# ADVANCED THEORY OF INTEGRAL

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## 1. Covering theorems

1.1. **Terminology.** Disjointed=pairwise disjoint.

1.2. **Notation.** Let $X$ be a metric space, $x \in X$ and $r > 0$. We write $|y-x|$ instead of $\rho(x,y)$ even if there is no linear structure on $X$. We denote by $\overline{B}(x,r)$ the closed ball $\{y : |x-y| < r\}$, no matter whether $\overline{B}(x,r) = \overline{B(x,r)}$ or not. Note that always $\overline{B}(x,r) \supset \overline{B(x,r)}$.

1.3. **Lemma.** *Let $X$ be a separable metric space and $\mathcal{V}$ be a system of closed balls. Then $\mathcal{V}$ contains a maximal disjointed subsystem $\mathcal{V}'$ and this $\mathcal{V}'$ is at most countable.*

*Proof.* We consider a countable dense set $\{x_k\}$. We start with the empty system $\mathcal{V}_0$ and at the $k$-the step we try add to $\mathcal{V}_{k-1}$ a ball which is disjoint from the balls of $\mathcal{V}_{k-1}$ and contains $x_k$. If such a ball $B_k$ exists, we set $\mathcal{V}_k = \mathcal{V}_{k-1} \cup \{B_k\}$, otherwise we let $\mathcal{V}_k = \mathcal{V}_{k-1}$. At the end we have a system $\mathcal{V}' = \bigcup_k \mathcal{V}_k$ with the desired properties. □

1.4. **Remark.** A similar assertion (without the countability claim) holds also in nonseparable spaces, but then this requires the full strength of the axiom of choice.

1.5. **Theorem.** *Let $X$ be a separable metric space and $E \subset X$. Let $\mathcal{V}$ be a system of closed balls in $X$ covering $E$. Suppose that*

$$R := \sup\{r : \overline{B}(x,r) \in \mathcal{V}\} < \infty.$$

*Then there exists a countable disjointed subsystem $\mathcal{V}'$ of $\mathcal{V}$ such that*

$$E \subset \bigcup_{\overline{B}(x,r) \in \mathcal{V}'} B(x,5r).$$

*Proof.* We may assume that $R = 1$. We set

$$\mathcal{V}_m = \{\overline{B}(x,r) \in \mathcal{V} :\ 2^{-m-1} < r \le 2^{-m}\}.$$

We start with an empty system. At the $m$-the step, $m \ge 0$, we select a maximal disjointed subsystem $\mathcal{V}'_m$ from

$$\{\overline{B}(x,r) \in \mathcal{V}_m :\ \overline{B}(x,r) \text{ does not intersect any } B \in \mathcal{V}'_0 \cup \cdots \cup \mathcal{V}'_{m-1}\}.$$

We set

$$\mathcal{V}' = \bigcup_{m=0}^{\infty} \mathcal{V}'_m.$$

Let $z \in E$. Then there exists $p$ and $\overline{B}(x,r) \in \mathcal{V}_p$ such that $x \in \overline{B}(x,r)$. If $\overline{B}(x,r)$ is not selected in $\mathcal{V}'_p$, this must have the reason that there exists $m \le p$ and $\overline{B}(y,\rho) \in \mathcal{V}'_m$ such that $\overline{B}(x,r) \cap \overline{B}(y,\rho) \ne \emptyset$. Thus,

$$|z-y| \le |z-x| + |x-y| \le r + (\rho + r) < 5\rho.$$

Indeed,

$$r \le 2^{-p} \le 2 \cdot 2^{-m-1} < 2\rho.$$

It follows that $z \in B(y, 5\rho)$, so that the covering property is satisfied. □

1.6. **Lemma.** *Let $G \subset \mathbb{R}^n$ be an open set with $|G| < \infty$. Let $E \subset G$ and $\mathcal{V}$ be a system of closed balls in $\mathbb{R}^n$ covering $E$. Then there exists a finite subsystem $\mathcal{V}'$ of $\mathcal{V}$ and an open set $G' \subset G$ such that*

$$|G'| \le (1 - 6^{-n})|G|$$

*and*

$$E \subset G' \cup \bigcup_{B \in \mathcal{V}'} B. \tag{1}$$

*Proof.* Using Theorem 1.5, we find a disjointed finite or infinite sequence $\{\overline{B}(x_j, r_j)\}$ of balls from the system $\mathcal{V}$ such that

$$E \subset \bigcup_j B(x_j, 5r_j).$$

Set

$$G'' = G \cap \bigcup_j B(x_j, 5r_j).$$

Then $G''$ is an open set containing $E$. Since the balls $B(x_j, r_j)$ are disjointed, we have

$$\sum_j |B(x_j, r_j)| \le \Big|\bigcup_j B(x_j, r_j)\Big| \le |G| < \infty$$

and thus the sum $\sum_j |B(x_j, r_j)|$ is finite or converges. On the other hand,

$$\sum_j |B(x_j, r_j)| = 5^{-n} \sum_j |B(x_j, 5r_j)| \ge 5^{-n} \Big|\bigcup_j B(x_j, 5r_j)\Big| \ge 5^{-n}|G''|.$$

We find $m$ such that

$$\sum_{j=1}^{m} |B(x_j, r_j)| > 6^{-n}|G''|$$

and set

$$G' = G'' \setminus \bigcup_{j=1}^{m} \overline{B}(x_j, r_j), \qquad \mathcal{V}' = \{\overline{B}(x_1, r_1), \dots, \overline{B}(x_m, r_m)\}.$$

Then $G'$ is open, (1) holds and

$$|G'| \le |G''| - |G'' \setminus G'| \le |G''| - 6^{-n}|G''| \le (1 - 6^{-n})|G|.$$

□

1.7. **Theorem** (Vitali)**.** *Let $E \subset \mathbb{R}^n$. Let $\mathcal{V}$ be a system of closed balls in $\mathbb{R}^n$. Suppose that for each $x \in E$,*

$$\inf\{r > 0 \colon \exists\, z \in \mathbb{R}^n,\ x \in B(z,r) \in \mathcal{V}\} = 0. \tag{2}$$

*Then there exists a countable disjointed subsystem $\mathcal{V}'$ of $\mathcal{V}$ such that*

$$\Big| E \setminus \bigcup_{B \in \mathcal{V}'} B \Big| = 0. \tag{3}$$

*Proof.* Since we can split $\mathbb{R}^n$ into a disjointed countable union of bounded open sets up to a Lebesgue null set, we may assume that $E$ is contained in a bounded open set $G_0$. Denote $\tau = 1 - 6^{-n}$. Lemma 1.6 enables us construct by induction a nested sequence $G_0 \supset G_1 \supset G_2 \supset \dots$ of open sets and an increasing sequence $\{\mathcal{V}_k\}$ of finite subfamilies of $\mathcal{V}$ so that

$$E \subset G_k \cup \bigcup_{B \in \mathcal{V}_k} B, \text{ and}$$

$$|G_k| \le \tau^k |G_0|.$$

Indeed, in the $k$-th step we select only from those balls from $\mathcal{V}$ which are contained in $G_{k-1}$. The assumption (2) guarantees that these balls still cover $E \cap G_{k-1}$. Finally, we set

$$\mathcal{V}' = \bigcup_k \mathcal{V}_k.$$

From the construction it easily follows that

$$E \setminus \bigcup_{B \in \mathcal{V}'} B \subset \bigcap_k G_k,$$

so that (3) holds. □

## 2. $k$-dimensional measures on metric spaces

2.1. **Definition** (Hausdorff measure and spherical measure)**.** Let $X$ be a metric space and $k \ge 0$ be a real number. For $E \subset X$ we set

$$\mathcal{H}^k_\delta(E) = \inf\Big\{\sum_{j=1}^{\infty} \boldsymbol{\alpha}_k \big(\tfrac12 \operatorname{diam} E_j\big)^k \colon \bigcup_{j=1}^{\infty} E_j \supset E, \operatorname{diam} E_j \le \delta\Big\}, \quad \delta > 0,$$

$$\mathcal{H}^k(E) = \sup_{\delta > 0} \mathcal{H}^k_\delta(E) \quad (= \lim_{\delta \to 0+} \mathcal{H}^k_\delta(E)\ ).$$

and

$$\mathcal{K}^k_\delta(E) = \inf\Big\{\sum_{j=1}^{\infty} \boldsymbol{\alpha}_k \big(\tfrac12 \operatorname{diam} B_j\big)^k \colon B_j \text{ are balls, } \bigcup_{j=1}^{\infty} B_j \supset E, \operatorname{diam} B_j \le \delta\Big\}, \quad \delta > 0,$$

$$\mathcal{K}^k(E) = \sup_{\delta > 0} \mathcal{K}^k_\delta(E) \quad (= \lim_{\delta \to 0+} \mathcal{K}^k_\delta(E)\ ).$$

Here

$$\boldsymbol{\alpha}_k = \frac{\pi^{k/2}}{\Gamma(\frac{k}{2} + 1)}.$$

If $k$ is integer, then the constant $\boldsymbol{\alpha}_k$ has the meaning of the volume of the unit ball in $\mathbb{R}^k$. The set function $E \mapsto \mathcal{H}^k(E)$ is called the *$k$-dimensional (outer) Hausdorff measure*. It is quite straightforward to verify that $\mathcal{H}^k$ satisfies the axioms of outer measure. The restriction of $\mathcal{H}^k$ to the $\sigma$-algebra of all $\mathcal{H}^k$-measurable sets (in the sense of Carathéodory's construction) is a measure for which we use the same symbol. If we integrate with respect to the Hausdorff measure $\mathcal{H}^k$ we assume $\mathcal{H}^k$-measurability of the integrand.

The set function $E \mapsto \mathcal{H}^k_\infty(E)$ is called the $k$-dimensional Hausdorff content. It is an important set function which however does not obey the additivity properties of measures.

The set function $\mathcal{K}^k$ is called the *$k$-dimensional (outer) spherical measure*.

2.2. **Observations.** (a) *The Hausdorff content in $\mathbb{R}^n$ is not additive on Borel sets (except the case $k = n$).*
(b) $\mathcal{H}^k(E) = 0$ *iff* $\mathcal{H}^k_\infty(E) = 0$.
(c) *For each $E \subset X$, $\mathcal{H}^k(E) \le \mathcal{K}^k(E) \le 2^k \mathcal{H}^k(E)$.*

2.3. **Definition** (Distant sets)**.** Let $(X,\rho)$ be a metric space. We call sets $E, F \subset X$ to be *distant* if there exists $\delta > 0$ such that for each $x \in E$, $y \in F$ we have

$$\rho(x,y) > \delta.$$

2.4. **Definition** (Metric measure)**.** Let $\gamma$ be an outer measure on a metric space $X$. We say that $\gamma$ is an *metric measure*, if

$$E, F \subset X \text{ distant} \implies \gamma(E \cup F) = \gamma(E) + \gamma(F). \tag{4}$$

2.5. **Observation.** *$\mathcal{H}^k$ and $\mathcal{K}^k$ are metric measures.*

2.6. **Theorem.** *Let $\gamma$ be a metric measure on a metric space $X$. Then the $\sigma$-albegra of all $\gamma$-measurable sets in $X$ contains all Borel sets.*

*Proof.* We give the proof in a separable space. The reader interested in the nonseparable case can easily guess the necessary modification of the proof.

It would be clearly sufficient to prove that open balls are $\gamma$-measurable, since open balls generate the Borel $\sigma$-algebra (here we use the separability). Let $B = B(z,r)$ be an open ball in $\mathbb{R}^m$ and let $\{r_j\}_{j=1}^{\infty}$ be a sequence of positive radii such that $r_j \nearrow r$. The $\gamma$-measurability of $B$ in the sense of Carathéodory's definition means that for each "test set" $T \subset \mathbb{R}^m$ we have

$$\gamma(T \cap B) + \gamma(T \setminus B) \le \gamma(T). \tag{5}$$

In fact, then the equality holds in (5) since the converse inequality is easy and already stated above. It is enough to consider a set $T \subset \mathbb{R}^k$ with $\gamma(T) < \infty$. Denote

$$\begin{aligned} P_j &= T \cap \Big(B(z, r_{j+1}) \setminus B(z, r_j)\Big), \quad j = 1, 2, \dots, \\ P_0 &= T \cap B(z, r_1). \end{aligned}$$

Then the sets $P_0$, $P_2$, $P_4, \dots$ are pairwise distant, so that by (4) (together with an induction argument)

$$\sum_{j=0}^{q} \gamma(P_{2j}) = \gamma\Big(\bigcup_{j=0}^{q} P_{2j}\Big) \le \gamma(T)$$

for all $q \in \mathbb{N}$. Similarly $\sum\limits_{j=0}^{q} \gamma(P_{2j+1}) \le \gamma(T)$. Hence we observe that the series $\sum\limits_{j=0}^{\infty} \gamma(P_j)$ is convergent. Since for each $q \in \mathbb{N}$, the sets $\bigcup\limits_{j=0}^{q} P_j$ and $T \setminus B$ are distant, we have

$$\gamma\Big(\bigcup_{j=0}^{q} P_j\Big) + \gamma\Big(T \setminus B\Big) \le \gamma\Big(\bigcup_{j=0}^{q} P_j \cup (T \setminus B)\Big) \le \gamma(T)$$

and thus

$$\begin{aligned} \gamma(T \cap B) &\le \gamma\Big(\bigcup_{j=0}^{q} P_j\Big) + \gamma\Big(\bigcup_{j=q+1}^{\infty} P_j\Big) \\ &\le \gamma(T) - \gamma(T \setminus B) + \sum_{j=q+1}^{\infty} \gamma(P_j). \end{aligned}$$

Letting $q \to \infty$ we obtain

$$\gamma(T \cap B) \le \gamma(T) - \gamma(T \setminus B),$$

which is (5). $\square$

2.7. **Theorem.** *Let $X$, $Y$ be metric spaces and $E \subset X$. Let $f \colon E \to Y$ be a Lipschitz mapping, $\operatorname{lip}_E f = \beta$. Then*

$$\mathcal{H}^k(f(E)) \le \beta^k \mathcal{H}^k(E)\,.$$

*Proof.* Let us consider a sequence $\{E_j\}$ such that $E \subset \bigcup_j E_j$. We have $\operatorname{diam} f(E_j) \le \beta \operatorname{diam} E_j$ and thus

$$\begin{aligned} \mathcal{H}^k_{\beta\delta}(f(E)) &\le \sum_j 2^{-k} \boldsymbol{\alpha}_k (\operatorname{diam} f(E_j))^k \\ &\le \sum_j 2^{-k} \boldsymbol{\alpha}_k (\beta \operatorname{diam} E_j)^k \end{aligned}$$

and passing to the infimum over all coverings we obtain

$$\mathcal{H}^k_{\beta\delta}(f(E)) \le \beta^k \mathcal{H}^k_\delta(E).$$

Letting $\delta \to 0+$ we conclude the proof. □

2.8. **Corollary** (Invariance with respect to isometry)**.** *Let $E \subset X$ and $f\colon E \to Y$ be an isometric mapping. Then*

$$\mathcal{H}^k(f(E)) = \mathcal{H}^k(E)\,.$$

2.9. **Remark.** The argument used in the proof of Theorem 2.7 obviously fails for the spherical measure. To save the situation we would need the following property of the spaces: If $E \subset B(w,\rho)$ in $X$ and $f : E \to Y$ is $\beta$-Lipschitz, then there exists $z \in Y$ such that $f(E) \subset B(z, \beta\rho)$.

However if $X$ is the unit disc in $\mathbb{R}^2$, $E = Y$ be the unit circle and $f$ is the identity mapping, then $f(E)$ is not contained in any ball of radius 1 in $Y$, because the centre that would make the job in $Y$ is missing.

In the rest of this section, we will show that the desirable property is satisfied if both the spaces $X$ and $Y$ are Euclidean. This part is included inly as a curiosity, we will not need it in the sequel.

2.10. **Lemma.** *Let $E \subset \mathbb{R}^n$ be a compact set. Then there exist a unique ball $\overline{B}(z,r)$ containing $E$ with the smallest possible diameter. Moreover,*

$$z \in \operatorname{conv}\{y \in E :\ |y-z| = r\}.$$

*Proof.* Let

$$r_0 = \inf\{t > 0 :\ \exists w \in \mathbb{R}^n \text{ such that } E \subset \overline{B}(w,t)\}.$$

Let $\{\overline{B}(w_j,t_j)\}$ be a minimizing sequence, then by compactness there exists a convergent subsequence and a limit ball $\overline{B}(z,r)$ with $r = r_0$, and containing $E$. Suppose that there exist two minimal balls $\overline{B}(z,r)$ and $\overline{B}(z',r)$ containing $E$. Then

$$E \subset \overline{B}(z,r) \cap \overline{B}(z',r) \subset \overline{B}(\tfrac12(z+z'), r')$$

with

$$r' = \sqrt{r^2 - \left(\tfrac12|z-z'|\right)^2} < r,$$

a contradiction. So there is a unique ball $\overline{B}(z,r)$ containg $E$ with the smallest possible diameter.

Now, suppose that $z$ does not belong to the convex hull of $A := \{y \in E :\ |y-z| = r\}$. Since $A$ is compact, $\operatorname{conv} A$ is also compact. There exists a hyperplane that separates $A$ and $z$. Without loss of generality,

$$z = -\lambda \mathbf{e}_1 \quad \text{with } \lambda > 0,$$
$$y \cdot \mathbf{e}_1 > 0, \qquad y \in A.$$

Let us consider the compact set $K = \{y \in E :\ y \cdot \mathbf{e}_1 \le 0\}$. Then $K \subset B(z,r)$, therefore there exists $\delta > 0$ such that $B(y,\delta) \subset B(z,r)$ for every $y \in K$. Also,

$$y \in E,\ y\cdot \mathbf{e}_1 \ge 0,\ 0 < t < \lambda \quad \Longrightarrow\ y - t\mathbf{e}_1 \in B(z,r).$$

Hence for sufficiently small $t > 0$ we have

$$E - t\mathbf{e}_1 \subset B(z,r), \quad \text{which means} \quad E \subset B(z + t\mathbf{e}_1, r),$$

which contradicts minimality of $r$. □

2.11. **Lemma.** *Let $E \subset \overline{B}(0,1) \subset \mathbb{R}^m$ be a compact set. Let $f : E \to \mathbb{R}^n$ be a 1-Lipchitz mapping. Then there exists $z \in \mathbb{R}^n$ such that*

$$f(E) \subset \overline{B}(z,1).$$

*Proof.* We find the smallest ball $\overline{B}(z,r)$ containing $f(E)$. Without loss of generality we may assume that $z = 0$. By Lemma 2.10,

$$0 \in \operatorname{conv}\{f(x) :\ x \in E,\ |f(x)| = r\}.$$

Therefore there exist $x_i \in E$, $y_i \in f(E)$ and $\lambda_i \in [0,1]$ such that

$$y_i = f(x_i), \quad \sum_i \lambda_i = 1, \quad \sum_i \lambda_i y_i = 0, \quad |y_i| = r.$$

Since $f$ is 1-Lipschitz, we have

$$\sum_{i,j} \lambda_i\lambda_j|y_i - y_j|^2 \le \sum_{i,j} \lambda_i\lambda_j|x_i - x_j|^2.$$

Since $|x_i| \le 1$ and $|y_i| = r$, after a routine computation we obtain

$$(r^2 - 1)\sum_{i,j} \lambda_i\lambda_j + |\sum_i \lambda_i x_i|^2 \le |\sum_i \lambda_i y_i|^2 = 0.$$

It follows that $r \le 1$. ☐

2.12. **Corollary.** *Let $E \subset \mathbb{R}^m$. Let $f\colon E \to \mathbb{R}^n$ be a Lipschitz mapping,* $\operatorname{lip}_E f = \beta$. *Then*

$$\mathcal{K}^k(f(E)) \le \beta^k\mathcal{K}^k(E)\,.$$

*Proof.* Using Lemma 2.11, we can proceed as in the proof of Theorem 2.7. ☐

2.13. **Notes.** The Hausdorff measure has been introduced by Carathéodory (1914) and Hausdorff (1919). It was invented to measure the "area" (or "length" if $k = 1$) of $k$-dimensional sets (as $k$-dimensional surfaces and their parts) in $n$-dimensional spaces. There are also alternative ways to introduce $k$-dimensional measures in $n$-dimensional spaces and their outputs are also different. The Hausdorff measure is seemingly the most traditional $k$-dimensional measure. Moreover, it has the advantage that it measures well also sets with fractal structure (which are much different from $k$-dimensional surfaces and their parts), and that the dimension parameter $k$ may be noninteger. However, this last point of view is not the topic of these lectures.

