AMERICAN INSTITUTES FOR RESEARCH®

What the United States Can Learn From Singapore's World-Class Mathematics System

(and what Singapore can learn from the United States):

An Exploratory Study

1000 THOMAS JEFFERSON STREET, NW | WASHINGTON, DC 20007-3835

What the United States Can Learn From Singapore's World-Class Mathematics System

(and what Singapore can learn from the United States):

An Exploratory Study

Alan Ginsburg
United States Department of Education
Policy and Program Studies Service (PPSS)

Steven Leinwand
American Institutes for Research

Terry Anstrom
American Institutes for Research

Elizabeth Pollock
American Institutes for Research

Principal Editor
Elizabeth Witt

TABLE OF CONTENTS

TABLE OF CONTENTS (CONTINUED)

LIST OF EXHIBITS

LIST OF EXHIBITS (CONTINUED)

LIST OF EXHIBITS (CONTINUED)

ACKNOWLEDGMENTS

EXECUTIVE SUMMARY

INTRODUCTION

Singaporean students ranked first in the world in mathematics on the Trends in International Mathematics and Science Study-2003; U.S. students ranked 16th out of 46 participating nations at grade 8 (Mullis, et al., 2004). Scores for U.S. students were among the lowest of all industrialized countries. Because it is unreasonable to assume that Singaporean students have mathematical abilities inherently superior to those of U.S. students, there must be something about the system that Singapore has developed to teach mathematics that is better than the system we use in the United States.

This exploratory study compares key features of the Singapore and U.S. mathematics systems in the primary grades, when students need to build a strong mathematics foundation. It identifies major differences between the mathematics frameworks, textbooks, assessments, and teachers in Singapore and the United States. It also presents initial results from four pilot sites that introduced the Singapore mathematics textbook in place of their regular textbooks.

Analysis of these evidentiary streams finds Singaporean students more successful in mathematics than their U.S. counterparts because *Singapore has a world-class mathematics system with quality components aligned to produce students who learn mathematics to mastery.* These components include Singapore's highly logical national mathematics framework, mathematically rich problem-based textbooks, challenging mathematics assessments, and highly qualified mathematics teachers whose pedagogy centers on teaching to mastery. Singapore also provides its mathematically slower students with an alternative framework and special assistance from an expert teacher.

The U.S. mathematics system does not have similar features. It lacks a centrally identified core of mathematical content that provides a focus for the rest of the system. Its traditional textbooks emphasize definitions and formulas, not mathematical understanding; its assessments are not especially challenging; and too many U.S. teachers lack sound mathematics preparation. At-risk students often receive special assistance from a teacher's aide who lacks a college degree. As a result, the United States produces students who have learned only to mechanically apply mathematical procedures to solve routine problems and who are, therefore, not mathematically competitive with students in most other industrialized countries.

The experiences of several of the U.S pilot sites that introduced the Singapore mathematics textbooks without the other aspects of the Singaporean system also illustrate the challenges teachers face when only one piece of the Singapore system is replicated. Some pilot sites coped successfully with these challenges and significantly improved their students' mathematics achievement, but others had great difficulty. Professional training improved the odds of success, as did serving a stable population of students who were reasonably able with mathematics. These mixed results further reinforce the comparative findings that the U.S. will have to consider making comprehensive reforms to its school mathematics system if we are to replicate the Singaporean successes.

The U.S. mathematics system has some features that are an improvement on Singapore's system, notably an emphasis on 21st century thinking skills, such as reasoning and communications, and a focus on applied mathematics. However, if U.S. students are to become successful in these

areas, they must begin with a strong foundation in core mathematics concepts and skills, which, by international standards, they presently lack.

EXPLORATORY METHODOLOGY

Carrying out in-depth analyses on systems as different as those in Singapore and the United States poses serious methodological challenges. Singapore has a centralized mathematics system, with detailed and consistent implementation procedures. This makes analysis of the separate components of their system relatively straightforward. Characterizing the decentralized U.S. mathematics system, in contrast, is difficult. We elected to rigorously study the components of the U.S. system by selecting typical examples from the wide variety available in each component area:

- **Standards:** The United States has no national standards, but many states' standards use the National Council of Teachers of Mathematics (NCTM) framework as a model. We used the NCTM standards in our analyses as a proxy for states that use a grade-band (e.g., K–2, 3–5), rather than a grade-by-grade structure in their standards. However, because many states are currently shifting to a grade-by-grade structure in response to NCLB, we supplemented our analysis by also examining standards from seven states (Exhibit A) that organize content grade by grade. These states are home to approximately one-third of all U.S. students.

- **Textbooks:** We limited our analysis to one traditional and one nontraditional U.S. mathematics textbook.

- **Assessments:** We used sample assessment items from the federally supported National Assessment of Educational Progress (NAEP) and from assessments from the same seven states whose standards we examined in our comparative analysis.

- **Teachers:** For analyses of teacher quality in the United States, we drew from national surveys on teacher education and from teacher preparation standards. We also examined sample problems from teacher licensing exams.

Exhibit A. The Average Number of Topics per Grade in Selected U.S. State Mathematics Frameworks Compared With Singapore's

	Avg. No. of Topics per Grade	Ratio to Sing.
Singapore	15	—
California	20	1.3
Florida	39	2.6
Maryland	29	1.9
New Jersey	28	1.9
N. Carolina	18	1.2
Ohio	26	1.7
Texas	19	1.3

In evaluating the results from the four Singapore textbook pilot sites – Baltimore, Maryland; Montgomery Country, Maryland; North Middlesex, Massachusetts; and Paterson, New Jersey – we relied on data previously collected by the districts rather than on uniform data collected specifically

for this study. Because different sites used different sites assessments, usually the state assessment, results are not completely comparable. The reader should be mindful of study limitations in all areas of comparison.

PREFERRED FEATURES OF THE SINGAPORE MATHEMATICS SYSTEMS

Our key findings show the advantages conferred by components of Singapore's mathematics system in comparison to similar components in the U.S. system.

Frameworks

A mathematically logical, uniform national framework that develops topics in-depth at each grade guides Singapore's mathematics system. The U.S. system, in contrast, has no official national framework. State frameworks differ greatly; some resemble Singapore's, whereas others lack Singapore's content focus.

Singapore's framework, shown in Exhibit B, lays out a balanced set of mathematical priorities centered on problem solving. It includes an emphasis on computational skills along with more conceptual and strategic thinking processes. The framework covers a relatively small number of topics in-depth and carefully sequenced grade-by-grade, following a spiral organization in which topics presented at one grade are covered in later grades, but only at a more advanced level. Students are expected to have mastered prior content, not repeat it.

Exhibit B. Singapore's Mathematics Framework
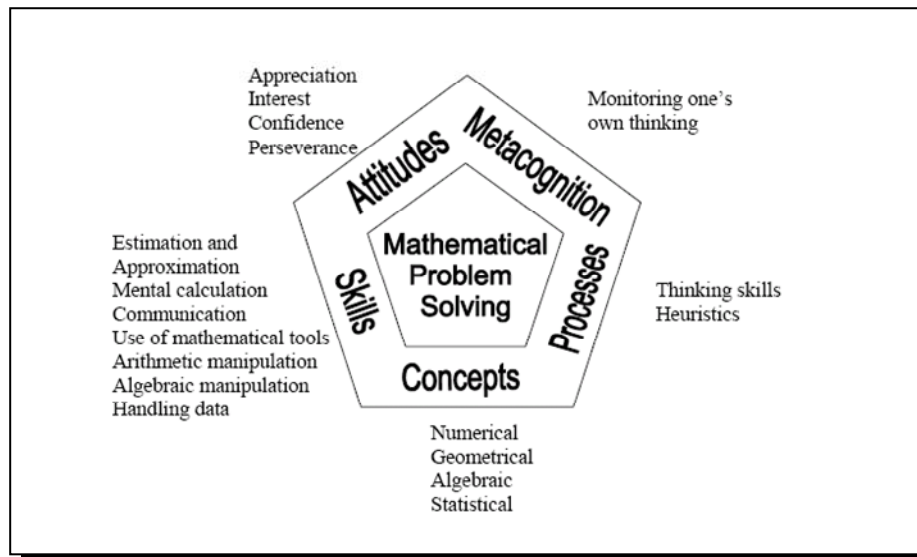
The NCTM framework, while emphasizing higher order, 21st century skills in a visionary way, lacks the logical mathematical structure of Singapore's framework. It identifies content only within broad grade bands (e.g., K–2, 3–5) and only in general terms, thus providing inadequate content guidance to educators.

The seven state frameworks we examined exhibit varying degrees of focus, although none is as focused as Singapore's. Exhibit A shows that three of the states, California, North Carolina, and

Texas, have frameworks that are similar to Singapore's, within 30 percent, in the average number of topics covered per grade. Two of these states, North Carolina and Texas, were praised in the 1990s as states where education reform had been particularly successful. Both states' NAEP mathematics scores improved significantly. The similarity between these states and Singapore suggests a correlation between focused frameworks and good test performance.

By contrast, the frameworks of Florida, Maryland, New Jersey, and Ohio exceeded Singapore's average numbers of topics per grade by 70 to 160 percent. If Singapore's excellent test performance is evidence that its curriculum exposes students to about the right number of topics per grade, then these states' test performance suggests they cover too many topics and should reduce breadth of coverage and deepen topic instruction.

Singapore recognizes that some students may have more difficulty in mathematics and provides them with an alternative framework; the U.S. frameworks make no such provisions.

Singapore's alternative mathematics framework for lower performing students covers all the mathematics topics in the regular framework, but at a slower pace and with greater repetition. Singapore also provides its slower students with extra help from well-trained teachers. NCTM and the states we examined provide no alternative framework for slower mathematics students. Moreover, such students are often unofficially tracked into slower mathematics courses, but unlike in Singapore, these students are seldom taught all the required mathematics material. Evaluations have shown that they frequently receive their extra help from teacher's aides who lack college degrees.

Textbooks

Singapore's textbooks build deep understanding of mathematical concepts through multistep problems and concrete illustrations that demonstrate how abstract mathematical concepts are used to solve problems from different perspectives. Traditional U.S. textbooks rarely get beyond definitions and formulas, developing only students' mechanical ability to apply mathematical concepts.

There is a clear difference in how Singapore and traditional U.S. textbooks develop mathematical concepts. The Singapore texts are rich with problem-based development in contrast to traditional U.S. texts that rarely get much beyond exposing students to the mechanics of mathematics and emphasizing the application of definitions and formulas to routine problems. While such books are filled with real-world illustrations, these seem to serve mainly to show students that mathematics concepts have real-world representations. The illustrations make virtually no contribution to helping students understand how to use the mathematics to solve real-world problems.

The Singapore illustrations also feature a concrete to pictorial to abstract approach. Many students who have difficulty grasping abstract mathematical concepts would benefit from visual representations of mathematical ideas. As part of this approach, the Singapore illustrations demonstrate how to graphically decompose, represent, and solve complicated multistep problems.

Another hindrance to the development of U.S. students' mathematical understanding is the U.S. texts' lack of focus. Singapore's textbooks follow its mathematically logical national framework, but U.S. textbooks must serve multiple state markets. To do so, they find it necessary to cover almost twice as many topics per grade so that all topics from many states' frameworks can be covered. Consequently, individual topic coverage in U.S. textbooks is much shorter and less

comprehensive than what is found in Singaporean texts. In fact, Singapore students are expected to complete about one thorough lesson focused on a single topic per week, while U.S. students are expected to complete about one lesson on a narrowly focused topic each day.

Finally, both Singaporean textbooks and U.S. textbooks "spiral" mathematical content – returning in successive years to the same concepts. However, while the spiral in U.S. textbooks includes significant repetition and reteaching of the same content in two or three consecutive grades, the Singapore textbooks assume that what was previously taught was learned. In other words, Singapore textbooks do not repeat earlier-taught content, because students are taught to mastery the first time around.

Assessments

The questions on Singapore's high-stakes grade 6 Primary School Leaving Examination (PSLE) are more challenging than the released items on the U.S. grade 8 National Assessment of Education Progress (NAEP) and the items on the grade 8 state assessments.

Singapore's grade 6 assessment contains almost double the percentage of constructed-response items as the U.S. grade 8 NAEP and a much higher proportion than that of state assessments. This is an important difference because constructed-response questions generally are more suitable for demonstrating students' higher-level cognitive process in mathematics.

Overall, Singapore's grade 6 assessment also contains a much greater percentage of items that could be characterized as more difficult than either the U.S. 8th grade NAEP or any of the state assessments we examined. These differences are in part the result of NAEP's policy of not including items with very high (or very low) p-values. Many PSLE problems require using multiple steps, solving for an intermediate unknown, or using a nonroutine solution that goes beyond a simple application of a definition or formula. Singapore's most challenging questions are designed to help Singapore identify the best students. These are more difficult than the most challenging questions on the state grade 8 assessments as well as on NAEP.

As a way to hold schools as well as students accountable for performance, Singapore uses a measure of each school's *value-added* contribution to student achievement. The U.S. Adequate Yearly Progress (AYP) measure in No Child Left Behind does not.

A value-added measure of school performance looks at the growth in student outcomes after adjusting for the initial performance. Singapore aggregates individual student test results on its national grades 6 and 10 exams by school. It then compares the expected growth in school outcomes, adjusted for a school's students' initial grade 6 performance, with actual growth to obtain a value-added indicator of a school's performance. Schools that perform above expectations are recognized and rewarded.

The U.S. requirements for AYP under NCLB hold each school accountable for annual growth toward the goal of having all students reach proficiency on state assessments. NCLB allows students to leave schools that have a record of poor performance and are in need of improvement and move to schools with high-performing students. However, a high-performing school that takes students who are low performers is penalized because it will have to make greater gains to meet AYP targets. Schools in this situation may be discouraged from taking low-performing students; Singapore's value-added measure of school progress removes this disincentive.

Teachers

Singaporean elementary school teachers are required to demonstrate mathematics skills superior to those of their U.S. counterparts before they begin teacher training. At every phase of pre- and post-service training, they receive better instruction both in mathematics content and in mathematics pedagogy.

Singapore's teachers must take a stringent examination before being accepted to education school, and while they are students, they are paid a teacher's salary. By contrast, the SAT mathematics scores of entering U.S. elementary education majors are among the lowest of all college students.

After content-driven pre-service preparation, Singaporean teachers are encouraged to continue to improve their knowledge and skills through 100 hours of required annual professional training. U.S. education majors, in contrast, take fewer formal mathematics courses than the average college graduate. The major U.S. teacher screening and licensing exams, the PRAXIS I and II, consist only of multiple-choice questions that, based on released items, appear far easier than items from the exam that Singapore gives to 6th graders. An alternative version of the PRAXIS II (10140) poses more challenging mathematics problems, consistent with having teachers demonstrate higher-order thinking skills, but currently no state requires prospective elementary teachers to pass this more difficult test. After entering the profession, U.S. elementary school teachers typically spend only about a quarter of the 100 hours per year that their Singaporean counterparts spend on professional development activities. The most common form of professional development in the United States is the short-term workshop, widely admitted to be ineffective for changing practice.

AREAS OF STRENGTHS IN THE U.S. MATHEMATICS SYSTEM COMPARED WITH SINGAPORE'S SYSTEM

Although the U.S. mathematics program is weaker than Singapore's in most respects, the U.S. system is stronger than Singapore's in some areas.

The U.S. frameworks give greater emphasis than Singapore's framework does to developing important 21st century mathematical skills such as representation, reasoning, making connections, and communication.

However, to develop these skills in students, the U.S. frameworks need to do a better job of integrating them with rigorous mathematics content.

The U.S. places a greater emphasis on applied mathematics, including statistics, probability, and real-world problem analysis.

The U.S. mathematics frameworks stress data analysis and probability, whereas the Singapore framework treats statistics in a strictly theoretical way. *Everyday Mathematics*, the nontraditional textbook we examined, uses a problem-based learning approach, which presents multistep real-world mathematics problems. Such applications give students practice in understanding how to apply mathematics in practical ways. However, the *Everyday Mathematics* lessons use real-world applications without providing the foundation of the strong conceptual topic development found in Singapore's textbooks. Even though Singapore's textbooks would benefit from

more real-world applications, their emphasis on conceptual development of mathematics and problem-based learning make them superior to U.S. textbooks overall.

PILOT SITE FINDINGS: MIXED RESULTS

The two pilot sites (out of four) that had both a stable population of higher performing students and a clear staff commitment to support the introduction of the Singapore mathematics textbooks produced sizeable improvements in student outcomes.

In North Middlesex, Massachusetts, the school system of about 5,000 was selected by the state education agency to pilot the Singapore textbooks. Over two years, the percentage of those students who participated in the Singapore pilot and scored at the advanced level on the grade 4 Massachusetts assessment increased by 32 percent over two years. The pilot schools had strong district and staff support. Over two years, Baltimore's Ingenuity Project increased the proportion of its students who scored at the 97th percentile or above by 17 percent. The Ingenuity Project serves gifted Baltimore students and can select highly skilled teachers capable of teaching the mathematical reasoning underlying the challenging Singapore problems.

The two other Singapore pilot sites, which in one case had uneven staff commitment to the project and in the other case had a more transient, lower income population, produced uneven or disappointing results.

- The Montgomery County outcomes were positively correlated with the amount of *professional training* the staff received. Two Singapore pilot schools availed themselves of extensive professional development and outperformed the controls; two other pilot schools had low staff commitment coupled with low exposure to professional training and were actually outperformed by the controls. Professional training is important in helping teachers understand and explain the nonroutine, multistep problems in the Singapore textbooks. Teachers also need preparation to explain solutions to Singapore problems, which often require students to draw on previously taught mathematics topics, which the Singapore textbook, in contrast to U.S. textbooks, does not reteach.

- The Paterson, New Jersey, school, with an *annual student turnover of about 40 percent,* fared no better on the New Jersey grade 4 test than the district average over two years. Having such a high student turnover meant that many 4th graders were exposed to the Singapore mathematics textbook for the first time - by definition, not a fair test of the cumulative effects of exposure to the textbook.

Several sites also had difficulties because the Singapore textbooks did not match their state's mathematics priorities.

The most serious mismatch occurred in Paterson, where grade 4 teachers supplemented the Singapore mathematics textbook with their U.S. textbook to cover a few topics, notably statistics and probability, that were on their grade 4 state assessment but not in the Singapore grade 4 textbook. Unfamiliar with the pedagogy laid out in Singaporean Teachers' Guides, several sites were also concerned that the Singapore textbooks did not stress written communication skills by requiring students to explain their answers.

The challenges in using the Singaporean textbooks, such as the lack of teacher preparation, the discrepancies between the topics on the state assessments and the topics in the textbooks at particular grades, and the lack of prior student exposure to the Singapore curriculum, are not challenges faced in Singapore where mathematics textbooks and teacher preparation are aligned to the content in the common framework and where students are held accountable for learning all topics to mastery as they go along.

CONCLUSION

Reform Options

Each component of Singapore's educational system is designed to enhance the mathematical proficiency of students and their teachers. If the United States is to reform its mathematics system so that it more closely resembles Singapore's successful system, the country needs to consider several options for improving each of the components of the system. The options are organized by how much change from current practice would be required and, hence, by how difficult it would be to gain political acceptance for them.

Tinkering Options: Improve or extend existing reforms. States could revise their frameworks to better match Singapore's content grade by grade and strengthen implementation of NCLB reforms for highly qualified teachers to ensure that teachers who meet the NCLB standards actually demonstrate that they understand mathematics content and how to teach it. The federal government could work with the states to produced a national bank of mathematics test items to encourage greater comparability across the states.

Leveraging Options: Use market leverage to bring about improvement. Professional organizations could develop an independent and objective textbook rating system that assesses the depth of mathematics content in textbooks, much as the American Association for the Advancement of Science has already piloted in the sciences.

Program Strengthening Options: Stay within the current U.S. education structure but substantially strengthen the mathematical depth and rigor of the current components of the U.S. mathematical system. U.S. textbooks could be reorganized so that they closely conform to the logical topic organization, rich problem-based approach, and varied pictorial representations of mathematical concepts found in Singaporean texts. Eighth-grade student assessments and teacher-licensing exams could be strengthened so that, at a minimum, they are at least as challenging as Singapore's grade 6 student assessment.

Systemic Reform Options: Strengthen features of the U.S. mathematics system so that it more closely resembles Singapore's integrated, national mathematics system. Such steps might include introducing a national mathematics framework, a national mathematics assessment, and value-added accountability measures of school performance.

Further Validation of Exploratory Findings

Our exploratory results have identified key differences between the U.S. and Singapore mathematics systems. These differences suggest potentially significant reforms that could improve the U.S. mathematics system, but these findings require further validation from larger, more

scientific studies. The suggested reforms need more thorough analyses and, ideally, small-scale introduction prior to going to scale. Only through such further study can we build on our exploratory findings to assess whether adopting the features that have produced a quality mathematics system for Singapore would significantly improve the performance of the U.S. mathematics system and better meet the challenging performance goals set by NCLB.

CHAPTER 1. INTRODUCTION

PURPOSE

On the 2003 Trends in International Mathematics and Science Study (TIMSS, 2003) assessment, eighth-grade students from the United States, as a group, scored near the bottom among students from industrialized nations on mathematics results, whereas students from Singapore, a small country with a population about the size of Chicago, achieved the top average score. This exploratory study examines factors that may explain why students in Singapore perform so much better in mathematics than students in the United States. Looking at the big picture, this paper compares features of both the Singapore and U.S. mathematics systems. It also examines the experiences of four U.S. school systems that sought to replicate Singapore's success by piloting the use of Singapore's mathematics textbooks. Using both international comparisons and lessons from the pilot sites, the study suggests reforms the United States should consider as it works to improve the mathematics performance of its students, while also retaining the effective features of the U.S. mathematics system.

When this study began, its purpose was narrower, seeking only to evaluate changes in student outcomes in four U.S. pilot sites that introduced Singapore textbooks into their mathematics programs in an attempt to replicate the strong mathematics performance of Singapore's students. Previous analyses of TIMSS data showed that the U.S. mathematics curriculum exposes students to many more topics at each grade than are taught in countries, such as Singapore, with higher mathematical performance (Schmidt, Houang, and Cogan, 2002). Because Singapore was the highest scoring TIMSS nation on mathematics, we expected that the use of the Singapore textbooks in the four U.S. pilot sites would expose their students to a substantially different mathematical curriculum. We wanted to determine whether or not this curriculum produced gains in students' mathematics performance.

However, the reactions of teachers and staff from the pilot sites to the Singapore curriculum exposed the challenges they faced in making the curriculum work. Teachers liked the rich content and multistep problems in the Singapore textbooks, but they also talked about the difficulties in implementing them. Although some difficulties were relatively superficial, others were structural, stemming from differences between the content covered by state frameworks and assessments and the content covered by the Singapore textbooks at the same grade level. This discrepancy raised questions about how the Singapore and U.S. mathematics frameworks compare in how they organize and specify foundational mathematics content in the primary grades.

During our initial visits to the Singapore pilot sites, one teacher discussed an additional challenge that U.S. teachers faced in using the Singapore mathematics textbook. During a focus group discussion, the teacher said, "I never realized that I do not understand math until I had to teach mathematics from the Singapore textbooks."[1] Other teachers in the focus group agreed that the depth of explanations and the challenging multistep problems in the Singapore textbooks required them to understand mathematical concepts in greater depth than they were accustomed to. They also found that they had to teach students the meaning of the mathematics being taught, as opposed to simply providing mechanical explanations.

---

[1] This statement was made in a group interview with teachers from the Montgomery County (MD) Public School System using the Singapore mathematics textbook.

The teachers' open admission that they lacked adequate preparation in the foundations of mathematics to teach the Singapore mathematics curriculum caused us to extend our analysis. We began to look at differences in the mathematics knowledge and training that teachers in Singapore and the United States bring to the classroom. We also looked at the problems on the assessments that students take to measure mathematical knowledge to see whether students in Singapore were required to demonstrate greater mathematical understanding than U.S. students at the same grade level.

We also asked representatives from the Singapore Ministry of Education what they saw as the key reasons for their mathematics system's success. They pointed first to their mathematics syllabus (i.e., framework), which clearly identifies mathematical priorities and content grade by grade. The mathematical framework is the foundation for all the other major components of the Singaporean system. This resonates with U.S. research that suggests in high-performing education systems, all core system components—content frameworks, curricula, assessments, and teacher preparation and training—are aligned and focused on producing solid outcomes for all students (Grissmer, Flanagan, Kawata, and Williamson, 2000; Newmann, Smith, Allensworth, and Bryk, 2001; Smith and O'Day, 1991).

To respond to these discussions, our study shifted from merely assessing the results from the U.S. textbook pilots to developing a broad comparison of the coherence and quality of the Singapore and U.S. systems for delivering mathematics instruction. The U.S. pilot results remain an important part of the process of understanding why Singapore students do so well. Studying the challenges the pilot sites faced in transferring only the Singapore mathematics textbooks helped us better understand the importance of looking at all the major components of a mathematics system and at the system as a whole.

Another advantage of a broader study comparing the Singapore and U.S. mathematics systems is that the findings provide information that informs the implementation of the No Child Left Behind Act (NCLB). NCLB establishes new, historic national accountability provisions that require states to assess student mathematics performance annually in grades 3-8 and once in high school and to gauge schools' improvement on the basis of these assessments. NCLB also requires states to have a highly qualified teacher in every classroom by 2005-06. These new provisions move the United States away from its tradition of highly decentralized school systems. Knowing more about the mathematics assessment, accountability, and teacher preparation provisions in the high-performing, highly centralized Singapore mathematics system gives the U.S. federal and state education agencies information that can help them implement NCLB reforms.

