra Universalis 32* (1994), 545–608.

[28] Blok, W., and Pigozzi, D. On the structure of varieties with equationally definable principal congruences IV. *Algebra Universalis 31* (1994), 1–35.

[29] Blok, W., and Pigozzi, D. Abstract algebraic logic and the deduction theorem. *Bulletin of Symbolic Logic*. To appear.

[30] Blok, W., and Raftery, J. G. Ideals in quasivarieties of algebras. In *Models, algebras and proofs*, X. Caicedo and C. H. Montenegro, Eds., vol. 203 of *Lecture Notes in Pure and Applied Mathematics.* Marcel Dekker, New York, 1999, pp. 167–186.

[31] Bloom, S. L. Some theorems on structural consequence operations. *Studia Logica 34* (1975), 1–9.

[32] Bou, F. Subintuitionistic logics. Master Thesis, Universitat de Barcelona, 2002.

[33] Brown, D. J., and Suszko, R. Abstract logics. *Dissertationes Math.* (*Rozprawy Mat.*) *102* (1973), 9–42.

[34] Büchi, J. R., and Owens, T. M. Skolem rings and their varieties. In *The collected works of J. Richard Büchi*, S. Maclane and D. Siefkes, Eds. Springer-Verlag, Berlin and New York, 1990, pp. 161–221.

[35] Casanovas, E., Dellunde, P., and Jansana, R. On elementary equivalence for equality-free logic. *Notre Dame Journal of Formal Logic 37*, 3 (1996), 506–522.

[36] Celani, S., and Jansana, R. A closer look at some subintuitionistic logics. *Notre Dame Journal of Formal Logic.* To appear.

[37] Church, A. *Introduction to Mathematical Logic, vol. I.* Princeton University Press, Princeton, New Jersey, 1956.

[38] Cignoli, R., and Monteiro, A. Boolean elements in Łukasiewicz algebras. *Proceedings of the Japan Academy 41* (1965), 676–680.

[39] Czelakowski, J. Reduced products of logical matrices. *Studia Logica 39* (1980), 19–43.

[40] Czelakowski, J. Equivalential logics, I, II. *Studia Logica 40* (1981), 227–236 and 355–372.

[41] Czelakowski, J. Logical matrices and the amalgamation property. *Studia Logica 41* (1982), 329–342.

[42] Czelakowski, J. Filter distributive logics. *Studia Logica 43* (1984), 353–377.

[43] Czelakowski, J. Algebraic aspects of deduction theorems. *Studia Logica 44* (1985), 369–387.

[44] Czelakowski, J. Local deductions theorems. *Studia Logica 45* (1986), 377–391.

[45] Czelakowski, J. Consequence operations: Foundational studies. Reports of the research project "Theories, models, cognitive schemata", Institute of Philosophy and Sociology, Polish Academy of Sciences, Warszawa, 1992.

[46] Czelakowski, J. *Protoalgebraic Logics*, vol. 10 of *Trends in Logic, Studia Logica Library.* Kluwer, Dordrecht, 2001.

[47] Czelakowski, J. The Suszko operator. Part I. In this volume.

[48] Czelakowski, J., and Dziobiak, W. A deduction theorem schema for deductive systems of propositional logics. In *Special Issue on Algebraic Logic*, W. Blok and D. Pigozzi, Eds. *Studia Logica 50* (1991), 385–390.

[49] Czelakowski, J., and Elgueta, R. Local characterization theorems for some classes of structures. *Acta Scientiarum Mathematicarum (Szeged) 65* (1999), 19–32.

[50] Czelakowski, J., and Jansana, R. Weakly algebraizable logics. *The Journal of Symbolic Logic 65*, 2 (2000), 641–668.

[51] Czelakowski, J., and Malinowski, G. Key notions of Tarski's methodology of deductive systems. *Studia Logica 44* (1985), 321–351.