## 3. Comparison of $n$-dimensional measures in $\mathbb{R}^n$

In this section we shall compare the (outer) measures $\mathcal{H}^n$, $\mathcal{H}^n_\infty$, $\mathcal{K}^n$, $\mathcal{K}^n_\infty$, and $\mathcal{L}^n$ in $\mathbb{R}^n$. The last one is the Lebesgue measure

$$\mathcal{L}^n(E) = \inf\Big\{\sum_{j=1}^\infty \ell(Q_j)\colon Q_j \text{ are intervals, } \bigcup_{j=1}^\infty Q_j \supset E.\Big\}.$$

Here $\ell(Q)$ is the elementary volume of the $n$-dimensional interval $Q$. We write also $|E| = \mathcal{L}^n(E)$.

If $E \subset \mathbb{R}^n$ and $\{B_j\}$ is a sequence of balls that cover $E$, then

$$\mathcal{L}^n(E) \subset \sum_j \mathcal{L}^n(B_j) = \sum_{j=1}^\infty \boldsymbol{\alpha}_k\big(\tfrac12 \operatorname{diam} B_j\big)^n$$

and passing to the infimum over all such covering we obtain

$$\mathcal{L}^n(E) \le \mathcal{K}^n_\infty(E) \le \mathcal{K}^n(E).$$

Also we observe that

$$\mathcal{H}^n_\infty(E) \le \mathcal{K}^n_\infty(E)$$

and

$$\mathcal{H}^n_\infty(E) \le \mathcal{H}^n(E) \le \mathcal{K}^n(E)$$

In order to obtain that all these outer measures are equal, we need to prove that

$$\mathcal{K}^n(E) \le \mathcal{L}^n(E). \text{ and } \mathcal{L}^n(E) \le \mathcal{H}^n_\infty(E)$$

The following lemma yields a very rough and temporary estimate.

3.1. **Lemma.** *Let $E \subset \mathbb{R}^n$. Then $\mathcal{K}^n(E) \le 5^n\mathcal{L}^n(E)$. In particular, every $\mathcal{L}^n$-null set is a $\mathcal{K}^n$-null set.*

*Proof.* Let $G$ be an open set containing $E$ and $\delta > 0$. Using Theorem 1.5 we find a system $\mathcal{W}$ of disjointed balls contained in $G$ such that $\operatorname{diam} B < \delta$ for each $B \in \mathcal{W}$ and

$$E \subset \bigcup_{B(x,r)\in\mathcal{W}} B(x, 5r).$$

Then

$$\mathcal{K}^n_{10\delta}(E) \le \sum_{B(x,r)\in\mathcal{W}} |B(x,5r)| = 5^n \sum_{B(x,r)\in\mathcal{W}} |B(x,r)| = 5^n\Big|\bigcup_{B(x,r)\in\mathcal{W}} B(x,r)\Big| \le 5^n\mathcal{L}^n(G).$$

Letting $\delta \to 0$ we obtain $\mathcal{K}^n(E) \le 5^n\mathcal{L}^n(G)$ and passing to the infimum over $G$ we conclude $\mathcal{K}^n(E) \le 5^n\mathcal{L}^n(E)$. ☐

3.2. **Lemma.** *Let $G \subset \mathbb{R}^n$ be an open set and $\delta > 0$. Then there exists a countable disjointed system $\mathcal{W}$ of balls in $G$ such that the diameters of balls in $\mathcal{W}$ are less than $\delta$ and*

$$\Big|G \setminus \bigcup_{B(x,r)\in\mathcal{W}} B\Big| = 0.$$

*Proof.* This is an immediate application of Theorem 1.7, namely we apply it to the system of all balls in $G$ with diameter less than $\delta$. □

3.3. **Theorem.** *Let $E \subset \mathbb{R}^n$. Then $\mathcal{K}^n(E) \le \mathcal{L}^n(E)$.*

*Proof.* Let $G$ be an open set containing $E$ and $\delta > 0$. Using Lemma 3.2 we find a a countable disjointed system $\mathcal{W}$ of balls in $G$ such that the diameters of balls in $\mathcal{W}$ are less than $\delta$ and

$$\Big|G \setminus \bigcup_{B(x,r)\in\mathcal{W}} B\Big| = 0.$$

Denote by $W$ the union of all balls in $\mathcal{W}$. Then

$$\mathcal{K}^n_\delta(E) \le \mathcal{K}^n_\delta(G) \le \mathcal{K}^n_\delta(G \setminus W) + \mathcal{K}^n_\delta(W)$$

Obviously $\mathcal{K}^n_\delta(W) \le |W| = |G|$ and by Lemma 3.1, $\mathcal{K}^n_\delta(G \setminus W) \le 5^n|G \setminus W| = 0$. Hence $\mathcal{K}^n(W) \le \mathcal{L}^n(G)$ and passing to the infimum over $G$ we conclude $\mathcal{K}^n(E) \le \mathcal{L}^n(E)$. □

3.4. **Definition** (Steiner symmetrization)**.** If $x \in \mathbb{R}^n$, the reflection of $x$ with respect to the $i$-th variable is defined as

$$R_i(x) = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n) = x - 2x_i\mathbf{e}_i.$$

Let $E \subset \mathbb{R}^n$ be a convex set. We write

$$R_i(E) = \{r_i(x) : \ x \in E\}.$$

The invariance $R_i(E) = E$ has the geometrical meaning of the symmetry with respect to the hyperplane $\{x \in \mathbb{R}^n : \ x_i = 0\}$. We define the *Steiner symmetrization* of $E$ with respect to the $i$-th variable as the set

$$S_i(E) = \Big\{x \in \mathbb{R}^n : \ \exists\, a \in \mathbb{R}, \ x + a\mathbf{e}_i \in E, \ R_i(x) + a\mathbf{e}_i \in E\Big\}.$$

3.5. **Lemma** (Properties of the Steiner symmetrization)**.**

(a) $R_i(S_i(E)) = S_i(E)$.
(b) $R_j(E) = E \implies R_j(S_i(E)) = S_i(E)$.
(c) $E$ *convex* $\implies S_i(E)$ *convex.*
(d) $\operatorname{diam}(S_i(E)) \le \operatorname{diam} E$
(e) $\mathcal{L}^n(S_i(E)) = \mathcal{L}^n(E)$.

*Proof.* (a) and (b) are elementary. For (c) and (d) it is important to observe that it remains to consider the situation in the plane, the details are left as an exercise. The part (e) follows from the Fubini theorem. □

3.6. **Lemma** (Isodiametric inequality)**.** *Let $E \subset \mathbb{R}^n$. Then*

$$\mathcal{L}^n(E) \le 2^{-n}\boldsymbol{\alpha}_n\,(\operatorname{diam} E)^n.$$

*Proof.* Since the diameter of a sets does not vary if we pass to its closed convex hull, we may assume that $E$ is closed and convex. Using Lemma 3.5 we deduce that the set

$$S(E) = S_n(\dots(S_2(S_1(E))))$$

is convex and balanced (=symmetric with respect to the origin), $\operatorname{diam} S(E) \le \operatorname{diam}(E)$ and $\mathcal{L}^n(S(E)) = \mathcal{L}^n(E)$. Let $r = \frac{1}{2}\operatorname{diam}(S(E))$. Then $S(E) \subset B(0, r)$ and

$$\begin{aligned}\mathcal{L}^n(E) = \mathcal{L}^n(S(E)) \le \mathcal{L}^n(B(0, r)) &= \boldsymbol{\alpha}_n r^n = 2^{-n}\boldsymbol{\alpha}_n(\operatorname{diam} S(E))^n\\ &\le 2^{-n}\boldsymbol{\alpha}_n(\operatorname{diam} E)^n.\end{aligned}$$

□

3.7. **Theorem.** *If $E \subset \mathbb{R}^n$, then $\mathcal{L}^n(E) \le \mathcal{H}^n_\infty(E)$.*

*Proof.* Let $E_j$, $j = 1, 2, \dots$ be subsets of $\mathbb{R}^n$ such that

$$E \subset \bigcup E_j.$$

Then by the isodiametric inequality (Lemma 3.6)

$$\mathcal{L}^n(E) \le \sum_j \mathcal{L}^n(E_j) \le 2^{-n}\boldsymbol{\alpha}_n \sum_j (\operatorname{diam} E_j)^n.$$

Thus $\mathcal{L}^n(E) \le \mathcal{H}^n_\infty(E)$. □

3.8. **Notes.** The $k$-dimensional measures in $\mathbb{R}^n$ are particularly important for $k$ being a positive integer. The concept of a $k$-dimensional measure can be introduced by axioms. Let us consider an outer measure $\gamma$ on $\mathbb{R}^n$. We can say that $\gamma$ is a $k$-dimensional measure, if e.g. the following set of axioms holds: each Borel set is $\gamma$-measurable, $\gamma((0,1)^k \times \{0\}^{n-k}) = 1$ and $\gamma(f(E)) \le \beta^k(\gamma(E))$ for each $E \subset \mathbb{R}^n$ and $\beta$-Lipschitz mapping $f : E \to \mathbb{R}^n$. The idea of axiomatic setting for $k$-dimensional measures is due to Kolmogorov (1933). There are various methods how to construct $k$-dimensional measures and the results are really different measures; even the measures $\mathcal{H}^k$ and $\mathcal{K}^k$ are diferent for $k < n$. However, all $k$-dimensional measures coincide on the $\sigma$-algebra generated by $\mathcal{C}^1$-surfaces.

Let us outline a construction of a $k$-dimensional measure based on ideas entirely different from the constructions of the measures $\mathcal{H}^k$ and $\mathcal{K}^k$. If $G \subset \mathbb{R}^k$ is an open set and $\varphi : G \to \mathbb{R}^n$ is an injective Lipschitz mapping, we may define

$$\gamma(\varphi(G)) = \int_G |J\varphi(t)|\,dt,$$

where $J\varphi$ is the $k$-dimensional Jacobian of $\varphi$. This definition does not depend on the choice of the parametrization $\gamma$ (this requires some proof). Now, by covering one can produce an outer measure. This measure has a poor system of null sets, it is sometimes convenient to "compile" the measure with adding the Hausdorff null set to the covering system.

## 4. Area and coarea formula – introduction

4.1. **Jacobians.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $f = (f_1, \dots, f_d) : \Omega \to \mathbb{R}^d$ be a weakly differentiable mapping. Then $\nabla f(x)$ forms the *Jacobi matrix* of $f$ at $x$, i.e. the matrix

$$\Big(\frac{\partial f_i}{\partial x_j}(x)\Big)_{\substack{i=1,\dots,d\\ j=1,\dots,n}} \in \mathbb{R}^{d\times n}.$$

Let $I(m,k)$ be the set of all increasing multiindices from $\{1,\dots,m\}^k$, i.e., $\alpha = (\alpha_1,\dots,\alpha_k) \in I(m,k)$ if $\alpha_j$ are integers, $1 \le \alpha_1 < \dots < \alpha_k \le m$. If $\alpha \in I(d,k)$ and $\beta \in I(n,k)$, we define the partial Jacobians

$$J^{\alpha\beta} f(x) = \frac{\partial\big(f_{\alpha_1},\dots,f_{\alpha_k}\big)}{\partial\big(x_{\beta_1},\dots,x_{\beta_k}\big)}(x) = \det\Big(\frac{\partial f_{\alpha_i}}{\partial x_{\beta_j}}(x)\Big)_{i,j=1,\dots,k}.$$

We introduce the *$k$-dimensional Jacobian* of $f$ at $x$ as the "multimatrix"

$$J_k f(x) := \Big(J^{\alpha\beta} f(x)\Big)_{\substack{\alpha\in I(d,k)\\ \beta\in I(n,k)}} \in \mathbb{R}^{I(d,k)\times I(n,k)}.$$

We write $Jf(x) = J_n f(x)$. If $n = d$, then, of course, $Jf(x)$ can be identified with an ordinary real number because the dimension of $\mathbb{R}^{I(n,n)\times I(n,n)}$ is one.

The points of $\{x \in \Omega : \ J_k f(x) = 0\}$ (i.e. where the rank of $\nabla f(x)$ is less than $k$) are called the *$k$-singular points ($k$-singularities)* for $f$. A mapping $f : \Omega \to \mathbb{R}^d$ is termed *$k$-regular* if it is $\mathcal{C}^1$ and the set of $k$-singular points for $f$ is empty.

Let us note that for an arbitrary index set $\Gamma$, we introduce the scalar (or inner) product $\cdot$ and the norm $|\,.\,|$ on $\mathbb{R}^\Gamma$ as

$$x \cdot y = \sum_{\gamma\in\Gamma} x_\gamma y_\gamma, \qquad x, y \in \mathbb{R}^\Gamma,$$

$$|x| = \sqrt{x\cdot x}, \qquad x \in \mathbb{R}^\Gamma.$$

This gives sense to the norm $|J_k f(x)|$.

Given a linear mapping $A$ we denote by $J_k A$ the $k$-dimensional Jacobian of $A$. The symbol does not refer to a point at which the Jacobian is computed, because linear mappings have constant derivatives and thus also constant Jacobians.

4.2. **Area formula.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $\Omega' \subset \Omega$ and $f : \Omega \to \mathbb{R}^d$ be a Sobolev mapping, $d \geq n$. We say that the area formula holds for $f$ on $\Omega'$ if for each measurable set $E \subset \Omega'$ we have

$$\int_E |Jf(x)|\, dx = \int_{\mathbb{R}^d} \mathcal{N}(f, y, E)\, d\mathcal{H}^n(y) \tag{6}$$

where $\mathcal{N}(f, y, E)$ is the number of points in the set $E \cap f^{-1}(y)$ (the *multiplicity function*). The statement that (6) is valid includes $\mathcal{H}^n$-measurability of the multiplicity function. If $\Omega'$ is not mentioned we understand that $\Omega' = \Omega$.

4.3. **Coarea formula.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $\Omega' \subset \Omega$ and $f : \Omega \to \mathbb{R}^d$ be a Sobolev mapping, $d \leq n$. We say that the coarea formula holds for $f$ on $\Omega'$ if for for each measurable set $E \subset \Omega'$ and a.e. $y \in \mathbb{R}^d$, the set $E \cap f^{-1}(y)$ is $\mathcal{H}^{n-d}$-measurable and the equality

$$\int_E |J_d f(x)|\, dx = \int_{\mathbb{R}^d} \mathcal{H}^{n-d}(E \cap f^{-1}(y))\, dy \tag{7}$$

holds. The statement that (7) is valid includes also measurability of the function

$$y \mapsto \mathcal{H}^{n-d}(E \cap f^{-1}(y)).$$

If $\Omega'$ is not mentioned we understand that $\Omega' = \Omega$.

4.4. **Remark.** Later on we shall show that the area formula and the coarea formula hold whenever $f$ is $\mathcal{C}^1$, see Section 6. Even more general criteria will be presented later. In the exercises below we will assume the knowledge of the $\mathcal{C}^1$-statement.

4.5. **Remark.** Notice that $|J_1 f| = |\nabla f|$ which simplifies the area formula if $n = 1$ (=integration along curves) and the coarea formula if $d = 1$ (=scalar case). However, if we want to formulate the area formula for $n = 1$ only it is quite unusual to use the symbol $\nabla f$ and and $f'$ is customarily used instead. This is not in conflict with the above conventions because Sobolev functions on one-dimensional intervals have a.e. differentiable continuous representatives.

4.6. **Advanced change of variables.** Validity of area or coarea formula for a function $f$ implies that $f$ is a legitimate transformation for change of variables. Advanced formulae on change of variables are obvious consequences which can be obtained from (6) and (7) in a very routine way (characteristic function, simple functions,...). If $d = n$, both area and coarea formula imply the classical theorem on change of variables.

4.7. **Theorem** (Change of variables by area formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be a Sobolev mapping, $d \geq n$. Suppose that the area formula holds for $f$. Let $E \subset \Omega$ be a measurable set and $u : E \to \mathbb{R}$ a measurable function. Then*

$$\int_E u(x)\, |Jf(x)|\, dx = \int_{\mathbb{R}^d} \Big( \sum_{x \in E \cap f^{-1}(y)} u(x) \Big)\, d\mathcal{H}^n(y)$$

*provided the integral on the left makes sense.*

4.8. **Theorem** (Change of variables by coarea formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be a Sobolev mapping, $d \leq n$. Suppose that the coarea formula holds for $f$. Let $E \subset \Omega$ be a measurable set and $u : E \to \mathbb{R}$ be a measurable function. Then*

$$\int_E u(x)\, |J_d f(x)|\, dx = \int_{\mathbb{R}^d} \Big( \int_{E \cap f^{-1}(y)} u(x)\, d\mathcal{H}^{n-d}(x) \Big)\, dy \tag{8}$$

*provided the integral on the left makes sense.*

## 5. Integration on linear subspaces

The simplest case of area or coarea formulae is that of linear mappings. However, already here we must settle all algebraical difficulties connected with this topic. In this section we use temporarily the bold font for vectors.

5.1. **Linear mappings.** We will not much distinguish between linear mappings and their representing matrices.

We denote by $I$ the identity mapping of $\mathbb{R}^n \to \mathbb{R}^n$.

If $A : \mathbb{R}^n \to \mathbb{R}^d$ is a linear mapping, recall that the *norm* of $A$ is defined by

$$\|A\| = \sup\{|A\boldsymbol{x}| : \ \boldsymbol{x} \in \mathbb{R}^m, \ |\boldsymbol{x}| \le 1\}.$$

The *determinant* of a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ is the determinant of its representing matrix.

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. The *adjoint* of $A$ is defined as the linear mapping $A^* : \mathbb{R}^m \to \mathbb{R}^n$ satisfying $\boldsymbol{x} \cdot (A^*\boldsymbol{y}) = (A\boldsymbol{x}) \cdot \boldsymbol{y}$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$. The matrix representing $A^*$ is the transposed matrix to the matrix representing $A$.

It can be verified that the norm of $A$ is the square root of the maximal eigenvalue of $A^*A$.

A linear mapping $Q : \mathbb{R}^n \to \mathbb{R}^m$ is termed *orthogonal* if $(Q\boldsymbol{x}) \cdot (Q\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. This is the case iff $Q^*Q = I$. Another characterization is that this is a linear mapping which preserves the norm (or euclidean distance).

A linear mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is termed *symmetric* (or selfadjoint) if $S^* = S$.

A linear mapping $D : \mathbb{R}^n \to \mathbb{R}^n$ is termed *diagonal* if its representing matrix is diagonal.

A linear mapping $P : \mathbb{R}^n \to \mathbb{R}^n$ is termed *positive definite* if $P\boldsymbol{x} \cdot \boldsymbol{x} > 0$ for each $\boldsymbol{x} \in \mathbb{R}^n$.