METHODOLOGY

Comparing one's procedures with those of high performers as a way of identifying effective practices is a common business strategy, and one that has been used in education for more than three decades since it was popularized by the effective schools movement (Edmonds, 1979). Edmonds wrote about the features of schools that effectively serve concentrations of low-income children, out-performing schools with similar student populations. In mathematics, the TIMSS analyses used similar methods to show that schools in the United States teach substantially more mathematics topics at each grade than do high-performing countries (Schmidt, Houang, and Cogan, 2002), leading to suggestions that the United States pare down topic exposure at each grade in order to deepen content coverage. This study uses a similar technique, examining the features of Singapore's high-

performing system in-depth to understand how these features work, both alone and together, as a means of identifying practices that the United States can use to improve the mathematics performance of its students.

Carrying out in-depth analyses on systems as different as those in Singapore and the United States poses serious methodological difficulties. Singapore has a centralized mathematics system with detailed and consistent implementation procedures, so looking at the separate components of the system was relatively straightforward. Characterizing the 50-state, decentralized U.S. mathematics system, in contrast, is difficult. School systems in the United States select from many available textbooks, and each state has different content standards, assessments, and requirements for teacher certification and training. Because our resources for this exploratory study were limited, we could not reasonably examine every permutation of the complicated U.S. system. Instead we elected to study the elements of the U.S. system by selecting representative examples from the wide variety of what is available in each component area:

- **Standards:** The United States has no national standards, but with only a few exceptions, states used the National Council of Teachers of Mathematics (NCTM) framework as a model in developing state mathematical standards. We used the NCTM standards in our analyses as a proxy for states that use a grade-band, rather than a grade-by-grade, structure in their standards. Because many states use a grade-by-grade structure, we supplemented our analysis of the grade-band NCTM standards with an examination of the standards of seven states that organize content grade by grade. These states are home to approximately one-third of all students in the United States.

- **Textbooks:** We limited our analysis to one traditional and one nontraditional U.S. mathematics textbook.

- **Assessments:** We used sample assessment items published by the federally supported National Assessment of Educational Progress (NAEP) in our comparative analysis. We also drew sample test items from the assessments in the same seven states whose standards we examined.

- **Teachers:** To analyze teacher quality in the United States, we drew from national surveys on teacher education and from teacher preparation standards. We also examined sample problems from teacher licensing exams.

These analyses, although not comprehensive, give a sense of the variation visible in all parts of the U.S. mathematics system.

The evaluation of the four Singapore pilot sites also presented problems in that we had to rely on data the four districts had collected before this study began rather than on uniform data collected specifically for this exploratory study. Existing student outcome data from the four sites were used to measure improvements on state assessments over two years. The scores of students in the Singapore pilot were compared with district or state average gains or with improvements on national norms, but because different assessments were used in the different sites, results are not completely comparable. Ideally, an experimental study would have randomly assigned students to Singapore textbook or regular textbook control classrooms to eliminate the influence of non-textbook factors affecting test scores. Only two of the four sites surveyed teachers to assess their impressions of the Singapore

mathematics textbooks, and although teachers' responses were informative, the questions from the two sites were not the same.

REPORT ORGANIZATION

This exploratory report on the Singapore and U.S. mathematics systems and the pilot study is organized as follows:

- Chapter 2 describes Singapore's mathematics system and the methodology used to compare its key features with those of the U.S. mathematics system.

- Chapters 3-6 compare the mathematics frameworks, textbooks, assessments, and teacher preparation and training programs used in Singapore and the United States. Each section concludes with a discussion of the implications of the comparisons for strengthening the U.S. system by adopting effective features of Singapore's mathematics system.

- Chapter 7 presents the findings from the four pilot U.S. sites using the Singapore mathematics textbooks and is based on student outcomes and teacher survey results.

- Chapter 8 summarizes the implications of the Singapore – U.S. comparisons for reform actions in the United States. We look at what actions should be considered by organizations at all levels involved in the provision of U.S. mathematics, including states, local school systems, textbook publishers, teacher education institutions, national education organizations, and the federal government. This chapter also includes a discussion of additional studies that might be undertaken to strengthen and expand on the findings from this exploratory study of the Singapore and U.S. mathematics systems.

CHAPTER 2. STUDY METHODOLOGY FOR ASSESSING SINGAPORE'S EDUCATION SYSTEM

SINGAPORE'S MATHEMATICS SUCCESS

Founded as a British trading colony in 1819, Singapore joined Malaysia in 1963 but withdrew two years later and became an independent city-state. Its resident population is about 4.1 million, slightly larger than Los Angeles or Chicago. Singapore is a multiracial, multireligious, multilingual urban society. The largest ethnic group is Chinese (77 percent), followed by Malay (14 percent) and Indian (8 percent). In 1970, Singapore's per capita Gross Domestic Product (GDP) was about $300. By 2000, its per capita income was about $25,000, one of the highest in the world. Singapore's economic growth is described "as a modern miracle because it has built its success on only one resource, its people" (MariMari, 2003). Singapore's emphasis on education is seen as a major reason for its economic success.

Singapore's superior performance on the Trends in International Mathematics and Science Studies is one indicator of its education systems effectiveness. In 1999, Singapore's eighth-grade students earned the top average score among students from the 38 countries participating in TIMSS-R. Forty-six percent of Singapore's students were among the top 10 percent of all test takers, five times the 9 percent of U.S. students. Even a Singaporean student in the bottom quartile of Singaporean students outperformed more than two-thirds of U.S. students (Mullis, et al., 2000). In 2003, Singapore's eighth-grade students retained the top average score among student from 46 countries (Mullis, et al., 2004).

Singapore's students performed well in all mathematics areas, scoring at or near the top in all five TIMSS mathematics content areas: 1) fractions and number sense; 2) measurement; 3) data representation, analysis, and probability; 4) geometry; and 5) algebra (Exhibit 2–1). U.S. students, in contrast, scored significantly lower in all five content areas. Only in data, including statistics and probability, was the achievement gap relatively small. (Mullis, et al., 2004)

Exhibit 2–1. Singapore and U.S. Eighth Grade Percent Correct Items on TIMSS, 1999 and 2003

Mathematics Content Area	Singapore		U.S.		Difference (Singapore – U.S)	
	1999 (% correct)	2003 (% correct)	1999 (% correct)	2003 (% correct)	1999 (% correct)	2003 (% correct)
Overall	76	74	50	51	26	23
Number	80	78	54	54	26	24
Measurement	76	74	40	42	36	32
Geometry	73	71	44	45	29	26
Data	81	79	68	72	13	7
Algebra	69	69	47	50	22	19
Overall Rank	(1)*	(1)**	(18)*	(15)**	(17)	(14)
*out of 38 countries **out of 46 countries Source: Mullis, Martin, Gonzalez, and Chrostowski (2002) and Mullis, Gonzales, and Chrostowski (2004)						

How can a small nation, which was only recently among the world's poorest countries, achieve such high mathematics scores? What factors explain Singapore's world-class performance in

mathematics? Answering these questions is a first step toward assessing the transferability of its successful program to the United States.

OVERVIEW OF SINGAPORE'S EDUCATION SYSTEM

Singapore has a highly centralized education system controlled and coordinated by its Ministry of Education. The Ministry has implemented a national curriculum, developed a syllabus that guides instruction in all required subjects in all schools, and instituted uniform high-stakes assessments at the critical end of both primary and secondary school. Singapore's education system (see Exhibit 2–2) consists of six years of primary education and four or five years of secondary education (Ministry of Education, Singapore, 2003). At the primary level, pupils undertake a four-year foundation stage in primary grades 1–4, followed by a two-year orientation stage in primary grades 5 and 6. Singapore and the United States have a similar age-grade correspondence in the primary grades; fourth graders are typically nine years old. The emphasis during the foundation stage is on basic literacy and numeracy. Eighty percent of the curriculum time is used for instruction in English, the student's mother tongue (Chinese, Malay, or Tamil), and mathematics. Science is not taught until primary grade 3.

Exhibit 2–2. The Structure of Singapore's Education System
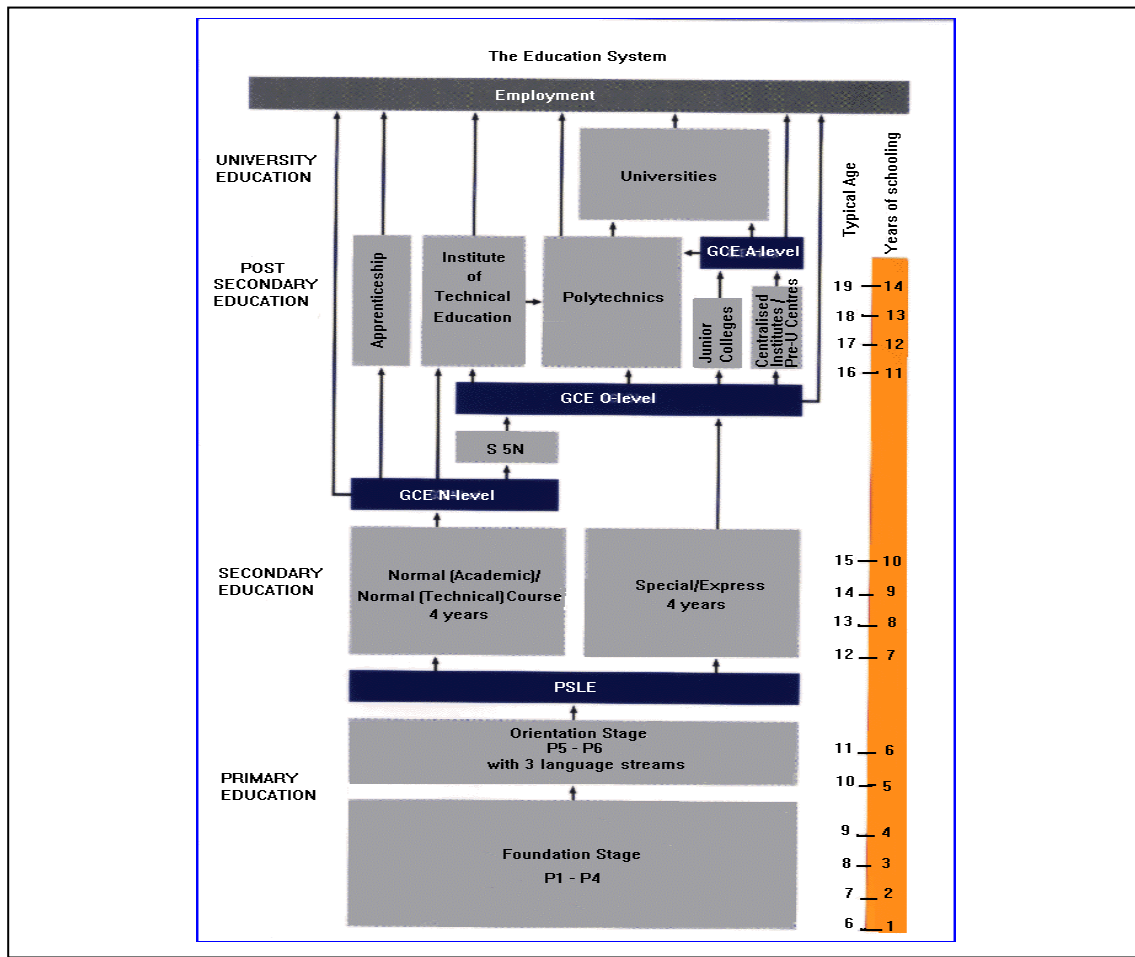
Before they begin the orientation stage of primary school, pupils are assessed. On the basis of their abilities, they are placed in one of three streams – EM1, EM2, or EM3. In the EM1 and EM2 streams, in which about 90 percent of the pupils enroll, students continue to learn English, their mother tongue, mathematics, and science. EM1 pupils may study higher Malay, higher Chinese, or higher Tamil as their mother tongue. The remaining 10 percent of pupils are placed in the slower EM3 stream where they take foundation English, basic mother tongue, and foundation mathematics. The foundation mathematics program offers lower-achieving students the same mathematics topics offered to EM1 and EM2 students, but over a more extended time and with extra assistance.

At the end of primary grade 6, pupils take the Primary School Leaving Examination (PSLE), which assesses their abilities for placement in a secondary school program that suits their "learning, pace, abilities, and inclinations" (Ministry of Education, Singapore, 2000a). Pupils are then admitted to the special, express, or normal stream for four years of secondary education. Students in the express and special streams, which have a high-level language curriculum, complete a college preparatory course and take the rigorous Joint Cambridge University (England) and Singapore O-level college entrance examination at the end of their fourth year. Students in the normal stream complete a less rigorous curriculum and take the Singapore-Cambridge General Certificate of Education Normal (N-level) examination. Somewhat more than three-quarters of all secondary students take the O-level exam and the remaining students take the N-level exam (Ministry of Education, Singapore, 2003a).

Throughout primary and secondary school, student advancement is tied to performance. A Ministry of Education's mission statement makes this clear:

> *Every child must be encouraged to progress through the education system as far as his ability allows. Advancement must always depend on performance and merit to ensure equal opportunity for all. (Ministry of Education, Singapore, 2003b)*

Singapore also recognizes that not all children proceed at a rapid academic pace and that some children require special assistance:

> *Every child should be taught at a pace he can cope with. Each should be stimulated to excel according to his individual aptitudes. The system must be flexible, to cope with pupils who mature mentally, physically, emotionally and socially at different rates. (Ministry of Education, Singapore, 2003b)*

In practice, this approach means that Singapore relies on early high-stakes testing, but it also holds teachers responsible for the success of all children and ensures that teachers devote more attention, rather than less, to students with greater academic needs.

SINGAPORE-U.S. POPULATION DIFFERENCES

Several theories have been put forth to explain Singapore's mathematics success (Colvin, 1997; Viadero, 2000). These theories can be grouped into those that focus on perceived differences between the populations of Singapore and the United States and those that look specifically at Singapore's education system.

One explanation focuses on differences between the populations of the United States and Singapore. Singapore is small and fairly homogeneous and has highly motivated students. Therefore,

the argument goes, Singapore's mathematics experience may not be transferable to the United States where these conditions do not apply. Although population differences do matter, they are not great enough to make Singapore's mathematic success impossible to reproduce in the U.S. education system.

Arguments about Singapore's homogeneity, for example, are not persuasive. Some believe that Singapore is successful because it educates a comparatively homogeneous population that is unlike the multiethnic U.S. population. It is true that Singapore's student population is not as diverse as the U.S. student population, but to characterize Singapore as homogeneous is misleading. Singapore has three major ethnic groups. About three-fourths of Singapore's population is Chinese, but almost a quarter is Malay or Indian. Like the United States, Singapore experienced serious ethnic strife in the 1960s. Singapore accommodates its heterogeneous population by practicing principles of multiracialism and meritocracy. It practices true bilingualism in grades 1 through 3 when, although English is the primary language of instruction, children from each major ethnic group also study their home languages. Singapore does remarkably well academically even though many students are receiving instruction primarily in a language other than what they speak at home, something at which the United States has been less successful.

Singapore's 1999 TIMSS scores confirm that its minority students do well. Singapore broke out the 1999 TIMSS scores for its Malay and Chinese populations (Ministry of Education Singapore, 2000b). Although 96 percent of Chinese students performed better than the international eighth-grade mathematics average, Malaysian Singaporeans also did very well, with 83 percent scoring above the international average. Scores for Singapore's Indian minority population were not available, but typically, students of Indian background outperform their Malaysian peers by a small margin. By comparison, in the United States, half of black eighth-grade students achieved no better than the bottom quarter of all international test takers (NCES, 2000).

Another unsatisfactory explanation for Singapore's success focuses on the remarkable work ethic of Singapore's students rather than on Singapore's mathematics program. Singaporean students are hardworking, but if Singapore's success is attributable only to work ethic, how can we account for the fact that its high achievement is a comparatively recent development? On the Second International Science Study in the mid-1980s, Singaporean fourth graders scored only 13th out of 15 participating nations, and Singaporean eighth graders did no better than their U.S. counterparts, tying for 13th among 18 nations (Medrich and Griffith, 1992). In response to these poor scores, Singapore's Ministry of Education re-engineered and strengthened the education system, reforming both the science and mathematics curriculum. Singapore's eighth-grade students are now among the highest-achieving students in the world in both science and mathematics.

This is not to say that Singaporean students do not work hard; they do. This is partly attributable to the high value that Singaporean families place on education and to a culture in which knowing mathematics is as important as knowing how to read well. But value differences are not the only reason that Singaporean students work harder. They also receive more homework than U.S. students. Two-thirds of Singaporean eighth graders were assigned at least 30 minutes of mathematics homework at least twice a week, compared with only 25 percent of U.S. eighth graders (Mullis, et al., 2000). Population differences are not what prevent U.S. teachers from emulating Singapore's more stringent homework policies.

Another reason given for Singapore's success is that it is small. Its population of 4 million, compared with 290 million in the United States, this argument claims, allows Singapore to

implement a centralized education system that is not replicable in the United States. Federal policies in the United States may not allow a centralized curriculum like Singapore's, but smaller units of government in the United States could enforce greater centralization. *No Child Left Behind* assessment provisions that require annual testing in grades 3-8 are an impetus for all states to move toward grade-by-grade standards, and some U.S. states, such as California and Massachusetts, have recently developed or are in the process of developing mathematics frameworks that look much like Singapore's. Larger U.S. local education agencies, including some low-performing urban school systems, are about the same size as Singapore, and new research suggests that these districts could reap academic benefits by ensuring that students learn high, uniform levels of content, as students in Singapore do (Snipes, Doolittle, and Herlihy, 2002). Moreover, U.S. states and school systems do not have to replicate all the features Singapore's mathematics system; they can selectively adapt features of the Singapore system that fit well with their own frameworks.

Although size, homogeneity, and student motivation certainly play a role in Singapore's mathematics success, they fall far short of completely explaining it. It is, therefore, worthwhile to carefully analyze Singapore's mathematics education system and how it compares with the U.S. education system.

STUDY METHODOLOGY

This comparative study focuses on mathematical frameworks, textbooks, assessments, and teachers to determine how these major features operate in the mathematics system of Singapore and the United States (Grissmer, et al., 2000; Smith and O'Day, 1991). Exhibit 2–3 shows how the system components work together. The mathematical framework outlines the content that the curriculum is intended to cover and sets priorities for processes that students are to learn. Textbooks delineate the available curriculum, and assessments measure what is most valued by the system. Ultimately, the quality of the teachers determines the quality of mathematics instruction received. Noticeable differences in these characteristics may help explain why student performance in mathematics is poorer in the United States than in Singapore. Collectively these four elements exert considerable influence over the content and quality of classroom instruction, and our analysis of these elements tells us something about classroom instruction, even in the absence of direct observation. For each of the four components we asked a series of analytic questions to address how they work, both individually and together.

Mathematics frameworks (i.e., syllabus). Singapore's mathematics framework defines expectations about what Singaporean students should know and be able to do in mathematics. Singapore's well-defined syllabus describes mathematical topics and outcomes grade by grade within broad mathematical strands. Although the United States has no similarly legislated national mathematics standards, the National Council of Teachers of Mathematics (NCTM) standards, which are organized by broad grade-bands (e.g., K–2, 3–5), have been widely used by states in developing their own mathematics standards. The NCTM standards were, however, developed prior to the passage of *No Child Left Behind*, which requires assessing students each year in grades 3 through 8. Because grade-by-grade assessments are now required, many states are shifting to grade-specific content standards that let administrators and teachers know the expectations for student performance at each grade. For this study, we compared the Singapore standards with both the NCTM standards, which stand in for state standards organized by grade-bands, and seven sets of state standards that are organized grade by grade. We compared the Singapore, NCTM, and selected state frameworks with respect to the following questions:

- What are the overarching mathematical processes set out in the standards?

- How do the standards structure mathematical content in terms of organization and specificity?

- What mathematical content do the standards expose students to, and do individual sets of standards cover mathematics topics that are not addressed by other standards?

- How do the standards address the needs of diverse students who progress in mathematical understanding at different rates?

Exhibit 2–3. Analytical Framework to Compare Singapore and U.S. Mathematics Systems

Mathematical Frameworks
- Overarching Process Priorities
- Content Organization and Specificity
- Content Coverage
- Provisions for Diverse Mathematics Students

Textbooks
- Overall Content
- Lessons
- Problems

Assessments
- Content Areas
- Question Type
- Question Difficulty

Teachers
- Entry
- Preparation
- Certification
- Professional Training
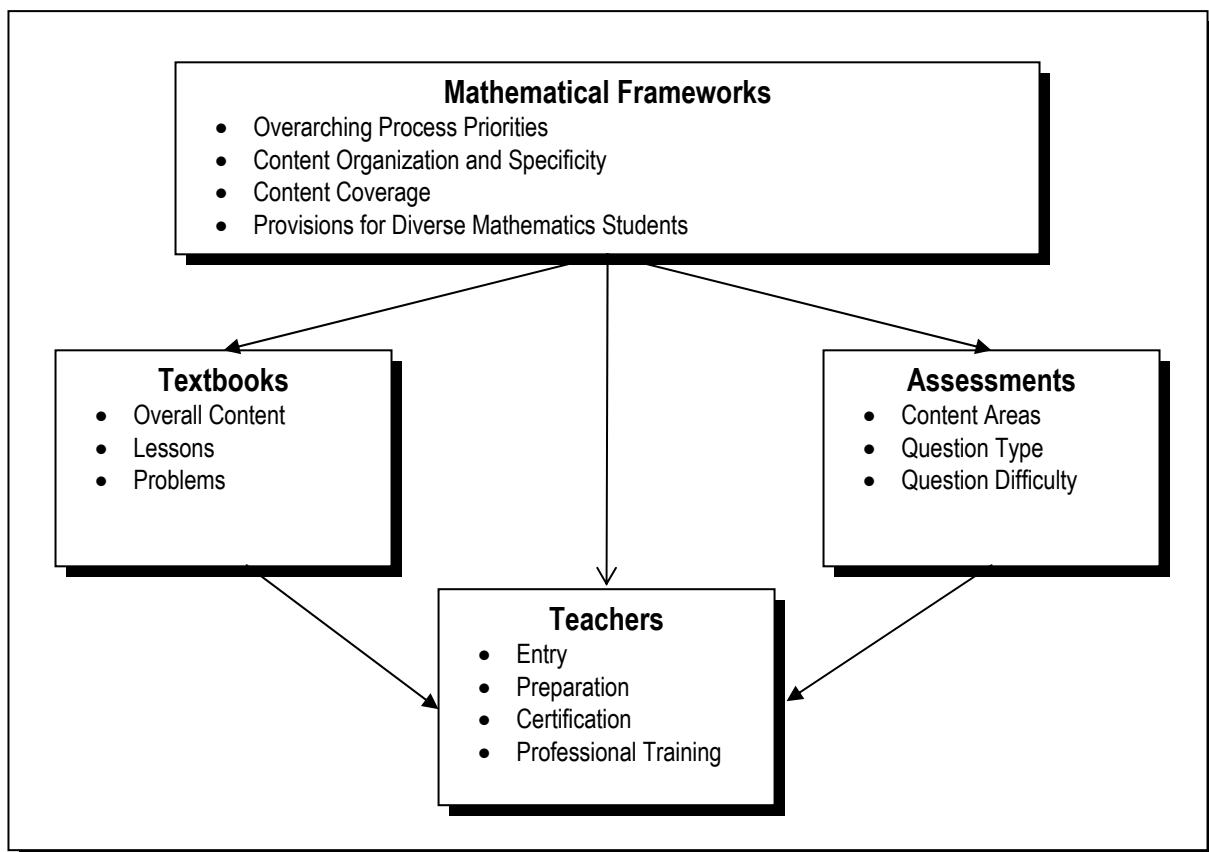
Textbooks. The Singapore mathematics textbooks certainly look different from U.S. textbooks, despite the fact that both are written in English. The Singaporean textbooks are thinner than their U.S. counterparts, use many fewer words, and are more obviously mathematical in content. But do fewer words deliver more content? How do the U.S. and Singaporean textbooks compare in pedagogical approach? This section compares the Singapore mathematics textbook with a traditional U.S. mathematics textbook and a nontraditional U.S. textbook at three levels of textbook organization:

- At the textbook level, how do the textbooks compare in their structure and content coverage across the grades?

- At the lesson level, how do the textbooks compare in their treatment of selected topics across grades?

- At the problem or exercise level, how do the textbooks compare in their presentation of mathematically challenging exercises?

Assessments. Singapore's end-of-year mathematics assessments, the Primary 4 Examination and the Primary School Leaving Exam (PSLE), are required by the Ministry of Education and are used to place students in different learning streams. Each school develops its own Primary 4 exam, whereas the PSLE is uniform across schools. Singaporean students know the importance of these exams and take them very seriously. Students also sit for a uniform exam at the end of secondary school, around grade 10. This study focuses on the items in Singapore's grade 6 PSLE.

In the United States, *No Child Left Behind* has expanded the use of state assessments to assess whether schools make adequate yearly progress. Although the United States' use of assessments for school accountability is different from Singapore's use of assessments for individual student placement, many U.S. school systems are adopting remediation programs, including required summer school, for students who fail the state assessments. NCLB also requires that state assessment results be compared with the results for the federally administered National Assessment of Educational Progress (NAEP, 2004a), which was previously used only for informational purposes. This study compares Singapore's PSLE, selected state assessments, and NAEP to determine the following:

- Overall, how do assessments compare with respect to content areas covered, question type, and the mathematical difficulty of questions?

- How do difficult assessment items on similar topics compare on each test? Are items deemed difficult on a U.S. test as challenging as difficult items on the PSLE?

Teachers. Singapore gives its teachers much of the credit for its education success. Singapore's teachers "lie at the heart of all we do in education" (Ministry of Education, Singapore, 2001c). Although Singaporean teachers receive a solid foundation in basic mathematics, the majority of primary school mathematics teachers do not have a four-year college degree. In Singapore, quality teaching is supported in ways other than through a formal four-year degree program, in contrast to the United States, where a four-year degree is required. In this study, we compare the Singapore and U.S. teacher pipelines in terms of those who enter education school, teacher preparation, certification, and ongoing professional training to answer the following questions:

- How are students with an interest in becoming teachers selected for entrance into education schools, and are incentives offered to able candidates?