[52] Czelakowski, J., and Pigozzi, D. Amalgamation and interpolation in abstract algebraic logic. In *Models, algebras and proofs*, X. Caicedo and C. H. Montenegro, Eds., no. 203 in Lecture Notes in Pure and Applied Mathematics Series. Marcel Dekker, New York and Basel, 1998, pp. 187–265.

[53] Czelakowski, J., and Pigozzi, D. Fregean logics. *Annals of Pure and Applied Logic*, to appear.

[54] Czelakowski, J., and Pigozzi, D. Fregean logics with the multiterm deduction theorem and their algebraization. Preprint 433, Centre de Recerca Matemàtica, Bellaterra (Barcelona), February 2000.

[55] Dellunde, P. A finitary 1-equivalential logic not finitely equivalential. *Bulletin of the Section of Logic 24*, 3 (1995), 120–122.

[56] Dellunde, P. *Contributions to the model theory of equality-free logic.* Ph. D. Dissertation, University of Barcelona, 1996.

[57] Dellunde, P. Equality-free logic: the method of diagrams and preservation theorems. *Logic Journal of the I.G.P.L. 7*, 6 (1999), 717–732.

[58] Dellunde, P., and Jansana, R. Some characterization theorems for infinitary universal Horn logic without equality. *The Journal of Symbolic Logic 61*, 4 (1996), 1242–1260.

[59] Diskin, Z. Abstract universal algebraic logic (Part I and Part II). *Proceedings of the Latvian Academy of Sciences, section B, 50* (1996), 13–30.

[60] D'Ottaviano, I., and Da Costa, N. Sur un problème de Jaśkowski. *C. R. Acad. Sci. Paris, Série A, 270* (1970), 1349–1353.

[61] Došen, K., and Schroeder-Heister, P., Eds. *Substructural Logics*, vol. 2 of *Studies in Logic and Computation.* Oxford University Press, 1993.

[62] Dummett, M. *The interpretation of Frege's philosophy.* Harvard University Press, Cambridge, Massachusetts, 1981.

[63] Elgueta, R. *Algebraic model theory for languages without equality.* Ph. D. Thesis, University of Barcelona, 1994.

[64] Elgueta, R. Subdirect representation theory for classes without equality. *Algebra Universalis 40* (1998), 201–246.

[65] Elgueta, R. Freeness in classes defined without equality. *The Journal of Symbolic Logic 64* (1999). 1159–1194.

[66] Elgueta, R. Review of *Protoalgebraic logics* by J. Czelakowski. In this volume.

[67] Elgueta, R., and Jansana, R. Definability of Leibniz equality. *Studia Logica 63* (1999), 223–243.

[68] Fiadeiro, J., and Sernadas, A. Structuring theories in consequence. In *Recent trends in data type specification*, D. Sannella and A. Tarlecki, Eds., vol. 332 of *Lecture Notes in Computer Science.* Springer-Verlag, New York, 1988, pp. 44–72.

[69] Font, J. M. On the Leibniz congruences. In *Algebraic Methods in Logic and in Computer Science*, C. Rauszer, Ed., no. 28 of *Banach Center Publications.* Polish Academy of Sciences, Warszawa, 1993, pp. 17–36.

[70] Font, J. M. Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the I.G.P.L. 5*, 3 (1997), 413–440.

[71] Font, J. M. On the contributions of Helena Rasiowa to mathematical logic. *Multiple-Valued Logic 4* (1999), 159–179.

[72] Font, J. M., and Jansana, R. On the sentential logics associated with strongly nice and semi-nice general logics. *Bull. of the IGPL 2* (1994), 55–76.

[73] Font, J. M., and Jansana, R. *A general algebraic semantics for sentential logics*, vol. 7 of *Lecture Notes in Logic*. Springer-Verlag, 1996. 135 pp. Presently distributed by the Association for Symbolic Logic.

[74] Font, J. M., and Jansana, R. Leibniz filters and the strong version of a protoalgebraic logic. *Archive for Mathematical Logic 40* (2001), 437–465.