5.2. **Lemma** (Decomposition of a linear mapping)**.** *Let $n \le d$. Let $A : \mathbb{R}^n \to \mathbb{R}^d$ be an $n$-regular linear mapping. Then*

$$A = QDP,$$

*where $P : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear mapping, $D : \mathbb{R}^n \to \mathbb{R}^n$ is a positive definite diagonal linear mapping and $Q : \mathbb{R}^n \to \mathbb{R}^d$ is an orthogonal linear mapping.*

*Proof.* The mapping $A^*A$ is symmetric and positively definite, hence there exists an orthonormal basis $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k)$ of the space $\mathbb{R}^n$ consisting of eigenvectors of $A^*A$. Thus there are $\lambda_i > 0$ such that

$$A^*A\boldsymbol{u}_i = \lambda_i^2 \boldsymbol{u}_i, \qquad i = 1, \dots, n.$$

Then $Q$, $D$ and $P$ will be constructed as linear mappings transforming bases into bases: $P(\boldsymbol{u}_i) = \mathbf{e}_i$, $D(\mathbf{e}_i) = \lambda_i \mathbf{e}_i$, $Q(\lambda_i \mathbf{e}_i) = A(\boldsymbol{u}_i)$. We have

$$\begin{aligned}(Q\mathbf{e}_i) \cdot (Q\mathbf{e}_j) &= \frac{A\boldsymbol{u}_i}{\lambda_i} \cdot \frac{A\boldsymbol{u}_j}{\lambda_j} = \frac{A^*A\boldsymbol{u}_i}{\lambda_i\lambda_j} \cdot \boldsymbol{u}_j \\ &= \begin{cases} \boldsymbol{u}_i \cdot \boldsymbol{u}_j = 1, & i = j, \\ \frac{\lambda_i^2}{\lambda_i\lambda_j}\, \boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0, & i \ne j. \end{cases}\end{aligned}$$

which shows that $Q$ is orthogonal. The rest is routine. □

5.3. **Theorem** (Cauchy-Binet formula)**.** *Let $A, B : \mathbb{R}^n \to \mathbb{R}^d$ be linear mappings, $d \ge n$. Then*

$$JA \cdot JB = \det(B^*A).$$

*In particular,*

$$|JA| = \sqrt{\det(A^*A)}.$$

*Proof.* Consider $2n$-linear forms

$$\Phi(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n, \boldsymbol{w}_1, \dots, \boldsymbol{w}_n) = \sum_{\alpha \in I(d,n)} \det(v_{i\alpha_q})_{i,q=1}^n \det(w_{j\alpha_q})_{j,q=1}^n \,, \tag{9}$$

$$\Psi(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n, \boldsymbol{w}_1, \dots, \boldsymbol{w}_n) = \det\Big(\boldsymbol{v}_i \cdot \boldsymbol{w}_j\Big)_{i,j=1}^n \,, \tag{10}$$

where we write

$$\boldsymbol{v}_i = \sum_{j=1}^d v_{ij}\mathbf{e}_j, \quad \boldsymbol{w}_i = \sum_{j=1}^d w_{ij}\mathbf{e}_j.$$

We shall shortly verify that $\Phi$ and $\Psi$ yield the same output if $\boldsymbol{v}_i$ and $\boldsymbol{w}_i$ are selected from the canonical basis. Hence the multilinearity implies that $\Phi$ and $\Psi$ coincide. The proof the follows applying the result to the vectors $\boldsymbol{v}_i = A(\mathbf{e}_i)$, $\boldsymbol{w}_i = B(\mathbf{e}_i)$, $i = 1, \dots, n$.

Pick $\boldsymbol{v}_i$ and $\boldsymbol{w}_i$ from the canonical basis vectors. If $\boldsymbol{v}_{i_1} = \boldsymbol{v}_{i_2}$, where $i_1, i_2 \in \{1, \dots, n\}$ are distinct, then all determinants $\det(v_{i\alpha_j})_{i,j=1}^n$ vanish and since

$$\boldsymbol{v}_{i_1} \cdot \boldsymbol{w}_j = \boldsymbol{v}_{i_2} \cdot \boldsymbol{w}_j, \qquad j = 1, \dots, d,$$

also the determinant in (10) vanishes. We may then assume that $\boldsymbol{v}_i$ are pairwise distinct, thus there exists $\beta \in I(d,n)$ such that $\boldsymbol{v}_i$ are just a permutation of $\mathbf{e}_{\beta_1}, \dots, \mathbf{e}_{\beta_n}$. If among the vectors $\boldsymbol{w}_j$ there does not occur $\boldsymbol{v}_{i_0}$, $i_0 \in \{1, \dots, n\}$, then $\Phi$ yields zero, as

$$\det(w_{i\beta_j})_{i,j=1}^n = 0$$

and for $\alpha \neq \beta$ we have

$$\det(v_{i\alpha_j})_{i,j=1}^n = 0.$$

Also $\Psi$ yields zero, as in the matrix

$$\Big(\boldsymbol{v}_i \cdot \boldsymbol{w}_j\Big)_{i,j=1}^n$$

the $i_0$-th row vanishes. It remains the case that up to a permutation, $\boldsymbol{w}_j$ are $\mathbf{e}_{\beta_1}, \dots, \mathbf{e}_{\beta_n}$. Then on the right of (9) there is only one nonvanishing term corresponding to the multiindex $\beta$. By the theorem on the product of determinants, we have

$$\Phi(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n, \boldsymbol{w}_1, \dots, \boldsymbol{w}_n) = \det\Big(\sum_{q=1}^n v_{i\beta_q} w_{j\beta_q}\Big).$$

Taking into account that both $\boldsymbol{v}_i$ and $\boldsymbol{w}_j$ are selected from $\{\mathbf{e}_{\beta_1}, \dots, \mathbf{e}_{\beta_n}\}$, we infer

$$\sum_{q=1}^n v_{i\beta_q} w_{j\beta_q} = \boldsymbol{v}_i \cdot \boldsymbol{w}_j, \qquad i,j = 1, \dots, n.$$

Thus we have verified $\Phi(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n, \boldsymbol{w}_1, \dots, \boldsymbol{w}_n) = \Psi(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n, \boldsymbol{w}_1, \dots, \boldsymbol{w}_n)$ which finishes the proof. □

5.4. **Theorem** (Area formula for a linear mapping)**.** *Let $n \leq d$. Then the area formula holds for any $n$-regular linear mapping $A : \mathbb{R}^n \to \mathbb{R}^d$.*

*Proof.* Consider first the case of a positive definite diagonal linear mapping $D : \mathbb{R}^n \to \mathbb{R}^n$. Then

$$D\boldsymbol{x} = (d_1 x_1, \dots, d_n x_n), \qquad x \in \mathbb{R}^n,$$

where $d_1, \dots, d_n$ are positive real numbers. For each $n$-dimensional interval $I = (a_1, b_1) \times \dots \times (a_n, b_n)$ we have

$$D(I) = (d_1 a_1, d_1 b_1) \times \dots \times (d_n a_n, d_n b_n)$$

and elementary geometry gives

$$\mathcal{L}^n(D(I)) = d_1 \dots d_n \, \mathcal{L}^n(I) = JD \, \mathcal{L}^n(I).$$

From the construction of the Lebesgue measure it is clear that it follows

$$L^n(D(E)) = |JD| \, \mathcal{L}^n(E)$$

for each $E \subset \mathbb{R}^n$.

Consider now the general case of $A$. By Lemma 5.2 there is a decomposition of $A$ as

$$A = QDP,$$

where $P : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear mapping, $D : \mathbb{R}^n \to \mathbb{R}^n$ is a positive definite diagonal linear mapping and $Q : \mathbb{R}^n \to \mathbb{R}^d$ is an orthogonal linear mapping. Let $E \subset \mathbb{R}^n$ be an arbitrary set. Then by Corollary 2.8, equality of Lebesgue and Hausdorff measure and the previous step we have

$$\mathcal{H}^n(A(E)) = \mathcal{H}^n(Q(D(P(E)))) = \mathcal{H}^n(D(P(E))) = \mathcal{L}^n(D(P(E))) = JD\mathcal{L}^n(P(E)) = JD\mathcal{L}^n(E).$$

It remains to show that $JD = |JA|$. Obviously

$$A^*A = (QDP)^*(QDP) = P^*D^*Q^*QDP = P^*D^*DP.$$

By the theorem on product of determinants we have

$$\det(A^*A) = \det P^* \det(D^*D) \det P = \det(D^*D).$$

(Recall that for orthogonal $P : \mathbb{R}^n \to \mathbb{R}^n$ we have $\det P^* = \det P \in \{+1, -1\}$). Hence by the Cauchy-Binet formula (Theorem 5.3), $JD = |JD| = |JA|$. This finishes the proof. □

5.5. **Lemma** (Decomposition of a linear mapping, $n \geq d$)**.** . *Let $A : \mathbb{R}^n \to \mathbb{R}^d$ be a $d$-regular linear mapping, $n \geq d$. Then $A = L\Pi Q$, where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal mapping, $\Pi : \mathbb{R}^n \to \mathbb{R}^d$ is the projection $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_d)$ and $L : \mathbb{R}^d \to \mathbb{R}^d$ is a $d$-regular linear mapping.*

*Proof.* Let $\operatorname{Ker} A = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = 0\}$. Let $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n)$ be an orthonormal basis of $\mathbb{R}^n$ such that $(\boldsymbol{u}_{d+1}, \dots, \boldsymbol{u}_n)$ is an orthonormal basis of $\operatorname{Ker} A$. Then $Q$ and $L$ will be constructed as linear mappings transforming bases into bases: $Q\boldsymbol{u}_i = \mathbf{e}_i$, $L\mathbf{e}_i = A\boldsymbol{u}_i$. □

5.6. **Theorem** (Coarea formula for a linear mapping). *Let $n \geq d$. Then the coarea formula holds for any $d$-regular linear mapping $A : \mathbb{R}^n \to \mathbb{R}^d$.*

*Proof.* Let $E \subset \mathbb{R}^n$ be measurable. By Corollary 2.8, for each $\boldsymbol{y} \in \mathbb{R}^d$ we have

$$\begin{aligned}\mathcal{H}^{n-d}(E \cap A^{-1}(\boldsymbol{y})) &= \mathcal{H}^{n-d}\big(E \cap Q^{-1}(\Pi^{-1}(L^{-1}(\boldsymbol{y})))\big)\\ &= \mathcal{H}^{n-d}\big(Q(E) \cap \Pi^{-1}(L^{-1}(\boldsymbol{y}))\big).\end{aligned}$$

Let $\boldsymbol{z} = L^{-1}\boldsymbol{y}$. Then, by the area formula (Theorem 5.4) applied to $L$, the equality of Hausdorff and Lebesgue measure, the Fubini theorem and Corollary 2.8 we have

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{H}^{n-d}(E \cap A^{-1}(\boldsymbol{y}))\, d\boldsymbol{y} &= \int_{\mathbb{R}^d} \mathcal{H}^{n-d}\big(Q(E) \cap \Pi^{-1}(L^{-1}(\boldsymbol{y}))\big)\, d\boldsymbol{y}\\ &= \int_{\mathbb{R}^d} \mathcal{H}^{n-d}\big(Q(E) \cap \Pi^{-1}(\boldsymbol{z})\big)|J_d L|\, d\boldsymbol{z}\\ &= \int_{\mathbb{R}^d} \mathcal{L}^{n-d}\Big(\Big\{\boldsymbol{w} \in \mathbb{R}^{n-d} :\ (\boldsymbol{z}, \boldsymbol{w}) \in Q(E)\Big)\Big\}\, |J_d L|\, d\boldsymbol{z}\\ &= |J_d L|\ \mathcal{L}^n(Q(E)) = |J_d L| \mathcal{L}^n(E).\end{aligned}$$

It remains to show that $|J_d L| = |J_d A|$. Using that $Q^*$ and $\Pi^*$ are orthogonal mappings we observe that

$$AA^* = (L\Pi Q)(L\Pi Q)^* = L\Pi Q Q^* \Pi^* L^* = L\Pi\Pi^* L^* = LL^*.$$

The dual version of the Cauchy-Binet formula then yields that

$$|J_d A|^2 = \det(AA^*) = \det(LL^*) = |J_d L|^2.$$

□

## 6. Area and coarea formula: $\mathcal{C}^1$-case

In this section we establish area and coarea formulae for $\mathcal{C}^1$-mappings. First we consider the regular case and then we use a trick to handle also the singularities.

We start with the case $d \geq n$.

6.1. **Lemma.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}^d$ be a $\mathcal{C}^1$-mapping and $x_0 \in \Omega$. If $Jf(x_0) \neq 0$ then for each $\varepsilon > 0$ there exists a neighborhod $U \subset \Omega$ of $x_0$ such that for all $x, x' \in U$ we have*

$$(1 - \varepsilon)|Ax' - Ax| \leq |f(x') - f(x)| \leq (1 + \varepsilon)|Ax' - Ax| \tag{11}$$

*where $A = f'(x_0)$.*

*Proof.* Choose $\varepsilon > 0$. Since $A$ is $n$-regular, there exists $\lambda > 0$ such that

$$|Ax| \geq \lambda|x|, \qquad x \in \mathbb{R}^n.$$

We find a ball $U$ centered at $x_0$ and contained in $\Omega$ such that for each $x \in U$ we have $\|f'(x) - A\| < \lambda\varepsilon$. Then for $x, x' \in U$ we get

$$\begin{aligned}\Big|f(x') - f(x) - A\,(x' - x)\Big| &= \Big|\int_0^1 \frac{d}{d\xi}\Big(f(x + \xi(x' - x)) - A\,(x + \xi(x' - x))\Big)\, d\xi\Big|\\ &= \Big|\int_0^1 [f'(x + \xi(x' - x)) - A]\,(x' - x)\, d\xi\Big|\\ &\leq \lambda\varepsilon|x' - x| \leq \varepsilon|A\,(x' - x)|.\end{aligned}$$

Hence

$$\begin{aligned}\Big|f(x') - f(x)\Big| \leq \Big|A\,(x' - x)\Big| + \Big|f(x') - f(x) - A\,(x' - x)\Big| &\leq |Ax' - Ax| + \varepsilon|Ax' - Ax|\\ &= (1 + \varepsilon)|Ax' - Ax|\end{aligned}$$

and similarly

$$\Big|f(x') - f(x)\Big| \geq \Big|A\,(x'-x)\Big| - \Big|f(x') - f(x) - A\,(x'-x)\Big| \geq |Ax' - Ax| - \varepsilon|Ax' - Ax| \\ = (1-\varepsilon)|Ax' - Ax|$$

□

6.2. **Lemma.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be a $\mathcal{C}^1$-mapping. Let $x_0 \in \Omega$ with $Jf(x_0) \neq 0$. Then given $\varepsilon \in (0,1)$ there exists a neighborhood $U$ of $x_0$ such that for each measurable set $E \subset U$, $f(E)$ is $\mathcal{H}^n$-measurable and*

$$(1-\varepsilon)^{n+1} \int_E |Jf(x)|\,dx \leq \mathcal{H}^n(f(E)) \leq (1+\varepsilon)^{n+1} \int_E |Jf(x)|\,dx. \tag{12}$$

*Proof.* Let $A = f'(x_0)$. We write $f$ as

$$f = (f \circ A^{-1}) \circ A.$$

Choose $\varepsilon > 0$ and find a neighborhood $U$ of $x_0$ such that the conclusion of Lemma 6.1 holds and moreover

$$(1+\varepsilon)^{-1}|JA| \leq |Jf(x)| \leq (1-\varepsilon)^{-1}|JA|, \qquad x \in U. \tag{13}$$

Let $E \subset U$ be measurable. Then by Theorem 5.4 we have

$$\mathcal{H}^n(A(E)) = |JA|\mathcal{L}^n(E). \tag{14}$$

Further, Lemma 6.1 in fact says that

$$\operatorname{lip}_{A(E)} f \circ A^{-1} \leq 1 + \varepsilon,$$
$$\operatorname{lip}_{f(E)} A \circ f^{-1} \leq (1-\varepsilon)^{-1}.$$

By Theorem 2.7 and (14) then

$$\mathcal{H}^n(f(E)) = \mathcal{H}^n(f \circ A^{-1}(A(E))) \leq (1+\varepsilon)^n \mathcal{H}^n(A(E)) = (1+\varepsilon)^n |JA|\mathcal{L}^n(E). \tag{15}$$

With the aid of (13) it follows

$$\mathcal{H}^n(f(E)) \leq (1+\varepsilon)^n \int_E |JA|\,dx \leq (1+\varepsilon)^{n+1} \int_E |Jf(x)|\,dx.$$

Similarly one can verify the second inequality. It remains to show that $f(E)$ is $\mathcal{H}^n$-measurable. We can write

$$E = N \cup \bigcup_{j=1}^{\infty} F_j$$

where $F_j$ are compact and $\mathcal{L}^n(N) = 0$. By the above computation we have also $\mathcal{H}^n(f(N)) = 0$. The sets $f(F_j)$ are $\mathcal{H}^n$-measurable because in view of continuity of $f$ they are compact. Hence

$$f(E) = f(N) \cup \bigcup_{j=1}^{\infty} f(F_j)$$

is $\mathcal{H}^n$-measurable. □

6.3. **Lemma** (Area formula: regular case)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be an $n$-regular $\mathcal{C}^1$-mapping. Then the area formula holds for $f$.*

*Proof.* Choose $\varepsilon \in (0,1)$ and a measurable set $E \subset \Omega$. We cover $\Omega$ by balls $B_j$ such that $f$ restricted to $B_j$ is one-to-one and for each $j = 1, 2, \dots$ and each measurable $F \subset B_j$ we have (12). We set

$$\begin{aligned} E_1 &= E \cap B_1, \\ E_2 &= E \cap (B_2 \setminus B_1), \\ E_3 &= E \cap (B_3 \setminus (B_1 \cup B_2)), \\ &\dots \end{aligned} \tag{16}$$

Then the sets $E_j$ are pairwise disjoint and $E = \bigcup_{j=1}^{\infty} E_j$. By (12) we have

$$(1-\varepsilon)^{n+1} \int_{E_j} |Jf(x)|\,dx \leq \mathcal{H}^n(f(E_j)) \leq (1+\varepsilon)^{n+1} \int_{E_j} |Jf(x)|\,dx.$$

Taking into account that the sets $f(E_j)$ are $H^n$-measurable (by Lemma 6.2) and $f$ is one-to-one in $B_j$, we rewrite $\mathcal{H}^n(f(E_j))$ as

$$\mathcal{H}^n(f(E_j)) = \int_{\mathbb{R}^d} \mathcal{N}(f, y, E_j)\, d\mathcal{H}^n(y).$$

Summing over $j$ we obtain

$$(1-\varepsilon)^{n+1} \int_E |Jf(x)|\, dx \le \int_{\mathbb{R}^d} \mathcal{N}(f, y, E)\, d\mathcal{H}^n(y) \le (1+\varepsilon)^{n+1} \int_E |Jf(x)|\, dx.$$

Letting $\varepsilon \to 0$ we obtain the assertion. □

6.4. **Lemma** (Sard's theorem). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be a $\mathcal{C}^1$-mapping. Let $Z = \{x \in \Omega : Jf(x) = 0\}$. Then $\mathcal{H}^n(f(Z)) = 0$.*

*Proof.* Since the matter is local, we may assume that $f'$ is bounded in $\Omega$ and that $Z$ is bounded. Choose $\varepsilon \in (0, 1)$ and consider the mappings $f_\varepsilon : \Omega \to \mathbb{R}^d \times \mathbb{R}^n$, $\Pi : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d$ defined by

$$f_\varepsilon(x) = (f(x), \varepsilon x), \qquad \Pi(y_1, \dots, y_{d+n}) = (y_1, \dots, y_d).$$

Then

$$f = \Pi \circ f_\varepsilon,$$

the mapping $f_\varepsilon$ is $n$-regular and one-to-one and the mapping $\Pi$ is Lipschitz with $\operatorname{lip} \Pi = 1$. The Jacobi matrix of $f_\varepsilon$ at $x \in \Omega$ is

$$\begin{pmatrix} \frac{\partial f^{(1)}}{\partial x_1} & \dots & \frac{\partial f^{(1)}}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f^{(d)}}{\partial x_1} & \dots & \frac{\partial f^{(d)}}{\partial x_n} \\ \varepsilon & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \varepsilon \end{pmatrix}.$$

We easily estimate that there is a constant $C$ such that

$$|Jf_\varepsilon(x)|^2 \le |Jf(x)|^2 + C^2 \|f'(x)\|^{2n-2} \varepsilon^2.$$

In particular, if $x \in Z$, then

$$|Jf_\varepsilon(x)| \le C_{17}\, \varepsilon, \tag{17}$$

where $C_{17}$ is a multiple of the upper bound for $\|f'(x)\|$ over $Z$. Hence

$$\mathcal{H}^n(f(Z)) = \mathcal{H}^n(\Pi(f_\varepsilon(Z))) \le \mathcal{H}^n(f_\varepsilon(Z)) \le \int_Z |Jf_\varepsilon(x)|\, dx \le C_{17}\, \varepsilon \mathcal{L}^n(Z).$$

Letting $\varepsilon \to 0$ we obtain the assertion. □

6.5. **Remark.** It is quite technical to prove that for $d = n$ we have

$$Jf = 0 \text{ on } Z \implies \mathcal{L}^n(f(Z)) = 0$$

using Lebesgue measure only. The above proof is more elegant, however, it requires the knowledge of $n$-dimensional integration in $\mathbb{R}^{2n}$.