- What pre-service preparation do mathematics teachers receive?

- How are teachers certified through licensing examinations? How difficult is the mathematics content on these examinations?

- What induction programs are available for new teachers, and what professional development opportunities are available for experienced mathematics teachers?

Finally, in addition to making international comparisons at the system level and to looking at the four system components in detail, our exploratory analyses examine experiences and outcomes in four pilot U.S. sites that adopted Singapore mathematics textbooks.

CHAPTER 3. SINGAPORE, NCTM, AND STATE MATHEMATICS FRAMEWORKS

CONTEXT AND METHODOLOGY

This chapter examines how the mathematics program delineated in Singapore's mathematics framework compares with programs laid out in the NCTM framework and in selected U.S. state mathematics frameworks. Singapore's well-defined, national framework, which has led to outstanding TIMSS performance since 1995, sets out a more specific, challenging, and mathematically logical program than the NCTM or state frameworks. Singapore's national framework identifies the overarching mathematics processes (i.e., competencies) and specific mathematics content that students should learn at each grade.

The United States has no official national mathematics standards. We have chosen to look at the National Council of Teachers of Mathematics (NCTM) standards because they are the closest U.S. approximation to national standards. First published in 1989 and revised in 2000 under the title *Principles and Standards for School Mathematics*, the NCTM standards establish broad national priorities for what children should know and be able to do in mathematics. They are intended to serve multiple purposes:

Principles and Standards supplies guidance and vision while leaving specific curriculum decisions to the local level. This document is intended to—

- set forth a comprehensive and coherent set of goals for mathematics for all students from prekindergarten through grade 12 that will orient curricular, teaching, and assessment efforts during the next decades;

- serve as a resource for teachers, education leaders, and policymakers to use in examining and improving the quality of mathematics instructional programs;

- guide the development of curriculum frameworks, assessments, and instructional materials;

- stimulate ideas and ongoing conversations at the national, provincial or state, and local levels about how best to help students gain a deep understanding of important mathematics. (NCTM, 2000, p. 6)

This statement indicates that *Principles and Standards* is intended to be both a visionary document that tries to "orient curricular, teaching and assessment efforts during the next decades" and a traditional framework that serves the traditional functions of a national standards document by guiding detailed development of state standards, assessments, and textbooks. Evaluations of state standards do indeed show that the NCTM standards have influenced the design of state standards. A National Research Council (NRC) study indicates that the mathematics standards from most states were either adapted from NCTM standards or taken from NCTM verbatim (NRC, 2001, p. 34). The NCTM, itself, has concluded that its standards

have influenced state standards and curriculum frameworks (Council of Chief State School Officers 1995; Raimi and Braden 1998), instructional materials (U.S. Department of Education, 1999), teacher education (Mathematical Association of America 1991), and classroom practice (Ferrini-Mundy and Schram 1997). (NCTM, 2002, p. 5)

NCTM has given itself a difficult task, however, in trying to create a single document that is simultaneously a visionary and strategic document and a guide for states in developing their own frameworks. A visionary framework that is strategic and emphasizes new mathematics reform goals is difficult to make compatible with a framework that serves a more traditional aim of identifying and focusing content in a balanced, specific, and detailed way.

In addition to looking at the NCTM standards, we also look at selected state frameworks. Although the NCTM standards have influenced the development of many state standards, some states are now moving away from them, necessitating that we look at some state frameworks as well. One reason that states may be moving away from the NCTM model is poor U.S. performance in the middle grades on the TIMSS international assessments. Because of these results, some states, such as California, used high-performing Singaporean and Japanese mathematics systems as models in rethinking their mathematics standards.

The states are also revising their frameworks to respond to NCLB provisions requiring them to conduct annual mathematics assessments in grades 3–8. As states adjust to the grade-by-grade assessments required under NCLB, they are under increasing pressure to define explicit mathematical expectations for each grade. NCTM organizes its intended content by broad grade-bands, a structure not easily correlated with grade-by-grade assessments. Consequently, by 2003, 26 states had structured their mathematics standards around grade levels rather than grade-bands (see appendix: Exhibit B3–1). Massachusetts typifies state concerns over the need to respond to the tough NCLB assessment provisions. Its 2004 *Supplement to the Massachusetts Curriculum Framework* shows how it has altered its standards to accommodate NCLB provisions:

> In 2003, when work on the Supplement began, Massachusetts students were assessed in mathematics at grades 4, 6, 8, and 10. However, the federal No Child Left Behind (NCLB) Act requires annual testing in mathematics at each grade from grades 3 through 8, beginning with a first operational test in spring 2006. Therefore, Department staff, working with committees of educators and mathematicians, drafted grade-level standards for grades 3, 5, and 7, as presented in this Supplement. These grade-level standards were approved by the Board of Education on March 30, 2004. (Driscoll, 2003)

Because state standards are in flux, we needed to look at standards from both grade-specific states and grade-band states in making our comparisons to Singapore's framework. Because resources for this exploratory study were too small to allow us to look at standards from all states, we have used the NCTM grade-band standards as a rough proxy for states that still use a grade-band approach in their standards, and we have chosen seven state frameworks that use a grade-specific approach similar Singapore's to stand in for all states that organize their standards by grade. These frameworks, from California, Florida, Maryland, New Jersey, North Carolina, Ohio, and Texas, are also important in their own right because they collectively affect about one-third of students in the United States. Our study focuses on the primary grades because Singaporean mathematics textbooks are most often used in the United States in elementary schools and because the primary grades provide a foundation for future mathematical learning.

We compare the Singapore, NCTM, and the grade-level state frameworks in four areas:

- The *overarching process and content priorities* for preparing primary students mathematically that are emphasized in the curriculum frameworks;

- The *organization* of mathematical content, especially logical sequencing and specificity;

- The *coverage of mathematical content*, including numbers of mathematical topics and outcomes addressed and topics that some, but not all, standards cover; and

- The provisions that address the needs of *diverse students* who learn mathematics at different rates.

OVERARCHING PROCESS AND CONTENT PRIORITIES

At the broadest level, elementary-level frameworks establish the overarching mathematical processes and content priorities that students need to be proficient in beginning mathematics. For the purposes of this study, we define *process priorities* as the key ways that students should be able to use mathematical knowledge. Examples include representing a problem mathematically, reasoning through the logic of a solution, and communicating mathematical content. We use the term *content priorities* to describe the core subject matter of mathematics that students should learn. These include such things as the concept of a number, the meaning of addition or multiplication, and the statistical measures of central tendencies. An effective elementary-grade framework must identify both the essential process and content priorities that are the foundation of mathematical understanding.[2]

Process Priorities

Singapore's process priorities are identified in its conceptual framework (Exhibit 3-1). The central priority is mathematical problem solving:

Exhibit 3–1. Singapore's Mathematics Framework

---
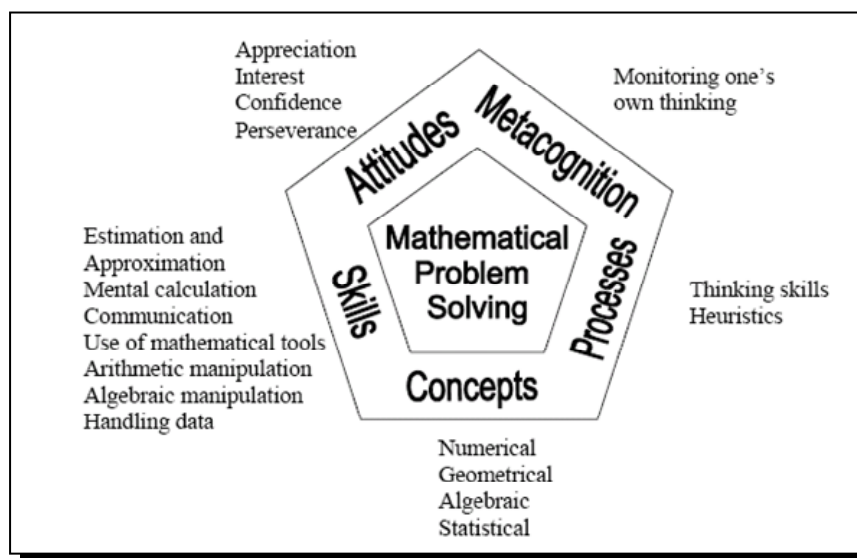
[2] Discussing the features of frameworks presents difficulties because there is no common terminology for describing them. The same term may be used differently in different frameworks. For example, Singapore's framework uses the term process to mean strategic problem solving rather than to characterize framework components that explain how mathematics is carried out. NCTM's framework uses processes more generally, a definition we have adopted for this paper. It should also be noted that the Singapore Mathematics Framework shares much in common with the five strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) discussed in *Adding It Up* (NRC, 2001).

The primary aim of the mathematics curriculum is to enable pupils to develop their ability in mathematical problem solving. Mathematical problem solving includes using and applying mathematics in practical tasks, in real life problems and within mathematics itself. (Ministry of Education, Singapore, 2001a)

Singapore's framework for developing students' problem-solving capabilities identifies five categories: concepts (i.e., content), which we examine in the next section, and four process priorities. The process priorities are skills, processes (i.e., problem-solving strategies), metacognition, and attitudes:

- *Skills* are defined as "the topic-related manipulative skills that pupils are expected to perform when solving problems" (Ministry of Education, Singapore, 2001). These include procedural fluency in estimation and approximation, mental calculation, communication, use of mathematical tools, arithmetic manipulation, algebraic manipulation, and handling of data.

- *Processes* are defined as problem-solving strategies, including ways of thinking about problems (e.g., induction, deduction) and heuristic strategies for formulating problems (e.g., use a diagram or model, work backward, simplify the problem, look for patterns, make a systematic list).

- *Metacognition* is defined to include abilities such as monitoring one's own thinking, checking alternative ways of performing a task, and checking the reasonableness of the answer.

- *Attitudes* are defined to include such things as finding joy in doing mathematics, appreciating the beauty and power of mathematics, showing confidence in using mathematics, and persevering in solving problems.

NCTM's framework also identifies five core mathematical processes, but they are not the same five identified by Singapore. The NCTM process priorities are problem solving, reasoning and proof, communication, connection, and representation (2000):

- *Problem solving* is the ability to "apply and adapt a variety of appropriate strategies" and to "monitor and reflect on the process of mathematical problem solving."

- *Communication* is the ability to use language to communicate mathematical ideas and explain problem solutions.

- *Reasoning and proof* cover logical thinking skills, including making and investigating mathematical conjectures, developing and evaluating mathematical arguments, and using many kinds of reasoning and methods of proof.

- *Representation* is the ability to "apply mathematical translations to solve problems," moving from abstract concepts to symbols, expressions, or diagrams, and the ability to "use representations to model and interpret physical, social, and mathematical phenomena."

American Institutes for Research®

- *Connections* are the abilities to "understand how mathematical ideas interconnect" and "apply mathematics in contexts outside of mathematics."

The Singapore and NCTM categorizations of mathematical processes highlight the frameworks' different mathematical emphases. Singapore's framework, by elevating skills as a distinct component, places a higher priority on computation and mental arithmetic than the NCTM standards do. In Singapore, instilling procedural skills is a primary goal even in an age of calculators and computers. The original 1989 NCTM standards, in contrast, were criticized by mathematicians and educators for reducing emphasis on computation skills. Although the 2000 revision gives greater emphasis to computation, NCTM still does not elevate arithmetic and other mathematical procedures to the same level as does Singapore.

NCTM's framework instead emphasizes higher-order mathematical processes. Whereas Singapore has a single "process" category for strategic problem-solving skills, NCTM has three: reasoning and proof, representation, and connections, in addition to problem solving. NCTM makes communication of mathematical ideas a priority, but Singapore does not give it much emphasis and includes communication only as one on a list of skills. Research supports NCTM's elevation of communication as a fundamental priority in that communication builds understanding and sharp thinking essential to the learning process (Slavin, 1995; Webb, 1992). Collectively, the NCTM process priorities are consistent with the emphasis on teaching those skills, including information analyses, systems thinking, and communication, that are essential job skills in a digital workplace (Partnership for the 21st Century, 2004).

The frameworks in each of the seven states examined in this study also identify process priorities. Exhibit 3–2 compares process priorities in the seven states' frameworks with a list of Singapore and NCTM processes, problem solving being the only priority identified by both Singapore and NCTM. The states' process priorities are more similar to NCTM's than they are to Singapore's. All the states, except Florida, include one or more of the NCTM strategic processes in their frameworks. Only North Carolina includes a process that addresses Singapore's interest in computational skills.

The National Research Council states that pitting conceptual understanding against computation facility is a "false dichotomy" (2001). Overall, Singapore's framework emphasizes a balanced set of mathematical processes, recognizing the need to support students' conceptual understanding with computational skills and strategic problem-solving abilities. The NCTM list is also logical and useful in that it calls attention to current thinking about the processes by which students acquire higher-order skills, such as conceptual reasoning and communicative competence. However, as a total conceptualization of mathematical priorities for what students should know and be able to do, NCTM's mathematical priorities do not give as much emphasis to skills, especially computational skills, as Singapore. Among the states reviewed, North Carolina appears to be on the right track in developing a combined process list that includes both Singapore's emphasis on traditional computational skills and NCTM's emphasis on higher-order processes.

Exhibit 3–2. A Comparison of Overarching Processes: Singapore, NCTM, and Selected State Standards

Process Priorities	Sing	NCTM	CA	FL	MD	NJ	NC	OH	TX
Problem Solving	X	X				X	X	X	X
Skills (e.g. computation)	X						X		
Processes (e.g., strategic problem solving including thinking skills and heuristics)	X								
Metacognition	X								
Attitudes	X						X		
Reasoning and Proof		X	X		X		X	X	X
Communications		X			X	X	X	X	X
Connections		X			X	X	X	X	
Representation		X				X		X	
Source: State mathematics standards from 2004.									

Content Priorities

In addition to process priorities, mathematics frameworks must specify content priorities. Content priorities, often called content standards, lay out the mathematical content that students should learn at each grade level. At their broadest level, which we examine here, the Singapore, NCTM, and state content standards consist of mathematical strands, such as numbers or geometry, that are the fundamental building blocks of introductory mathematics, and they prioritize mathematical content.

Singapore's conceptual mathematics framework, as seen in Exhibit 3–1, identifies four overarching mathematical content areas in the primary grades. These areas, which Singapore calls concepts, are numbers, geometry, algebra, and statistics (Ministry of Education, Singapore, 2001a):

- The *numbers* strand begins with whole numbers and extends to fractions, decimals, rates of speed, proportion, and percentages.

- The *geometry* strand begins with simple shapes, such as rectangles, squares, circles, and triangles, and later introduces more complicated shapes, such as semicircles and quarter circles. It then moves on to more complicated presentations of angles and three-dimensional figures.

- The *statistics* strand presents statistical graphs beginning with basic picture graphs. The *algebra* strand, which begins in Primary 6, is limited to a traditional conception of algebraic expressions involving relationships among variables. In contrast, NCTM includes numeric patterns in its definition of algebra and introduces algebraic ideas in the early grades while students are still learning to add and subtract.

The NCTM framework defines five content areas as priorities: numbers, algebra, geometry, measurement, and data analysis and probability. Exhibit 3–3 illustrates how the content areas are emphasized across grades. NCTM, like Singapore, expects that different content strands will not receive equal emphasis in each grade:

It is not expected that every topic will be addressed each year. Rather, students will reach a certain depth of understanding of the concepts and acquire certain levels of fluency with the procedures by prescribed points in the curriculum, so further instruction can assume and build on this understanding and fluency. (NCTM, 2000, p.30)

Exhibit 3–3. Content Areas in NCTM Framework Receive Different Emphasis Across Grades

Source: National Council of Teachers of Mathematics (2000)
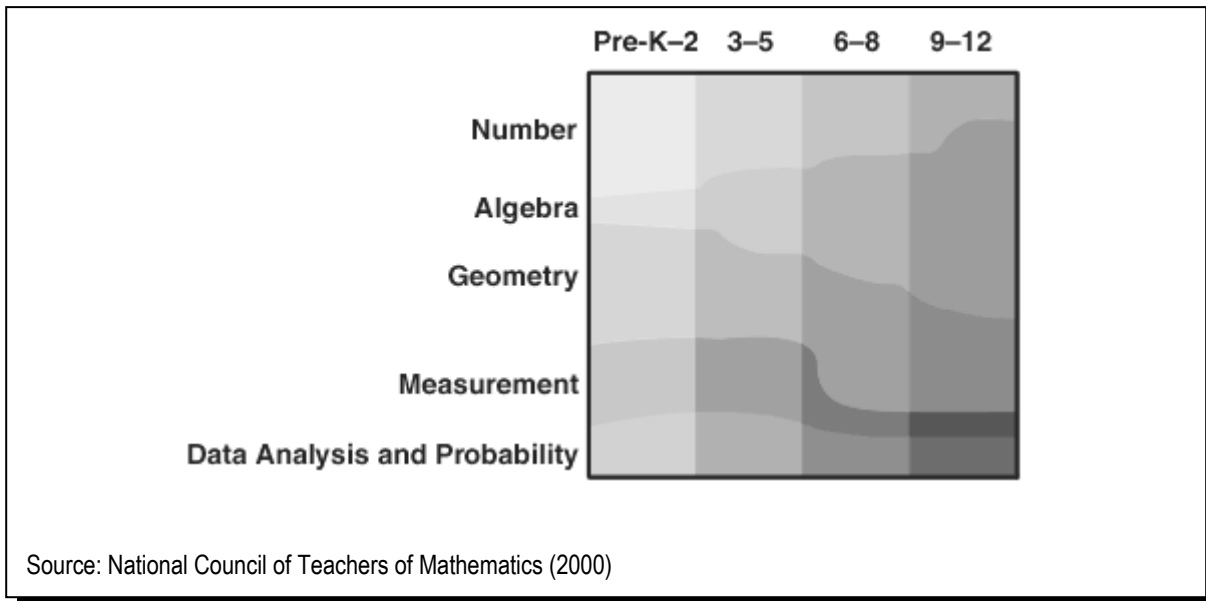
The NCTM framework identifies four content areas similar to those spelled out in the Singapore standards, but adds a fifth, measurement, which is intended to enable students to "understand measurable attributes of objects and the units, systems, and processes of measurement; apply appropriate techniques, tools, and formulas to determine measurements" (NCTM, 2000). Singapore includes measurement in its numbers strand.

NCTM also expands what Singapore includes in a simple statistics category into a data analysis and probability category, giving it greater emphasis. Singapore's statistics category concentrates on the development of statistical representations, such as pie charts. NCTM's data analysis strand is more extensive, covering the collection, organization, analysis, and display of data and the way data are used in making predictions. In addition, NCTM covers probability, the mathematical description of events of chance, which Singapore does not treat in primary school. In a digital world, the ability to make sense of and process quantitative information is an essential job skill, making NCTM's greater emphasis on measurement, data analysis, and probability more relevant than Singapore's narrower treatment of these topics.

Exhibit 3–4 compares the overarching content priorities in the selected seven states with those of Singapore and NCTM. The states resemble NCTM more closely than they do Singapore in how they treat measurement and data analysis and probability, the two areas of major content differences between Singapore and NCTM. Six states, like NCTM, make measurement a separate strand; California combines measurement with geometry. All the states cover probability. Six also include data analyses; Maryland, however, places data analyses under statistics. The states opt to

follow NCTM's strong emphasis on applied mathematics in their content priorities, which is preferable to Singapore's approach and consistent with a digital age's emphasis on mathematical problem solving.

Exhibit 3–4. A Comparison of Content Priorities: Singapore, NCTM, and Selected State Standards

Mathematical	Sing	NCTM	CA	FL	MD	NJ	NC	OH	TX
Content Priorities									
Numbers	X	X	X	X	X	X	X	X	X
Measurement		X	X[1]	X	X	X	X	X	X
Geometry	X	X	X[1]	X	X	X	X	X	X
Statistics	X		X		X				
Data Analyses and Probability		X	X	X	X[2]	X	X	X	X
Algebra	X	X	X	X	X	X	X	X	X[3]
Source: State mathematics standards from 2004									
[1]Combines measurement and geometry into a single content priority									
[2]Includes only a separate probability topic									
[3]Includes only number and geometric patterns and not relationships among variables, in grades 1–6									

CONTENT ORGANIZATION

Mathematical content standards are not a detailed curriculum; they are an outline of topics and how they should be sequenced. An effective organizational scheme for mathematical topics should do two things (American Federation of Teachers, 2001; Kendall, 2001):

(1) *Develop mathematical topics logically and sequentially* within each content area. The organization of topics should model how mathematical content should be grouped and ordered to optimize student understanding. Mathematics has a reasonable, natural order of content development. Number concepts are fundamental to all mathematics and should be the initial emphasis in the early grades. Simpler concepts, such as lines and squares, should be presented before more complex concepts, such as polygons and angles. Easier computations, such as one-digit multiplication, should be presented prior to more taxing computations that involve multidigit numbers.

(2) Describe content with *sufficient specificity to guide the development of mathematical content of textbooks, assessments, and teacher training.* In the United States, the organizational scheme should also provide guidance for developing state and local frameworks. If content descriptions are very general, they will not provide the content guidance necessary to determine content coverage. However, if content descriptions are too detailed, they run the risk of being rigid, not providing educators with flexibility to address the needs of students with different mathematical abilities or to respond to alternative curriculum designs.

Singapore's framework meets both of these criteria well. In each mathematical content area, and for each grade, Singapore's topic matrix, part of which is shown in Exhibit 3–5, lays out the mathematical topics to be covered (see Appendix: Exhibit B3–3 for Singapore's full topic matrix over grades 1–6). The matrix emphasizes students' understanding of numbers by explicitly and

carefully breaking large mathematical contents areas, the numbers strand in this example, into several mathematical substrands: whole numbers, fractions, decimals, averages, ratios and proportions, and percentages. The topic matrix shows the logical development of mathematical content; new topics build on prior mathematical content as students progress from grade to grade. Singapore calls this process of building and deepening content over successive grades a "spiral approach" (Ministry of Education, Singapore, 2001a).

Exhibit 3–5. Singapore Topic Matrix for Numbers—Primary 1 to 4 and Primary 5 and 6 (Normal Track)

P1	P2	P3	P4	P5	P6
NUMBERS: WHOLE NUMBERS					
1. Number notation and place values up to 100 2. Cardinal and ordinal numbers 3. Comparing and ordering 4. Addition and subtraction of numbers within 100 5. Multiplication of numbers whose product is not greater than 40 6. Division of numbers not greater than 20	1. Number notation and place values up to 1,000 2. Addition and subtraction of numbers up to 3 digits 3. Multiplication and division within the 2, 3, 4, 5, and 10 times tables	1. Number notation and place values up to 10,000 2. Addition and subtraction of numbers up to 4 digits 3. Multiplication tables up to 10×10 4. Multiplication and division of numbers up to 3 digits by a 1-digit number 5. Odd and even numbers	1. Number notation and place values up to 100 000 2. Approximation and estimation 3. Factors and multiples 4. Multiplication of numbers • up to 4 digits by a 1-digit number • up to 3 digits by a 2-digit number 5. Division of numbers up to 4 digits by a 1-digit number and by 10	1. Number notation and place values up to 10 million 2. Approximation and estimation 3. Multiplication and division of numbers up to 4 digits by a 2-digit whole number 4. Order of operations	
NUMBERS: FRACTIONS					
	1. Equal parts of a whole 2. Idea of simple fractions 3. Comparing and ordering like fractions	1. Equivalent fractions 2. Comparing and ordering unlike fractions	1. Addition and subtraction • like fractions • related fractions 2. Product of a proper fraction and a whole number 3. Mixed numbers and improper fractions	1. Addition and subtraction of • mixed numbers, • unlike fractions 2. Product of fractions 3. Concept of fraction as division 4. Division of a proper fraction by a whole number	
NUMBERS: DECIMALS					
				1. Multiplication up to 2 decimal places by a 2-digit whole number 2. Multiplication and division up to 3 decimal places by tens, hundreds, thousands	1. Number notation and place values up to 3 decimal places 2. Comparing and ordering 3. Addition and subtraction up to 2 decimal places 4. Multiplication and division up to 2 decimal places by 1-digit whole number 5. Conversion between decimals and fractions 6. Approximation and estimation 10
NUMBERS: AVERAGE/RATE/SPEED					
				1. Average 2. Rate	1. Time (24-hour clock) 2. Speed
NUMBERS: RATIO/ PROPORTION					
				1. Ratio	1. Ratio and direct proportion
NUMBERS: PERCENTAGES					
				1. Concept of percentage 2. Percentage of a quantity	1. One quantity as a percentage of another

Source: Ministry of Education, Singapore, 2001

The numbers strand illustrates the logic of the spiral approach. Singapore's organization builds and deepens content for whole numbers:

- Number notation and place value are introduced in first grade, followed by the ordering of numbers and then arithmetic for the four operations. The limits of arithmetic operations are specified – addition and subtraction only of numbers under 100, multiplication not yielding answers greater than 40, and division yielding answers not greater than 20.

- In second grade, number notation extends to 1,000, addition and subtraction to three digits, and multiplication and division in the 2, 3, 4, 5, and 10 tables.

- This process of building up successively more advanced number topics proceeds through grade 5, exposing students to numbers up to 10 million. Only after a solid grounding in the fundamentals of arithmetic are estimation (grade 4) and order of operations (grade 5) introduced as topics of study.

The topic matrix for other substrands in the numbers strand displays a similarly clear order of movement diagonally from the top left to the bottom right, graphically illustrating how Singapore builds new content on content already learned:

- The concept of fractions is introduced in grade 2, equivalent and ordering of fractions in grade 3, addition and subtraction of fractions in grade 4, and multiplication and division of fractions in grade 5.

- Fraction applications, such as decimals, average rates of speed, ratio and proportion, and percentage, are introduced only after the basics are learned.