[75] Font, J. M., Jansana, R., and Pigozzi, D. Fully adequate Gentzen systems and closure properties of full models. Manuscript, 1998.

[76] Font, J. M., Jansana, R., and Pigozzi, D. Fully adequate Gentzen systems and the deduction theorem. *Reports on Mathematical Logic 35*, (2001), 115–165.

[77] Font, J. M., and Rius, M. An abstract algebraic logic approach to tetravalent modal logics. *The Journal of Symbolic Logic 65*, 2 (2000), 481–518.

[78] Font, J. M., and Rodríguez, G. Algebraic study of two deductive systems of relevance logic. *Notre Dame Journal of Formal Logic 35*, 3 (1994), 369–397.

[79] Font, J. M., and Verdú, V. A first approach to abstract modal logics. *The Journal of Symbolic Logic 54* (1989), 1042–1062.

[80] Font, J. M., and Verdú, V. Algebraic logic for classical conjunction and disjunction. In *Special Issue on Algebraic Logic*, W. J. Blok and D. Pigozzi, Eds. *Studia Logica 50* (1991), 391–419.

[81] Gil, A. J. *Sistemes de Gentzen multidimensionals i lògiques finitament valorades. Teoria i aplicacions.* Ph. D. Thesis, University of Barcelona, 1996.

[82] Gil, A. J., and Rebagliato, J. Protoalgebraic Gentzen systems and the cut rule. In *Special Issue on Abstract Algebraic Logic. Part I*, J. M. Font, R. Jansana, and D. Pigozzi, Eds. *Studia Logica 65* (2000), 53–89.

[83] Gil, A. J., Rebagliato, J., and Verdú, V. A strong completeness theorem for the Gentzen systems associated with finite algebras. *Journal of Applied Non-Classical Logics 9*, 1 (1999), 9–36.

[84] Gil, A. J., Torrens, A., and Verdú, V. On Gentzen systems associated with the finite linear MV-algebras. *Journal of Logic and Computation 7*, 4 (1997), 473–500.

[85] Goguen, J. A., and Burstall, R. M. Introducing institutions. In *Proceedings of the Logic Programming Workshop*, E. Clarke and D. Kozen, Eds., vol. 164 of *Lecture Notes in Computer Science*. Springer-Verlag, New York, 1984, pp. 221–256.

[86] Gumm, H. P., and Ursini, A. Ideals in universal algebras. *Algebra Universalis 19* (1984), 45–54.

[87] Gyuris, V. *Variations of algebraizability.* Ph. D. Dissertation, University of Illinois at Chicago, 1999.

[88] Henkin, L., Monk, J. D., and Tarski, A. *Cylindric algebras, part II.* North-Holland, Amsterdam, 1985.

[89] Herrmann, B. *Equivalential logics and definability of truth.* Ph. D. Thesis, Freie Universität Berlin, 1993. 61 pp.

[90] Herrmann, B. Equivalential and algebraizable logics. *Studia Logica 57* (1996), 419–436.

[91] Herrmann, B. Characterizing equivalential and algebraizable logics by the Leibniz operator. *Studia Logica 58* (1997), 305–323.

[92] Hoogland, E. Algebraic characterizations of various Beth definability properties. In *Special Issue on Abstract Algebraic Logic. Part I*, J. M. Font, R. Jansana, and D. Pigozzi, Eds. *Studia Logica 65* (2000), 91–112.

[93] Hoogland, E. *Definability and interpolation: model-theoretic investigations.* Ph. D. Dissertation DS-2001-05, Institute of Logic, Language and Information, University of Amsterdam, 2001.

[94] Idziak, P., Słomczyńska, K., and Wroński, A. Equivalential algebras: a study of Fregean varieties. Version 1.1, 1997. To appear.

[95] Jansana, R. Abstract modal logics. *Studia Logica 55*, 2 (1995), 273–299.

[96] Jansana, R. Full models for positive modal logic. *Mathematical Logic Quarterly*, 48 (2002), 427-455.