6.6. **Theorem** (Area formula: the $\mathcal{C}^1$ case). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be a $\mathcal{C}^1$-mapping. Then the area formula holds for $f$.*

*Proof.* Set $Z = \{x \in \Omega : Jf(x) = 0\}$. Since $Jf$ is continuous, the set $\Omega \setminus Z$ is open. By Lemma 6.3, the area formula for $f$ holds on $\Omega \setminus Z$. By Lemma 6.4, $\mathcal{H}^n(f(Z)) = 0$ and thus the area formula for $f$ holds on $Z$. Therefore it holds on $\Omega$. □

Now we turn to the case $d \le n$.

6.7. **Lemma** (Coarea formula: regular case). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be a $d$-regular $\mathcal{C}^1$-maping, $d \le n$. Then the coarea formula holds for $f$.*

*Proof.* We already know the trick how make the matter local, cf. (16). Hence it is enough to verify that for each $x_0 \in \Omega$, the coarea furmula for $f$ holds on some neighborhood $U$ of $x_0$. Let $A = f'(x_0)$. Then $\nabla f^{(1)}(x_0), \dots, \nabla f^{(d)}(x_0)$ are linearly independent. We find a multiindex $\beta = (\beta_1, \dots, \beta_{n-d})$ such that

$$\nabla f^{(1)}(x_0), \dots, \nabla f^{(d)}(x_0), \mathbf{e}_{\beta_1}, \dots, \mathbf{e}_{\beta_{n-d}} \text{ are linearly independent.}$$

We write

$$\Pi(x) = (x_{\beta_1}, \dots, x_{\beta_{n-d}}),$$
$$\Phi(x) = (f(x), \Pi(x)).$$

Then $J_n\Phi(x_0) \neq 0$. We find a neighborhood $U$ of $x_0$ such that $\Phi$ is a diffeomorphism on $U$. By the local invertibility theorem there is a diffeomorphic inverse mapping $\Psi$ to $\Phi$ on $\Phi(U)$. The set $\Phi(E)$ is measurable and thus, by the Fubini theorem, for $\mathcal{L}^d$-a.e. $y$ we have measurability of

$$\{z \in \mathbb{R}^{n-d} : (y, z) \in \Phi(E)\}.$$

For such $y$, by Theorem 4.7 and Theorem 6.6, the set $E \cap f^{-1}(y)$ is $\mathcal{H}^{n-d}$-measurable and we have

$$\mathcal{H}^{n-d}(E \cap f^{-1}(y)) = \int_{\Pi(E\cap f^{-1}(y))} |J_{n-d}\Psi(y, \cdot)(z)|\, dz. \tag{18}$$

By the classical change of variables and the Fubini theorem

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{H}^{n-d}(E \cap f^{-1}(y))\, dy &= \int_{\mathbb{R}^d} \Big( \int_{\Pi(E\cap f^{-1}(y))} |J_{n-d}\Psi(y, \cdot)(z)|\, dz \Big)\, dy \\ &= \int_{\Phi(E)} |J_{n-d}\Psi(y, \cdot)(z)|\, dz\, dy \\ &= \int_E |J_{n-d}\Psi(f(x), \cdot)(\Pi(x))|\, |J\Phi(x)|\, dx, \end{aligned} \tag{19}$$

where it follows from the computation that the function

$$y \mapsto \mathcal{H}^{n-d}(E \cap f^{-1}(y))$$

is $\mathcal{L}^d$-measurable. The equation (19) is almost the coarea formula, but instead of the Jacobian $J_d f(x)$ we have some strange expression. We need to show the equality

$$|J_{n-d}\Psi(f(x), \cdot)((\Pi(x)))|\, |J\Phi(x)| = |J_d f(x)|. \tag{20}$$

Applying the formula (19) to the linear mapping $A = f'(x)$ in place of $f$ and specifying $E$ to be the unit interval $I = (0, 1)^n$, appealing to Theorem 5.6 and integrating with respect to $x'$ while $x$ remains fixed, we obtain

$$\begin{aligned} |J_{n-d}\Psi(f(x), \cdot)(\Pi(x))|\, |J\Phi(x)| &= \int_I |J_{n-d}\Psi(f(x), \cdot)(\Pi(x))|\, |J\Phi(x)|\, dx' \\ &= \int_{\mathbb{R}^d} \mathcal{H}^{n-d}(I \cap A^{-1}(y'))\, dy' \\ &= |J_d A|\, \mathcal{L}^n(I) = |J_d A|. \end{aligned}$$

This proves (20). □

6.8. **Lemma** (Eilenberg's inequality). *Let $T \subset \mathbb{R}^n \times \mathbb{R}^d$ and $d \le k \le n + d$. Then*

$$\int_{\mathbb{R}^d} \mathcal{H}^{k-d}\Big(T \cap (\mathbb{R}^n \times \{w\})\Big)\, dw \le C_{21}\mathcal{H}^k(T) \tag{21}$$

*where*

$$C_{21} = \frac{\boldsymbol{\alpha}_{k-d}\boldsymbol{\alpha}_d}{\boldsymbol{\alpha}_k}.$$

*Proof.* Choose $\delta > 0$ and consider a covering $\{T_j\}$ of $T$ such that $\operatorname{diam} T_j < \delta$ for all $j \in \mathbb{N}$. Let $X_j$ be the projection of $T_j$ to $\mathbb{R}^n$ and $W_j$ be the projection of $T_j$ to $\mathbb{R}^d$, i.e.

$$X_j = \{x \in \mathbb{R}^n :\ \exists\, w \in \mathbb{R}^d,\ (x, w) \in T_j\},$$
$$W_j = \{w \in \mathbb{R}^d :\ \exists\, x \in \mathbb{R}^n,\ (x, w) \in T_j\}.$$

Then $\operatorname{diam} X_j \le \operatorname{diam} T_j$ and $\operatorname{diam} W_j \le \operatorname{diam} T_j$. For any $w \in \mathbb{R}^d$ we have

$$T \cap (\mathbb{R}^n \times \{w\}) \subset \bigcup_j X_j \times (W_j \cap \{w\})$$

and thus

$$\mathcal{H}_\delta^{k-d}\Big(T \cap (\mathbb{R}^n \times \{w\})\Big) \le 2^{d-k}\boldsymbol{\alpha}_{k-d}\sum_j (\operatorname{diam} X_j)^{k-d}\chi_{W_j}(w). \tag{22}$$

On the other hand, by the isodiametric inequality (Lemma 3.6), for each $j$ we have

$$\mathcal{L}^d(W_j) \le 2^{-d}\boldsymbol{\alpha}_d(\operatorname{diam} W_j)^d. \tag{23}$$

Using the Levi theorem, (22) and (23) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{H}_\delta^{k-d}\Big(T \cap (\mathbb{R}^n \times \{w\})\Big)\,dw &\le \int_{\mathbb{R}^d}\Big(2^{d-k}\boldsymbol{\alpha}_{k-d}\sum_j (\operatorname{diam} X_j)^{k-d}\chi_{W_j}(w)\Big)\,dw\\
&= \sum_j \Big(\int_{\mathbb{R}^d} 2^{d-k}\boldsymbol{\alpha}_{k-d}(\operatorname{diam} X_j)^{k-d}\chi_{W_j}(w)\Big)\,dw\\
&= 2^{d-k}\boldsymbol{\alpha}_{k-d}\sum_j (\operatorname{diam} X_j)^{k-d}\mathcal{L}^d(W_j)\\
&\le 2^{-k}\boldsymbol{\alpha}_{k-d}\boldsymbol{\alpha}_d(\operatorname{diam} X_j)^{k-d}(\operatorname{diam} W_j)^d\\
&\le 2^{-k}\boldsymbol{\alpha}_{k-d}\boldsymbol{\alpha}_d(\operatorname{diam} T_j)^k.
\end{aligned}$$

Taking infimum over all admissible coverings $\{T_j\}$ gives

$$\int_{\mathbb{R}^d} \mathcal{H}_\delta^{k-d}\Big(T \cap (\mathbb{R}^n \times \{w\})\Big)\,dw \le \frac{\boldsymbol{\alpha}_{k-d}\boldsymbol{\alpha}_d}{\boldsymbol{\alpha}_k}\,\mathcal{H}_\delta^k(T) \le C_{21}\mathcal{H}^k(T).$$

Letting $\delta \to 0$ we conclude the proof. □

6.9. **Lemma.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a $\mathcal{C}^1$-mapping. Let $Z = \{x \in \Omega : J_d f(x) = 0\}$. Then*

$$\int_{\mathbb{R}^d} \mathcal{H}^{n-d}(f^{-1}(y) \cap Z)\,dy = 0.$$

*Proof.* Since the matter is local, we may assume that $f'$ is bounded in $\Omega$ and that $Z$ is bounded. Choose $\varepsilon > 0$ and consider the mapping $f_\varepsilon : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$f_\varepsilon(x, w) = f(x) + \varepsilon w.$$

The Jacobi matrix of $f_\varepsilon$ at $(x, w) \in \Omega \times \mathbb{R}^d$ is

$$\begin{pmatrix}
\frac{\partial f^{(1)}}{\partial x_1} & \dots & \frac{\partial f^{(1)}}{\partial x_n} & \varepsilon & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots\\
\frac{\partial f^{(d)}}{\partial x_1} & \dots & \frac{\partial f^{(d)}}{\partial x_n} & 0 & \dots & \varepsilon
\end{pmatrix}.$$

It follows that $J_d f_\varepsilon \ne 0$ and we easily estimate that there is a constant $C$ such that

$$|J_d f_\varepsilon(x, w)|^2 \le |J_d f(x)|^2 + C^2\|f'(x)\|^{2d-2}\varepsilon^2.$$

In particular, if $x \in Z$, then

$$|J_d f_\varepsilon(x, w)| \le C_{24}\,\varepsilon, \tag{24}$$

where $C_{24}$ is a multiple of the upper bound for $\|f'(x)\|$ over $Z$. Let $I = (0, 1)^d$. Given $y \in \mathbb{R}^d$, by Lemma 6.8

$$\int_I \mathcal{H}^{n-d}\Big((Z \times \{w\}) \cap f_\varepsilon^{-1}(y)\Big)\,dw \le C_{21}\mathcal{H}^n\Big((Z \times I) \cap f_\varepsilon^{-1}(y)\Big).$$

For any $w \in \mathbb{R}^d$ we clearly have

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{H}^{n-d}\Big((Z \times \{w\}) \cap f_\varepsilon^{-1}(y)\Big)\,dy &= \int_{\mathbb{R}^d} \mathcal{H}^{n-d}\Big(Z \cap f^{-1}(y - \varepsilon w)\Big)\,dy\\
&= \int_{\mathbb{R}^d} \mathcal{H}^{n-d}\Big(Z \cap f^{-1}(y)\Big)\,dy.
\end{aligned}$$

Using Fubini's theorem we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{H}^{n-d}\Big(Z\cap f^{-1}(y)\Big)\,dy &= \int_I\Big(\int_{\mathbb{R}^d} \mathcal{H}^{n-d}\Big((Z\times\{w\})\cap f_\varepsilon^{-1}(y)\Big)\,dy\Big)\,dw\\ &= \int_{\mathbb{R}^d}\Big(\int_I \mathcal{H}^{n-d}\Big((Z\times\{w\})\cap f_\varepsilon^{-1}(y)\Big)\,dw\Big)\,dy\\ &\le C_{21}\int_{\mathbb{R}^d}\mathcal{H}^n\Big((Z\times I)\cap f_\varepsilon^{-1}(y)\Big)\,dy.\end{aligned}$$

Using the coarea formula for $d$-regular mappings (Lemma 6.7) we can rewrite the previous estimate as

$$\int_{\mathbb{R}^d} \mathcal{H}^{n-d}\Big(Z\cap f^{-1}(y)\Big)\,dy \le C_{21}\int_{Z\times I} Jf_\varepsilon(x,w)\,dx\,dw \le C_{21}C_{24}\,\varepsilon\,\mathcal{L}^n(Z).$$

Letting $\varepsilon\to 0$ we obtain the assertion. □

6.10. **Theorem** (Coarea formula: the $\mathcal{C}^1$ case)**.** *Let $\Omega\subset\mathbb{R}^n$ be an open set and $f:\Omega\to\mathbb{R}^d$ be a $\mathcal{C}^1$-mapping, $d\le n$. Then the coarea formula holds for $f$.*

*Proof.* Set $Z=\{x\in\Omega: J_d f(x)=0\}$. Since $J_d f$ is continuous, the set $\Omega\setminus Z$ is open. By Lemma 6.7 and Lemma 6.9, the coarea formula for $f$ holds on $\Omega\setminus Z$ and on $Z$. Therefore it holds on $\Omega$. □

## 7. Lusin type approximation

7.1. **Lemma** (McShane; Lipschitz extension)**.** *Let $E\subset\mathbb{R}^n$ and and $u$ be a bounded Lipschitz function on $E$. Then there exists a function $v:\mathbb{R}^n\to\mathbb{R}$ such that*

$$v=u \text{ on } E,\qquad \sup_{\mathbb{R}^n}|v|\le\sup_E|u|,\qquad \operatorname{lip}_{\mathbb{R}^n} v\le \operatorname{lip}_E u.$$

*Proof.* Denote

$$C_0=\sup_E|u|,\quad C_1=\operatorname{lip}_E u.$$

Set

$$v(x)=\min\Big\{C_0,\ \inf\big\{u(z)+C_1|x-z|,\ z\in E\big\}\Big\}.$$

Then it is easy to verify that $v$ has the required properties. □

7.2. **Remark.** Notice that $W^{1,\infty}(\mathbb{R}^n)$ is exactly the set of all functions on $\mathbb{R}^n$ which have a bounded Lipschitz representative. For this representative $u$ we have

$$\|u\|_{1,\infty}=\sup|u|+\operatorname{lip}u.$$

7.3. **Lemma.** *Let $B=B(x_0,R)$ and $u\in W^{1,1}(B)$. If $x\in B$ is a Lebesgue point for $u$ and a point where $M\nabla u(x)<+\infty$ (where $M\nabla u$ is the Hardy-Littlewood maximal function of the function equal to $\nabla u$ on $B$ and $0$ elsewhere), then*
(a)

$$-\Big(u(x)-\fint_B u\Big)=\int_0^1\Big(\fint_{B_t}\nabla u(z)\cdot\frac{z}{t}\,dz\Big)\,dt,$$

(b)

$$\Big|u(x)-\fint_B u\Big|\le 2R\int_0^1\Big(\fint_{B_t}|\nabla u(z)|\,dz\Big)\,dt\le 2^{n+1}RM\nabla u(x),$$

*where*

$$B_t=B(x+t(x_0-x),\,tR).$$

*Proof.* We may assume that $x=0$. We set

$$\Phi(t)=\fint_{B_t}u(z)\,dz=\fint_{B(z,R)}u(ty)\,dy,\qquad t\in(0,1].$$

Then

$$\Phi(1)-\Phi(a)=\int_a^1\Big(\fint_{B(z,R)}\nabla u(ty)\cdot y\,dy\Big)\,dt=\int_a^1\Big(\fint_{B_t}\nabla u(z)\cdot\frac{z}{t}\,dz\Big)\,dt,\quad a\in(0,1). \tag{25}$$

This follows from derivation beyond the sign of integral if $u$ is smooth, the general case then follows by mollification technique. Hence

$$(26)\qquad \Big|u(x)-\fint_B u\Big| = |\Phi(1)-\Phi(0+)| \le 2R\int_0^1\Big(\fint_{B(z,R)}|\nabla u(ty)|\,dy\Big)\,dt = 2R\int_0^1\Big(\fint_{B_t}|\nabla u(z)|\,dz\Big)\,dt$$

if 0 is a Lebesgue point for $u$. Since

$$\Big|\fint_{B_t}|\nabla u(z)|\,dz\Big| \le 2^n\fint_{B(x,2Rt)}|\nabla u(z)|\,dz$$

we have at such a $x$

$$\Big|u(x)-\fint_B u\Big| \le 2^{n+1}RM\nabla u(x).$$

If moreover $M\nabla u(x)<+\infty$, then the integral on the right of (26) converges, and thus we may let $a\to 0+$ on the right of (25).