- The measurement stream, which in Singapore is part of the numbers content area, begins with simple concepts such as length, mass, time, and money and only later moves on to more complicated topics such as perimeter, area, and volume and to more complicated arithmetic such as the multiplication and division of money.

The Singapore framework provides an additional level of detail by laying out student outcomes for each topic, by grade. Simply identifying a mathematical topic is not sufficient to identify what students are expected to know about it. The topic matrix identifies comparing and ordering as a topic for Primary 1, but it also spells out expected student outcomes for that topic: students should be able to compare "two or more sets in terms of the difference in numbers" and "arrange numbers in increasing and decreasing order" (Ministry of Education, Singapore, 2001a). Exhibit 3–6 illustrates the depth of Singapore's outcome descriptions. In Primary 1, for the addition and subtraction topic in the numbers strand, students are to compare two numbers within 20 and commit to memory addition bonds up to $9 + 9$. Even in first grade, students solve word problems, although none involving more than one step.

The topic- and outcome-level descriptions in Singapore's content standards show how students' mathematical knowledge develops by presenting very specific and logically sequenced topics and outcomes. New content deepens and extends prior knowledge.

Exhibit 3–6. Singapore Topics and Outcomes for Addition and Subtraction, Primary 1

Topics	Outcomes
Addition and subtraction	Illustrate the meaning of "addition" and "subtraction." (Include comparing two numbers within 20 and finding how much greater/smaller.)
	Write mathematical statements for given situations involving addition and subtraction.
	Build up the addition bonds up to 9 + 9 and commit to memory. (Include writing number stories for each number up to 10.) Include sums such as the following: i) □ + 2 = 7 ii) 3 + □ = 12
	Recognize the relationship between addition and subtraction.
	Add and subtract numbers involving
	– 2-digit numbers and one – 2-digit numbers and tens – 2-digit numbers and 2-digit numbers
	Add 3 one-digit numbers.
	Carry out simple addition and subtraction mentally involving – 2-digit number and ones without renaming – 2-digit number and tens
	Solve 1-step word problems on addition and subtraction. (Use numbers within 20.)
Source: Ministry of Education Singapore (2001a)	

NCTM's standards, by contrast, adopt a more general, inclusive organization. NCTM organizes mathematical content in three broad grade-bands, K–2, 3–5, and 6–8, rather than grade by grade. Grade-bands allow state and local educational agencies more flexibility in identifying content suited to their own needs than Singapore's grade-specific structure does. What is left unspecified by NCTM, however, is how topics are to be presented to students within each band and in what sequence. In the 3–5 band, for example, a specific topic might be taught in third grade or it might be taught in fifth grade.

The organization of the NCTM standards also differs from the Singapore framework in that NCTM uses a set of common topics across all grades. Exhibit 3–7 shows NCTM's topic organization and outcomes for the numbers strand in the Pre-K–2 grade-band. Although the outcomes in the chart are only for grades K–2, the topics in the left-hand column are "goals that apply across all the grades" (NCTM, 2000). The three NCTM numbers goals—understanding numbers and number systems, understanding the meaning of operations, and computing fluently and estimating—are the same for every grade-band in the framework. Only the expected outcomes change. This structure, if followed faithfully, requires mathematics educators to address every topic at every grade-band level, even if this results in inappropriate sequencing and instruction in far more topics than are practicable in a single school year. Singapore's content organization, in contrast, is more flexible and adjusts, and limits, the topic categories to the mathematics appropriate for each grade.

Exhibit 3–7. NCTM Topics and Outcomes for Numbers and Operations, Grades Pre-K to Grade 2*

Instructional programs from prekindergarten through grade 12 should enable all students to	In prekindergarten through grade 2 all students should
<u>Understand numbers</u>, ways of representing numbers, relationships among numbers, and number systems	• count with understanding and recognize "how many" in sets of objects; • use multiple models to develop initial understandings of place value and the base-10 number system; • develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections; • develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers; • connect number words and numerals to the quantities they represent, using various physical models and representations; • understand and represent commonly used fractions, such as 1/4, 1/3, and 1/2.
<u>Understand meanings</u> of operations and how they relate to one another	• understand various meanings of addition and subtraction of whole numbers and the relationship between the two operations; • understand the effects of adding and subtracting whole numbers; • understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally.
<u>Compute fluently</u> and make reasonable estimates	• develop and use strategies for whole-number computations, with a focus on addition and subtraction; • develop fluency with basic number combinations for addition and subtraction; • use a variety of methods and tools to compute, including objects, mental computation, estimation, paper and pencil, and calculators.
*For the full set of NCTM standards Pre-K-2, see Appendix, Table B3–4.	

Although a common topic structure might seem to simplify the framework presentation, it has several effects that make it an unwieldy means for organizing content. First, core mathematical concepts that Singapore's framework groups together are scattered across more than one NCTM category. For example, NCTM's Pre-K–2 framework treats addition and subtraction in both the "understand numbers" category and the "compute fluently" category.

A second problem is that a single NCTM category may include a large number of topics. For example, the category "understand meanings" lumps multiplication and division in with addition and subtraction. Singapore, in contrast, treats addition, subtraction, multiplication, and division as fundamentally different categories, as seen in Exhibits 3–4 and 3–5. Singapore's organization of topics in the framework around mathematical concepts makes it easier for users to grasp what students are to learn about core mathematical ideas than does the organization of the NCTM standards.

A third problem with the organization of the NCTM framework is that it describes student outcomes in a much more general way than Singapore's framework does and uses less precise

mathematical terminology. For example, students are to "use multiple models to develop initial understandings of place value," but which models and understandings are not specified. Students are to "develop a sense of whole numbers and represent and use them in flexible ways," but it is unclear what this means. It is also unclear how states and communities are to respond to statements like these in determining instructional content. NCTM is forced into these general descriptions in part because their outcome descriptions must be applicable across multiple grades, which makes more precise statements impractical.

Because topics are mapped out in such a general way, the NCTM requirements risk exposing students to unrealistically advanced mathematics content in the early grades. NCTM's Pre-K–2 algebra topics, shown in Appendix: Exhibit B3–2, indicate that students should *understand* patterns, relations, and functions; *represent* and analyze mathematical situations and structures using algebraic symbols; *use* mathematical models to represent and understand quantitative relationships; and *analyze* change in various contexts, all in the very early grades. Students are exposed to these complicated mathematical topics in kindergarten and first grade at the same time they are learning basic addition and subtraction. Singapore, in contrast, considers algebraic concepts to be advanced rather than introductory mathematics, so algebra is not introduced until the sixth grade. Overall, NCTM's grade-band structure lacks the specificity needed to serve as a solid national content benchmark. The specificity and logic in Singapore's spiral approach offer a more effective, better-sequenced framework for a mathematical curriculum.

The seven state's frameworks further complicate matters in that they differ widely in how they organize content. Some use an organization similar to Singapore's, whereas others look more like the NCTM framework. Still others lack much obvious organization at all. The California, Maryland, and Florida frameworks are good examples of the range of approaches to content organization that the selected states take, as revealed by an examination of how they approach the numbers strand.

The California mathematics framework is modeled on Singaporean and Japanese frameworks. It is similar to the Singapore framework in that it is organized around a varying set of mathematical topics appropriate to the grades in which they are taught. Exhibit 3-8 shows the portion of the California standards that addresses content in the numbers strand. The numbers standard for grade 1 comprises three mathematical topics: understanding and using numbers; using addition and subtraction to solve problems; and using estimation for numbers involving the ones, tens, and hundreds places. This example reveals key points about how the California standards are organized:

- As in the Singapore framework, topics and outcomes are described in very specific mathematical terms.

- Content in grade 1 is limited to two-digit numbers. As in the Singapore framework, descriptions of content are precise and make limits clear.

- Addition and subtraction are grouped in one section, as in the Singapore framework.

- The 13 outcomes in Exhibit 3–8 show that students are expected to learn a balanced set of introductory numeric processes that include understanding of concepts, basic computation, memorization, and estimation. These expectations are similar to those expressed in the Singapore framework.

Exhibit 3–8. California's Number Strand, Grade 1

Topics	Outcomes
Students understand and use numbers up to 100.	1.1 Count, read, and write whole numbers to 100.
	1.2 Compare and order whole numbers to 100 by using the symbols for less than, equal to, or greater than (<, =, >).
	1.3 Represent equivalent forms of the same number through the use of physical models, diagrams, and number expressions (to 20) (e.g., 8 may be represented as 4 ± 4, 5 ± 3, $2 + 2 + 2 + 2$, $10 - 2$, $11 - 3$).
	1.4 Count and group object in ones and tens (e.g., three groups of 10 and 4 equals 34, or 30 ± 4).
	1.5 Identify and know the value of coins and show different combinations of coins that equal the same value
Students demonstrate the meaning of addition and subtraction and use these operations to solve problems.	2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.
	2.2 Use the inverse relationship between addition and subtraction to solve problems.
	2.3 Identify one more than, one less than, 10 more than, and 10 less than a given number.
	2.4 Count by 2s, 5s, and 10s to 100.
	2.5 Show the meaning of addition (putting together, increasing) and subtraction (taking away, comparing, finding the difference).
	2.6 Solve addition and subtraction problems with one- and two-digit numbers
	2.7 Find the sum of three one-digit numbers.
Students use estimation strategies in computation and problem solving that involve numbers that use the ones, tens, and hundreds places.	3.1 Make reasonable estimates when comparing larger or smaller numbers.
Source: California State Department of Education (1999)	

Maryland's standards are more similar to the NCTM standards in that Maryland organizes content around a standard set of mathematical categories across elementary grades. Exhibit 3–9 shows the number computation category for grades 1 and 2. This organization has some of the same problems seen in the NCTM framework. Maryland's standards tend to repeat the same content in slightly different ways to ensure that content in each category is taught at each grade level, producing excessive repetition. In grade 1, for example, students "identify the concept of inverse operation to addition and subtraction"; in grade 2, students "apply the concept of inverse operation to addition and subtraction." Basically, in both grades 1 and 2, students cover the inverse relationship between addition and subtraction, whereas the Singapore framework covers this topic only once, in first grade. This allows Singaporean students to move on to the next step, whereas students in Maryland repeat what they have already learned. Overall, there does not appear to be a compelling advantage to the

Maryland common-topic organization, and there are some clear disadvantages, excessive repetition being the most obvious.

Exhibit 3–9. Maryland's Number Topic and Outcomes, Grades 1 and 2

Grade 1	Grade 2
C. Number Computation 1. Analyze number relations and compute a. Develop strategies for addition and subtraction basic facts such as: counting on, counting back, making ten, doubles, and doubles plus one. b. Solve a given word problem based on addition or subtraction situation c. Identify the concept of inverse operation to addition and subtraction	C. Number Computation 1. Analyze number relations and compute a. Demonstrate proficiency with addition and subtraction basic facts using a variety of strategies b. Add no more than 3 whole number addends with no more than 2 digits in each addend and a sum of no more than 100 c. Subtract whole numbers with no more than 2 digits in the minuend or the subtrahend d. Solve word problems based on addition or subtraction situations e. Write word problems for addition and subtraction situations f. Add and subtract money amounts up to $1 g. Apply the concept of inverse operations to addition and subtraction h. Build equal groups to model multiplication i. Build groups that share equally for division 2. Estimation a. Determine the reasonableness of sums and differences
Source: Maryland Voluntary State Curriculum – Mathematics PreK–3. Available at http://mdk12.org/mspp/standards/math/index.html	

Maryland also presents all four mathematical operations in its "number computation" category; Singapore breaks out addition and subtraction from multiplication and division, an approach that more clearly shows the reciprocal mathematical relationship for each critical pair of mathematical operations. In other respects, however, Maryland's statements about content are specific, much like Singapore's concrete descriptions of intended mathematical content. Grade 2 Maryland students, for example, are asked to add no more than three whole numbers of one or two digits that have sums of less than 100. In grade 3, they progress to sums to 1,000 and add three-digit numbers. In essence, Maryland keeps track of its requirements for the different operations and builds them across the grades in a manner similar to Singapore's spiral approach.

Florida, unlike California and Maryland, provides no clear organizational scheme for topics within a mathematical content area. Instead, Florida simply presents one very long, unstructured list of mathematical outcomes without any apparent topic groupings. Exhibit 3–10 shows an unstructured list of 30 outcomes for grade 1 in Florida's numbers strand. These 30 outcomes compare with only 13 in California. One reason for the large number of outcomes is that 10 of 30 address students' ability to demonstrate mathematical understanding through the use of concrete materials or manipulatives. In Florida, the manipulation of concrete objects seems to become an end in itself rather than a means to understand abstract mathematical concepts.

Exhibit 3–10. Florida's Number Strand, Grade 1

Number Sense, Concepts, and Operations

- uses one-to one correspondence to count objects to 100 or more.

- reads and writes numerals to 100 or more.

- uses ordinal number 1th – 10th or higher.

- compares and orders whole numbers to 100 or more using concrete materials, drawings, number lines, symbols (<, =, >), and vocabulary such as equal to, more than, or less than.

- represents real-world applications of whole numbers, to 100 or more, using concrete materials, drawings, and symbols.

- represents, explains, and compares fractions (one half, one fourth, three fourths) as part of a whole and part of a set using concrete materials, drawings, and real-life situations.

- knows that the total of equivalent fractional parts makes a whole (for example, two halves equal one whole).

- represents equivalent forms of the same number, up to 20 or more, through the use of concrete materials (including coins), diagrams, and number expressions (for example, 16 can be represented as 8+8, 10+6, 4+4+4+4, 20-4, 17-1).

- counts orally to 100 or more by 2s, 5s, and 10s with or without a hundred chart.

- uses concrete materials, pictures, and symbols to show the grouping and place value of numbers to 100 or more.

- counts forward and backward by one beginning with any number less than 100.

- counts forward by tens from any number less than 10 using a hundred chart.

- knows place value patterns and uses zero as a place holder (for example, trading 10 ones for 1 ten).

- knows the place value of a designated digit in whole numbers to 100.

- demonstrates knowledge of the meaning of addition (putting together, increasing) and subtraction (taking away, comparing, finding the difference) using manipulatives, drawings, symbols, and story problems.

- solves basic addition facts using concrete objects and thinking strategies, such as count on, count back, doubles, doubles plus one, and make ten.

- describes the related facts that represent a given fact family up to 18 (for example, 9+3=12, 12-9=3, 12-3=9).

- knows how to use the commutative and associative properties of addition in solving problems and basic facts.

- adds and subtracts two-digit numbers without regrouping (sums to 100) using models, concrete materials, or algorithms.

- poses and solves simple number problems by selecting the proper operation (for example, finding how many students are sitting at tables one and two).

- uses concrete objects to solve number problems with one operation.

American Institutes for Research®

Exhibit 3–10. Florida's Number Strand Grade 1 (Continued)

Number Sense, Concepts, and Operations

- describes thinking when solving number problems.

- writes number sentences associated with addition and subtraction situations.

- knows appropriate methods (for example, concrete materials, mental mathematics, paper and pencil) to solve real-world problems involving addition and subtraction.

- uses a calculator to explore addition, subtraction, and skip counting.

- uses the language of estimation and approximation to identify and describe numbers in real-world situations (for example, about, near, closer to, between).

- estimates the number of objects, explains the reasoning for the estimate, and checks the reasonableness of the estimate by counting.

- makes reasonable estimates when comparing larger or smaller quantities.

- estimates reasonable answers to basic facts (e.g., Will 7+8 be more than 10?).

- demonstrates and builds models to show the difference between odd and even numbers using concrete objects or drawings.

Source: Florida Department of Education (2004). Available at http://www.firm.edu/doe/menu/sss.htm

---

Perhaps because Florida's content organization lacks a clear rationale and structure, it introduces many advanced topics much earlier than Singapore does. First graders in Florida cover odd and even numbers, which Singapore does not introduce until third grade. Similarly, Florida's first graders are exposed to symbolic notation for greater than and less than, which Singapore does not cover until middle school. Treating these advanced topics in the first grade takes time away from the essential introductory mathematics that students need to learn first. Further, these topics may be impossible to teach well, given that students who do not know basic mathematical concepts may be unable to understand them.

Taken together, the three state examples represent three very different approaches to organizing mathematical content. These differences help characterize U.S. mathematical weaknesses and strengths. Singapore's grade-specific, mathematically logical spiral organization has much to offer compared with NCTM's grade-band structure, but simply organizing state frameworks by grade is not enough. Florida's standards, although organized by grade, offer little of Singapore's obvious care and attention to sequencing and to the number of topics that students can reasonably learn in a year. Maryland's use of common topic categories may not be conducive to teaching the topics best suited for each grade. California's approach, however, is similar to Singapore's spiral organization of topics and has the advantage of reducing the number of topics that students are expected to learn. Both California's and Maryland's structures are preferable to Florida's apparent lack of structure, but none of the state frameworks examined quite reaches Singapore's level of care and planning.

CONTENT COVERAGE

Number of Topics and Outcomes

The number of mathematics topics and outcomes covered in the various frameworks is an indicator of the amount of mathematics that students are expected to learn in the elementary grades. If Singapore's framework presents enough content to allow students to learn core mathematical ideas with depth and understanding, and Singapore's performance on TIMSS in 1995, 1999, and 2003 suggests that it does, then U.S. frameworks should not excessively exceed Singapore's in content coverage if we wish to match Singapore's success. We examined Singapore's framework and the selected state frameworks to see how they compare in terms of numbers of topics and outcomes included for each grade level.[3]

We followed a specific methodology in comparing topics and outcomes. We looked first at Singapore's Primary Mathematics Syllabus. This document defines the content that students are to learn in grades 1 through 6, grouping content by grade level. It further divides content areas, such as number, into topics such as number notation and place value, addition and subtraction, and word problems. Specific outcomes are then listed for student achievement. The Primary Mathematics Syllabus identifies 93 topics that are subdivided into about 235 outcomes that students are expected to achieve by the end of Primary 6.

Because many topics build from grade to grade and because outcomes are often taught in different grades in Singapore than in the selected states, we elected to make our comparisons on the basis of outcomes grouped by topics rather than by grade levels. We grouped Singapore's outcomes by content area and topic. We then combined all outcomes for a given topic from across the grades so that all topics within a content area were grouped together and so that topics repeated in multiple grades were treated as a single topic. All outcomes were listed for each topic, by grade. This organization shows the progression of content and the development, or lack thereof, of mathematical understanding across the grades.

To present a clearer mathematical picture, we combined topics within content areas when they were closely related. For example, money, addition and subtraction of money, and multiplication and division of money were combined to form one topic. In this way, the 93 Singapore topics were combined into 40 grouped topics. Separate charts were developed for each state, comparing its framework with Singapore's. Each chart matches the content from the state's framework to a common Singapore listing of topics and outcomes.

Exhibit 3–11 summarizes the comparisons of numbers of topics and outcomes addressed in Singapore and the seven selected states. Singaporean students in grades 1–6 are exposed to 40 mathematical topics. Three U.S. states have similar topic exposure: California has 42 topics, North Carolina has 41, and Texas has 40. California's close correspondence to Singapore is not surprising given that it used the Singapore framework as a model in its recent revision of its standards. The relatively curtailed number of topics covered in North Carolina and Texas is also unsurprising, given that these two states are among the few that have aligned high-stakes assessments to their standards for more than a decade (Grissmer, et al., 2000). Because students are assessed on the content in the

---

[3] Because NCTM's grade-band structure makes it difficult to determine the grade levels at which content is taught and because NCTM's generalized statements about content make it difficult to compare content coverage, we have not included NCTM standards in this comparison.

standards, these states have had a greater incentive than others to ensure that their standards do not expose students to an excessive amount of content. Another possible explanation for the relatively low number of topics covered in these three states is that all three organize their frameworks around a topic structure that varies with the mathematics taught at each grade instead of using a structure that fixes topics across grades. Tying the organization of content to the natural progression of mathematics may discourage introducing advanced topics early and curtail excessively repeating topics across grades, both of which drive up content exposure.

Exhibit 3–11. Analysis Comparing Singapore and U.S. Content Exposure: Topics and Outcomes: Grades 1–6

	Organization of Standards Across Grades, Within a Strand (1)	Total No. of Topics (2)	Avg. No. of Grades/Topic		Avg. No. of Topics/Grade		Total No. of Out-comes (7)	Avg. No. of Outcomes/ Grade	
			No. (3)	Ratio to Sing. (4)	No. (5)	Ratio to Sing. (6)		No. (8)	Ratio to Sing. (9)
Singapore	Variable Topics	40	2.3	—	15	—	232	39	—
California	Variable Topics	42	2.9	1.3	20	1.3	305	51	1.3
Florida	No Topic Organization	54	4.2	1.8	39	2.6	640	107	2.7
Maryland	Fixed Topics	46	3.8	1.7	29	1.9	415	69	1.8
New Jersey	Fixed Topics	50	3.4	1.5	28	1.9	336	56	1.4
N. Carolina	Variable Topics	41	2.6	1.1	18	1.2	217	36	.9
Ohio	Fixed Topics	48	3.3	1.4	26	1.7	370	62	1.6
Texas	Variable Topics	40	2.8	1.2	19	1.3	265	44	1.1

At the high end in topic exposure is Florida with 54 topics in grades 1–6. Florida students are expected to learn one-third more mathematical topics than their peers in Singapore, California, North Carolina, and Texas. Ohio, New Jersey, and Maryland fall in the mid-range among states in terms of topic coverage, exposing students to about one-fifth to one-fourth more topics than Singapore does.

We also computed the average number of grades in which each topic is taught, an indicator of the duration of topic exposure (columns 2 and 3, Exhibit 3–11). Repeating topics across grades helps ensure that students practice the topic enough to learn it well, but repetition also reduces the instructional time available to learn new content. It also appears to add significantly to the total number of topics to which students are exposed in each grade. Singapore's average duration, 2.3 grades per topic, is lower than the average for any of the states, but the three states closest to Singapore in topic duration are the same three most like it in number of topics covered. Topic duration is 2.9 grades in California, 2.6 grades in North Carolina, and 2.8 grades in Texas. Florida is again the state most unlike Singapore, with a topic duration of 4.2 grades, 80 percent greater than in Singapore. Maryland, New Jersey, and Ohio are again in the mid-range, with a topic typically repeated between 40 to 70 percent more than in Singapore.

Combining the information on the number and duration of topics yields an estimated average number of topics taught per grade (columns 5 and 6). At the low end of the range, North Carolina teaches about 20 percent more topics per grade than Singapore. California and Texas cover about 30 percent more topics than Singapore. At the high end, Florida's framework covers 160 percent more topics than Singapore's. This is a huge difference in content exposure; it is hard to imagine how Florida's teachers can teach so many more topics with close to the depth that Singapore achieves.

Exhibit 3–11 also compares the total number of student outcomes in all grades with the average number of outcomes per grade (columns 7–9). Singapore identifies 232 distinct mathematical outcomes over the course of six grades, averaging 39 outcomes per grade. North Carolina covers only 217 outcomes over grades 1–6, an average of 36 outcomes per grade. California and Texas are again among the states most like Singapore in number of topics; Florida is at the other extreme with 107 outcomes per grade, or 2.7 times Singapore's number. Maryland, New Jersey, and Ohio are again in the mid-range of states on exposure in terms of student outcomes.

Our results show a clear and very interesting pattern: the amount of mathematical content that state standards expect students to learn is associated with the structures that states use to organize their standards (column 1 of Exhibit 3–11). The three states most like Singapore in having relatively low topic and outcome exposure are the same ones that use a variable structure in which the topics reflect the mathematical content emphasized at each grade. The middle group of states in terms of content exposure relies, for the most part, on a fixed topic organization that uses common mathematical topics across grades. A common topic structure across grades may encourage introducing some topics early and retaining topics later in order to ensure that each topic is addressed at each grade level. Florida, at the high end in content exposure, has no internal topic organization in its mathematical strands. This lack of organization apparently inclines Florida's standards toward excessive content inclusion.

Overall, Singapore's framework has significantly fewer topics, and repeats its topics fewer times, than four of the seven selected states. Consequently, students in these four states probably spend less time on each topic in a given year, which likely makes the coverage they do receive more superficial than what Singaporean students receive. If Singapore has established a good balance between breadth and depth of content as demonstrated by its high student performance, one would suspect that the three states whose standards most resemble Singapore's would also perform well. Although California's framework is too new to profitably assess its impact on student outcomes, North Carolina and Texas have been singled out as the states that have improved their mathematics scores the most during the last decade, after adjusting for population characteristics (Grissmer, et al., 2000). Other states would do well to reexamine the scope of the content they expect students to learn to assess whether mathematical content overload is occurring, because it is unlikely that U.S. students can learn significantly more mathematical outcomes than Singaporean students.

Mathematics Topics Not Covered by Frameworks

In addition to looking at content that Singaporean and state frameworks cover, it is instructive to look at what they choose not to cover. Singapore's framework includes in the primary grades some mathematical topics that the states' frameworks consistently do not cover, and vice versa. There are reasons to believe that not covering these topics tells us something about what Singapore and the United States value in mathematics instruction.

To identify topics that Singapore, but not the states, covers, we first compared the list of Singapore's topics with each state's list of topics and counted all the places where they matched. Once we had counted the matches, we computed the percentage of Singapore topics that had no counterparts in state frameworks and the percentage of each state's topics that had no counterpart in Singapore's framework; these percentages appear in Exhibit 3–12.[4]

---

[4] This topic matching is not as precise as it may appear to be. Because of differences in how the same or similar topics might be characterized in different frameworks, we had to estimate whether topic coverage overlapped.

Exhibit 3–12. Comparison of the Percentage of Mathematical Topics Unique in Singapore and State Mathematical Frameworks

State	Percent of Singapore Topics Only in Singapore and Not in the State Framework (%)	Percent of State Topics Only in the State Frameworks and Not in Singapore (%)
California	10	14
Florida	3	28
Maryland	10	22
New Jersey	8	26
N. Carolina	13	15
Ohio	3	19
Texas	15	15

Given that the state frameworks cover more mathematical content than Singapore's framework, it is not surprising that most of the selected states cover at least 90 percent of Singapore's topics. The state frameworks, given their expanded topic coverage, include a higher percentage of mathematical topics not covered in Singapore's framework. Once again, North Carolina and Texas most closely resemble Singapore in content coverage, and Florida is the most different, matching Singapore's topics only 72 percent of the time. If Singapore's content is adequate to achieve world-class standards, the large number of Florida's mathematical topics may be diluting the curricula.