[97] Jansana, R. Leibniz filters revisited. *Studia Logica*. To appear.

[98] Köhler, P., and Pigozzi, D. Varieties with equationally definable principal congruences. *Algebra Universalis 11* (1980), 213–219.

[99] Lewin, R. A., Mikenberg, I. F., and Schwarze, M. G. On the algebraizability of annotated logics. *Studia Logica 59* (1997), 359–386.

[100] Łoś, J. *O matrycach logicznych*, vol. 19 of *Ser. B.* Prace Wrocławskiego Towarzystwa Naukowege, 1949.

[101] Łoś, J., and Suszko, R. Remarks on sentential logics. *Indag. Math. 20* (1958), 177–183.

[102] Łukasiewicz, J. *Selected Works, edited by L. Borkowksi.* Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1970.

[103] Łukasiewicz, J., and Tarski, A. Untersuchungen über den Aussagenkalkül. *C. R. Soc. Sci. Lettr. Varsovie, Cl. III 23* (1930), 30–50. English translation in [102]: *Investigations into the sentential calculus*, 131–152.

[104] McKinsey, J. C. C. A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. *The Journal of Symbolic Logic 6* (1941), 117–134.

[105] McKinsey, J. C. C., and Tarski, A. Some theorems about the sentential calculi of Lewis and Heyting. *The Journal of Symbolic Logic 13* (1948), 1–15.

[106] Meseguer, J. General logics. In *Logic Colloquium'87*, H. D. Ebinghaus et al. Ed. North-Holland, Amsterdam, 1989, pp. 275–329.

[107] Monk, J. D. The contributions of Alfred Tarski to algebraic logic. *The Journal of Symbolic Logic 51* (1986), 899–906.

[108] Monteiro, A. La sémisimplicité des algèbres de Boole topologiques et les systèmes déductifs. *Revista de la Unión Matemática Argentina 25* (1971), 417–448.

[109] Monteiro, A. Sur les algèbres de Heyting simétriques. *Portugal. Math. 39* (1980)), 1–237.

[110] Pałasińska, K. *Deductive systems and finite axiomatization properties.* Ph. D. Dissertation, Iowa State University, Ames, Iowa, 1994.

[111] Pałasińska, K. Sequent calculi and quasivarieties. *Reports on Mathematical Logic 34* (2000), 107–131.

[112] Pałasińska, K. Finite basis theorem for filter-distributive multidimensional protoalgebraic deductive systems. In this volume.

[113] Paoli, F. *Substructural logics: a primer.* vol. 13 of *Trends in Logic, Studia Logica Library.* Kluwer, Dordrecht, 2002.

[114] Pigozzi, D. Finite basis theorem for relatively congruence-distributive quasivarieties. *Transactions of the Amer. Math. Soc. 310* (1988), 499-533.

[115] Pigozzi, D. Fregean algebraic logic. In *Algebraic Logic*, H. Andréka, J. D. Monk, and I. Németi, Eds., vol. 54 of *Colloq. Math. Soc. János Bolyai.* North-Holland, Amsterdam, 1991, pp. 473–502.

[116] Pigozzi, D. Abstract algebraic logic: past, present and future. A personal view. In *Workshop on Abstract Algebraic Logic*, J. M. Font, R. Jansana, and D. Pigozzi, Eds., no. 10 in Quaderns. Centre de Recerca Matemàtica, Bellaterra, January 1998, pp. 122–138.

[117] Pigozzi, D. Abstract Algebraic Logic In *Encyclopaedia of Mathematics, Supplement III*, M. Hazewinkel, Ed., Kluwer, Dordrecht, 2001, pp. 2–13.

[118] Prucnal, T., and Wroński, A. An algebraic characterization of the notion of structural completeness. *Bulletin of the Section of Logic 3* (1974), 30–33.

[119] Pynko, A. Definitional equivalence and algebraizability of generalized logical systems. *Annals of Pure and Applied Logic 98* (1999), 1–68.