□

7.4. **Lemma.** *Let $u\in W^{1,1}(\mathbb{R}^n)$ and $\varepsilon>0$. There is $v\in W^{1,1}(\mathbb{R}^n)\cap W^{1,\infty}(\mathbb{R}^n)$ such that*

$$(27)\qquad \|v\|_{1,1}\le C_{27}\|u\|_{1,1},$$

$$(28)\qquad \|v\|_{1,\infty}\le \frac{C_{28}}{\varepsilon}\|u\|_{1,1}$$

*and*

$$\mathcal{L}_n(\{v\ne u\})\le\varepsilon.$$

*The constants $C_{27}, C_{28}$ depend only on $n$.*

*Proof.* Choose $a>0$ and set

$$G_0=\{|u|>a\},\quad G_1=\{M\nabla u>a\}.$$

Then by the Chebyshev inequality

$$(29)\qquad \mathcal{L}_n(G_0)\le a^{-1}\int_{G_0}|u|\,dx\le\frac{\|u\|_1}{a}$$

and by the Hardy-Littlewood maximal theorem

$$(30)\qquad \mathcal{L}_n(G_1)\le\frac{5^n}{a}\|\nabla u\|_1.$$

Let $N$ be the set of zero measure consisting of all points which are not Lebesgue points for $u$. Set $G=G_0\cup G_1\cup N$. Then the choice

$$(31)\qquad a=5^n\,\frac{\|u\|_{1,1}}{\varepsilon}$$

implies by (29) and (30)

$$(32)\qquad \mathcal{L}_n(G)<\varepsilon.$$

Obviously

$$(33)\qquad |u(x)|\le a,\qquad x\notin G.$$

Let $x,y\notin G$. Consider the ball $B=B(z,R)$ where $z=\frac12(x+y)$ and $R=|y-x|$. Then $x,y\in B$. By Lemma 7.3, we have

$$|u(x)-u_B|\le 2CR\,M\nabla u(x),\quad |u(y)-u_B|\le 2CR\,M\nabla u(y),$$

and thus

$$(34)\qquad |u(y)-u(x)|\le 2CR\big(MDu(x)+MDu(y)\big)\le 4CR\,a=4C\,a\,|y-x|,\qquad x,y\notin G.$$

By (31), (33), (34) and Lemma 7.1, there exist a Lipschitz function $v:\mathbb{R}^n\to\mathbb{R}$ and a constant $C_{28}$ such that

$$v=u \text{ on } \mathbb{R}^n\setminus G$$

and

$$(35)\qquad \|v\|_{1,\infty}\le(1+4C)a\le C_{28}\frac{\|u\|_{1,1}}{\varepsilon}.$$

We have $\nabla v = \nabla u$ a.e. in $\mathbb{R}^n \setminus G$ (see TDPP). Using (32) and (35) we estimate

$$\begin{aligned} \|v\|_{1,1} &\le \int_{\mathbb{R}^n\setminus G} (|v| + |\nabla v|)\,dx + \int_G (|v| + |\nabla v|)\,dx \\ &\le \int_{\mathbb{R}^n\setminus G} (|u| + |\nabla u|)\,dx + 2\|v\|_{1,\infty}\mathcal{L}^n(G) \qquad (36)\\ &\le \Big(1 + 2C_{28}\Big)\|u\|_{1,1}. \end{aligned}$$

Hence the function $v$ has the required properties. □

7.5. **Lemma.** *Let $u \in W^{1,1}(\mathbb{R}^n)$ a $\varepsilon > 0$. Then there exist functions $\tilde{u} \in W^{1,1}(\mathbb{R}^n)$ and $w \in \mathcal{C}^1(\mathbb{R}^n)$ such that*

$$\begin{aligned} \|\tilde{u}\|_{1,1} &\le \frac{1}{4}\|u\|_{1,1}, \\ \|w\|_{1,\infty} &\le \frac{C_{28}}{\varepsilon}\|u\|_{1,1}, \end{aligned} \qquad (37)$$

*and*

$$\mathcal{L}_n(\{\tilde{u} + w \ne u\}) < \varepsilon.$$

*Proof.* Let $v$ be as in Lemma 7.4. Set

$$w = \psi_\delta * v, \quad \tilde{u} = v - w,$$

where $\delta > 0$ is so small that

$$\|\tilde{u}\|_{1,1} < \frac{1}{4}\|u\|_{1,1}$$

The existence of such $\delta$ is justified because

$$\|\psi_\delta * v - v\|_1 \to 0, \qquad \|\nabla(\psi_\delta * v) - \nabla v\|_1 \to 0.$$

By the convolution inequality and Lemma 7.4

$$\|w\|_{1,\infty} \le \|v\|_{1,\infty} \le \frac{C_{28}}{\varepsilon}\|u\|_{1,1}.$$

Finally

$$\mathcal{L}_n(\{\tilde{u} + w \ne u\}) = \mathcal{L}_n(\{v \ne u\}) < \varepsilon.$$

□

7.6. **Theorem** (Calderón and Zygmund)**.** *Let $u \in W^{1,1}(\mathbb{R}^n)$ and $\varepsilon > 0$. Then there exists $v \in \mathcal{C}^1(\mathbb{R}^n)$ such that*

$$\|v\|_{1,1} \le C_{27}\|u\|_{1,1}, \qquad (38)$$

$$\|v\|_{1,\infty} \le \frac{C_{28}}{\varepsilon}\|u\|_{1,1} \qquad (39)$$

*and*

$$\mathcal{L}_n\big(\{v \ne u\}\big) \le \varepsilon. \qquad (40)$$

*Proof.* A recurrent use of Lemma 7.5 yields sequences $\{u_j\}$ and $\{w_j\}$ of functions on $\mathbb{R}^n$ such that $u_0 = u$, $u_j \in W^{1,1}(\mathbb{R}^n)$ for $j \in \mathbb{N}$, $w_j \in \mathcal{C}_1(\mathbb{R}^n)$ for $j \in \mathbb{N}$ and

$$\begin{aligned} \|u_j\|_{1,1} &\le 4^{-j}\|u\|_{1,1}, \\ \|w_j\|_{1,\infty} &\le \frac{C_{28}}{2^{-j}\varepsilon}\|u_{j-1}\|_{1,1} \le \frac{C_{28}2^{-j}}{\varepsilon}\|u\|_{1,1} \end{aligned}$$

and

$$\mathcal{L}_n(\{u_j + w_j \ne u_{j-1}\}) < 2^{-j}\varepsilon.$$

Set

$$G = \Big(\bigcup_j \{u_j + w_j \ne u_{j-1}\}\Big) \cup \Big\{\sum_j u_j = \infty\Big\}.$$

Then

$$\mathcal{L}_n(G) < \varepsilon.$$

On the set $\mathbb{R}^n \setminus G$ we have

$$u_0 = u_1 + w_1 = u_2 + w_2 + w_1 = u_3 + w_3 + w_2 + w_1 = \cdots = \sum_j w_j.$$

The series $\sum_j w_j$ converges in $\mathcal{C}^1$ by the Weierstrass criterion. We define $v = \sum_j w_j$, this is a $\mathcal{C}^1$-function. The estimate (39) follows easily from the construction and (38) can be derived as in (36). □

## 8. Area and coarea formula for Sobolev functions

8.1. **Condition N.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $\Omega' \subset \Omega$. We say that a mapping $f : \Omega \to \mathbb{R}^d$ satisfies the (Lusin) condition N on $\Omega'$ if for each $E \subset \Omega'$ we have

$$\mathcal{L}^n(E) = 0 \implies \mathcal{H}^n(f(E)) = 0.$$

8.2. **Theorem** (Area formula with condition N)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\Omega' \subset \Omega$. Let $f : \Omega \to \mathbb{R}^d$ be a weakly differentiable mapping, $d \geq n$. Then the area formula for $f$ on $\Omega'$ holds if and only if $f$ satisfies the condition N on $\Omega'$.*

*Proof.* If the area formula holds for $f$, for each $E \subset \Omega'$ we have

$$\mathcal{H}^n(f(E)) \leq \int_{\mathbb{R}^d} \mathcal{N}(f, y, E)\, d\mathcal{H}^n(y) = \int_E |Jf|\, dx = 0.$$

This verifies the only if part. The if part is more delicated, however, we have Theorem 7.6 in hand which is a very strong tool for this purpose. Since the matter is local, we may assume that $\Omega = \mathbb{R}^n$ and $f \in W^{1,1}(\mathbb{R}^n)$. Consider $E \subset \Omega' \subset \mathbb{R}^n$. For any $j \in \mathbb{N}$ we find by Theorem 7.6 a mapping $f_j \in \mathcal{C}^1(\mathbb{R}^n)$ such that

$$\mathcal{L}^n(\{f_j \neq f\}) < 2^{-j}.$$

We know that

$$\nabla f_j = \nabla f \text{ a.e. in } \{f_j = f\}.$$

Let

$$\begin{aligned} E_j &= E \cap \{f = f_j\} \cap \{\nabla f = \nabla f_j\}, \\ F_j &= E_1 \cup \cdots \cup E_j, \end{aligned} \qquad j \in \mathbb{N}$$

and write

$$F_0 = \emptyset, \qquad N = \Omega' \setminus \bigcup_j E_j.$$

Then the sets $E_j \setminus F_{j-1}$ are pairwise disjoint and their union is $\Omega' \setminus N$. Then

$$\begin{aligned} \int_{E_j \setminus F_{j-1}} |Jf|\, dx = \int_{E_j \setminus F_{j-1}} |Jf_j|\, dx &= \int_{\mathbb{R}^d} \mathcal{N}(f_j, y, E_j \setminus F_{j-1})\, d\mathcal{H}^n(y) \\ &= \int_{\mathbb{R}^d} \mathcal{N}(f, y, E_j \setminus F_{j-1})\, d\mathcal{H}^n(y) \end{aligned}$$

and summing over $j$ we obtain

$$\int_{\Omega' \setminus N} |Jf|\, dx = \int_{\mathbb{R}^d} \mathcal{N}(f, y, \Omega' \setminus N)\, d\mathcal{H}^n(y). \tag{41}$$

On the other hand, the set $N$ has measure zero and thus the condition N guarantees

$$\int_N |Jf|\, dx = \int_{\mathbb{R}^d} \mathcal{N}(f, y, N)\, d\mathcal{H}^n(y). \tag{42}$$

Getting together (41) and (42) we obtain the assertion. □

8.3. **Condition co-N.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $\Omega' \subset \Omega$. We say that a mapping $f : \Omega \to \mathbb{R}^d$ satisfies the condition co-N on $\Omega'$ if for each $E \subset \Omega'$ we have

$$\mathcal{L}^n(E) = 0 \implies \int_{\mathbb{R}^d} \mathcal{H}^{n-d}(E \cap f^{-1}(y))\, dy = 0.$$

8.4. **Theorem** (Coarea formula with condition co-N)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\Omega' \subset \Omega$. $f : \Omega \to \mathbb{R}^d$ be a weakly differentiable mapping, $d \leq n$. Then the coarea formula for $f$ on $\Omega'$ holds if and only if $f$ satisfies the condition co-N on $\Omega'$.*

*Proof.* This can be shown by the same argument as in case of area formula (Theorem 8.2). □

## 9. Generalized Lipschitz functions

In this section, $\Omega$ stands for a fixed open set in $\mathbb{R}^n$. Let $E \subset \Omega$ and $f : \Omega \to \mathbb{R}^d$. The diameter of the image $f(E)$ is called the *oscillation* of $f$ over $E$ and denoted as $\operatorname{osc}_E f$.

9.1. **Generalized Lipschitz function.** Let $1 \le p < \infty$. A function $f : \Omega \to \mathbb{R}^d$ is said to be a generalized Lipschitz function of class $\mathrm{RR}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ (shortly $f \in \mathrm{RR}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$) if there exists a nonnegative function $\theta \in L^p_{\mathrm{loc}}(\Omega)$ such that

$$\Big(\frac{\operatorname{osc}_{B(z,r/2)} f}{r}\Big)^p \le \fint_{B(z,r)} \theta^p \, dx \tag{43}$$

for each $B(z,r) \subset \Omega$. The function $\theta^p$ is called a *weight* for $f$.

We write $f \in \mathrm{RR}^p(\Omega)$ if it has a weight $\theta^p \in L^1(\Omega)$.

9.2. **Remark.** We write the definition with $\theta^p$ to keep the analogy with the Poincaré inequalities. However, this notation may be somewhat inconvenient. Indeed a similar condition can be considered with a measure weight in place of $\theta^p$. Then of course, $\theta$ itself does not make a sense.

The use of a smaller ball instead of $B(z,r)$ on the left of (43) has the reason that then the class is much more stable, eg. it is invariant under a bilipschitz change of variables and does not depend on the choice of norm in $\mathbb{R}^n$.

9.3. **Observations.** (a) *For $1 \le p < q$ we have $\mathrm{RR}^q_{\mathrm{loc}}(\Omega; \mathbb{R}^d) \subset \mathrm{RR}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$.*

(b) *If $\Omega$ is connected, then $\mathrm{RR}^p(\Omega; \mathbb{R}^d)$ factorized by constants, is a Banach space with the norm*

$$\|f\|_{\mathrm{RR}^p} = \inf\big\{\|\theta\|_p : \ \theta^p \text{ is a weight for } f\big\}.$$

(c) *Each Lipschitz function is a $RR^p_{\mathrm{loc}}$-functions with a constant weight.*

(d) *Conversely, if $f$ is a $\mathrm{RR}^p_{\mathrm{loc}}$ function with a constant weight, then it is locally Lipschitz.*

(e) *Each $f \in \mathrm{RR}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ is locally bounded.*

9.4. **Remark.** If $\Omega \subset \mathbb{R}^1$ is an interval, then $f \in RR^p_{\mathrm{loc}}(\Omega)$ if and only if $f$ is locally absolutely continuous and $f' \in L^p_{\mathrm{loc}}(\Omega)$.

9.5. **Theorem.** *Each $f \in \mathrm{RR}^n_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ is continuous.*

*Proof.* From (43), for $p = n$ we obtain

$$\operatorname{osc}_{B(z,r/2)} f \le C\Big(\int_{B(z,r)} \theta^n \, dx\Big)^{1/n}$$

The right hand side tends to zero as $r \to 0+$. □

9.6. **Remark.** If $1 \le p < n$, then the characteristic function of a single point is an $\mathrm{RR}^p$-function, which, in contrast with the theory of $L^p$-spaces, is not identified with the zero function. Another example of a discontinuous $\mathrm{RR}^p$-function, $1 \le p < n$, is $x_1/|x|$.

9.7. **Theorem** (Weak differentiability)**.** *Let $f \in \mathrm{RR}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ with weight $\theta^p$. Then $f$ is weakly differentiable and*

$$|\nabla f| \le C_{44}\theta. \tag{44}$$

*Consequently, $\nabla f \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^{d\times n})$.*

*Proof.* See TDPP. □

The next result generalizes the Rademacher theorem on differentiation of Lipschitz functions.

9.8. **Theorem.** *Let $f \in \mathrm{RR}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$. Then $f$ is differentiable a.e. and $f' = \nabla f$ a.e.*

*Proof.* We may assume that $p = 1$. By the Lebesgue differentiation theorem, a.e. point $z \in \Omega$ is a simultaneous Lebesgue point for $f$, $\nabla f$ and $\theta$. We fix such a $z$ and prove that $\nabla f(z)$ is the differential of $f$ at $z$. We may assume that $f(z) = 0$ and $\nabla f(z) = 0$, otherwise we pass to the function $x \to f(x) - f(z) - \nabla f(z)(x - z)$. We choose $\varepsilon \in (0, \frac{1}{4})$ and (using Lebesgue point properties of $z$) find $\delta > 0$ so that

$$0 < \rho \le \delta \implies \begin{cases} \displaystyle\fint_{B(z,\rho)} |\nabla u(x)| \, dx \le \varepsilon^{n+1}, \\ \displaystyle\fint_{B(z,\rho)} |\theta(x) - \theta(z)| \, dx \le \varepsilon^n. \end{cases} \tag{45}$$

Let $y \in B(z, \delta/2)$ and $r = 2|y-z|$. Then $B(y, 2\varepsilon r) \subset B(z,r)$. We estimate

$$|f(y)| \le \Big| f(y) - \fint_{B(y,\varepsilon r)} f(x)\,dx \Big| + \fint_{B(y,\varepsilon r)} |f(x)|\,dx. \tag{46}$$

For the first term we have by (45)

$$\begin{aligned}
\Big| f(y) - \fint_{B(y,\varepsilon r)} f(x)\,dx \Big| &\le \operatorname{osc}_{B(y,\varepsilon r)} f \\
&\le \varepsilon\, r \fint_{B(y,2\varepsilon r)} \theta(x)\,dx \\
&\le \varepsilon\, r \Big( \theta(z) + \fint_{B(y,2\varepsilon r)} |\theta(x) - \theta(z)|\,dx \Big) \\
&\le \varepsilon\, r \Big( \theta(z) + 2^{-n}\varepsilon^{-n} \fint_{B(z,r)} |\theta(x) - \theta(z)|\,dx \Big) \\
&\le \varepsilon\, r \Big( \theta(z) + 1 \Big).
\end{aligned} \tag{47}$$

Because $z$ is a Lebesgue point for $f$, for the second term of (46) we have by Lemma 7.3 and (45)

$$\begin{aligned}
\fint_{B(y,\varepsilon r)} |f(x)|\,dx &\le \varepsilon^{-n} \fint_{B(z,r)} |f(x)|\,dx \\
&\le 2\varepsilon^{-n}\, r \sup_{0<\rho\le r} \fint_{B(z,\rho)} |\nabla f(x)|\,dx \\
&\le 2\varepsilon\, r.
\end{aligned} \tag{48}$$

From (46)–(48) we conclude that

$$|v(y)| \le C\varepsilon r \le C\varepsilon |y-z|.$$

This shows that $\nabla v(z) = 0$ as required. □

9.9. **Remark.** Theorem 9.7 and Theorem 9.8 are both statements on differentiation of generalized Lipschitz functions, but of a different nature. Notice that in general, weak differentiability does not imply differentiability a.e. (for $n > 1$ there exist weakly differentiable functions which are nowhere continuous and thus also nowhere differentiable independently on choice of a representative) and differentiability a.e. does not imply weak differentiability (Cantor's example).

9.10. **Theorem** (Embedding theorem for $p > n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $p > n$. Then $f$ has a continuous reprensentative for which, given any ball $B = B(z,r) \subset\subset \Omega$,*

$$\big(\operatorname{osc}_B f\big)^p \le C_{49} r^{p-n} \int_B |\nabla f|^p\,dx, \tag{49}$$

*where $C_{49}$ depends only on $n$ and $p$.*

*Proof.* Assume first that $f$ is smooth in $\mathbb{R}^n$ and consider a ball $B = B(z,r) \subset \mathbb{R}^n$. Denote

$$f_B = \fint_B f\,dy.$$

From Lemma 7.3,

$$\begin{aligned}
|f(x) - f_B| &\le C \int_0^1 \Big| \nabla u|_{B_t}\,dt \\
&\le Cr \int_0^1 \Big( \fint_{B_t} |\nabla u(y)|^p\,dy \Big)^{1/p} dt \\
&\le Cr \|\nabla u\|_{L^p(B)} \int_0^1 |B_t|^{-1/p}\,dt \\
&\le Cr^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B)} \int_0^1 t^{-n/p}\,dt.
\end{aligned} \tag{50}$$

Since $|f(x') - f(x)| \le |f(x') - f_B| + |f_B - f(x)|$ for any $x, x' \in B$, from (50) we obtain (49). Now, let $f \in W^{1,p}(\mathbb{R}^n)$. By mollification we obtain a sequence $\{f_j\}$ of smooth functions in $W^{1,p}(\mathbb{R}^n)$ such that $\|f_j - f\|_{1,p} < 2^{-j}$. Applying (49) to $f_j - f_{j+1}$ we observe that the sequence converges also locally uniformly, and thus its pointwise limit is a continuous representative of $f$. An easy passage to limit yields (49) for $f$. A standard localization argument proves the theorem for a general open domain. $\square$

9.11. **Remark.** Theorem 9.10 is a converse of Theorem 9.7 for $p > n$. It follows that $RR^p_{\mathrm{loc}}(\Omega)$ can be identified with $W^{1,p}_{\mathrm{loc}}(\Omega)$ for $p > n$.

9.12. **Remark.** If $n > 1$ and $1 \le p \le n$, there exists a continuous compactly suppported function $f \in W^{1,p}(\mathbb{R}^n)$ which does not belong to $RR^p_{\mathrm{loc}}(\mathbb{R}^n)$. Indeed, we find $f$ in the form

$$f = \sum_j f_j,$$

where $f_j$ are smooth functions supported in $B_j = B(x_j, r_j)$. We find first $r_j$ such that $\sum_j r_j^n < \infty$ whereas $\sum_j r_j^{n-p} = \infty$. Then we find $b_j \searrow 0$ such that still $\sum_j r_j^{n-p} b_j^p = \infty$. Then we find a bounded sequence $x_j$ such that the balls $B(x_j, 2r_j)$ are pairwise disjoint, and functions $f_j$ as above such that $\operatorname{osc}_{B_j} f_j = b_j$ and

$$\sum_j \int_{B_j} |\nabla f_j|^p < \infty.$$

Then the function $f$ has the required properties. Indeed, for any weight $\theta$ controlling $f$ we observe $\int_{B(x_j,2r_j)} \theta^p \ge r_j^{n-p} b_j^p$.