We also identified the specific topics found in the Singapore standards but not in a majority of the selected state standards and the topics present in a majority of the state standards but not in the Singapore standards. These topics are displayed in Exhibit 3–13. Singapore's use of multistep word problems stands out as an area that Singapore emphasizes and the states do not. Singapore also covers several more advanced content areas, including arithmetic computations on physical quantities and exposure to advanced geometric concepts, including cuboids, tessellations, and nets, that are not present in the states' frameworks.

The topics found only in the U.S. standards reflect greater attention to real-world applications and include statistics topics, such as data analysis and probability, and geometry topics, such as geometric coordinate systems and congruence and similarity. The states also cover patterns, a subtopic of algebra, in the early grades, whereas Singapore does not.

The prominence that Singapore gives to multistep word problems in its standards reflects its focus on understanding mathematical concepts for the purpose of problem solving. The states' inclusion of data and probability is consistent with the strong U.S. focus on 21st century skills. Overall, Singapore's inclusion of multistep word problems is an essential means of developing students' conceptual understanding and would be a welcome addition to the U.S. topic list. At the same time, the U.S. emphasis on data analysis and probability is consistent with its interest in real-world mathematical applications. This interest is something that Singapore would do well to adopt.[5]

---

[5] Although we were unable to systematically examine the overlap between the NCTM and Singapore frameworks, we do not find in NCTM a strong and clear emphasis on multistep word problems, and we do find a definite emphasis on data analysis and probability similar to what appears in the individual state frameworks.

Exhibit 3–13. Differences in Mathematical Content in Singapore (Primary 1–6) Standards and in a Majority of the State Standards

Content Area/Strand	Mathematical Content Only in Singapore Standards	Mathematical Content Only in State Standards
Numbers and Operations	• Multistep word problems (fractions, decimals) • Division of a proper fraction by a whole number • Percentage of one quantity as a percentage of another	• Properties of arithmetic operations
Measurement	• Time, speed and distance (computations and word problems). • Arithmetic manipulation of physical quantities (weight, length, mass, volume, money, compound units) • Unknown dimensions of rectangle • Volume of a cuboid	• Temperature
Statistics		• Statistical data analyses (data collection, analysis, prediction) • Probability (event likelihood, experiments)
Geometry	• Angles (sums, unknowns) • Tessellation and nets	• Coordinate systems (maps, distance, movement) • Congruence and similarity
Algebra	• Multistep word problems	• Patterns (sounds, shapes, numbers, repeating)

ADDRESSING EQUITY: CURRICULUM STANDARDS AND SUPPORT FOR THE SLOWER MATHEMATICS STUDENT

The topic structure in Singapore's framework is efficient because topics are not taught and retaught as students move through the primary grades. Instead of repeating topics that students have already learned, teachers simply reintroduce them as a foundation on which to build new mathematical content. This practice, however, may not be suitable for students who have more difficulty with mathematics. The Singapore system recognizes that students who have trouble with mathematics may not attain mastery by following Singapore's regular program of mathematics instruction and that these students may need special assistance to attain competence.

Beginning in grades 5 and 6, Singapore identifies its weaker students on the basis of a general examination of mathematics and language competency. These students receive special assistance and are taught according to a special fifth- and sixth-grade mathematics framework. This special framework mandates that students in the slower track

- receive approximately 30 percent more mathematics instruction than students in the regular track, and

- be exposed to the same mathematical content as students in the regular track, although at a slower pace.

The mathematics framework for students needing compensatory assistance adds review material to strengthen students' understanding of previously taught content. For example, topics on numbers and geometry taught in grade 4 are repeated at a faster pace in grade 5. The introduction of some new concepts such as ratios, rates, and averages, which are normally introduced in grade 5, are delayed until grade 6 for the weaker students (Ministry of Education, 2001a). What is important, however, is that because slower students spend extra time studying mathematics, topics usually taught in grades 5 and 6 do not have to be completely sacrificed to make room for repetition.[6]

To support the framework for slower students, Singapore has developed a *Learning Support Program* to help educators identify these students and provide them with extra help (Ministry of Education 2003c). Mathematics Support Teachers (MST), who receive on-the-job supervision and specialized training to ensure that they are professionally competent, deliver compensatory assistance.

In the United States, we expect all students to meet the standards in state frameworks, but the standards do not help teachers address the needs of slower students. In fact, U.S. standards do not acknowledge that students learn at different rates. *No Child Left Behind* addresses the needs of failing schools, but it does not directly require that failing students receive help. Although some research evidence supports the belief that students benefit when the curriculum is adjusted to match their ability levels (Loveless, 1999), a distinct alternative curriculum would raise concerns in the United States about potential harm to students from ability grouping. Singapore's approach differs from traditional ability grouping in that Singapore establishes a framework that requires students to master the same content as other students, not a watered-down curriculum as often happens in U.S. ability-grouped classrooms. Singapore also provides extra assistance from an expert teacher.

CONCLUSIONS

Singapore's framework offers a highly logical mathematical approach to setting mathematical priorities and content across the primary grades. This approach balances conceptual, computational, and strategic problem-solving skills. Mathematical topics and outcomes are precisely specified, and content is sequenced across grades with a spiral approach that limits topic repetition. The content that students are expected to learn in grades 1–6 establishes a strong foundation in numbers while also developing basic geometric and statistical concepts. Compared with the NCTM framework and the standards from the United States, Singapore's unique emphasis on multistep word problems, in particular, is consistent with its emphasis on promoting conceptual understanding through solving thoughtful problems. Singapore also provides a special framework for students who are slower mathematically that is backed by policies that provide the slower students with special assistance from expert teachers.

---

[6] Even though the special primary grade mathematics framework for slower students covers only grades 5 and 6, Singapore provides special mathematics services for slower students much earlier. Students take a readiness test of numeracy at the end of grade 1. Students whose scores qualify them for assistance receive an additional 2.5 to 3 hours of support each week in small groups, outside regular class time. Instruction covers the same topics taught in the regular class but at a pace suitable for students who learn mathematics more slowly.

Although Singapore's framework is in many ways superior to frameworks used in the United States, the U.S. frameworks do have some features that are preferable to features in Singapore's framework. The separate measurement strand and the emphasis on data analyses and statistics in the NCTM and state standards, which stress important mathematical skills that students need to develop in a digital world, elevate the importance of applied mathematics.

The state frameworks that we examined are more diverse than one would expect, given the consistent complaint that the U.S. mathematics system exposes students to too much content. Three states—California, North Carolina, and Texas—were similar to Singapore in topic and outcome exposure, and their frameworks organize content as Singapore does, by mathematical concepts appropriate to each grade. Two of these states, North Carolina and Texas, have improved their mathematics scores significantly during the last decade. Although there is no way to isolate the impacts of their mathematical frameworks, the empirical evidence suggests that having a framework that is focused and mathematically logical can help students perform more successfully.

These findings suggest the following actions, which should be considered as ways to strengthen U.S. mathematics frameworks by having them adopt features similar to the effective features of the Singapore framework, while building on their own existing strengths.

1. NCTM and the states should consider strengthening overarching process priorities like reasoning and making connections across mathematical topics. Doing so would better show how the process priorities apply to solving problems in the different content strands (e.g., numbers, measurement). A good example of this sort of application is the work that the College Board's SpringBoard project is doing to link NCTM process standards with NCTM content standards for grades 6–12. Under each of the five NCTM content strands, the project will identify outcomes for each one of the mathematical processes. For example, skills and concepts in geometry are distributed across problem solving, communication, representations, reasoning, and connections process priorities so that the processes are integrated with the mathematical content. Information about the project can be found at http://www.collegeboard.com/.

2. NCTM and the states should consider better balancing the mathematical processes in their standards by adding an additional process standard for addressing "computational fluency." Research supports developing students' ability to compute quickly and automatically so that the arithmetic does not get in the way of their focus on reasoning through the solutions to challenging mathematics problems (NRC, 2001).

3. NCTM and the states should consider strengthening the organization of their content standards in two ways:

   • By organizing them grade by grade rather than by grade-bands. The grade-bands cover too broad a content range to provide curriculum guidance for textbooks, instruction, or assessment design. Moreover, because NCLB requires grade-by-grade assessments, communities need to know what students are expected to learn.

   • By reorganizing their frameworks around mathematical concepts that are specific to each grade rather than uniform across grades. States that use a uniform topic structure, or worse, just list mathematical topics in an unstructured way, expose students to far more content in a grade than Singapore does, which encourages superficial coverage.

4. States with frameworks that have numbers of topics and outcomes substantially in excess of Singapore's frameworks should consider tightening up their standards. They should eliminate content that is inappropriate to a grade as well as excessive repetition of content across grades. State frameworks should, however, retain their emphasis on data analysis and probability, which is largely absent in Singapore's framework.

5. The U.S. federal government should consider strengthening NCLB so that students who are failing in mathematics receive the same type of high-quality assistance that Singapore provides its students who perform poorly in mathematics. NCLB currently initiates special actions for failing schools but has no similar provisions for individual students. States should consider formally adopting mathematical frameworks and supporting policies, like Singapore's, that provide slower students with additional class time in which to learn core introductory mathematics content.

6. The U.S. federal government, through NCLB, should consider developing and using a national mathematics framework. Creating and administering national mathematics standards would be new roles for the federal government, but they may be roles essential to improving the U.S. mathematics instruction system. Some Singaporean experts point to their national syllabus as the biggest system difference between their system and that of the United States. Further, it is hard to imagine how all 50 state systems can create similar high-quality mathematics frameworks without some central agreement on what content it is important to teach. With a few exceptions, there was broad overlap between Singapore's content and the content in a majority of U.S. states, which indicates that some consensus about what mathematics content is important may already have been reached. The overlap may make the creation of national mathematics standards more viable than previously thought. The development of a national mathematics framework could be part of NAEP (Finn and Hess, 2004) or developed by the National Academy of Sciences, which has already developed national science education standards (1996).

CHAPTER 4. SINGAPORE AND U.S. MATHEMATICS TEXTBOOKS

CONTENT AND METHODOLOGY

Although the mathematics content spelled out by standards and frameworks is a significant determiner of the content instruction for teachers, on a day-to-day basis the content of textbooks is more important (McKnight et al., 1987; Tyson and Woodward, 1989). It is the textbooks that transform the mathematical priorities, topics, and outcomes identified in the mathematics frameworks into a curriculum.

A good elementary mathematics textbook should provide rich mathematical content that is both aligned with the content in the frameworks and presented using sound pedagogical approaches (AAAS, 2000). It should be constructed so that students come out of the course understanding not only the mechanics of mathematics problem solving but also the mathematical concepts themselves. The textbook should also provide students with sufficient exercises so that they can practice the mathematical concepts they have learned.

The appearance of the Singapore mathematics textbooks does not initially inspire confidence that they can meet these standards. Unlike most traditional American mathematics texts, the Singapore books are paperbacks, with simple sketch drawings and no photographs. Only a sketch of a smiling boy or girl, for example, accompanies an idea box that provides hints on problem solving. To Americans, the Singapore books might look like American texts from the 1950s. Singapore texts for each grade consist of two textbooks and two workbooks, each about 125 pages and costing about $2 to $3 per book in Singapore, about one-fifth the price charged for most U.S. textbooks. The Singapore textbooks have none of the special features of American textbooks: there are no goal statements for learners at the beginning of lessons; no sections that explore relevance and applications; and no test questions organized to prepare students for state assessments, to promote reasoning, or to help students make real-world applications of mathematics content.

Instead, the Singapore texts offer clear and straightforward presentations of the mathematical concepts and topics outlined according to the national framework. The books also provide numerous problem sets. Textbooks explain mathematical concepts primarily through problems that illustrate concepts from a variety of different perspectives. This approach accords with research affirming the value of problem-based learning, which requires students to work through extensive problem sets that include routine and nonroutine applications in a wide variety of real-world contexts (Bransford, Brown, and Cocking, 2000).

The Singapore textbooks also feature mathematical explanations that begin with physical examples or picture representations and only later build up to more abstract concepts, a technique particularly helpful to students who have difficulty understanding abstract mathematics (Devlin, 2000; Maccini & Gagnon, 2000). The Singapore textbooks use pictures to develop heuristics that are particularly useful in helping students visualize how to break down and attack complex problems.

For U.S. textbook publishers, textbook design is more difficult than it is for the Singaporean publishers because U.S. publishers cannot organize their textbooks around a single mathematical framework. They must, instead, accommodate the standards of 50 states. Textbooks in the United States have been criticized for being thick in pages but thin in mathematics content, but publishers

have little choice because of the need to cover the mathematical topics from standards in a multistate market (Tyson-Bernstein, 1988). Although we have seen that the commonly held idea that the U.S. mathematics curriculum is a mile wide and an inch deep is not always true for U.S. frameworks, the textbook publishers' need to meet the curriculum needs of multiple states makes the criticism more likely to be true of U.S. textbooks. In general, the U.S. textbooks do a much worse job than the Singapore textbooks in clarifying the mathematical concepts that students must learn. Because the mathematics concepts in these textbooks are often weak, the presentation becomes more mechanical than is ideal. We looked at both traditional and nontraditional textbooks used in the United States and found this conceptual weakness in both.

This chapter examines textbooks from Singapore and the United States to develop a clearer picture of just how different they are in structure, approach, and content. The selection of the Singapore textbook for comparison was a simple choice: we examined the textbooks used by the U.S. school systems that participated in the Singapore mathematics pilot. *Primary Mathematics*, published in 2003 by the Curriculum, Planning and Publishing Division of the Singapore Ministry of Education and Times Media Private Limited, consists of two textbooks and two workbooks for each grade. Each textbook and workbook consists of approximately 100 pages. In general, we found the Singapore texts to be strong in their treatment of mathematical concepts and provide rich problem sets that give students many and varied opportunities to apply the concepts they have learned. If the Singapore texts are weak in any way, it is that the problems they provide, although numerous and varied, are not often based on real-world data and information, making the mathematics perhaps less applicable to real life than it should be.

School systems in the United States use many different mathematics textbooks, and we were in no position to look at all of them. Instead, we chose to look at two widely used textbooks. First, we looked at the Scott-Foresman Addison-Wesley Mathematics (2004) textbook series for grades 1–6. We chose this series because it is one of the four major traditional basal textbooks. Such traditional textbooks are currently used in approximately 80 percent of elementary classrooms in the United States (Malzahn, 2002). Relative to Singapore, we found this textbook series to be weak in almost every respect. Although the books present important mathematical concepts, they do so in a way that encourages a mechanical rather than a conceptual understanding of the content. The problem sets provide opportunities for students to plug what they have learned into exercises, but the exercises do not allow students to deepen their conceptual understanding of the material.

We also looked at the *Everyday Mathematics* textbook series (Everyday Learning Corporation, 2001) for grades 1–6. This series is a widely used nontraditional mathematics curriculum. Nontraditional mathematics programs emphasize the ability to reason and solve problems, and they reflect a constructionist view of learning, which posits that when children are given appropriate experiences, they have the ability to construct their own knowledge. *Everyday Mathematics* is driven by a comprehensive *Teacher's Manual and Lesson Guide* and supported by a *Student Journal* that serves as a workbook. Nontraditional texts like *Everyday Mathematics* are currently used in approximately 20 percent of elementary classrooms (Malzahn, 2002). *Everyday Mathematics* uses problem-based learning methods, so the problem sets are strong, especially in terms of how students are asked to apply mathematics to real-world situations, the one area in which the Singaporean texts are weak. However, although the problems are strong, the conceptual set up for the problems is weak. The texts do a relatively poorer job of systematically developing mathematical concepts.

We drew these conclusions about the three sets of textbooks on the basis of how they cover and present mathematical content at the textbook, lesson, and problem levels:

- At the **textbook** level, we examined the overall treatment of content in terms of how the textbooks organize content by chapter, lesson, and page. We analyzed the treatment of specific mathematical content in terms of how intensely the textbooks cover different content areas and topics. These comparisons address the emphasis and focus the textbook gives to different instructional content. We also looked at whether textbooks presented content in extended, in-depth lessons or in brief units that only superficially touched on concepts.

- Next, at the **lesson** level, we compared lessons or sets of lessons within the textbooks that focused on particular, important mathematical concepts. We examined lesson content for structure, coherence, and depth of mathematical content.

- Finally, at the **problem** level, we examined one or more of the most difficult problems from the textbooks to gauge the depth of mathematical understanding required to master the textbook material. We looked at problems that address number, geometry and statistics from fifth or sixth grade textbooks.

TEXTBOOK LEVEL

Textbook Organization

We first looked at the Singapore textbooks and workbooks, the Scott-Foresman student texts, and the *Everyday Mathematics* student journals to get an overall sense of how they address mathematical topics. Exhibit 4–1 summarizes the structural organization of the content in the three textbooks at grades 1, 3, and 6. The table shows the total number of chapters or units; the number of lessons in each textbook; and the number of pages, which we divided into three types. Development pages contain primarily instructional material. Exercise pages have as their primary purpose practicing the material presented in the development pages. Pages were designated as other when their purpose was introductory, they provided extra practice, or they contained features that did not relate directly to the lesson development. This analysis of how the physical space in the three textbooks is allocated reveals significant differences in the proportion of pages allocated to development and exercises and in the average number of pages for each lesson.

One difference between the textbooks is that the Singapore textbooks decrease in size, shrinking from 500 total pages at grades 1 and 3 to 400 pages at grade 6. The Scott-Foresman texts, in contrast, grow from 560 pages at grade 1 to more than 700 pages at grades 3 and 6, but it is the "other" pages, which represent extra practice, reviews, and assorted features, that account for much of the difference in the total page counts. Such "other" pages are not a factor in the *Everyday Mathematics* texts, in which every one of the 250 pages in the student journal is devoted either to the development of the lesson or to related exercises.

An apparent confirmation of the "mile wide, inch deep" issues in U.S. textbooks is that Singapore's materials average one-third as many lessons as the *Everyday Mathematics* textbooks and one-quarter as many lessons as the Scott-Foresman textbooks. This translates into an average lesson length of about 15 pages in the Singapore materials, compared with 2 pages in *Everyday Mathematics* and 4 pages in the Scott-Foresman books. It is hard to see how the same depth can be achieved in 2 to 4 pages as in 15. It is equally hard to imagine how a book with 150 lessons can present as focused a curriculum as a book with 30 lessons. In fact, Singapore students are expected to

complete about one thorough lesson focused on a single topic per week, while U.S. students are expected to complete about one lesson on a narrowly focused topic each day.

Exhibit 4–1. Textbook Space Organization: Chapters, Lessons, and Pages by Type

Textbook	Total Pages	No. of Chapters*	No. of Lessons	Avg. Pages/ Lesson	Pages of Develop-ment	Pages of Exercises	Other Pages
Grade 1							
Singapore	497	18	34	15	174 (35%)	261 (53%)	62 (12%)
Scott-Foresman	564	29	157	4	145 (26%)	169 (30%)	250 (44%)
Everyday Math	246	10	110	2	136 (55%)	102 (41%)	8 (3%)
Grade 3							
Singapore	496	14	42	12	150 (30%)	234 (47%)	112 (23%)
Scott-Foresman	729	32	164	4	187 (26%)	217 (30%)	325 (45%)
Everyday Math	257	11	120	2	137 (53%)	120 (47%)	—
Grade 6							
Singapore	402	11	24	17	107 (27%)	132 (33%)	163 (41%)
Scott-Foresman	740	33	158	5	202 (27%)	216 (29%)	322 (44%)
Everyday Math	411	10	113	4	299 (73%)	112 (27%)	—
Source: Ministry of Education, Singapore (2002); Scott-Foresman (2004); Everyday Learning Corporation (2001).							

The abundance of lessons in the U.S. textbooks puts pressure on teachers to teach them all in one school year, thus encouraging teachers to rush through material and teach concepts only superficially. Fewer lessons mean that teachers in Singapore are able to focus on a lesson and teach the concepts in greater depth before they move on to the next idea. A short 2- to 4-page lesson also provides limited opportunities for student to develop an in-depth understanding of a topic. Consequently, the topic must be taught in multiple grades in an attempt to achieve sufficient depth, a practice that is probably less effective than intended.

Textbook Content

We examined textbook content first in terms of how well the three textbooks emphasize the five NCTM mathematical strands. We then disaggregated content to examine topic exposure by grade. Interestingly enough, as Exhibit 4–2 shows, sharp differences do not appear when lessons are differentiated by content strands in grade 1; the table shows remarkable similarity among the three texts when the proportion of lessons that focuses on number, measurement, geometry, algebra, and data is examined. The Singapore text places a somewhat greater emphasis on numbers, whereas the U.S. texts have a smattering of algebra and data analyses and probability lessons. However, the similarity among the three texts lessens as they advance in grade level. Singapore allocates significantly more lessons than the United States to measurement at grades 3 and 6 and significantly more lessons to numbers at grade 6. This is balanced by the relatively greater attention in the U.S. materials to geometry and data at grade 3 and to algebra and data at grade 6.

Exhibit 4–2. Singapore, Scott-Foresman, and Everyday Mathematics Textbooks: Lessons by Content Strand/Area[1]

| | | | | | Data Analyses/ |
Textbook	Number	Measurement	Geometry	Algebra	Probability
Grade 1					
Singapore	79%	15%	3%	0%	3%
Scott-Foresman	68%	17%	7%	3%	5%
Everyday Math[*]	66%	17%	8%	2%	7%
Grade 3					
Singapore	60%	33%	5%	0%	2%
Scott-Foresman	67%	14%	8%	2%	9%
Everyday Math[*]	64%	11%	11%	0%	14%
Grade 6					
Singapore	63%	17%	13%	4%	4%
Scott-Foresman	51%	11%	11%	14%	14%
Everyday Math[*]	37%	4%	18%	19%	22%

[*]Review lessons that cut across mathematical strands were not included in this table.
Source: Ministry of Education, Singapore (2002); Scott-Foresman (2004); Everyday Learning Corporation (2001).

When we compared the textbooks by disaggregating how they treat topics by grade level, however, the differences among them become quite clear. Exhibit 4–3 displays a list of mathematics topics in the left-hand column; this list consists of the topics in the Singapore framework supplemented by the topics covered in a majority of the frameworks in the seven selected states.[7]

Singapore textbooks cover 13 to 15 mathematical topics in each of the first two grades and 17 to 19 topics in grades 5 and 6. The ordering of topics follows the spiral approach spelled out in Singapore's framework. The most basic mathematics topics, such as addition and multiplication, are introduced in the early grades, repeated across several grades to deepen content, and then phased out. Singapore's textbooks' treatment of statistics shows a clear pattern of movement from the upper left to lower right of the table, as the texts sequentially introduce a different type of statistical chart at each grade, as outlined in the framework. Singapore's textbooks introduce advanced topics such as approximation and estimation, volume, and 3-D solid figures only in the upper primary grades. Singapore's textbooks, like its framework, do not introduce algebra until grade 6.

The Scott-Foresman textbooks expose students to about twice as many topics at each grade as the Singapore textbooks—25 topics in grade 1, increasing to 34 in grade 6. One reason for the expanded topic exposure is that the Scott-Foresman textbooks introduce many somewhat-advanced mathematical topics in the early grades and then repeat these topics through the later grades. For example, the Scott-Foresman textbooks cover the geometric properties of plane shapes, 3-D solid figures, symmetry, congruence, and transformations in all six elementary grades. The Singapore textbooks, in contrast, do not introduce 3-D solid figures and symmetry until grade 4. Congruence and symmetry are not taught until middle school. When difficult mathematical topics are introduced too early, the textbook treatment of these topics is of necessity superficial, given students' limited ability to understand them.

---

[7] The tabular format in Exhibit 4-3 is similar to the format developed in conducting the TIMSS topics-by-grade comparisons (Schmidt, Houang, and Cogan, 2002).