[120] Rasiowa, H. *An algebraic approach to non-classical logics*, vol. 78 of *Studies in Logic and the Foundations of Mathematics.* North-Holland, Amsterdam, 1974.

[121] Rasiowa, H., and Sikorski, R. Algebraic treatment of the notion of satisfiability. *Fundamenta Mathematicae 40* (1953), 62–95.

[122] Rebagliato, J. *Sistemes de Gentzen algebritzables i el teorema de la deducció.* Ph. D. Dissertation, University of Barcelona, 1995.

[123] Rebagliato, J., and Verdú, V. On the algebraization of some Gentzen systems. In *Special Issue on Algebraic Logic and its Applications*, C. Rauszer, Ed., *Fundamenta Informaticae 18* (1993), 319–338.

[124] Rebagliato, J., and Verdú, V. A finite Hilbert-style axiomatization of the implication-less fragment of the intuitionistic propositional calculus. *Mathematical Logic Quarterly 40* (1994), 61–68.

[125] Rousseau, G. Sequents in many-valued logic I. *Fundamenta Mathematicae 60* (1967), 23–33.

[126] Shoesmith, D., and Smiley, T. *Multiple-conclusion logic.* Cambridge University Press, Cambridge, 1978.

[127] Surma, S. J. On the origin and subsequent applications of the concept of the Lindenbaum algebra. In *Logic, methodology and philosophy of science VI, Hannover 1979.* North-Holland, Amsterdam, 1982, pp. 719–734.

[128] Suszko, R. Abolition of the Fregean axiom. In *Logic Colloquium 1972–73*, R. Parikh, Ed., no. 453 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1975, pp. 169–239.

[129] Suszko, R. Congruences in sentential calculus. In *A report from the Autumn School of Logic* (Miedzygorze, Poland, November 21–29, 1977). Mimeographed notes, edited and compiled by J. Zygmunt and G. Malinowski.

[130] Tarski, A. Über einige fundamentale Begriffe der Metamathematik. *C. R. Soc. Sci. Lettr. Varsovie, Cl. III 23* (1930), 22–29. English translation in [133]: *On some fundamental concepts of metamathematics*, 30–37.

[131] Tarski, A. Grundzüge der Systemenkalküls. Erster Teil. *Fundamenta Mathematica 25* (1935), 503–526. English translation in [133]: *Foundations of the calculus of systems*, 342–383.

[132] Tarski, A. Grundzüge der Systemenkalküls. Zweiter Teil. *Fundamenta Mathematica 26* (1936), 283–301. English translation in [133]: *Foundations of the calculus of systems*, 342–383.

[133] Tarski, A. *Logic, Semantics, Metamathematics. Papers from 1923 to 1938*. Hackett Pub. Co., Indianapolis, Indiana, second edition, 1983. Edited by J. Corcoran.

[134] Torrens, A. Algebraic models of multidimensional deductive systems. Manuscript, 1991.

[135] Ursini, A. Sulle varietà di algebre con una buona teoria degli ideali. *Bolletino U. M. I. 6* (1972), 90–95.

[136] Ursini, A. On subtractive varieties I. *Algebra Universalis 31* (1994), 204–222.

[137] Verdú, V. Some algebraic structures determined by closure operators. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 31* (1985), 275–278.

[138] Voutsadakis, G. Categorical abstract algebraic logic: algebraizable institutions. *Applied Categorical Structures* 10 (6) (2002), 531-568.

[139] Voutsadakis, G. Categorical abstract algebraic logic: equivalent institutions. In this volume.

[140] Wójcicki, R. Logical matrices strongly adequate for structural sentential calculi. *Bulletin de l'Académie Polonaise des Sciences, Classe III, XVII* (1969), 333–335.

[141] Wójcicki, R. Matrix approach in the methodology of sentential calculi. *Studia Logica 32* (1973), 7–37.

[142] Wójcicki, R. *Theory of logical calculi. Basic theory of consequence operations*, vol. 199 of *Synthese Library.* Reidel, Dordrecht, 1988.

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