9.13. **Theorem** (Area formula for generalized Lipschitz functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in RR^n_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$, $d \ge n$. Then the area formula holds for $f$.*

*Proof.* In view of Theorem 8.2 and Theorem 9.7 it is enough to verify the condition N. Let $E \subset \Omega$ be a set of zero measure. Choose $\varepsilon > 0$. We find an open set $G \subset \Omega$ such that $E \subset G$ and

$$\int_G \theta^n \, dy < \varepsilon. \tag{51}$$

Consider $x \in E$. If we had

$$\int_{B(x,r)} \theta^n \, dy < 20^{-n} \int_{B(x,10r)} \theta^n \, dy$$

for all small $r > 0$, by iteration we would obtain

$$\lim_{r \to 0} \fint_{B(x,r)} \theta^n \, dy = 0$$

which would contradict that $\theta \ge 1$. Hence for each $x \in E$ there exists $r_x > 0$ such that $\overline{B}(x, r_x) \subset G$ and

$$\int_{B(x,10r_x)} \theta^n \, dy \le 20^n \int_{B(x,r_x)} \theta^n \, dy. \tag{52}$$

The system $\{B(x, r_x) : \ x \in E\}$ forms a covering of $E$ and therefore by the Vitali theorem (Theorem 1.5) there exists a pairwise disjoint sequence $\{B_j\}$ of balls $B_j = B(x_j, r_j)$ such that $x_j \in E$, $r_j = r_{x_j}$ and

$$E \subset \bigcup_j B(x_j, 5r_j).$$

Then

$$f(E) \subset \bigcup_j f(B(x_j, 5r_j)$$

and thus, using (52) and (51)

$$\begin{aligned}
\mathcal{H}^n_\infty(f(E)) &\le 2^{-n} \boldsymbol{\alpha}_n \sum_j (\operatorname{diam} f(B(x_j, 5r_j)))^n \le 2^{-n} \sum_j \int_{B(x_j,10r_j)} \theta^n \, dy \\
&\le 10^n \sum_j \int_{B(x_j,r_j)} \theta^n \, dy \\
&\le 10^n \int_G \theta^n \, dy < 10^n \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ we conclude the proof. $\square$

9.14. **Theorem** (Coarea formula for generalized Lipschitz functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in RR^d_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, $d \le n$. Then the coarea formula holds for $f$.*

*Proof.* In view of Theorem 8.4 and Theorem 9.7 it is enough to verify the condition co-N. Let $\theta^d$ be a weight for $f$, we may assume that $\theta \ge 1$. Let $E \subset \Omega$ be a set of zero measure. Choose $\varepsilon > 0$ and $\delta > 0$. We find an open set $G \subset \Omega$ such that $E \subset G$ and

$$\int_G \theta^d\,dy < \varepsilon. \tag{53}$$

Consider $x \in E$. As in the proof of Theorem 9.13 we find $r_x > 0$ such that $10r_x < \delta$, $\overline{B}(x,r_x) \subset G$ and

$$\int_{B(x,10r_x)} \theta^d\,dy \le 20^n \int_{B(x,r_x)} \theta^d\,dy. \tag{54}$$

The system $\{B(x,r_x):\ x \in E\}$ forms a covering of $E$ and therefore by the Vitali theorem (Theorem 1.5) there exists a pairwise disjoint sequence $\{B_j\}$ of balls $B_j = B(x_j,r_j)$ such that $x_j \in E$, $r_j = r_{x_j}$ and

$$E \subset \bigcup_j B(x_j,5r_j).$$

Given $y \in \mathbb{R}^d$, observe

$$\mathcal{H}^{n-d}_\delta(E \cap f^{-1}(y)) \le 2^{d-n}\boldsymbol{\alpha}_{n-d}\sum_j (\operatorname{diam} B(x_j,5r_j))^{n-d}\chi_{f(B(x_j,5r_j))}(y).$$

Integrating with respect to $y$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{H}^{n-d}_\delta(E \cap f^{-1}(y))\,dy &\le \int_{\mathbb{R}^d}\Big(2^{d-n}\boldsymbol{\alpha}_{n-d}\sum_j (\operatorname{diam} B(x_j,5r_j))^{n-d}\chi_{f(B(x_j,5r_j))}(y)\Big)\,dy\\ &= 2^{d-n}\boldsymbol{\alpha}_{n-d}\sum_j\Big(\int_{\mathbb{R}^d}(\operatorname{diam} B(x_j,5r_j))^{n-d}\chi_{f(B(x_j,5r_j))}(y)\,dy\Big)\\ &= 5^{n-d}\boldsymbol{\alpha}_{n-d}\sum_j r_j^{n-d}\mathcal{L}^d\big(f(B(x_j,5r_j))\big).\end{aligned}$$

Applying isodiametric inequality, (54) and (53) we can continue

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{H}^{n-d}_\delta(E \cap f^{-1}(y))\,dy &\le 5^{n-d}\boldsymbol{\alpha}_{n-d}2^{-d}\boldsymbol{\alpha}_d\sum_j r_j^{n-d}\big(\operatorname{diam} f(B(x_j,5r_j))\big)^d\\ &\le C20^{-n}\sum_j\int_{B(x_j,10r_j)}\theta^d\,dy \le C\sum_j\int_{B(x_j,r_j)}\theta^d\,dy\\ &\le C\int_G \theta^d\,dy \le C\varepsilon.\end{aligned}$$

□

9.15. **Remark.** The co-area formula also holds in Sobolev spaces $W^{1,p}$ with $p > d$ (Malý, Swanson and Ziemer). The area formula can fail for $W^{1,n}$ mappings and the coarea formula can fail for $W^{1,d}$ mappings.

## 10. Lebesgue density

10.1. **Lebesgue density and measure theoretic "topology".** Given a set $E \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, we define

$$\bar{d}(x,E) = \limsup_{r\to 0+}\frac{|B(x,r)\cap E|}{|B(x,r)|}.$$

A point $x \in \mathbb{R}^n$ is said to be a point of density for a set $E \subset \mathbb{R}^n$ if $\bar{d}(x,\mathbb{R}^n \setminus E) = 0$. The set of all points of density for $E$ is called the *measure theoretic interior* of $E$ and denoted as $\operatorname{int}_* E$. The *measure theoretic closure* of $E$ is

$$\operatorname{cl}_* E = \{x \in \mathbb{R}^n:\ \bar{d}(x,E) > 0\} = \mathbb{R}^n \setminus \operatorname{int}_*(\mathbb{R}^n \setminus E)$$

and finally the *measure theoretic boundary* of $E$ is

$$\partial_* E = \operatorname{cl}_* E \setminus \operatorname{int}_* E$$

All these sets are Borel sets (e.g. $\operatorname{int}_* E$ is $F_{\sigma\delta}$).

The so called *density topology* is closely related to these notions but there is still a difference between e.g. the density closure and measure theoretic closure. The density closure $\operatorname{cl}_d E$ of a set $E$ is the smallest density closed set which contains $E$, whereas $\operatorname{cl}_* E$ is the smallest density closed set which contains almost all of $E$. They are related by the formula $\operatorname{cl}_d E = E \cup \operatorname{cl}_* E$.

The following theorem is a version of the Lebesgue density theorem, which is an easy consequence of the theory of differentiation of measures. (Recall that this theory relies on Vitali covering theorems.)

10.2. **Theorem.** *Let $E \subset \mathbb{R}^n$ be a measurable set. Then $|\partial_* E| = 0$.*

## 11. Federer normal

11.1. **Federer normal.** Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $x \in \mathbb{R}^n$. A unit vector $\mathbf{n} \in \mathbb{R}^n$ is said to be a Federer normal to $\Omega$ at $x$ if

$$x \notin \operatorname{cl}_*\big(\Omega \bigtriangleup (x + \mathbb{H}_{\mathbf{n}})\big),$$

where $\mathbb{H}_{\mathbf{n}}$ is the halfspace

$$\mathbb{H}_{\mathbf{n}} = \{y \in \mathbb{R}^n : \ y \cdot \mathbf{n} \le 0\}$$

and $\bigtriangleup$ denotes the symmetric difference. The Federer normal is obviously unique if it exists.

Let $\mathbf{n}(x)$ is a Federer normal to $\Omega$ at $x$. We write in coordinates

$$\mathbf{n}(x) = (\mathrm{n}_1(x), \dots, \mathrm{n}_n(x)).$$

The set of all points of $\mathbb{R}^n$ where there exists a Federer normal to $\Omega$ is denoted by $\partial_F\Omega$. Obviously $\partial_F\Omega \subset \partial\Omega$.

A set $M \subset \mathbb{R}^n$ is called a *Lipschitz graph*, if there exists a Lipschitz function $\xi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$M = \{x \in \mathbb{R}^n : x_n = \xi(x_1, \dots, x_{n-1})\}.$$

We say that $M$ is a *rotated Lipschitz graph*, if there exists a linear isometric isomorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(M)$ is a Lipschitz graph.

11.2. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a measurable set. Then $\partial_F\Omega$ can be covered by a countable number of rotated Lipschitz graphs. More specifically, the set*

$$\partial_F^{n,+}\Omega = \{x \in \partial_F\Omega : \ \mathrm{n}_n(x) > 0\}$$

*can be covered by a countable number of Lipschitz graphs.*

*Proof.* Define

$$C_\alpha = \Big\{x \in \mathbb{R}^n : \frac{x_n}{|x|} > \cos\alpha\Big\}.$$

Since

$$\begin{aligned}
\partial_F^{n,+}\Omega = \bigcup_{p,q\in\mathbb{N}} \Big\{&x \in \partial_F\Omega \colon \mathrm{n}_n(x) > \sin\tfrac{1}{p}, \\
&|B(x,r) \cap (x + C_{1/p}) \cap \Omega| + |B(x,r) \cap (x - C_{1/p}) \cap \Omega^c| \le \varepsilon_p |B(x,r)| \text{ for each } 0 < r \le \tfrac{1}{q}\Big\}, \\
&\text{with } \varepsilon_p = \tfrac14\big(\tfrac12 \sin\tfrac{1}{2p}\big)^n
\end{aligned} \tag{55}$$

we may consider a part $\Gamma$ of $\partial_F\Omega$ which for given $p, q \in \mathbb{N}$ is contained in the set described on the right of (55) and moreover satisfies that $\operatorname{diam}\Gamma < 1/q$. Let $x, y \in \Gamma$, without loss of generality we may assume that $x_n \le y_n$. Assume that

$$y_n - x_n > \cos\tfrac{1}{2p}\ |y - x|. \tag{56}$$

Set

$$z = \tfrac12(x+y), \quad r = |x-y|, \quad \rho = \tfrac12\, r \sin\tfrac{1}{2p}.$$

An easy geometric argument shows that

$$\begin{aligned}
B(z,\rho) \cap \Omega &\subset B(x,r) \cap (x + C_{1/p}) \cap \Omega, \\
B(z,\rho) \cap \Omega^c &\subset B(y,r) \cap (y - C_{1/p}) \cap \Omega^c,
\end{aligned}$$

and thus

$$|B(z,\rho)| \le \varepsilon_p(|B(x,r)| + |B(y,r)| < |B(z,\rho)|,$$

so that (56) leads to a contradiction. Therefore, $\Gamma$ is contained in a Lipschitz graph with Lipschitz constant not exceeding $\operatorname{cotan}\frac{1}{2p}$. $\square$

## 12. Functions of bounded variation

Let $W \subset \mathbb{R}^n$ be an open set. Let $\mathcal{C}_c(W)$ be the set of all continuous functions on $W$ with a compact support in $W$ and $\mathcal{C}_b(W)$ be the Banach space of all bounded continuous functions on $W$ equipped with the supremum norm $\|\cdot\|_\infty$. We define the Banach space $\mathcal{C}_0(W)$ as the closure of $\mathcal{C}_c(W)$ in $\mathcal{C}_b(W)$. In particular, $\mathcal{C}_0(\mathbb{R}^n)$ is the set of all continuous functions $f$ on $\mathbb{R}^n$ such that

$$\lim_{|x|\to\infty} f(x) = 0,$$

THe dual space of $\mathcal{C}_0(W)$ is the space $\mathcal{M}(W)$ of all finite signed Radon measures on $W$. (the dual of $\mathcal{C}_0(W, \mathbb{R}^d)$ consists of vector measures).

If $\mu \in \mathcal{M}(W, \mathbb{R}^d)$, the variation of $\mu$ is the nonnegative Radon measure $|\mu|$ which acts on nonnegative functions $f \in \mathcal{C}_0(W)$ as

$$|\mu|(f) = \sup\{\mu(g) : \ g \in \mathcal{C}_0(W, \mathbb{R}^d), \ |g| \le f\}.$$

Then the norm of $\mu$, defined according to the general definition of dual norm, can be computed as

$$\|\mu\|_1 = |\mu|(W).$$

We define the space $BV(W)$ as the set of all functions $u \in L^1(W)$ whose distributional derivatives belong to $\mathcal{M}(W)$, with the norm

$$\|u\|_{BV} := \|u\|_1 + \|Du\|_1.$$

We reserve the symbol $Du$ for gradient being a measure, and $\nabla u$ for gradient being a function. We use the notation

$$\int_W g(x)|Du(x)|\,dx$$

for the integral of $g$ by the measure $|Du|$, although this is not integration by the Lebesgue measure.

Let us notice that the fundamental estimate of Sobolev functions by means of Riesz potential with all its consequences remains true also for $BV$-function, namely

$$⨍_B |u(y) - u(x)|\,dy \le C \int_B |y - x|^{1-n} |Du(y)|\,dy$$

holds whenever $B$ is a ball, $u \in BV(\mathbb{R}^n)$ and $x$ is a point of $B$ which is a Lebesgue point for $u$, also the statements on approximative differentiability a.e. Poincaré inequality and a Luzin type theorem (every $BV$ function coincides with a $\mathcal{C}^1$ function outside a set of small Lebesgue measure) are true.

12.1. **Proposition.** *If $u \in BV(\mathbb{R}^n)$, then*

$$\|u\|_{BV} = \inf_{\{u_j\}} \Big\{ \liminf_j \int_{\mathbb{R}^n} \big(|u_j| + |\nabla u_j|\big)\,dx \Big\},$$

*where $\{u_j\}$ runs over all sequences of functions from $\mathcal{D}(\mathbb{R}^n)$ converging to $u$ in $L^1(\mathbb{R}^n)$.*

*Proof.* For the $\le$ inequality, let us consider a sequence $\{u_j\}$ of $\mathcal{C}^1$-functions converging to $u$ in $L^1(\mathbb{R}^n)$. We may assume that

$$\liminf_j \int_{\mathbb{R}^n} \big(|u_j| + |\nabla u_j|\big)\,dx < \infty.$$

Then there is no problem with the convergence of the first term. For the second term notice that the norm of the dual space is always weakly* lower semicontinuous.

For the converse inequality, if $u$ has a compact support we consider the particular sequence obtained by mollification and notice that by the convolution inequality,

$$\int_{\mathbb{R}^n} |\nabla(\psi_\varepsilon * u)(x)|\,dx \le \int_{\mathbb{R}^n} (\psi_\varepsilon * |Du|)(x)\,dx \le \int_{\mathbb{R}^n} |Du|.$$

In the general case, we first multiply by a cut-off function and then mollify, the details are left to the reader. $\square$

12.2. **Proposition.** *Let $u \in BV(\mathbb{R}^n)$. Then the function*

$$u^y : \ x_i \mapsto u(y + x_i\mathbf{e}_i), \qquad x_i \in \mathbb{R}$$

*belongs to $BV(\mathbb{R})$ for a.e. $y \in \mathbb{H}_i := \{y \in \mathbb{R}^n : \ y_i = 0\}$, with $\|Du_y\|_1 = \|D_i u\|_1$.*

*Proof.* Consider a sequence $\{u_j\}$ of smooth functions such that $u_j \to u$ in $L^1$ and

$$\lim_j \int_{\mathbb{R}^n} \big(|u_j| + |\nabla u_j|\big)\, dx = \|u\|_{BV}.$$

Set

$$f_j(y) = \int_{\mathbb{R}} \big(|u_j(y + x_i\mathbf{e}_i)| + |\nabla u_j(y + x_i\mathbf{e}_i)|\big)\, dx_i, \qquad y \in \mathbb{H}_i.$$

Then by the Fatou lemma,

$$\int_{\mathbb{H}_i} \liminf_j f_j(y)\, dy \le \liminf_j \int_{\mathbb{H}_i} f_j(y)\, dy \le \|u\|_{BV}.$$

Take $y$ so that $\liminf_j f_j(y) < \infty$ and $u_j^y \to u^y$ in $L^1(\mathbb{R})$, where

$$u_j^y(x_i) = u_j(y + x_i\mathbf{e}_i),$$

The integrability of $\liminf_j f_j(y)$ and the Fubini theorem imply that a.e. $y$ has this property. Then, passing to a subsequence, $\{u_j^y\}_j$ is bounded in $BV(\mathbb{R})$. This implies an existence of a further subsequence such that $\nabla u_j^y$ converges weakly* in $\mathcal{C}_0^*(\mathbb{R})$. It is easily verified that then this weak* limit is $Du^y$ and that thus $u^y \in BV(\mathbb{R})$ with the claimed estimate of the norm. □

12.3. **Proposition.** *Suppose that $u \in BV(\mathbb{R})$. Then there is a representative of $u$ which is a $BV$-function in the classical sense, in particular it is a difference of two bounded monotone functions.*

## 13. Sets of finite perimeter

13.1. **Set of finite perimeter.** We say that $E \subset \mathbb{R}^n$ is a set of finite perimeter if $\chi_E \in BV(\mathbb{R}^n)$. The perimeter of $E$ is then $P(E) := \|D\chi_E\|_1$.

We say that $E$ is of locally finite perimeter in an open set $U \subset \mathbb{R}^n$ if $\chi_E \in BV_{\mathrm{loc}}(U)$ (the local version of the space $BV$),

Recall that $C_0^1(\mathbb{R}^n)$ is defined as the closure of $C_c^1(\mathbb{R}^n)$ in the $C^1$-norm.

13.2. **Observation.** *Suppose that $\Omega \subset \mathbb{R}^n$ be a set of finite measure.*

(a) *Let $P(\Omega) < \infty$ and $\nu = -D\chi_\Omega$. Then*

$$\nu(f) = \int_\Omega \operatorname{div} f\, dx, \qquad f \in \mathcal{C}_0^1(\mathbb{R}^n; \mathbb{R}^n). \tag{57}$$

(b) *Conversely, if $\nu$ is a vector Radon measure which satisfies* (57)*, then $\nu = -D\chi_\Omega$ in the sense of distributions, and thus $P(\Omega) < \infty$.*

13.3. **Remark.** Notice that Observation 13.2 can be regarded as a form of the divergence theorem, even as a characterization of the situations when the divergence theorem holds. However, for applications it is quite unclear what is the measure $\nu$ that we have to integrate by. Also the requirement of finite perimeter is less transparent and less easy to verify than e.g. assumptions in terms of the smoothness of the boundary.

## 14. Differentiability of the boundary

In the following theorem, we find the representation of $\nu_n$, where $\nu = (\nu_1, \dots, \nu_n) = -D\chi_\Omega$. A similar representation can be found for each $\nu_i$, only the notation is then more complicated. In our case, we identify $\mathbb{R}^n$ with the cartesian product $\mathbb{R}^{n-1} \times \mathbb{R}$ and write $x = (\tau, \gamma)$, where $\tau = (x_1, \dots, x_{n-1})$ and $\gamma = x_n$. We write $E \sim F$ if the measure of the symmetrical difference $E \triangle F$ is zero.

14.1. **Theorem.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a set of finite perimeter, $\nu = -D\chi_\Omega$. Then there exist measurable sets $A_m \subset \mathbb{R}^{n-1}$, $m = 1, 2, \dots$, measurable functions $a_m$, $b_m$: $A_m \to \mathbb{R}$ and measurable sets $E_{m,l} \subset A_m$, $l = 1, 2, \dots$, such that*

$$A_1 \supset A_2 \supset \dots, \tag{58}$$

$$a_1 < b_1 < a_2 < b_2 < \dots < a_m < b_m \quad \textit{on } A_m, \tag{59}$$

$$\{\gamma : (\tau, \varphi(E)) \in \Omega\} \sim \bigcup_{\{m:\, \tau \in A_m\}} (a_m(\tau), b_m(\tau)) \qquad \textit{for a.e. } \tau \in \mathbb{R}^{n-1}, \tag{60}$$

$$
A_m \sim \bigcup_l E_{m,l}, \tag{61}
$$

*and*

$$
a_m,\ b_m \text{ are Lipschitz on } E_{m,l}. \tag{62}
$$

*Moreover, we obtain the representation*

$$
\int_{\mathbb{R}^n} u\, d\nu_n = \sum_m \int_{A_m} \big(u(\tau, b_m(\tau)) - u(\tau, a_m(\tau))\big)\, d\tau \tag{63}
$$

*for each* $u \in \mathcal{C}_0^1(\mathbb{R}^n)$.