Exhibit 4–3. Mathematics Topics Coverage by Grade: Comparison of Singapore, Scott-Foresman, and Everyday Mathematics Textbooks

TOPIC	Singapore						Scott-Foresman						Everyday Math					
	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
# Notation & Place Value	X	X	X	X	X		X	X	X	X	X	X	X	X	X	X	X	X
Ordinals	X						X	X					X	?				
Compare & Order #'s	X	X	X	X			X	X	X	X	X	X	X	X	X	X	X	
Addition & Subtraction	X	X	X				X	X	X	X	X		X	X	X	X	X	
Odd & Even							X	X					X	?				
Multiplication & Division	X	X	X	X	X			X	X	X	X			X	X	X	X	X
Approximation & Estimation				X	X			X	X	X	X	X		X	X	X	X	
Factors & Multiples				X						X	X	X	X		?	X	X	X
Order of Operations					X	X					X	X					X	X
Word Problems*			X	X	X	X												
Integers											X	X				X	X	X	X
Fractions	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Decimals				X	X				X	X	X	X		X	X	X	X	X
Ratio & Proportion					X	X					X	X				X	X	X
Percentages					X	X					X	X				X	X	X
Length, Mass, Capacity	X	X	X				X	X	X	X	X		X	X	X	X	X	
Measurement Units & Conversion		X	X		X		X	X	X	X	X	X	X	X	X	X	X	
Compare & Compute With Measures	X	X	X	X			X	X	X	X	X	X	X	?	X	X		
Area & Perimeter		X	X	X	X	X	X	X	X	X	X	X		X	X	X	X	X
Volume				X	X	X		X	X	X	X	X			X	X	X	X
Time	X	X	X				X	X	X	X	X	X	X	X	X		X	
Money	X	X	X				X	X	X	X			X	X	X			
Basic 2-D Geometry	X	X					X	X					X	X				
Geometric Figures (2-D)				X		X	X	X	X	X	X	X	X	X	X	X	X	X
Lines, Curves, Surfaces		X		X		X	X	X	X	X	X	X	X	X	X	X		X
Angles			X	X	X	X			X	X	X	X	X	?	X	X	X	X
Properties of Plane Shapes				X		X	X		X	X	X	X		X	X	X	X	X
3-D Solid Figures				X		X	X	X	X	X	X	X	X	X	X	X	X	
Symmetry				X			X	X	X	X	X	X		X		X		X
Congruency & Similarity							X	X	X	X	X	X					X	X
Transformations							X	X	X	X	X	X				X	X	X
Tessellations				X								X					X	X
Constructions				X	X							X	X	?	X	X	X	X
Nets						X		X			X	X	X	?		X	X	
Coordinate Plane							X	X	X	X	X	X			X	X	X	X
Picture Graphs	X	X					X	X	X	X								
Bar Graphs			X				X	X	X	X	X	X	X	X	X	X	X	
Tables, Line Graphs, Pie Charts				X	X	X		X	X	X	X	X		?	X	X	X	X
Statistical Data										X	X	X			X	X	X	X
Average, Rate, Speed					X	X				X	X	X				X	X	X
Basic Patterns	X	X	X				X	X	X	X			X	X	X	X		
Algebraic Expressions						X			X	X	X	X				X	X	X
Algebraic Patterns									X	X	X	X					X	X
Probability							X	X	X	X	X	X			X	X	X	X
Total	**13**	**15**	**15**	**13**	**19**	**17**	**25**	**29**	**31**	**33**	**35**	**34**	**20**	**21**	**29**	**33**	**34**	**26**

*Only counted word problems when an entire lesson was devoted to word problems and titled "word problems"
Source: Singapore Ministry of Education (2001), Scott-Foresman (2004), and Everyday Mathematics (2001).

Average topic exposure in the *Everyday Mathematics* textbooks falls between the Singapore and the Scott-Foresman textbooks in the primary grades, although in the intermediate grades, topic exposure is similar to what appears in the Scott-Foresman textbooks. Compared with Scott-Foresman, *Everyday Mathematics* tends to introduce advanced topics at a later grade and to eliminate more basic topics (e.g., compare and order numbers) by grade 6. This allows the series to achieve a tighter curricular focus at both ends of the elementary grade span.

At the textbook level, analysis presents an interesting picture of the issue of curricular focus. Because U.S. textbooks are designed for multistate sales, mathematical content exposure must be broad so that the texts cover all topics in the standards of all the market states. This makes for texts that substantially exceed Singapore's in the average number of topics covered. This is particularly true of the traditional textbook, but it also applies to the nontraditional textbook. Covering more topics in each grade means less depth of coverage of any one topic and less emphasis on making sure that students have a clear conceptual grasp of the mathematical content, not just a mechanical understanding of it. In addition, states with very targeted frameworks, such as California, North Carolina, and Texas, may end up with textbooks that are not consistent with their frameworks.

LESSON LEVEL

The essence of how a textbook treats the development of mathematical understanding is most clearly viewed at the lesson level by examining individual lessons or topic-related sets of lessons. The textbook level comparison of the Singaporean and U.S. textbooks shows that Singapore's textbooks contain fewer lessons and a greater number of pages for each lesson than the U.S. textbooks. Although Singapore's textbooks cover only about half the number of topics in half the number of pages as the U.S. textbooks, the lessons in the Singapore textbooks are three to four times the length of lessons in the U.S. textbooks. Thus, the U.S. textbooks look as though they have been designed to provide smaller bites of mathematical content. The reasons for this fragmentation are not clear. Perhaps the small lesson length allows educators the flexibility of selecting lessons that match their state's content standards. Perhaps brief lessons offer a strategy that supports U.S. elementary school teachers, who, as a group, are not well trained in mathematics and may feel more comfortable with brief lessons that do not require them to understand and explain mathematics with much depth. Although we are uncertain as to its cause, this finding about lesson length provides insight into the degree to which the Singapore materials focus on depth rather than breadth, whereas the U.S. materials, both traditional and nontraditional, focus on breadth rather than depth.

Lessons in the Singapore textbooks are structured to provide depth and conceptual understanding. The textbooks use an approach that begins with concrete pictorial representations of content and later moves to abstract approaches to teach mathematics. Each lesson in the Singapore textbooks starts with an introduction to the topic, which uses diagrams and models to illustrate the basic concept, followed by specific learning tasks. The learning tasks use additional diagrams and pictures to show different ways of thinking about the concept, and they provide guided practice exercises so that students can become familiar with the topic. A workbook containing exercises to supplement the lessons accompanies each of the Singapore textbooks. The lessons include a large and varied number of exercises, many of which ask students to do nonroutine tasks and increasingly complex word problems. The U.S. textbooks provide much less depth than this for each topic.

To further examine lesson depth in Singaporean and U.S. textbooks, we compared the treatment of three specific topics: one topic common to the three grade 1 textbooks, one common to

the three grade 3 texts, and one common to the three grade 5 texts. In the first-grade texts, we looked at how the texts develop students' understanding of addition; we selected this topic because understanding the operation of addition is a key conceptual hurdle in the number strand of the K–6 program and a basis for much of what students learn later. In the third-grade texts, we looked at how multiplication facts are presented because facility with them enables much of the succeeding mathematics curriculum and they are a critical variable in developing students' self-confidence in their ability to do mathematics. In the fifth-grade texts, we looked at line graphs because although the Singapore materials do not emphasize data or statistics as much as the U.S. materials do, Singapore focuses on one type of graph to help review and reinforce a wide range of nondata mathematical understanding. By contrast, the U.S. materials scatter the focus across a range of different types of graphs.

Grade One Comparison: Understanding the Meaning of Addition

A key concept taught in first grade is the meaning of addition. Addition is taught in multiple lessons in all three texts. Students first develop a basic understanding of what it means to add (combine, join) and then develop proficiency with basic addition facts. The first three addition lessons in the Singapore textbook illustrate the meaning of addition and develop proficiency in addition facts up to 10:

- The first Singapore lesson, "Making Addition Stories," uses pictures and number stories to show that addition means "putting together." The lesson asks the student to create stories to go along with pictures that illustrate addition facts and write corresponding number sentences.

- The second lesson deepens understanding of addition facts, introducing the concept of "number bonds" (fact families) and illustrating the commutative property of addition.

- The third lesson, "Other Methods of Addition," illustrates other ways of thinking about addition, such as counting on and making 10, while also reinforcing addition facts.

- Exercises in the workbook use games and pictures to reiterate the different ways of thinking about addition illustrated in the lessons and to provide plenty of practice in solving problems using basic addition facts.

The Scott-Foresman textbook takes six lessons in its "Understanding Addition" section to teach basically the same material the Singapore textbook covers in its first addition unit. The first lesson, "Stories about Joining," uses pictures to illustrate that addition means "joining." The next lesson shows students how to use concrete materials to find sums. In the lesson titled "Using Numbers to Add," as in the Singapore textbook, pictures show students how to create number sentences to represent addition facts. Whole lessons are then devoted to illustrating the effect of zero in addition and to showing how addition can be represented vertically. The final lesson in this section deals with writing number sentences to represent pictures and with simple word problems that illustrate addition situations. Lessons that come slightly later in the textbook introduce the concepts of counting on, the commutative property of addition, and other methods of doing addition. In general, the Scott-Foresman textbook teaches the early concepts of addition in a way similar to the Singapore textbooks. Material is condensed into fewer, but longer, lessons in the Singapore textbooks, and the Singaporean text includes more practice exercises that help students achieve mastery in addition facts. The Scott-Foresman textbook, however, uses separate lessons that teach

smaller pieces of the material. This fragmentation makes it more difficult for students to see how all the small pieces of information about addition fit together into a conceptual whole.

The *Everyday Mathematics* textbook teaches early addition concepts by using a less focused approach. Several early lessons touch briefly on some of the key concepts, such as creating number stories, but do not emphasize addition facts. The concept of "more" is also introduced early on in connection with developing counting skills and finding patterns in numbers. Summing is also used in connection with money and measures. "Domino Dot Patterns and Domino Combinations" is the first lesson that introduces students to basic addition and subtraction facts. About 20 lessons later, "More Number Stories and Mental Arithmetic" illustrates the meaning of addition, as well as subtraction, by having students create number stories, illustrate them with pictures, and indicate the operation to be used. The following lesson asks students to express number stories using a number model. The next lesson, "Addition and Subtraction Games," provides interactive practice of addition and subtraction facts up to 10. Subsequent lessons introduce the idea of addition facts, illustrate using vertical and horizontal formats to write addition and subtraction problems, teach the commutative property of addition, and review addition facts with additional games. The key ideas of the meaning of addition are made in lessons with activities in which the concept and use of addition arise naturally, but they are nonconsecutive, making the entire concept of addition more difficult to grasp.

The three textbooks use relatively similar approaches to introduce the meaning of addition. Each book develops the meaning of addition, shows how to create number stories to represent addition situations, shows how to write number sentences, and teaches basic addition facts. However, Singapore and *Everyday Mathematics* place a greater emphasis on the essential ability to recall sums. Singapore achieves this through more practice exercises, and *Everyday Mathematics* uses games to provide the frequent practice necessary for students to master a skill. The three textbooks may use slightly different approaches to teach the meaning of addition, but the instructional lessons in each are consistent in imparting basic understanding of the meaning of addition to students.

In summary, the grade 1 lesson analysis reveals that although there are more similarities than differences among the three programs in the development of the meaning of addition, Singapore is by far the most focused in terms of systematic conceptual development. The problems and varied pictorial representations work together toward a unified presentation of the concept. In contrast, a fragmentation of approach and a scattered focus appear in both the Scott-Foresman and *Everyday Mathematics* lessons.

It should be noted that, as with the comparison of frameworks, the similarity between Singapore and U.S. textbooks is much stronger in the early grades than in the later grades. The next two examples bear this out.

Grade Three Comparison: Multiplication and Division Facts

To teach multiplication tables and corresponding division facts to 10, the grade 3 Singapore textbook devotes a lesson to multiplying and dividing by each number from 6 to 9. Singaporean students have already learned multiplication facts through 5 from the grade 2 textbook, so the grade 3 book focuses on multiplying by 6, 7, 8, and 9. For the purposes of this comparison, we examined the 5-page lesson titled "Multiplying and Dividing by 6," the 4-page lesson titled "Multiplying and Dividing by 7," and the accompanying 21 pages in the workbook.

The Scott-Foresman textbook also teaches multiplication and division facts to 10, but divides the material into separate lessons, "6 and 7 as Factors" and "Dividing with 6 and 7." The book devotes a single page of developmental material to multiplying by 6 and a single page to multiplying by 7. Later in the textbook, a single page is devoted to dividing by 6 and 7. The *Everyday Mathematics* textbook does not break down multiplication and division facts number by number but instead covers all multiplication and division facts in four lessons.

The Singapore lessons show how multiplication and division facts are interrelated, and they use models of different kinds to illustrate different ways of computing multiplication facts. Array models show how to break down multiplication facts with 6 and 7 into facts the student should already know. The commutative nature of multiplication and the relationship between multiplication and division are also illustrated. The lesson is then extended to show how to multiply and divide three-digit numbers by 6 or 7. Workbook exercises use games and word problems to reinforce the concepts from the chapter and facilitate committing multiplication and division facts to memory.

The Scott-Foresman lessons also use arrays to illustrate multiplication, but not with the same thoroughness and reinforcement as the Singapore textbooks. As in the Singapore text, the Scott-Foresman division chapter teaches division with 6 and 7 by asking students to apply the appropriate multiplication fact; however, the relationship between multiplication and division is not illustrated until this lesson. Practice exercises follow the lessons, but with less reiteration than in the Singapore textbook. Although the division chapter contains word problems similar to those found in the Singapore book, multiplication and division with three-digit numbers is not taught in either of these lessons, but in separate lessons much later in the textbook.

The *Everyday Mathematics* textbook, like the Singapore textbook, teaches multiplication and division facts in the same lessons. The textbook emphasizes the inverse relationship between the two operations and uses it to develop fact families, to illustrate the commutative property, and to show patterns in multiplication and division facts. Games are also used to provide practice with multiplication and division facts and to promote memorization of these facts. *Everyday Mathematics*, like the Scott-Foresman textbook, also teaches multiplication and division of a three-digit number by a one-digit number, but much later in the textbook.

Although the Singapore and *Everyday Mathematics* textbooks both focus on the importance of the inverse relationship between multiplication and division and provide practice and exercises that promote the memorization of multiplication and division facts, the Singapore book teaches each number as a separate entity and in more detail and depth than does *Everyday Mathematics* or the Scott-Foreman book. The Singapore book's emphasis on proficiency with multiplication and division facts in grade 3 promotes the higher-order computational skills that Singapore expects students to use in the later grades.

A major difference between the textbooks is that the Singapore text extends teaching multiplication and division facts into teaching multiplication and division of two-digit and three-digit numbers by a single-digit number. The two U.S. textbooks do not teach this concept until much later in the curricula. The Singapore textbook teaches multiplication and division facts in more depth than either of the U.S. textbooks by using multiple diagrams and models to show different ways of solving problems and by providing more practice exercises, including multistep, complex word problems, to allow students to hone their skills.

In summary, the grade 3 lesson analysis reinforces the grade 1 comparison. The Singapore materials employ a much broader range of representations (arrays, grids, strips, bundles, number sentences, and related facts) and problem situations (e.g., soldiers per tent, points per ring) both to develop conceptual understanding and to give students multiple models for practicing and memorizing facts. In addition, although all three programs eventually link multiplication and division through fact families or number bonds, only the Singapore materials systematically draw connections between multiplication and division from the very start, on a number-by-number basis.

Grade Five Comparison: Line Graphs

The only statistics content in the Singapore grade 5 textbooks and workbooks is a single lesson on line graphs. The lesson consists of one line graph made by using a table of data on attendance at a swimming pool over a 5-month period. Three similar line graphs show data on exchanging U.S. and Singapore dollars, which are accompanied by questions on the magnitude of increase and decrease, the average of the amounts, and how to read a linear line graph. This lesson is supplemented by exercises in the workbook containing six similar line graphs, each accompanied by three to five questions on interpreting the graphs and problems based on the data in the graphs. The Singapore treatment of line graphs is intensive, exposing students to the graphs and asking them to read and interpret them. The lesson on graphs also reinforces computational skills with questions about averages, increases, and decreases.

The Scott-Foresman grade 5 textbook includes line graphs in a chapter on data, graphs, and probability. One four-page lesson is devoted solely to line graphs, one page to how to read a line graph, one page to making a line graph, and two pages to practice exercises. Line graphs also appear later in the chapter in a four-page lesson titled "Choosing an Appropriate Graph" and in a two-page lesson titled "Writing to Compare," which uses line graphs to support writing a comparison of two graphs. The need to cover everything scatters the focus on line graphs in ways that give students a little experience with interpreting line graphs, constructing line graphs, choosing when to use a line graph, or writing comparisons on the basis of two related line graphs. Although one extension uses double line graphs, no connection is made between line graphs and averages, and far less attention is paid to describing amounts of increase or decrease than in the Singapore texts.

In the *Everyday Mathematics* grade 5 textbook, line graphs appear as part of three lessons: "Rules, Tables and Graphs," Parts 1 and 2, and "Reading Graphs." Instead of using line graphs to show trends or changes over time, the line graphs in these lessons are related to linear functions and to translating between tables and graphs using the context of rate. The focus on line graphs as linear functions leads into variables, formulas, and ordered pairs, all of which are more closely aligned with the algebra strand than the data strand. The questions in the chapters (e.g., How many yards apart are Eli and Sara after 7 seconds?) are commensurate in depth to what is found in the Singapore materials and expect more from students than what is found in the Scott-Foresman textbook.

Examining how these lessons are organized makes clear that the Singapore lessons and the *Everyday Mathematics* lessons are in many ways similar and that both are quite different from the lessons in traditional Scott-Foresman books. The Singapore and *Everyday Mathematics* books use line graph topics to make connections and reinforce prior learning; the Scott-Foresman text merely shows what a line graph is and how to create one. The differences in the treatment of line graphs reinforce the contention that the Singapore textbooks teach topics in greater depth than do traditional U.S. textbooks. The one lesson on line graphs in the Singapore textbook is thorough, builds on

previously learned experiences, and uses in-depth problems that enable students to develop a deeper understanding of the topic.

The grade 5 lesson analysis reveals the greatest differences among the three lesson comparisons. The similarities found between Singaporean and U.S. texts in grade 1 diminished at grade 3, and significant differences emerge by grade 5, a pattern also observed in the comparison of frameworks. The Singapore design, which focuses on just line graphs at grade 5, allows an in-depth treatment that is impossible when line graphs are presented as only one part of a larger focus on data, graphs, and probability (Scott-Foresman's approach) or of a larger focus on the graphs of linear functions (as in *Everyday Mathematics*). Once again, the development of an understanding of line graphs in the Singapore materials is supported by multiple representations and contexts and by a series of questions that reinforce an array of skills and concepts that are important components of interpreting data. The result is a much richer presentation of the topic in the Singapore materials when compared with the Scott-Foresman textbook and, to a lesser degree, when compared with *Everyday Mathematics.*

PROBLEM LEVEL

Summary data about textbook structure and content and descriptions of lessons tell only part of the story about mathematics textbooks in Singapore and the United States. Examining the rigor of mathematics problems the books expect students to be able to solve provides another perspective about the mathematical content of the textbooks.

To assess the depth the books require from students in mathematical problem solving, we examined difficult exercises from the problem sets for three topics from the fifth- or sixth-grade textbooks.[8] We selected fifth- or sixth-grade exercises because they show the level of difficulty that students are expected to meet at or near the end of their elementary coursework. The problems we chose were not the most difficult ones available, but they represent the content and problems illustrated in the lesson explanations for that topic.

We compared textbook exercises in three advanced mathematical topic areas:

- **Volume:** In grade 5, geometry is an advanced topic in the geometry strand. The Singaporean and U.S. textbooks provide nearly equal support for this topic over the grades.

- **Pie charts:** In Singapore, this is a grade 6 topic in the statistics strand, but the topic generally receives greater emphasis in U.S. texts.

- **Ratios:** In grade 6, ratios are an advanced topic in the numbers strand. Both Singapore and the U.S. texts emphasize them about equally in the early grades, but Singapore gives them more emphasis in the later grades.

---

[8] Because *Everyday Mathematics* lessons are structured around problem-based learning, the problem sets we selected were imbedded in the *Everyday Mathematics* lessons. The problems from the Singapore and Scott-Foresman texts are part of the explanatory materials in the chapter, not the problems that students are expected to solve themselves, but in all cases, the problems selected are very similar to the exercise problems that students are expected to do on their own.

We characterized each exercise in three ways: the approximate number of steps to arrive at a solution; whether the solution strategy involved solving for an unknown intermediate variable; and whether the solution strategy involved merely a routine application of a formula or definition (e.g., a straightforward, obvious, or commonplace application of mathematical knowledge) or, alternatively, a nonroutine strategy or approach to solving the problem (e.g., a more nontraditional, less obvious application requiring problem-solving strategies).

Volume of a Prism Exercise, Grade 5 Geometry Strand

Singapore Textbook Volume Problems

- About 3 steps

- Unknown intermediate variable

- Routine application of formula or definition

Problem: A rectangular tank, 30 cm long and 20 cm wide, is filled with water to a depth of 8 cm. When a stone was put in, the water level rose to 11 cm. Find the volume of the stone. (Assume that the stone was completely under water.)

Problem: A rectangular tank, 30 cm long and 18 cm wide, contained some water and a stone. When the stone was taken out, the water level dropped by 2 cm. Find the volume of the stone. (Assume that the stone was completely under water.)

Discussion: The solutions to these problems are similar. The student must find the difference in the height of the water, multiply the height difference by the length and width of the tank, and equate that volume with the volume of the stone. These problems show how the Singapore books teach conceptual understanding by juxtaposing different applications of the same concept. In this instance, students must recognize that the problem in which the water rises when the stone is put into the water is conceptually the same as the problem in which the water falls when the stone is taken out.

Scott-Foresman Textbook Volume Problem

- About 2 steps

- Unknown intermediate variable

- Routine application of formula or definition

Problem: Matt built five rectangular prisms with unit cubes. Each prism had a base that was 4 by 5 units. If he started with 2 layers and continued to add a layer each time he built a new prism, how many cubes did he use for the largest prism?

Discussion: This problem requires finding the height of the fifth prism by realizing that its height is one more than the total number of prisms built (i.e., 6 units). This result is multiplied by the length and width of the base to find the volume. Thus the problem uses a pattern and a set of prisms that grow layer by layer to teach the concept of volume in a straightforward manner that does not involve the important concept of displacement found in the Singapore problems.

- About 4 steps

- No unknown intermediate variable

- Nonroutine application of formula/definition

Problem:

1. Use your triangular prism and triangular pyramid.

    a. Fill the pyramid so that the material is level with the top. Empty the material into the prism.

    b. Fill the pyramid again and empty the material into the prism.

    c. Repeat until the prism is full to the top.

    d. It takes about _3_ pyramids of material to fill the prism.

2. Use your 4-sided prism and 4-sided pyramid.

    a. Repeat the steps in problem 1.

    b. It takes about _3_ pyramids of material to fill the prism.

3. Use your can and the cone you just made.

    a. Fill the cone to the level shown by your pencil mark. Empty the material into the can.

    b. Fill the cone again and empty the material into the can.

    c. Repeat until the can is filled to the top.

    d. It takes about _3_ cones of material to fill the can.

4. To calculate the volume of any prism or cylinder, you multiply the area of the base by the height. How would you calculate the volume of a pyramid or cone?

Sample answer: You would multiply the area of the base by the height and divide this number by 3.

Discussion: This problem provides students with instructions in how to use physical objects to construct the volume formula for different-shaped prisms. This problem is much like ones that might be carried out in a physical science laboratory. Here, students are given the procedures to follow to carry out mathematical experiments. This formula is then used in some real-world volume problems. What is notable in this presentation is the hands-on approach used to actually derive a formula and compare the volume of a prism and a pyramid that have the same base.

Summary: All three problems require multiple steps to find solutions to routine problems in finding the volume of prisms. The Singapore and Scott-Foresman problems are quite similar in that both involve computing an unknown height and multiplying this result by the area of the base. The major distinction is that Singapore presents two different looks at its problem about measuring the volume of the stone. In one case the stone is added to raise the height of the water and in the other case it is removed to lower the water height. This difference allows students to develop additional conceptual knowledge by getting them to see that the two problems are related. *Everyday Mathematics* presents a different type of problem approach altogether in which students actually derive the formula for the volume from physical experiments. This constructivist approach is sometimes cast as requiring students to invent the formula – a claim not supported by the approach used in *Everyday Mathematics.*

Although all three problems involve the routine use of the volume formula, the Singapore approach uses a real-world application to develop this concept, whereas the *Everyday Mathematics* approach relies on literally constructing an understanding of why pyramid volumes are 1/3 of prism volumes. In both cases, the approach is more likely to support deep conceptual development than the Scott-Foresman approach.

Pie Chart Problems, Grade 6 Statistics Strand

Singapore Textbook Pie Chart Problem

- About 6 steps

- Unknown intermediate variable

- Nonroutine application of formula or definition

Problem: See Exhibit 4–4.

Exhibit 4–4. Singapore Textbook Pie Chart Problem

The pie chart represents the amount of money collected by various stalls at a funfair.

(a) What fraction of the total amount of money was collected by the games stalls?

(b) What was the total amount of money collected by the various stalls?

(c) How much money was collected by the music stalls?