*Proof.* From Proposition 12.2 we infer that for a.e. $\tau \in \mathbb{R}^{n-1}$ there are intervals $a_1(\tau) < b_1(\tau) < \cdots < a_p(\tau) < b_p(\tau)$ with $p = p(\tau)$ such that

$$
\{\gamma : (\tau, \gamma) \in \Omega\} \sim \bigcup_{m=1}^{p} (a_m(\tau), b_m(\tau)).
$$

Indeed, recall that if a $BV$-function on line is simultaneously a characteristic function, then it has a representative which is a union of a finite family of intervals. Set

$$
A_m = \{\tau :\ p(\tau) \ge m\}.
$$

The function $p$ is measurable, because by Proposition 12.2,

$$
2p(\tau) = \|D_n \chi_\Omega(\tau, \cdot)\|_1.
$$

It follows that the sets $A_m$ are measurable. For a fixed $m$, the functions $a_m$, $b_m$ are measurable. Indeed, by induction, it follows from the Fubini theorem that, given $c \in \mathbb{R}$, the sets

$$
\begin{aligned}
\{\tau \in \mathbb{R}^{n-1} :\ a_1(\tau) < c\} &= \Big\{\tau \in \mathbb{R}^{n-1} :\ \int_{-\infty}^{c} \chi_\Omega(\tau, \gamma)\, d\gamma > 0, \Big\},\\
\{\tau \in \mathbb{R}^{n-1} :\ b_1(\tau) < c\} &= \Big\{\tau \in \mathbb{R}^{n-1} :\ \int_{a_1(\tau)}^{c} (1 - \chi_\Omega(\tau, \gamma))\, d\gamma > 0\Big\},\\
\{\tau \in \mathbb{R}^{n-1} :\ a_2(\tau) < c\} &= \Big\{\tau \in \mathbb{R}^{n-1} :\ \int_{b_1(\tau)}^{c} \chi_\Omega(\tau, \gamma)\, d\gamma > 0\Big\},\\
&\dots
\end{aligned}
$$

are measurable. We have proven (58)–(60). Now, we can divide $A_m$ into measurable sets $P_{m,q}$, $q = 1, 2, \dots$ in such a manner that for each $q$ there exists and interval $(c, d)$ (depending on $m$ and $q$) such that

$$
\begin{aligned}
&a_m(\tau) < c < b_m(\tau) < d \qquad \text{for each } \tau \in P_{m,q},\\
&d < a_{m+1}(\tau) \qquad \text{for each } \tau \in P_{m,q} \cap A_{m+1}.
\end{aligned}
$$

Now, set

$$
\ell(\tau) = \ell_{m,q}(\tau) = c + \int_c^d \chi_\Omega(\tau, \gamma)\, d\gamma.
$$

Then $\ell(\tau) = b_m(\tau)$ for each $\tau \in P_{m,q}$. By Proposition 12.1, there are smooth function $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi_k \to \chi_\Omega$ in $L^1(\mathbb{R}^n)$ and

$$
\|\varphi_k\|_{BV} \to \|\chi_\Omega\|_{BV}.
$$

Set

$$
\psi_k(\tau) = c + \int_c^d \varphi_k(\tau, \gamma)\, d\gamma.
$$

Then

$$
\frac{\partial \psi_k}{\partial \tau_j}(\tau) = \int_c^d \frac{\partial \varphi_k}{\partial y_j}(\tau, \gamma)\, d\gamma,
$$

and thus

$$
\int_{\mathbb{R}^{n-1}} |\nabla \psi_k|\, d\tau \le \int_{\mathbb{R}^n} |\nabla \varphi_k|\, dx.
$$

It follows that the sequence $\{\psi_k\}$ is bounded in $BV(\mathbb{R}^{n-1})$. By the Rellich compact embedding theorem, passing if necessary to a subsequence, we may achieve that $\psi_k$ converge in $L^1$ and a.e., and the limit cannot be anything else than $\ell$. Now, the function $\ell$ belongs to $BV(\mathbb{R}^{n-1})$ and thus by the Lusin-type approximation theorem, there exist $\mathcal{C}^1$-functions $\ell_{m,q,s} : \mathbb{R}^{n-1} \to \mathbb{R}$, $s \in \mathbb{N}$, and measurable sets $E_{m,q,s} \subset \mathbb{R}^{n-1}$ such that, for each $m, q \in \mathbb{N}$,

$$P_{m,q} \sim \bigcup_p E_{m,q,s},$$

and, for each $l \in I$,

$$\ell_{m,q,s} = \ell_{m,q} = b_m \quad \text{on } E_{m,q,s}.$$

Rearranging for each $m$ the sequence $\{E_{m,q,s}\}_{q,s}$ into a single sequence $\{E_{m,l}\}_l$, we obtain (61) and (62).

It remains to prove (63). The $n$-th coordinate of $\nu$ can be characterized by the following consequence of (57):

$$\int_{\mathbb{R}^n} u\, d\nu_n = \iint_\Omega \frac{\partial u}{\partial \gamma}(\tau,\gamma)\, d\tau\, d\gamma, \qquad u \in C_0^1(\mathbb{R}^n).$$

Using the Newton-Leibniz rule and the Fubini theorem we obtain

$$\begin{aligned}\int_{A_m} \big(u(\tau, b_m(\tau)) - u(\tau, a_m(\tau))\big)\, d\tau &= \int_{A_m} \Big(\int_{a_m}^{b_m} \frac{\partial u}{\partial \gamma}(\tau,\gamma)\, d\gamma\Big)\, d\tau \\ &= \iint_{\{(\tau,\gamma)\colon\, a_m(\tau)<\gamma<b_m(\tau)\}} \frac{\partial u}{\partial \gamma}(\tau,\gamma)\, d\tau\, d\gamma\end{aligned}$$

and summing over $m$ we conclude

$$\sum_m \int_{A_m} \big(u(\tau, b_m(\tau)) - u(\tau, a_m(\tau))\big)\, d\tau = \iint_\Omega \frac{\partial u}{\partial \gamma}(\tau,\gamma)\, d\tau\, d\gamma,$$

which proves (63). □

14.2. **Lemma.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a set of finite perimeter, $\nu = -D\chi_\Omega$. Let $A \subset \mathbb{R}^{n-1}$ be a measurable set, $I = (c,d) \subset \mathbb{R}$ be an interval, $b : \mathbb{R}^{n-1} \to I$ be a Lipschitz functions. Suppose that*

$$\begin{aligned}\{(\tau,\gamma) \in A \times I \colon \gamma < b(\tau)\} &\sim (A \times I) \cap \Omega,\\ \{(\tau,\gamma) \in A \times I \colon \gamma > b(\tau)\} &\sim (A \times I) \cap \Omega^c,\end{aligned}$$

*Then*

(a) *for a.e. $\tau \in A$, $(\tau, b(\tau)) \in \partial_F\Omega$ and*

$$\mathrm{n}(\tau, b(\tau)) = \frac{\big(-\frac{\partial b(\tau)}{\partial \tau_1}, \dots, -\frac{\partial b(\tau)}{\partial \tau_{n-1}},\, 1\big)}{\sqrt{1 + |\nabla b(\tau)|^2}}.$$

(b) *For $\mathcal{H}^{n-1}$-a.e. $x \in \{(\tau,\gamma) \in A \times I \colon \gamma = b(\tau)\}$ we have $x \in \partial_F\Omega$.*

(c) *for each $u \in \mathcal{C}_0(\mathbb{R}^n)$,*

$$\int_E u(\tau, b(\tau))\, d\tau = \int_{\{(\tau,b(\tau))\colon\, \tau\in E\}} u(x)\, \mathrm{n}_n(x)\, d\mathcal{H}^{n-1}(x).$$

*Proof.* (a) Let $\tau$ be a point of density of $A$ and of differentiability of $b$ and $x := (\tau, b(\tau))$. Denote $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{\mathbf{n-1}}) = \nabla b(\tau)$, this is understod as a linear form on $\mathbb{R}^{n-1}$. Consider the vector

$$\mathbf{n} = \frac{\big(-\beta_1, \dots, -\beta_{n-1},\, 1\big)}{\sqrt{1 + |\boldsymbol{\beta}|^2}}$$

Then

$$\mathbb{H}_{\mathbf{n}} = \{y \in \mathbb{R}^n \colon y_n < \boldsymbol{\beta}(y_1, \dots, y_{n-1})\}$$

We have

$$\Omega \triangle (x + \mathbb{H}_{\mathbf{n}}) \subset (A \times I)^c \cup \Big\{(\tau', \gamma') \in A \times I :\ |\gamma' - \beta(\tau' - \tau)| \le |b(\tau') - \beta(\tau' - \tau)|\Big\}.$$

Since $x$ is a density point of $A \times I$ and

$$b(\tau') - \beta(\tau' - \tau) = o(\tau' - \tau),$$

the claim follows.

(b) Consider the Lipschitz mapping $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined as $\varphi(\tau) = (\tau, b(\tau))$. Then the assertion of (b) follows from (a) using the Theorem 2.7).

(c) It is easily obtained from (a) and (b) using the area formula (Theorem 9.13) for the mapping $\varphi$. $\square$

14.3. **Lemma.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a set of finite perimeter, $\nu = -D\chi_\Omega$. Let $A \subset \mathbb{R}^{n-1}$ be a measurable set, $I = (c,d) \subset \mathbb{R}$ be an interval, $\ell\colon \mathbb{R}^{n-1} \to I$, $b\colon \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz functions. Suppose that*

$$\{(\tau,\gamma) \in A \times I\colon \gamma < b(\tau)\} \sim (A \times I) \cap \Omega,$$
$$\{(\tau,\gamma) \in A \times I\colon \gamma > b(\tau)\} \sim (A \times I) \cap \Omega^c,$$
$$\Gamma := \{(\tau,\gamma) \in A \times I\colon \gamma = \ell(\tau)\} \subset \partial_F\Omega.$$

*Then $b = \ell$ a.e. in $A$.*

*Proof.* Let $\tau$ be a point of density of $A$ and $x := (\tau, \ell(\tau))$. Suppose that e.g. $b(\tau) > \ell(\tau)$. Then there is a neighborhood $V$ of $\tau$ such that $b > \ell$ on $V$. We see that $x$ is a density point of

$$\big\{(\tau',\gamma') \in A \times I : \gamma' < b(\tau')\big\} \lesssim \Omega$$

and thus there cannot exist a Federer normal to $\Omega$ at $x$. $\square$

14.4. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a set of finite perimeter and $\nu = -D\chi_\Omega$. Then*
(a) $|\nu| = \mathcal{H}^{n-1}\lfloor(\partial_F\Omega)$,
(b) $\mathbf{n} = \frac{d\nu}{d|\nu|}$.

*Proof.* It follows from Theorem 14.1 and Lemma 14.2 that the integration over $\nu$ consist of integration over the graphs of functions of type $\gamma = a(\tau)$, $\gamma = b(\tau)$ with respect to the Hausdorff measure with weight $\mathbf{n}$. We also know that the above mentioned graphs can be considered as parts of the Federer boundary $\partial_F\Omega$. It remains to show that the integration is performed over (almost) all of $\partial_F\Omega$. However, we know by Theorem 11.2, Theorem 14.1 and Lemma 14.3 that the Federer boundary can be also decomposed into Lipschitz graphs and these are included in graphs of the functions of type $\gamma = a(\tau)$, $\gamma = b(\tau)$. $\square$

14.5. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a set of finite perimeter. Then $\partial_F\Omega \subset \partial_*\Omega$ and $\mathcal{H}^{n-1}(\partial_*\Omega \setminus \partial_F\Omega) = 0$.*

*Proof.* The inclusion $\partial_F\Omega \subset \partial_*\Omega$ is obvious. Suppose that $x \in \partial_*\Omega$. Then there exists $k \in \mathbb{N}$ such that

$$\limsup_{r\to 0+} \frac{|B(x,r) \cap \Omega|}{|B(x,r)|} > \frac{1}{k} \quad \text{and} \quad \limsup_{r\to 0+} \frac{|B(x,r) \setminus \Omega|}{|B(x,r)|} > \frac{1}{k}, \tag{64}$$

Let $M_k$ be the set of all points $x$ for which (64) holds and $G$ be an open set containing $M_k \setminus \partial_F\Omega$. The collection of balls

$$\mathcal{V} := \Big\{B(x,r) :\ x \in M_k \setminus \partial_F\Omega,\ B(x,r) \subset G,\ \ \frac{|B(x,r) \cap \Omega|}{|B(x,r)|} > \frac{1}{k},\ \frac{|B(x,r) \setminus \Omega|}{|B(x,r)|} > \frac{1}{k}\Big\}$$

is then a fine covering of $M_k \setminus \partial_F\Omega$. Indeed, notice that if for a radius $r'$ we have

$$\frac{|B(x,r') \cap \Omega|}{|B(x,r')|} > \frac{1}{k}$$

and for another radius $r''$ we have

$$\frac{|B(x,r'') \cap \Omega|}{|B(x,r'')|} < 1 - \frac{1}{k}$$

then, by the Darboux property of the quotient, there is a $r$ between $r'$ and $r''$ where we have both. Thus, by the Vitali type theorem, there exists a sequence $\{B(x_j,r_j)\}_j$ of pairwise disjoint balls from $\mathcal{V}$ such that

$$M_k \setminus \partial_F\Omega \subset \bigcup_j B(x_j, 5r_j). \tag{65}$$

If $B(x,r) \in \mathcal{V}$, using the Poincaré inequality we obtain

$$r^n \le C \inf_{c\in\mathbb{R}} \int_{B(x,r)} |\chi_\Omega - c|\, dy \le Cr \int_{B(x,r)} |D\chi_\Omega| = Cr|\nu|(B(x,r)). \tag{66}$$

By (65) and (66),

$$\mathcal{H}^{n-1}_\infty(M_k \setminus \partial_F\Omega) \le C \sum_j r_j^{n-1} \le C \sum_j |\nu|(B(x_j,r_j)) \le C|\nu|(G).$$

Since $G \supset M_k \setminus \partial_F \Omega$ was arbitrary and, by Theorem 14.4, $|\nu|(\mathbb{R}^n \setminus \partial_F \Omega) = 0$, we conclude

$$\mathcal{H}^{n-1}(M_k \setminus \partial_F \Omega) \le C|\nu|(M_k \setminus \partial_F \Omega) = 0.$$

□

## 15. Characterization of sets of finite perimeter

15.1. **Lemma.** *Suppose that $K_0 \subset \mathbb{R}^{n-1}$ is a compact set of positive measure and $a_0 < b_0$ are real numbers. Suppose that*

$$K_0 \times [a_0, b_0] = V_1 \cup V_2,$$

*where $V_1$ and $V_2$ are measurable subsets of $\mathbb{R}^n$ of positive measure. Suppose that for each $\tau \in K$, the one dimensional measure of both sets*

$$\{\gamma \in (a_0, b_0) : (\tau, \gamma) \in V_1\}, \quad \{\gamma \in (a_0, b_0) : (\tau, \gamma) \in V_2\}$$

*is strictly positive. Then $\operatorname{cl}_* V_1 \cap \operatorname{cl}_* V_2 \ne \emptyset$.*

*Proof.* We extend the definition: $V_j = V_1$ if $j$ is odd, $V_j = V_2$ if $j$ is even. A set of type $K \times [a, b]$, where $K$ is a compact subset of $\mathbb{R}^{n-1}$ and $(a, b) \subset \mathbb{R}$ is an open interval, is said to be a *good cylinder* if $|K| > 0$ and for each $\tau \in K$,

$$|\,\{\gamma \in (a, b) : (\tau, \gamma) \in V_1\}| > 0 \quad \text{and} \quad |\{\gamma \in (a, b) : (\tau, \gamma) \in V_2\}| > 0.$$

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let us denote $Q(x, r) = [x_1 - r, x_1 + r] \times \dots \times [x_n - r, x_n + r]$. We construct recurrently a sequence $\{Q_j\}$ of cubes, $Q_j = Q(x_j, r_j)$, $x_j = (\tau_j, \gamma_j)$, and a sequence $\{T_j\}$ of good cylinders of the form $T_j = K_j \times [a_j, b_j]$. We find a cube $Q_0$ such that $T_0 := K_0 \times [a_0, b_0] \subset Q_0$. Suppose that all the objects that we construct are already determined and $T_{j-1} \subset Q_{j-1}$. Using the Fubini theorem we observe that

$$|V_2 \cap T_{j-1}| > 0, \quad |V_1 \cap T_{j-1}| > 0.$$

We find a point $x'_j$, which is a density point of $V_j \cap T_{j-1}$. We find a cube $Q'_j$ centered at $x'_j$ of diameter at most $1/j$ such that $Q'_j \subset Q_{j-1}$ and

$$|V_j \cap T_{j-1} \cap Q'_j| > \tfrac{3}{4}\,|Q'_j|.$$

Now we distinguish two cases. If $Q'_j \cap T_{j-1}$ contains a good cylinder, we define $T_j$ as this good cylinder and $Q_j := Q'_j$. Otherwise $Q'_j \cap T_{j-1} \cap V_j$ has the structure

$$Q'_j \cap T_{j-1} \cap V_j \sim H_j \times [a'_j, b'_j],$$

where $H_j$ is a measurable subset of $\mathbb{R}^{n-1}$ and

$$|H_j \cap \Pi(Q'_j)| > \tfrac{3}{4}\,|\Pi(Q'_j)|$$

Here $\Pi$ denotes the projection $x \mapsto (x_1, \dots, x_{n-1}) : \mathbb{R}^n \to \mathbb{R}^{n-1}$. Find the largest interval $[a''_j, b''_j] \subset [a_{j-1}, b_{j-1}]$ with the property that

$$|(H_j \times [a''_j, b''_j]) \setminus V_j| = 0.$$

Then, by the definition of goodness of $T_{j-1}$,

$$[a''_j, b''_j] \ne [a_{j-1}, b_{j-1}]$$

and thus there exists $[a_j, b_j] \subset [a_{j-1}, b_{j-1}]$ which is as long as $[a'_j, b'_j]$ and contains "both pieces of $[a''_j, b''_j]$ and of its complement" in such a way that

$$|V_j \cap Q_j| > \tfrac{1}{2}\,|Q_j|,$$

where

$$Q_j = \Pi(Q'_j) \times [a_j, b_j],$$

and $Q_j$ contains a good cylinder. This completes one step of the construction. If the entire construction is established, we observe that the intersection of the cubes $Q_j$ contains exactly one point, which belongs to $\operatorname{cl}_* V_1 \cap \operatorname{cl}_* V_2$. □

15.2. **Theorem.** *Suppose that $\Omega \subset \mathbb{R}^n$ be a measurable set of finite measure. The $\Omega$ has a finite perimeter if and only if*

$$\mathcal{H}^{n-1}(\partial_* \Omega) < \infty.$$

*Proof.* Necessity follows from Theorem 14.5, we will prove the sufficiency. Suppose that $\mathcal{H}^{n-1}(\partial_*\Omega) < \infty$. Then, by the Eilenberg inequality (Lemma 6.8),

$$\int_{\mathbb{R}^{n-1}} \#\{\gamma : \ (\tau, \gamma) \in \partial_*\Omega\} \, dt < \infty.$$

We claim that for almost all $\tau \in \mathbb{R}^{n-1}$, $\chi_\Omega(\tau, \cdot)$ is a.e. constant between the points of $\{\gamma : \ (\tau, \gamma) \in \partial_*\Omega\}$. Having this claim for granted, we observe that $\chi_\Omega$ is $BV$ along all lines parallel to coordinate axes and

$$\int_{\mathbb{R}^{n-1}} \|\chi_\Omega(\cdot, \gamma)\|_{BV} \, d\gamma < \infty,$$

similarly for rermaining coordinates. A routine calculation involving integration by parts and Fubini theorem then shows that $\chi_\Omega \in BV(\mathbb{R}^n)$ as required. To prove the claim, we proceed as in the proof of Theorem 14.1 and, assuming that the conclusion is not true, find a compact set $K_0$ and an interval $[a_0, b_0]$ such that

$$\partial_*\Omega \cap \Big(K_0 \times [a_0, b_0]\Big) = \emptyset$$

and $\chi_\Omega(\tau, \cdot)$ is essentially nonconstant on $[a_0, b_0]$ for each $\tau \in K_0$. Then an application of Lemma 15.1 leads to a contradiction. □