(d) What was the ratio of the money collected by the food stalls to the money collected by the handicraft stalls?
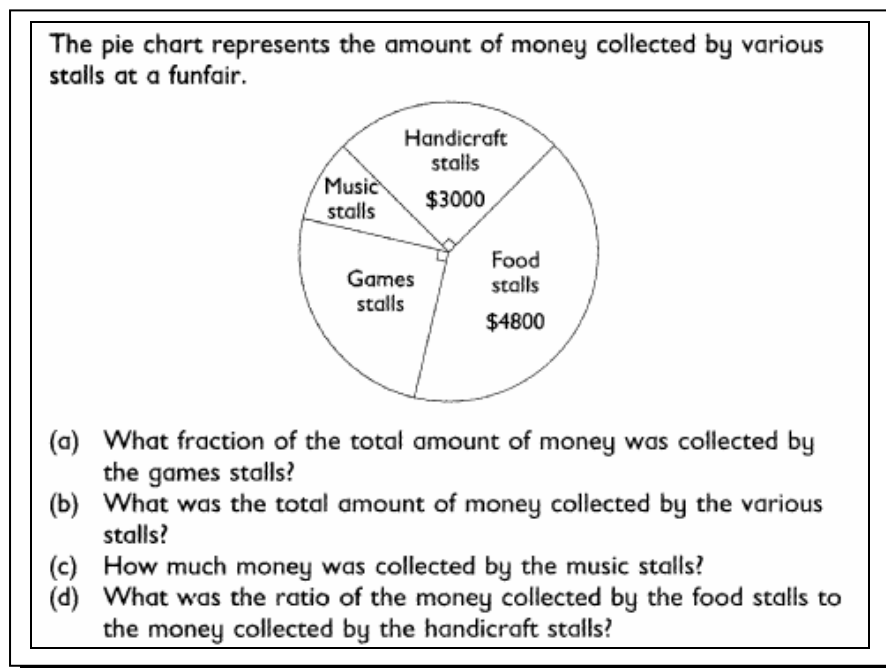
Discussion: The Singapore pie chart problem describes the amount of money collected at various stalls at a fair. This problem illustrates how the Singapore textbook forces students to draw deeply on their understanding of different types of mathematical knowledge. To solve the problem, students must realize that the amounts of money collected by different stalls are in proportion to the size of the angles in the slices of the pie chart. The games stall slice of the pie chart is a right angle, so its money equaled the handicraft stalls $3000, one quarter of the total. The sum of the games stalls slice plus the handicraft stalls slice is 180 degrees. This means that the sum of the angles of the remaining two slices, music stalls and food stalls, must also be 180 degrees. The total amount of money collected by these two stalls is also $6000, so the music stalls collected $1200. The final part of the problem requires an application of the ratio definition and simplification of ratios.
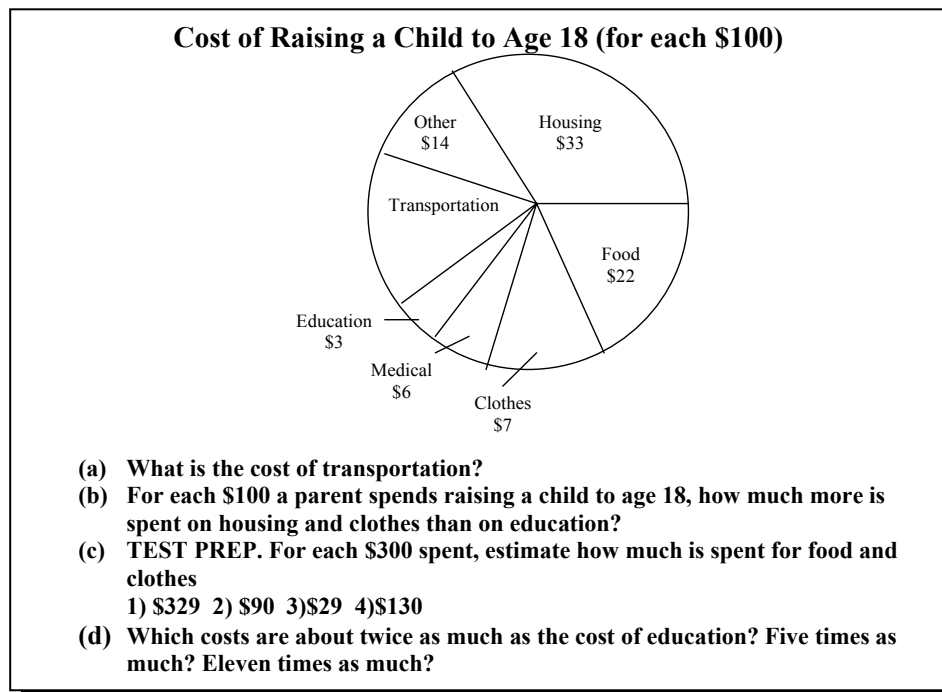
Scott-Foresman Textbook Pie Chart Problem

- About 4 steps

- No unknown intermediate variable

- Routine application of formula or definition

Problem: See Exhibit 4–5.

Exhibit 4–5. Scott-Foresman Pie Chart Problem

Cost of Raising a Child to Age 18 (for each $100)

Other $14
Housing $33
Transportation
Food $22
Education $3
Medical $6
Clothes $7

(a) **What is the cost of transportation?**
(b) **For each $100 a parent spends raising a child to age 18, how much more is spent on housing and clothes than on education?**
(c) **TEST PREP. For each $300 spent, estimate how much is spent for food and clothes**
   1) $329 2) $90 3)$29 4)$130
(d) **Which costs are about twice as much as the cost of education? Five times as much? Eleven times as much?**

Discussion: The pie chart describes how the different costs of raising a child are distributed for every $100 spent. The amount for one component, transportation, is not shown. This amount is simply obtained by subtracting the known amounts from the $100 total. The Scott-Foresman problem then asks several interpretative questions about the pie chart graph, requiring the application of the proportionality concept to a new total amount and an understanding of how to compare multiples of amounts. As such, this problem is much more straightforward and involves a far narrower scope of the mathematics curriculum than what the Singapore materials require.

American Institutes for Research®

Everyday Mathematics Textbook Pie Chart Problem

- 1 or 2 steps

- No unknown intermediate variable

- Routine application of formula or definition

Problem: See Exhibit 4–6.

Exhibit 4–6. Everyday Mathematics Pie Chart Problem

Discussion: This problem has students use a percent circle, in the dotted box, to physically measure the size of angles in different pie charts and interpret the results. The questions about pie charts are straightforward. The first three questions involve reading numbers off the chart. The next two questions are about computing differences to find the amount that is greater. The final question is open-ended and has students interpret results. Once again, this problem is much more straightforward and involves a far narrower scope of the mathematics curriculum than what the Singapore materials require.

Summary: The Singapore pie chart problem requires students to apply their prior geometry knowledge about angles and the area of circles to this multistep problem. The Scott-Foresman example is a straightforward problem in reading a pie chart, which requires students only to know that the sum of the values of the pieces in the pie chart must equal the total value. It does not require using the geometric properties of circles, which are the basis of understanding the construction of circle graphs. In fact, the traditional U.S. textbook, unlike the Singapore textbook, never explains to the student that pie charts are constructed so that the size of the angles of the pie slices are in proportion to the size of the numbers they represent. The problems all require only one step for which the problem-solving strategy is straightforward; they demand little, if any, conceptual understanding from students. The *Everyday Mathematics* example, unlike the Scott-Foresman problem, requires that students physically measure the size of the angles for different pieces of the pie chart and then has students use these measurements to estimate the values of each slice. Students learn by this mathematical demonstration the relationship between pie chart angles and quantities. However, the pie chart problems are quite straightforward and not nearly as challenging as Singapore's example with its intermediate unknowns and use of geometric knowledge. The *Everyday Mathematics* approach teaches more about pie charts and in a real way than does the Scott-Foresman example, but it would benefit from a pie chart problem, like Singapore's, that requires the development of deeper conceptual knowledge.

Ratio Problems, Grade 6 Numbers Strand

<u>Singapore Textbook Ratio Problems</u>

- 1 to 3 steps

- Some problems with unknown intermediate variables

- Some nonroutine applications of formula or definition

Problem: See Exhibit 4–7, parts a, b, and c.

Discussion: The Singapore textbook discusses ratios in three lessons: ratio and fraction, ratio and proportion, and changing ratios. Exhibit 4–7 presents an exercise from each section. These exercises are clear examples of how Singapore uses its concrete visual representations to build student understanding of concepts and to develop problem-solving approaches.

Problem 4-7a, on ratio and fraction, applies the concept of ratios to a geometric problem about the characteristics of rectangles. The example is interesting in several respects. It draws on students' prior knowledge about rectangles and perimeter, building on these concepts by making students apply this knowledge to a new topic instead of simply repeating the original content. It also builds connections between the concepts of ratios and the sides of geometric figures, and it uses ratios of three numbers. Finally, it requires students to simplify large numbers, thus reinforcing computational skills.

Problem 4-7b, on ratios and proportions, asks students to use information provided about ratios to find amounts of total paint used and amounts of individual paint colors. The example illustrates how the Singapore book promotes conceptual understanding by presenting a concept from multiple perspectives. In the first case, the amount of one component of a ratio is known, and it is used to find the amount of the other component. In the second case, only the total is known, making students use the concept of a ratio to find the amount of a component.

Exhibit 4–7. Singapore Textbook Ratio Problems

a. Ratio and Fraction

A rectangle measures 60 cm by 40 cm. Find the ratio of the length to the breadth to the perimeter of the rectangle.

60 : 40 : 200 =   :   :

40 cm

60 cm

Perimeter = 60 + 40 + 60 + 40
= 200 cm

b. Ratio and Proportion

Mr Muthu mixed red paint and blue paint in the ratio 3 : 2 for a painting job.
(a) If he used 12 litres of red paint, how many litres of blue paint did he use?

12 ℓ

Red

Blue

?

(b) If he made 10 litres of paint for the painting job, how many litres of red paint did he use?

?

Red

Blue

10 ℓ

c. Changing Ratios

John and Minghua had an equal amount of money each. After John spent $25 and Minghua spent $18, the ratio of John's money to Minghua's money was 2 : 3. How much money did each boy have at first?

?

John

Minghua

John

Minghua

$25

$18

Workbook Exercise 14

The example also illustrates the Singapore book's visual approach to problem representation. Students learn to translate all the information in the problem, including the unknown, into a graphical display to understand how the different pieces of the problem are related.

The problem is also important in that it builds in complexity. The number of liters of blue paint is solvable as a simple proportion because students know the ratio of red to blue paint and the amount of red paint used. However, the second part of the problem introduces the idea of the total amount of paint used; here students must understand that the known ratio of red to blue paint can be used to determine the amount of red paint used. The units of the bars that represent the problem make it clear that five units should equal 10 liters or that there is a 2-liter equivalent for each unit.

Problem 4-7c, on changing ratios, is particularly challenging in that the ratio changes over the course of the problem. This problem illustrates how geometric representation clarifies the relationships between the different pieces of information in the problem and the unknown. Further, developing a solution to the problem requires a sophisticated understanding of how to use information about the before and after values of a ratio to set up graphically as an equality (i.e., 2 units + 25 equals 3 units plus 18). Finally, the problem introduces algebraic ideas and uses them to solve for the two unknown amounts of money that each boy initially had.

- 1 or 2 steps

- No unknown intermediate variable

- Routine application of formula or definition

Problem: See Exhibit 4–8.

Exhibit 4–8. Scott-Foresman Ratio Problems

a. Ratio

Give a ratio comparing the number of male soccer players and the total number of players in lowest terms.

b. Ratio and Proportion

A 24 ft. tall statue of the Sioux Indian chief Sitting Bull, in Denmark, is made entirely of Lego blocks. If the real Sitting Bull stood 6 ft. and his head was 0.875 ft. tall, find the height of the statue's head.

c. Similar Figures

| ΔA | ΔB |
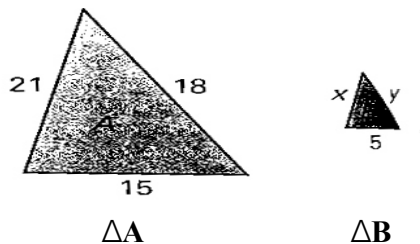
ΔA ~ ΔB Find the length of the side labeled x

Discussion: The Scott-Foresman book has 11 lessons on ratios, compared with Singapore's 3 lessons. We look at three problems. The ratio problem is similar to the Singapore problem on ratio and fractions. The second problem, on ratio and proportion, is identical to the second Singapore ratio problem. The Scott-Foresman book has no section comparable to Singapore's section on changing ratios, so we chose a section on similar triangles that is unique to the Scott-Foresman book. These sample problems are among the more difficult presented in the chapter explanations.

Problem 4-8a shows a ratio by comparing the number of male soccer players in a photograph to the total number of players of both sexes. This problem uses a real-world example, with pictures, and involves the application of a straightforward definition of ratio in which the student simply must know to divide the total number of male players by the total number of players (i.e., 2/6). The problem also requires the simplification of ratios using easy numbers (1/3 = 2/6).

Problem 4-8b uses ratio as a proportion to compute an unknown quantity, the height of the head of a Sioux Indian statue, given the total height of the statue, the height of the head, and the total height of the real Sioux Indian person. This problem uses a real-world example, with a picture, and requires a mechanical application of a cross-product formula for proportions.

Problem 4-8c applies the rules for proportionality to similar triangles and asks students to find an unknown length of one side of a triangle, given that the lengths of the sides of a similar triangle are known. This problem, which is part of a unit on similar triangles that is embedded in the ratio chapter, uses a straightforward application of the cross-product formula for two ratios to develop an equation for the unknown side. It also requires the students to use mental mathematics.

As such, the Scott-Foresman problems rely far less on pictorial cues to strengthen conceptual development. Each of these three problems requires only a straightforward application of the definition of ratio or the use of a formula. In short, these problems apply, at a rather simple level, the concept of ratio, whereas the three problems in the Singapore materials apply, develop, and extend the concept of ratio, resulting in a far more rigorous treatment.

Everyday Mathematics Ratio Problem

- 1 to 4 steps

- Some problems with unknown intermediate variables

Problem: See Exhibit 4–9.

Discussion: Three types of ratios are illustrated in a sequence of problem-based learning examples that express ratios from different perspectives. Problems 1–9, on the left, use ratios of cards in a deck. Problems 1–3 use ratios that compare the entire deck to parts of the deck. These problems involve one or two steps and some interpretation on the part of students. Problem 2 expresses a ratio in words, six out of nine cards, and requires students to translate the phrase into a numeric expression and simplify it. Problem 3 requires students to translate a ratio expressed as a percentage into a fraction. Problems 4–9 compare different parts of the total to one another rather than to the total number of cards. Problem 4 provides a part-to-part ratio and the quantity of one part; students must find the other part. Problem 5 gives the total and the ratio of the parts and asks students to compute the parts. Problems 6, 7, and 8 are more complicated, and Problem 9 is challenging only as far as it involves a decimal in the ratio. The problems identify starting amounts for two quantities to which new amounts are added in a specific ratio. Students must find the end amounts of each quantity.

Exhibit 4–9. Everyday Mathematics Ratio Problems

Exhibit 4-9: *Everyday Mathematics* **Ratio Problems**

Part-to-Whole and Part-to-Part Ratios

Solve the following problems. You may use a deck of cards to help you.

1. You have 24 cards. One out of 4 cards is faceup, and the rest are facedown.

   How many cards are faceup? ___6___ Facedown? ___18___

2. You have 30 cards. Six out of 9 cards are faceup, and the rest are facedown.

   How many cards are faceup? ___20___ Facedown? ___10___

3. You have 16 cards. Twenty-five percent of the cards are faceup, and the rest are facedown.

   How many cards are faceup? ___4___ Facedown? ___12___

4. You have 18 face-up cards. The ratio of face-up cards to face-down cards is 1 to 2.

   How many cards are faceup? ___18___ Facedown? ___36___

5. You have 20 cards. The ratio of face-up cards to face-down cards is 1.5 to 1.

   How many cards are faceup? ___12___ Facedown? ___8___

6. You have 5 face-up cards and no face-down cards. You add some face-down cards so that 1 in 3 cards is faceup. How many cards are there now?

   ___15___

7. You have 5 face-up cards and 15 face-down cards. You add some face-up cards so that 2 out of 5 cards are faceup. How many cards are there now?

   ___25___

8. You have 8 face-up cards and 12 face-down cards. You add some face-up cards so that $\frac{2}{3}$ of the cards are faceup.

   How many cards are faceup? ___24___ Facedown? ___12___

Challenge

9. The ratio of face-up cards to face-down cards is 0.3 to 1. Fewer than 10 cards are faceup. At least 21 cards are facedown.

   How many cards are faceup? ___9___ Facedown? ___30___

---

2. Keara's dad built her a scale model of the house pictured at the right. The scale model was built to a scale of 1 to 12. Every length in the scale model is $\frac{1}{12}$ the actual size.

   height 27 ft width 18 ft length 36 ft
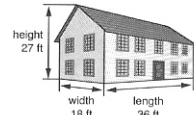
   a. Find the dimensions of the scale model.

   length = ___3___ feet width = ___1.5___ feet height = ___2.25___ feet

   b. Find the area of the first floor.

   scale model = ___4.5___ ft² actual house = ___648___ ft²

   c. Find the following ratios.
   Ratio of length of actual house to length of scale model = ___12___ to 1.
   Ratio of first-floor area of actual house to first-floor area of scale model = ___648___ to ___4.5___ or ___144___ to 1.

   d. Compare the ratio of the lengths to the ratio of the areas. Are they the same? ___No___
   How many times greater is the ratio of the areas than the ratio of the lengths? ___12 times___

The problem on the right in Exhibit 4-9 introduces ratios as proportions. Understanding is developed through a real-world example of a scale model of a house. Students have to calculate the actual dimensions of the house. Students also have to compare the dimensions of the scale model to the actual dimensions of house with respect to length and area, demonstrating that area increases by the square of the scale of the model.

Thus the *Everyday Mathematics* approach, using cards and the house, offers a more interesting real-world setting for the development of the concept of ratio than is found in the Singapore materials. However, like the Scott-Foresman approach, there is little attempt to use these applications to develop and extend the underlying concept of ratio or the mathematics of changing ratios.

***Summary*:** The Singapore textbook ratio problems require students to understand how to analyze ratios; they must apply ratios to a total amount to find its components or to ratios that change during the course of the problem. The problems from the Singapore text also require students to analyze initial information and to represent that information in the mathematical form of a problem and to reason through a number of solution steps. The explanations develop a pictorial approach to

provide visual heuristics to assist the student in reasoning through complex ratio problems involving multiple steps and unknowns.

Collectively, the three examples of ratio problems from the Scott-Foresman book require only straightforward applications of the definition or a formula. They have none of the depth or the pictorial representations that focus on the meaning of ratio found in the Singapore problems. Moreover, whereas the Scott-Foresman applications often present students with real-world problems, for the most part, the addition of the real-world context does not add much in the way of mathematical understanding. For example, the Sitting Bull statue pictured in Scott-Foresman (as well as the scale model of the house pictured in *Everyday Mathematics*) serves only as a context for the routine use of ratio and does nothing to support the conceptual understanding of ratios as comparisons of quantities that emerges from the graphics used in the Singapore materials.

The ratio problems from the *Everyday Mathematics* textbook offer students more interesting and relevant real-world constructions and applications and are generally more challenging. The problem involving the multidimensional scale model of a house is far more complex than the one-dimensional scale used for an American Indian statue in the Scott-Foresman book. However, *Everyday Mathematics* suffers in comparison with the Singapore texts because it never requires students to complete more complicated mathematics problems that involve completing many steps and solving for unknowns. It also does not systematically develop conceptual mathematical understanding through step-by-step problem development, as Singapore does. Although the Singapore text is superior to both U.S. texts in a number of respects, it could be improved by introducing some of the more complicated mathematical experiments and real-world examples used in the *Everyday Mathematics* textbook.

CONCLUSION

At each grade, and in each chapter and lesson, the Singapore textbooks present mathematical content that is tightly aligned with the topics and outcomes in the Singapore framework. Lessons are of a substantial page length and expose students to an extensive development of mathematical concepts. Students' conceptual understanding is developed through lessons that explain concepts through problem-based learning exercises that illustrate how concepts are applied from different perspectives in both routine and nonroutine ways. The concrete and pictorial illustrations in the Singapore texts use visual explanations to help students understand abstract mathematical concepts, and these visual aids give students the tools to formulate, represent, and reason through different types of complex problems. The extensive and challenging problem sets in the Singaporean books reinforce strong conceptual understanding and support procedural fluency.

The U.S. textbooks' use of physical textbook space, in contrast, does not allow the extended conceptual development of mathematical topics found in the Singaporean textbooks. They cannot develop the same type of content depth when they contain three to five times the number of lessons as the Singaporean books and when lesson length is only 2 to 4 pages, compared with 12 to 17 pages in the Singaporean texts. The U.S. textbooks expose students at each grade to nearly double the number of topics, which means that these topics must be taught much more superficially if teachers are to get through them in a year. All these factors lend credence to the criticism that mathematics instruction in the United States values breadth of coverage at the expense of depth. The traditional Scott-Foresman series is particularly weak. Even its most difficult problems are quite routine and

involve only two steps, in contrast to the more challenging and less routine problems presented in the U.S. nontraditional textbook and in the Singapore texts.

Despite the generally poor showing of both U.S. textbooks in terms of teaching mathematical concepts, the nontraditional U.S. textbook, *Everyday Mathematics*, does do one thing much better than the Singaporean texts do. It poses interesting real-world problems and uses illustrations that demonstrate the practical application of mathematical topics, which the Singapore textbooks do not do. These real-world examples are connected to the development of essential mathematical ideas, in contrast to the traditional U.S. mathematics textbook that used illustrations more to motivate students than to improve their understanding of real-world mathematical complexities.

These findings suggest several reforms that should be considered to strengthen U.S. mathematics textbooks:

1. States and school systems should consider comparing the content of approved textbooks with state frameworks and develop rubrics that identify which chapters and lessons line up with which items in the state standards. It is unfair to U.S. teachers to provide them with textbooks that are too thick and too inclusive because publishers must, for reasons of marketability, include content to fulfill the standards in multiple states. A sound road map that matches textbook content with each state's mathematical frameworks would help teachers systematically concentrate instruction on priorities identified by their state.

2. At root, the relatively weak traditional textbooks are a statement about what the textbook customers in the United States demand. States and school systems should use their considerable market power to influence textbook publishers to publish books that contain fewer topics that are developed through extended lessons that use problem-based learning. States such as California, North Carolina, and Texas, which have, as discussed in chapter 3, developed mathematically logical and focused frameworks like Singapore's, have sufficient market size to influence how publishers design textbooks.

3. Textbook publishers should create a market niche by producing mathematically strong textbooks, similar to Singapore's, that focus on a limited number of topics at each grade, that use rich problem-based learning to promote mathematical understanding, and that have extensive and challenging problem sets. The nontraditional textbook we examined proves that a textbook can develop a sizeable market share by offering something different that guides students' development of mathematical understanding through the systematic application of problem-based learning in real-world contexts. However, even one of the best U.S. nontraditional textbooks fails to expose students to the multistep, nonroutine problems found in the Singapore textbooks.

4. Professional organizations should develop a schema for rating textbooks along the lines developed and applied by AAAS (2000) in its assessment of the quality of middle school mathematics and science textbooks. Such a system, perhaps developed by NCTM, should include a very public set of criteria and scoring protocols, the periodic implementation of ratings, and the dissemination of results for each criteria to guide customers' and textbook publishers' decisions about what to buy and what to produce.

5. Textbook publishers, authors, and purchasers should find ways to increase the number, variety, and overall use of pictorial representations directly tied to concepts in textbooks;

increase the use of mathematical connections, particularly between and among mathematical concepts; and increase the number of multistep and nonroutine problems in U.S. textbooks to enhance the ability of these texts to develop stronger conceptual understanding and problem-solving capability.

CHAPTER 5. SINGAPORE AND U.S. MATHEMATICS ASSESSMENTS

CONTEXT

In general, state and national standards represent the intended curriculum, and textbooks represent the curriculum that is actually available to teachers on a daily basis. However, high-stakes state and national assessments most closely represent what is valued; thus, these tests strongly influence the implemented curriculum. As a measure of what is valued, an analysis of assessments provides important insights into the content that nations and states believe is important to teach and learn (Dossey, 1997). In Singapore, as we have seen, there is no appreciable difference between the curriculum as it appears in the framework and in textbooks, so it is no surprise that Singapore's assessments are also aligned. This is much less the case in the United States, where content gaps among frameworks, textbooks, and assessments are all too common and where, given the large number of topics that U.S. mathematics textbooks cover at each grade level, teachers base their selection of what to teach on their expectations for what will be on the assessments (Pedulla, Abrams, Madaus, Russell, Ramos, and Miao, 2003).

Assessments need to be evaluated according to the purpose they serve. Singapore's assessments are used primarily to evaluate individual student progress and to place students in educational streams according to their performance and abilities, but Singapore also uses assessments to judge the school's contribution to student performance. Singapore uses a value-added measure of contribution, and it rewards schools that perform substantially better than the entering scores of their students would indicate.

In contrast to Singapore's long tradition of national assessments, high-stakes assessments in the United States are a relatively new phenomenon and occur at the state rather than at the national level. Unlike Singapore, the United States uses its assessments primarily to hold schools, not students, accountable. NCLB's state-by-state annual testing requirements in grades 3–8 are intended to introduce a national school accountability strategy into the highly decentralized U.S. state assessment process and to increase the influence of assessments on classrooms. A comparison with Singapore's long-standing accountability system provides insights into how well a U.S. national accountability approach is able to succeed when it is embedded in a decentralized structure not designed for nationwide accountability.

An examination of the content of Singapore's assessments and its assessment process provides insight into expectations for students in a country with high levels of mathematics achievement. It also helps the United States assess whether its own assessments are sufficiently demanding. To the extent that Singapore's treatment of mathematical content in its frameworks and textbooks is more mathematically demanding than the comparable U.S. treatment, one would expect that Singapore's assessments would also be more mathematically demanding, an assumption that is borne out by our examination. Similarly, to the extent that the U.S. treatment of mathematical topics emphasizes a student's ability to explain his or her answers and solve examples of real-world problems, we also found that U.S. assessments are more likely to emphasize these skills.

We reached these conclusions by comparing Singaporean and U.S. assessment questions to answer the following questions:

- Are the differences between Singaporean and U.S. curriculum standards and textbooks also present in their assessments?

- At the system level, how do the purposes and the design of the Singaporean and U.S. assessment systems compare?

- At the test level, how do U.S. and Singaporean assessments compare in content balance, item type, and difficulty?

- At the individual question level, how do assessment questions that measure identical topics differ in mathematical difficulty?

CHARACTERISTICS OF ASSESSMENT SYSTEMS

Singapore and the United States have very different assessment systems. Singapore's assessments are administered primarily for the purposes of pupil accountability and placement. Singapore, despite strong Ministry of Education control over classroom content, does not have a common, annual exam at most primary grades. Singapore requires pupils to sit for a placement exam at the end of grade 4, but each school separately develops this exam. The grade 6 *Primary School Leaving Exam* (PSLE) is the only uniform nationwide exam administered in the primary grades. The PSLE is a high-stakes examination used to place the student "in one of the appropriate secondary school courses which matches his learning pace, ability and inclinations" (Ministry of Education, Singapore, 2003d). Pupils also take a rigorous national exit examination (O-level) at the end of secondary school, typically in grade 10, to demonstrate mastery of academic subjects. Mathematics achievement is a major component of the grade 4 exam, the PSLE, and the O-level exam.

In addition to using assessments for individual pupil placement and accountability, Singapore uses its grades 6 and 10 scores to reward high-performing schools that have achieved better-than-expected performance on value-added measures of school outcomes. The value-added measure of school performance quantifies the school's contribution to student outcomes. This is necessary, and desirable, because simple comparisons of the average outcomes of pupils at a school are not a good measure of school effectiveness; they do not account for differences in the entry performance levels of students. Schools may have higher student outcomes because they serve more able students, not because they are doing a better job of teaching students. A school's value-added performance is traditionally measured by the average absolute gain in its pupils' scores over two grades, which accounts for differences in initial scores.