## 16. Divergence theorem

16.1. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a set of finite perimeter and $f \in \mathcal{C}^1_0(\mathbb{R}^n; \mathbb{R}^n)$. Then*

$$\int_{\partial_*\Omega} f \cdot \mathbf{n} \, d\mathcal{H}^{n-1} = \int_\Omega \operatorname{div} f \, dx.$$

*Proof.* This is a summary of preceding results Theorem 14.4 and Theorem 14.5. □

16.2. **Corollary.** *Let $W \subset \mathbb{R}^n$ be an open set and $\Omega \subset W$ be a set of locally finite perimeter in $W$. Let $f \in \mathcal{C}^1_c(W; \mathbb{R}^n)$. Then*

$$\int_{W \cap \partial_*\Omega} f \cdot \mathbf{n} \, d\mathcal{H}^{n-1} = \int_\Omega \operatorname{div} f \, dx. \tag{67}$$

*Proof.* Let $B(x, r) \subset\subset W$. Then $\partial_*(\Omega \cap B(x, r)) \subset \partial B(x, r) \cup (\partial_*\Omega \cap B(x, r))$ and thus this is a set of finite $\mathcal{H}^{n-1}$-measure. In view of Theorem 15.2 it follows that $\Omega \cap B(x, r)$ is a set of finite perimeter and (67) holds for $f$ supported in such a ball. The general case follows now by a partition of unity. □

## 17. Surface integration

17.1. **Surface and surface integral.** Let $X$ be a metric space and $k$ be a positive integer. We define a *$k$-dimensional (parametric) surface* as a locally Lipschitz mapping $\varphi : E \to X$, where $E \subset \mathbb{R}^k$ is Lebesgue measurable. Suppose that $g_1, \dots, g_k : X \to \mathbb{R}$ are locally Lipschitz functions and $g = (g_1, \dots, g_k)$. Further, let $f : X \to \mathbb{R}$ be measurable with respect to $\mathcal{H}^k$. We define the *surface integral* of the integrand $f\, dg_1 \dots dg_k$ as

$$\int_\varphi f\, dg_1 \dots dg_k = \int_E f(\varphi(t))\, J(g \circ \varphi)(t)\, dt.$$

17.2. **Theorem** (Canonical change of variables)**.** *Let $X$ be a metric space, $E \subset \mathbb{R}^k$ be a measurable set, $\varphi : E \to X$ be a surface, $g = (g_1, \dots, g_k) : X \to \mathbb{R}^k$ be locally Lipschitz and $f : X \to \mathbb{R}$ be $\mathcal{H}^k$-measurable. Then*

$$\int_\varphi f\, dg_1 \dots dg_k = \int_{\mathbb{R}^k} \sum_{t \in E \cap g^{-1}(y)} \operatorname{sgn} J(g \circ \varphi)(t)\, f(\varphi(t))\, dy_1 \dots dy_k.$$

*Proof.* This is a routine consequence of the theorem on change of variables by the area formula (Theorem 4.7). □

## 18. Surrounding: Euclidean case

18.1. **Jacobian.** If $\varphi : E \to \mathbb{R}^n$ is an $(n-1)$ dimensional surface, we define

$$\mathbf{J}\varphi = \frac{\partial \varphi}{\partial t_1} \times \dots \times \frac{\partial \varphi}{\partial t_{n-1}}.$$

18.2. **Lemma.** *Let $W \subset \mathbb{R}^n$ $(n > 1)$ be an open set and $\Omega \subset W$ be of locally finite perimeter in $W$. Let $\varphi : E \to W$ be a $n-1$-dimensional surface. Suppose that $\mathcal{H}^{n-1}(\varphi(E) \setminus \partial_F \Omega) = 0$. Then $\mathbf{J}\varphi(t)$ is a real multiple of $\mathbf{n}(\varphi(t))$ for a.e. $t \in E$.*

*Proof.* We can ignore preimages of null sets, because by the area formula the Jacobian $\mathbf{J}\varphi$ vanishes a.e. on them. Now, let us assume that there is a measurable $A \subset \mathbb{R}^{n-1}$, and interval $I = (c, d) \subset \mathbb{R}$ and a Lipschitz function $\ell \colon \mathbb{R}^{n-1} \to I$ such that

$$\varphi(E) \subset \{(\tau, \gamma) \in A \times I \colon \gamma = \ell(\tau)\} \subset \partial_F \Omega$$

Then

$$\varphi_n(t) = \ell(\varphi_1(t), ..., \varphi_{n-1}(t)), \qquad t \in E. \tag{68}$$

By the area formula (and Rademacher theorem), for a.e. $t \in E \cap \{\mathbf{J}\varphi \neq 0\}$ we observe that $\ell$ is differentiable at $\varphi(t)$. Let us denote

$$\psi(\tau) = (\tau, \ell(\tau))\,, \qquad \mu = (\varphi_1, \dots, \varphi_{n-1}).$$

We may rewrite (68) as

$$\varphi = \psi \circ \mu.$$

By Lemma 14.3 and 14.2,

$$\mathbf{n}(\psi(\tau)) = \frac{\mathbf{J}\psi(\tau)}{|\mathbf{J}\psi(\tau)|} \quad \text{for a.e. } \tau \in A.$$

By the chain rule and the rule for multiplication of determinants,

$$\mathbf{J}\varphi(t) = J\mu(\psi(t))\mathbf{J}\psi(t)$$

for a.e. $t \in E \cap \{\mathbf{J}\varphi \neq 0\}$, this proves the claim in the special case.

Similarly we proceed if the image of $A$ lies in a rotated graph of above mentioned type, i.e. if a different than last coordinate is expressed explicitly in terms of the remaining ones. By Theorems 14.1 and 11.2 this describes piecewise all of $\partial_F \Omega$. The general case follows now by a decomposition argument. □

18.3. **Surrounding: Euclidean case.** Let $W \subset \mathbb{R}^n$ $(n > 1)$ be an open set and $\Omega \subset W$ be of locally finite perimeter in $W$. Let $\varphi : E \to W$ be an $n-1$-dimensional surface. We say that $\varphi$ *surrounds* (ohraničuje) $\Omega$ in $W$ if $\varphi$ is one-to-one,

$$\mathcal{H}^{n-1}(\partial_* \Omega \triangle \varphi(E)) = 0.$$

and $\mathbf{J}\varphi(t)$ is a positive multiple of $\mathbf{n}(\varphi(t))$ for a.e. $t \in E$.

18.4. **Theorem.** *Suppose that $W \subset \mathbb{R}^n$ $(n > 1)$ be an open set and $\Omega \subset W$ be of locally finite perimeter in $W$. Then there exists an $n-1$-dimensional surface which surrounds $\Omega$ in $W$.*

*Sketch of the proof.* Pieces of this surface are obtained in Theorem 14.1, they can be put together into a single surface (cf. proof of Theorem 19.4). The proof that such a surface surrounds $\Omega$ is based on Lemma 14.2, Lemma 14.3, Theorem 14.4 and Theorem 14.5. □

18.5. **Theorem.** *Suppose that $W \subset \mathbb{R}^n$ $(n > 1)$ be an open set and $\Omega \subset W$ be of locally finite perimeter in $W$. Let $\varphi : E \to W$ be a $n-1$-dimensional surface. Suppose that $\varphi$ surrounds $\Omega$ in $W$. Let $f, g_1 \dots g_{n-1} : W \to \mathbb{R}$ be locally lipschitz function and $f$ has a compact support in $W$. Then*

$$\int_\varphi f\, dg_1 \dots dg_{n-1} = \int_\Omega \det(\nabla f, \nabla g_1, \dots, \nabla g_{n-1})\, dx.$$

*Proof.* Suppose first that $f, g$ are smooth. Then

$$\operatorname{div}\big(f\ \nabla g_1 \times \cdots \times \nabla g_{n-1}\big) = \det(\nabla f, \nabla g_1, \dots, \nabla g_{n-1}).$$

By Corollary 16.2

$$\int_{\partial_* \Omega \cap W} f\ \nabla g_1 \times \cdots \times \nabla g_{n-1} \cdot \mathbf{n}\, d\mathcal{H}^{n-1} = \int_\Omega \det(\nabla f, \nabla g_1, \dots, \nabla g_{n-1})\, dx$$

On the other hand, by the definition of surrounding and the Cauchy-Binet formula (Theorem 5.3)

$$\begin{aligned}
&\int_{\partial_* \Omega \cap W} f\ \nabla g_1 \times \cdots \times \nabla g_{n-1} \cdot \mathbf{n}\, d\mathcal{H}^{n-1} \\
&\quad = \int_E (f\ \nabla g_1 \times \cdots \times \nabla g_{n-1})(\varphi(t)) \cdot \mathbf{n}(\varphi(t)) |J\varphi(t)|\, dt \\
&\quad = \int_E f(\varphi(t))\, (\nabla g_1 \times \cdots \times \nabla g_{n-1})(\varphi(t)) \cdot \mathbf{J}\varphi(t)\, dt \\
&\quad = \int_E f(\varphi(t))\ \det(\nabla (g \circ \varphi)(t))\, dt = \int_\varphi f\, dg_1 \dots dg_{n-1}.
\end{aligned}$$

The general case follows by a routine approximation argument. □

## 19. Stokes theorem

19.1. **Atlas, Lipschitz manifold.** Let $X$ be a separable metric space and $\mathcal{A}$ be a system of bilipschitz mappings (charts) $\mu : U_\mu \to \mathbb{R}^n$, where $U_\mu \subset X$ is open. Suppose that all images $\mu(U_\mu)$ are open in $\mathbb{R}^n$ and

$$\bigcup_{\mu \in \mathcal{A}} U_\mu = X.$$

Then $\mathcal{A}$ is called a *Lipschitz atlas* and $(X, \mathcal{A})$ is called a ($n$-dimensional) *Lipschitz manifold.* We say that $(X, \mathcal{A})$ is *oriented* if

$$J(\nu \circ \mu^{-1})(t) > 0 \text{ for a.e. } t \in \mu(U_\mu \cap U_\nu)$$

whenever $\mu, \nu \in A$. The definition is justified by the following theorem.

19.2. **Theorem.** *Suppose that $G \subset \mathbb{R}^n$ be a connected open set and $f : G \to \mathbb{R}^n$ be a bilipschitz mapping. Then $f(G)$ is open and either $Jf > 0$ a.e. ($f$ is sense preserving), or $Jf < 0$ a.e. ($f$ is sense reversing).*

*Proof.* See [LM]. Notice that $f$ can be extended as bilipschitz to $\overline{G}$ and that the signum of the Jacobian corresponds to the signum of

$$\deg(f, G, p), \qquad p \in f(G).$$

□

19.3. **Parametrization.** Let $X$ be a Lipschitz manifold. Let $G \subset \mathbb{R}^n$ be an open set and $\varphi : G \to X$ be a locally Lipschitz one-to-one mapping. We say that $\varphi$ is a

- *parametrization* of $X$ if $\varphi(G) = X$,
- *generalized parametrization* of $X$ if $X \setminus \varphi(G)$ is a null set (i.e. the image under any chart from the atlas is Lebesgue null).
- *local parametrization* in general.

A local parametrization $\varphi$ is said to be *positive* if $X$ is oriented and

$$J(\mu \circ \varphi) > 0 \text{ a.e. on } \varphi^{-1}(U_\mu) \quad \text{for each } \mu \in \mathcal{A}.$$

19.4. **Theorem.** Let $X$ be an oriented Lipschitz manifold of dimension $n \geq 1$. Then there exists a positive generalized parametrization $\varphi$ of $X$.

*Sketch of the proof.* Pieces of the parametrization are obtained from inversions of charts. We obtain bilipschitz $\varphi_j : G_j \to X$ such that $G_j$ are bounded, $\mathcal{L}^n(\nabla G_j) = 0$ and $\varphi_j(G_j)$ cover $X$. Now we set

$$\tilde{G}_i = G_i \setminus \bigcup_{j<i} \overline{\varphi_i^{-1}(\varphi_j(G_j))}$$

to make the images of the pieces disjoint, but still covering almost all of $X$. After a change of variable by shifts of $\tilde{G}_j$ we may assume that $\tilde{G}_j$ are disjoint, so that we can define

$$\varphi = \varphi_j \text{ on } G_j.$$

□

19.5. **Integration over manifolds.** Let $X$ be an oriented Lipschitz manifold, $M \subset X$ be a $\mathcal{H}^n$-measurable set, $g : X \to \mathbb{R}^n$ be localy Lipschitz and $f : M \to \mathbb{R}$ be $\mathcal{H}^n$-measurable. We define

$$\int_M f \, dg_1 \dots dg_n = \int_\varphi f \lfloor_M \, dg_1 \dots dg_n$$

where $f\lfloor_M$ is just $f$ extended by 0 outside $M$ and $\varphi$ is a positive generalized parametrization of $X$. This integral does not depend on the choice of $\varphi$.

19.6. **Perimeter.** Let $X$ be an oriented Lipschitz manifold and $\Omega \subset X$. We say that $\Omega$ has a *locally finite perimeter* in $X$ if for each $\mu \in \mathcal{A}$ the set $\mu(\Omega \cap U_\mu)$ has a locally finite perimeter in $\mu(U_\mu)$.

19.7. **Surrounding.** Let $X$ be an oriented $n$-dimensional Lipschitz manifold ($n > 1$), $\Omega \subset X$ be of locally finite perimeter and $\varphi : E \to X$ be a $n-1$-dimensional surface. We say that $\varphi$ *surrounds* $\Omega$ if $\mu \circ \varphi$ surrounds $\mu(\Omega)$ in $\mu(U_\mu)$ for each $\mu \in \mathcal{A}$.

19.8. **Lemma.** *Let $X$ be an oriented $n$-dimensional Lipschitz manifold ($n > 1$), $\Omega \subset X$ be of locally finite perimeter and $\varphi : E \to X$ be a $n-1$-dimensional surface which surrounds $\Omega$. Suppose that $(f, g_1, \dots, g_{n-1})$ is an $n$-tuple of locally Lipschitz functions on $X$ such that $f$ has a compact support in $X$. Then the formula*

$$\int_\varphi f \, dg_1 \dots dg_{n-1} = \int_\Omega df \, dg_1 \dots dg_{n-1} \tag{69}$$

*is valid.*

*Sketch of the proof.* If the support is obtained in $U_\mu$ for some $\mu \in \mathcal{A}$, this can be obtained using formulas on change of variables. The general case follows by partition of unity. □

19.9. **Remark.** In what follows, we will be rather interested in "global" validity of (69), i.e., we will want to relax the assumption concerning the compact support. The previous lemma leaves untreated some interesting cases, e.g. if $X$ is the cone $\{x_1^2 + x_2^2 = x_3^2,\ x_3 > 0\}$ and $0 \in \overline{\Omega \cup \varphi(E)}$. Our main result will be the following.

19.10. **Stokes Theorem.** *Let $X$ be an oriented $n$-dimensional Lipschitz manifold ($n > 1$), $\Omega \subset X$ be of locally finite perimeter and $\varphi : E \to X$ be a $(n-1)$-dimensional surface which surrounds $\Omega$. Suppose that $(f, g_1, \dots, g_{n-1})$ is an $n$-tuple of locally Lipschitz functions on $X$ such that $f$ has a compact support in $X$, consider the mapping $g = (g_1, \dots, g_{n-1})\colon X \to \mathbb{R}^{n-1}$. Suppose that for a.e. $y \in \mathbb{R}^{n-1}$ the set $\Omega \cap g^{-1}(y)$ is relatively compact in $X$. Then the formula*

$$\int_\varphi f \, dg_1 \dots dg_{n-1} = \int_\Omega df \, dg_1 \dots dg_{n-1}$$

*is valid provided both the integrals converge.*

*Proof.* Suppose first that $f$ has a compact support. Define a function $\sigma$ on $\varphi(E)$ as

$$\sigma(\varphi(t)) = \operatorname{sgn} J(g\circ\varphi)(t).$$

Recall that

$$\int_\varphi f\,dg_1\ldots dg_{n-1} = \int_{\mathbb{R}^{n-1}} \Big(\sum_{x\in\varphi(E)\cap g^{-1}(y)} f(x)\sigma(x)\Big)\,dy,$$

whereas, by the coarea formula,

$$\int_\Omega df\,dg_1\ldots dg_{n-1} = \int_{\mathbb{R}^{n-1}} \Big(\int_{\Omega\cap g^{-1}(y)} df\Big)\,dy,$$

where $\Omega\cap g^{-1}(y)$ is oriented by $*(dg_1\wedge dg_2\wedge\cdots\wedge dg_{n-1})$. Let $\psi\in\mathcal{D}(\mathbb{R}^{n-1})$ and set

$$f_\psi(x) = f(x)\psi(g(x)).$$

Then

$$df_\psi(x) = df(x)\psi(g(x)) + f(x)\,d\psi(g(x)).$$

Since

$$d\psi(g(x))\wedge dg_1\wedge\cdots\wedge dg_{n-1} = 0,$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \Big(\sum_{x\in\varphi(E)\cap g^{-1}(y)} f(x)\sigma(x)\Big)\psi(y)\,dy &= \int_{\mathbb{R}^{n-1}} \Big(\sum_{x\in\varphi(E)\cap g^{-1}(y)} f_\psi(x)\sigma(x)\Big)\,dy\\ &= \int_\varphi f_\psi\,dg_1\ldots dg_{n-1} = \int_\Omega df_\psi\,dg_1\ldots dg_{n-1}\\ &= \int_\Omega \psi\circ g\,df\,dg_1\ldots dg_{n-1} = \int_{\mathbb{R}^{n-1}} \Big(\int_{\Omega\cap g^{-1}(y)} df\Big)\,\psi(y)\,dy.\end{aligned}$$

It follows that

$$\sum_{x\in\varphi(E)\cap g^{-1}(y)} f(x)\sigma(x) = \int_{\Omega\cap g^{-1}(y)} df \quad \text{for a.e. } y\in\mathbb{R}^{n-1}.$$

Now we will consider the general case, only, since $f$ is integrable, we may assume that $f\geq 0$. Let $\{\omega_j\}$ be a partition of unity on $X$ and $\eta_j$ be the partial sums of $\omega_j$. We apply the preceding computation to $f\eta_k$, pass to the limit and obtain

$$\begin{aligned}\int_\Omega df\,dg_1\ldots dg_{n-1} &= \int_{\mathbb{R}^{n-1}} \Big(\int_{\Omega\cap g^{-1}(y)} df\Big) = \int_{\mathbb{R}^{n-1}} \lim_k \Big(\int_{\Omega\cap g^{-1}(y)} (\eta_k\,df + f\,d\eta_k)\Big)\,dy\\ &= \int_{\mathbb{R}^{n-1}} \lim_k \Big(\sum_{x\in\varphi(E)\cap g^{-1}(y)} f(x)\sigma(x)\eta_k(x)\Big)\,dy \qquad (70)\\ &= \int_{\mathbb{R}^{n-1}} \Big(\sum_{x\in\varphi(E)\cap g^{-1}(y)} f(x)\sigma(x)\Big)\,dy = \int_\varphi f\,dg_1\ldots dg_{n-1}.\end{aligned}$$

The first equality in (70) is the coarea formula, here passing from local to global we use the existence of the integral on the left. The second equality is an obvious consequence of the hypothesis on relative compactness of the preimages $\Omega\cap g^{-1}(y)$. The third equality is the relatively compact case. In the fourth equality we use that $f\geq 0$ and $\eta_k\nearrow 1$. $\square$