Singapore adds to the traditional value-added measurement by using an adjusted measure that takes into account individual pupils' gain scores, as well as schools' gain scores. One expects that pupils with higher initial scores will make higher than average gains on two successive assessments than pupils with lower initial scores. To adjust for initial pupil differences, Singapore's value-added measure uses students' scores on the PSLE to predict their scores on grade 10 examinations, as shown in Exhibit 5–1. Singapore identifies all the pupils who earned a particular score on the PSLE, shown in column 1, and averages all their grade 10 scores. This average is the predicted score for pupils with the common PSLE score. This process is repeated to obtain a predicted grade 10 score for each PSLE score, as shown in column 2. A school then computes its net value-added score by the difference between the average actual grade 10 scores of its pupils (column 6) and the average of their predicted scores (column 4). If the net difference is positive, the school is doing better than

expected; if negative, it has not met expectations. Schools with the highest net scores receive performance bonuses (Tan, 1995). This type of accountability is more fair to schools than the accountability commonly practiced in the United States because U.S. accountability systems generally do not take students' starting scores into consideration.

Exhibit 5–1. Singapore's Value-Added Method of Rewarding School Success

Initial Gr. 6 Scores (1)	Predicted Test Scores in Gr. 10 Given Students' Gr. 6 Scores (2)	No. Pupils in School A by Each Initial Grade 6 Score (3)	Predicted Avg. Gr. 10 Score for Students in School A (4)	Actual Gr. 10 Scores for Students in School A, Given Their Gr. 6 Initial Scores (5)	Actual Avg. Gr. 10 Score for Students in School A (6)
99	215	19	215 x 19	210	210 X 19
98	210	17	210 x 17	207	207 X 17
97	208	20	208 x 20	205	205 X 20
…	…	…	…	…	…
71	165	11	165 X 11	155	155 X 11
70	163	10	163 X 10	145	145 X10
Average Test Score			Predicted Avg. Gr. 10 Score for School A = 201		Actual Avg. Gr. 10 Score for School A =190
			Net value added score 190 -201 = −11		

The United States, with much less national control over classroom content than in Singapore, has a great deal of testing, but primarily at the state rather than the national level. At the national level, the only U.S. assessment, the National Assessment of Educational Progress (NAEP), is currently a low-stakes examination administered at grades 4, 8, and 12 to evaluate national student performance and progress over time. NCLB now requires each state to participate in the reading and mathematics portions of NAEP in grades 4 and 8 as a condition of receiving federal education support (U.S. Congress, 2001). Although only a few thousand students in each state actually take NAEP, the results will be representative of the state. NAEP has been in place since the 1970s, but this is the first time that states are being required to compare state assessment results with NAEP results. In effect, NAEP is becoming a *de facto* national benchmark test for states, but it is not a high-stakes test for individual students or schools.

It is at the state rather than the national level that U.S. assessments serve a high-stakes accountability function. NCLB accountability provisions set national conditions that drive the states' accountability provisions, but the NCLB's approach to accountability is substantially different from Singapore's. NCLB provisions focus on holding schools accountable for student performance; in contrast, Singapore's assessment system places greatest emphasis on student performance for purposes of placement. Core NCLB accountability provisions require that all students in grades 3–8 be assessed annually in mathematics and reading and that each state set three performance levels, basic, proficient, and advanced, in each subject. Schools are held annually accountable for improving the proportion of their students who achieve at proficient or advanced levels on state assessments.

Because NCLB's goal for school improvement is that all students in every school be proficient by 2013–2014 (U.S. Congress, 2002), the proficient performance band has received almost all the attention in how state results are reported. Requiring schools to progress in terms of the proportion of their students achieving proficiency on state assessments is a worthwhile goal, but it does not measure a school's value-added performance in the way Singapore's system does. The U.S. system makes few allowances for low- or high-performing students. Schools that serve students with lower initial scores are put in the uncomfortable position of having to demonstrate a higher than average rate of improvement to achieve full proficiency of their students. The system also discourages higher performing schools from enrolling lower performing students, who might lower their average scores. These schools, unlike schools in Singapore, receive no adjustment in what is expected of them for having to educate lower performing students.

Another factor that differentiates the U.S. and Singaporean assessment systems is that each U.S. state chooses its own test and sets its own definition of proficiency. Separate state-by-state assessments have produced a lack of comparable results across states (U.S. Department of Education, 1999). NCLB's requirement that states participate in NAEP at grades 4 and 8 is intended to be a way for states to compare themselves against the uniform NAEP results and voluntarily modify their own assessments to achieve comparability (U.S. Department of Education, 2002).

The current lack of comparability in assessment results across states is seen in Exhibit 5–2, which compares the study's seven selected states on the percentage of their students who attain proficiency or higher on NAEP with the percentage achieving proficient or higher scores on each state's test. California and Florida have nearly identical NAEP results, but only 38 percent of California's students are proficient or better compared with 56 percent in Florida. Texas scores below Maryland on NAEP, but on state tests, the percentage of Texas students who are proficient or better is considerably higher than in Maryland. Because Singapore uses a common national assessment at grades 6 and 10, it does not have a problem of nonuniform assessments in which a student's results depend on the state in which the student was assessed. Singapore's uniform assessment process is clearly preferable on equity grounds and because it measures the value that schools add to what students already know.

Comparisons of Assessments

It is not only the assessments systems of the United States and Singapore that differ. Our comparison of sample items from Singapore's grade 6 PSLE, the grade 8 NAEP mathematics test, and various state assessments shows that the tests themselves also differ in structure, content covered, and difficulty.

Exhibit 5–2. Percent Proficient on NAEP Grade 8 Math and State Assessments

Selected States for Singapore-U.S. Analyses	NAEP Grade 8 Scores (% Students Proficient or Above) [1] (2003)	% Grade 8 Students Proficient or Above on State Assessments[2] (2004)
California	22	38
Florida	23	56
Maryland	30	46
New Jersey	33	67 [3]
North Carolina	32	N.A.
Ohio	30	No grade 8 test
Texas	25	57[4]
[1]NAEP 2003 Mathematics Assessment. Available at http://nces.ed.gov/nationsreportcard/mathematics/results2003/ [2]Education Week State Progress Reports as of September 2004. Available online at http://www.edweek.org/ew/vol-24/gallery/ayp.xls [3]Year is 2002 [4]Percent achieving at panel's recommended level		

The Singapore Ministry of Education provided us with a sample of mathematics questions of varying difficulty and covering a range of mathematical topics from the grade 6 PSLE, the only primary level exam that all Singaporean students take. We asked the Ministry to group the questions into three levels of difficulty: easy questions, which almost all students would answer correctly; moderately difficult questions, which average students would answer correctly; and difficult questions, which only the best students would answer correctly. We compared these questions with items from NAEP, which serves as a national benchmark for state-administered tests. Because NAEP is not administered in grade 6, we drew items from the grade 8 mathematics exam; items were drawn from the publicly available NAEP Data Tool (2004a). We compared items on the same mathematical topics by using items that were designated *hard* from the grade 8 NAEP and items from a range of difficulty levels on Singapore's grade 6 PSLE.[9] It should also be noted that since the NAEP items were drawn from the pool of released items, they represent items used over a span of years and may not adequately represent currently used items.

Selecting questions from state assessments for comparison purposes was more complicated. We focused on the seven states whose frameworks we analyzed in chapter 3. Because each state now has its own exam to fulfill federal assessment requirements under NCLB and because the stakes for these assessments are rising, states increasingly post sample test items on their websites, making it easier to examine actual or representative forms of state assessments. We determined which of the seven states had one or more sample assessments for grades 6 or 8 posted on the Internet so that we could match these items with PSLE or NAEP items. We eliminated two of the seven states: California because it has no published items available for these grades and Maryland because its assessment was in a state of transition while we carried out this study. We did find items from the following assessments:

- Florida Grade 8 FCAT (Florida Comprehensive Assessment Test)

---

[9] This Singapore-NAEP comparison is similar in purpose to a previous study that compared grade 8 NAEP questions with grade 8 questions on a national assessment from Japan, which, like Singapore, has performed well on TIMSS. The study concluded that the Japanese items were more mathematically difficult than NAEP items on similar topic (Dossey, 1997, p. 37).

- New Jersey Grade 8 Middle School Statewide Assessment

- North Carolina Grade 6 and Grade 8 End-of-Grade Test

- Ohio Grade 6 Proficiency Test

- Texas Grade 6 and Grade 8 TAKS (Texas Assessment of Knowledge and Skills)

We took the released items from these tests, the Singapore grade 6 PSLE, and the grade 8 NAEP and coded and compared each item by the following characteristics:

- Item type (multiple choice, short answer, enhanced short answer)

- Content area (number, measurement, geometry, data, algebra)

- Mathematical ability (conceptual, procedural, problem solving)

- Attribute (real-world context, solving for an unknown, multistep)

Although significant insights can be garnered from these comparisons, it is important to reiterate that the U.S. tests, both NAEP and the state assessments, are not entirely comparable with the Singapore test. The Singapore test is a placement test with very high stakes for the individual student. Because it is used to track students, it must contain some very difficult items so that schools can identify students who are capable of advanced work. In fact, unlike most U.S. assessments, where the fairness and developmental appropriateness of each item is a primary criterion for inclusion on the test, it is unlikely that Singapore expects all students to answer all questions correctly. The U.S. state tests, in contrast, are not placement tests. They are generally not used for tracking, but rather serve as accountability measures to check whether students are meeting a preset standard whose rigor varies significantly from state to state. NAEP, too, is not a placement test. But unlike the state tests, it assesses long-term trends and currently has little accountability impact. Although student motivation certainly plays a role on the Singapore and state tests, again, NAEP is a low-stress, low-stakes examination. Nonetheless, when seeking answers to the question about what mathematics content is valued, our analysis sheds important light on the similarities and differences between Singapore and the United States.

Exhibit 5–3 compares the grade 6 Singapore PSLE exam, the sample released assessments from the five selected states that release items for grades 6 or 8, and those from the grade 8 NAEP. The most striking difference between the Singapore assessments and the state tests is the balance of item types. Whereas 52 to 100 percent of the items on U.S. assessments are multiple choice (MC), only 31 percent of the items on the Singapore assessment are multiple choice. The Singapore assessment has more short answer (SA) items, which are essentially fill-in-the-blank questions, than the state tests and significantly more enhanced short-answer items (SA+), which require students to show their work and explain their reasoning in addition to providing an answer. More than one-quarter of the Singapore items are enhanced short answer. The Texas assessment uses only multiple-choice questions, making it the most unlike Singapore's, despite the similarity of its framework to Singapore's. Although U.S. standards and rhetoric speak of problem solving, reasoning, and communication, the assessments reveal far more attention to getting the right answers than those in Singapore.

Exhibit 5–3. Comparison of Assessment Items, by Type and Content Area, for Singapore, Selected States, and NAEP (Number and Percent of Items)

	Item Type**						Content Area									
	MC		SA		SA+		Number		Meas		Geom		Alg		Data	
Singapore – Gr. 6	15	31%	20	42%	13	27%	23	48%	10	21%	9	19%	3	6%	3	6%
Florida – Gr. 8	13	52%	12	48%	0	0%	8	32%	5	20%	4	16%	4	16%	4	16%
New Jersey – Gr. 8	34	85%	0	0%	6	15%	15	38%	5	13%	4	10%	9	23%	7	18%
N. Carolina – Gr. 6	80	100%	0	0%	0	0%	26	33%	16	20%	12	15%	16	20%	10	13%
N. Carolina – Gr. 8	80	100%	0	0%	0	0%	35	44%	14	17%	6	8%	12	15%	16	16%
Ohio – Gr. 6	34	74%	4	7%	8	17%	18	39%	6	13%	5	11%	8	17%	9	20%
Texas – Gr. 6	46	100%	0	0%	0	0%	24	52%	6	13%	7	15%	3	7%	6	13%
Texas – Gr. 8	50	100%	0	0%	0	0%	19	38%	8	16%	9	18%	5	10%	9	18%
NAEP* – Gr. 4	115	64%	66	36%	0	0%	75	41%	31	17%	29	16%	27	15%	19	10%
NAEP* – Gr. 8	129	60%	68	29%	0	10%	52	32%	30	15%	37	20%	48	20%	30	14%

*NAEP data based on entire item pool
**(MC – Multiple Choice, SA – Short Answer, SA+ – Extended Short Answer)

Singapore's assessments, like its framework and textbooks, focuses proportionately more on the number, measurement, and geometry strands and proportionately less on the data and algebra strands than do U.S. assessments. U.S. assessments, like Singapore's, do give the most emphasis to the numbers strand, but there is a much greater balance among the remaining four strands in the United States than in Singapore. Nowhere is this difference as great as in the algebra strand, including patterns and functions, to which Singapore devotes only 6 percent of its items compared with 23 percent of the New Jersey items.

Comparing assessments also reinforces earlier findings regarding the relative emphasis that Singapore and the United States give to each of the three mathematical abilities, conceptual understanding, procedural knowledge and problem solving, as seen in Exhibit 5–4. These three areas are the primary foci of NAEP in that they gauge the general mental abilities associated with mathematics. Conceptual understanding is demonstrated when students apply concept definitions, relations, or representations. Students demonstrate procedural knowledge when they select and apply appropriate procedures and show the ability to reason through a situation. Finally, problem solving requires students to apply and connect their mathematical knowledge of concepts, their procedural knowledge, and their reasoning and communication skills to solve problems. Whereas most of the U.S. assessments, with the notable exception of the Ohio assessment, tend toward a one-third, one-third, one-third balance among these three processes, Singapore's assessment subordinates conceptual understanding and focuses in a balanced manner on procedural skills and problem solving. This allocation is consistent with Singapore's mathematical priorities, which stress both procedural and problem-solving skills.

Exhibit 5–4. Comparison of Assessment Items, by Mathematical Ability and Attribute, for Singapore, Selected States, and NAEP (Number and Percent of Items)

| | Mathematical Ability | | | | | | Attribute | | | | |
	Conceptual		Procedural		Problem Solving		Real-World Context		Multistep		Finding an Intermediate Unknown	
Singapore – Gr. 6	9	19%	21	44%	18	38%	27	56%	12	25%	9	19%
Florida – Gr. 8	8	32%	11	44%	6	24%	17	68%	3	12%	2	8%
New Jersey – Gr. 8	15	38%	13	33%	12	30%	26	65%	13	33%	0	0%
N. Carolina – Gr. 6	21	26%	32	41%	27	33%	38	48%	6	8%	0	0%
N. Carolina – Gr. 8	26	30%	50	57%	11	13%	39	48%	4	5%	4	5%
Ohio – Gr. 6	5	11%	19	41%	22	48%	27	59%	8	17%	2	4%
Texas – Gr. 6	15	33%	14	30%	17	37%	32	70%	3	7%	1	2%
Texas – Gr. 8	18	36%	20	40%	12	24%	35	70%	3	6%	1	2%
NAEP* - Gr. 4	73	40%	50	28%	58	32%	91	50%	28	15%	7	4%
NAEP* - Gr. 8	74	38%	59	30%	64	32%	92	47%	42	21%	15	8%
*NAEP data based on entire item pool												

Exhibit 5–4 also shows that Singapore expects greater rigor and depth in mathematical knowledge; Singapore's assessment includes significantly more multistep items and significantly more items requiring finding an unknown than do most U.S. tests. Multistep items test students' understanding of mathematics by requiring students to reason and complete multiple steps to obtain the correct solution, and they often require students to use information from an initial step to solve other parts of the problem. These differences are in part the result of NAEP and state policy of generally not including items with very high (or very low) p-values and there are significant differences in the demands made of test items.

Overall, Singapore's assessments have more of the earmarks of a challenging assessment than do the U.S. assessments. Singapore's test questions are less likely to be multiple choice and more likely to require students to produce short or extended answers. The questions are also more likely to involve solving for intermediary unknowns in developing solutions.

In-Depth Comparisons of Individual Assessment Items, by Topic

The significant differences we found in the overall mathematical depth and complexity of the items on the assessments are even clearer when in-depth comparisons are made among items intended to assess equivalent mathematical topics. Individual items addressing common mathematical topics were selected from the Singapore PSLE, NAEP, and the upper elementary state assessments to examine the difficulty in the mathematics assessed.

We selected items on nine mathematics topics, representing all five NCTM content strands, from the sample PSLE items made available to us by the Singapore Ministry of Education, which classified these items by level of difficulty: easy, moderate, and hard. All students are expected to solve the easy items. The average student should be able to solve those of moderate difficulty. The better students should be able to solve hard items. We selected items from all three levels of difficulty.

American Institutes for Research®

For each of the nine selected Singapore items, we selected a comparable item from the released items from the grade 8 NAEP. NAEP classifies items as easy, medium, and hard on the basis of the percentage of students answering a problem correctly. For purposes of comparison with the Singapore questions, we used only items that NAEP classifies as hard, meaning that fewer than 40 percent of test takers answered them correctly. We selected only the most difficult questions so that the U.S. questions would have the best chance to be at least as difficult as Singapore's in terms of the demands made on students' mathematical skills, especially given that some of the selected Singapore questions were deemed by Singapore to be only moderately difficult.

Finally, we selected items on the same topics from the state tests, selecting one grade 6 and one grade 8 item for each topic. Unlike Singapore and NAEP, the states do not rate their items by difficulty. We attempted to select the most difficult items from the state tests, looking at the number of steps and the routineness of the solution.

Each item in the nine problem sets was coded by the approximate number of solution steps required to get a scorable result, the need to solve for an intermediate unknown, and the routine nature of the problem, as three measures of the level of challenge. Problems that require more steps demand greater reasoning skills. Solving for an intermediate unknown means that all the information required to solve the problems is not stated up front in the problem. Instead the student has to determine how to derive the needed information as part of developing the solution. Nonroutine problems do not have obvious solutions from the statement of the problem, such as mechanically applying a definition or formula to find the answer.

The process used to approximate the number of steps to solve a problem is exemplified by the first problem discussed, wherein students need to

- first, find the area of the larger semicircle using the formula $\frac{1}{2} \pi r^2$ to get 32π;

- second, determine that radius of the smaller semicircle to be 4 (an intermediate unknown);

- calculate the area of the smaller semicircle to be 8π; and

- sum the areas of the two semicircles to arrive at an answer of 40π.

1. Area

Singapore Grade 6 (moderate)

Question: This figure is made up of two semicircles. Q is the centre of the larger semicircle of radius 8 cm.
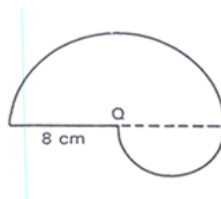
The area of the figure is _____ cm^2.

(1) 12π (2) 40π (3) 48π (4) 80π

- About 4 steps

- Intermediate unknown (radius of smaller circle)

- Nonroutine

Solution: The Singapore test item expects students to find the area of a figure made up of semicircles when the radius of the smaller semicircle is not immediately evident. Students need to understand that because Q is the center of the larger semicircle, the radius of the smaller semicircle is half the radius of the larger semicircle. Students also need to know how to use the formula for the area of a circle.

Texas Grade 6

Question: A family put a rectangular patio in their backyard and planted grass in the rest of the yard. The rectangular backyard is 100 feet by 80 feet, and the patio is 13 feet by 8 feet. What is the area of the backyard that is planted with grass?

Backyard

80 ft

8 ft ◆ 13 ft

100 ft
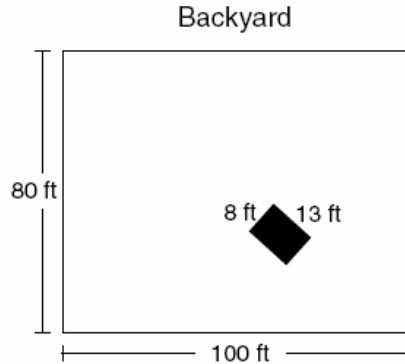
(F) 402 sq ft   (G) 7,896 sq ft   (H) 8,000 sq ft   (J) 8,104 sq ft

- About 3 steps

- No intermediate unknown

- Routine

Solution: Solving requires computing the areas of two rectangles and recognizing that the area of the patio has to be subtracted from the area of the yard.

New Jersey Grade 8

Question: The floor of an entranceway and corridor in an office building is to be covered with vinyl flooring. Find the number of square yards of flooring that will be needed. Use the diagram provided in your answer folder to show how you found the area of the floor. Show all your work.
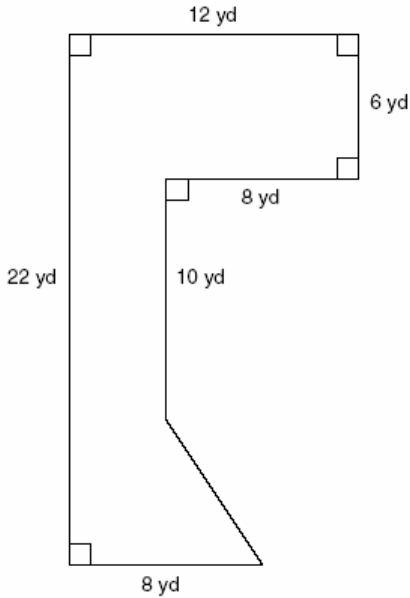
- About 6 steps

- Intermediate unknown (dimensions and areas of composite parts)

- Nonroutine

Solution: Students must break the figure up into four smaller figures. They must use given information about the lengths of some of the figure's sides to determine the length of unknown sides. Students must know the properties of squares and triangles to reach the final answer.

NAEP Grade 8 (Hard: 29 percent correct)
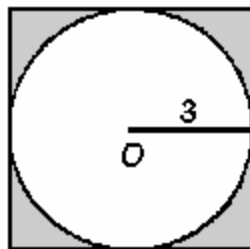
Question: In the figure above, a circle with center O and radius of length 3 is inscribed in a square. What is the area of the shaded region?

(A) 3.86   (B) 7.73   (C) 28.27   (D) 32.86   (E) 36.00

Did you use a calculator on this question? __Yes __No

- About 4 steps

- Intermediate unknown (area of the circle)

- Routine

Solution: The problem involves computing the unknown length of the side of the square by doubling the radius of the circle. Otherwise it requires straightforward computation: computing the area of a square, computing the area of the circle, and subtracting the area of the circle from the area of the square.

Discussion of Area Problems: Singapore's moderately difficult grade 6 problem and the New Jersey grade 8 problem have the most steps and are the most complicated to solve. The New Jersey problem has a number of parts and requires students to compute and use unknowns. The NAEP item also involves unknowns, but the computation is straightforward. The grade 6 Texas problem is the easiest, having the fewest steps, no unknowns, and a straightforward application of the formula for the area of a rectangle. Overall, the Singapore and the New Jersey problems are the most complicated because the pieces of the solution require strategic thinking about how to break the geometric figures in the diagram into the appropriate areas that must be computed. The Singapore problem is somewhat more complex because the figures are only half the areas computed by the circle formula, whereas the New Jersey problem requires only computing areas of rectangles or triangles. The areas in the NAEP and Texas problems are obvious from the diagram.

2. Volume

Singapore Grade 6 (easy)

Question: Global pours an equal amount of water into two empty P and Q tanks shown below.

If tank P is half-filled, what is the height of the water level in Tank Q?

A) 5 cm   B) 6 cm   C) 8 cm   D) 10 cm

- About 4 steps

- Intermediate unknown (amount of water)

- Nonroutine

Solution: Solving the problem requires students to know that the volume differences for a given height of water are related to the differences in the unknown areas of the bottom of the containers and that they must solved for these unknowns.

Ohio Grade 6

Question: For a fundraiser, the school is planning to sell popcorn at the next open house. The two bags they will use to sell popcorn are shown below. They would like to charge $1.00 for the small bag of popcorn. How much do you think they should charge for the large bag of popcorn based

on the volumes of the two bags? In the **Answer Booklet** explain your answer in detail and give reasons to support your answer.
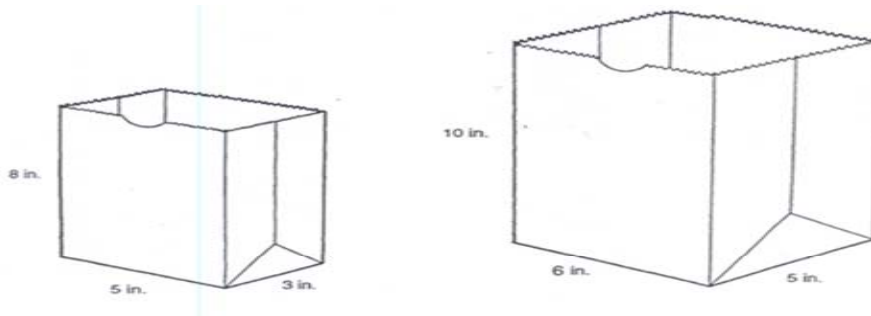
- About 3 steps

- No intermediate unknown

- Routine

Solution: The problem is a straightforward application of the formula for volume, although it does add some difficulty by requiring students to use the ratio of the larger to smaller volume to find the price.

Texas Grade 8

Question: A shipping company sells two types of cartons that are shaped like rectangular prisms.
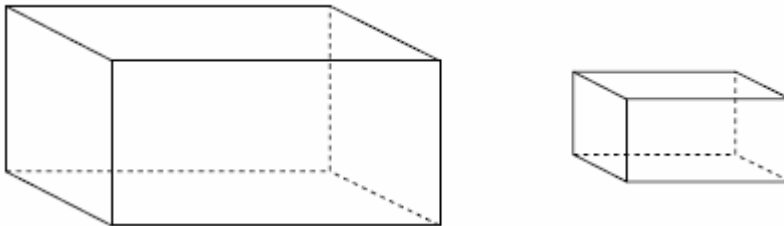
The larger carton has a volume of 720 cubic inches. The smaller carton has dimensions that are half the size of the larger carton. What is the volume, in cubic inches, of the smaller carton?

A* 90 in.^{3} **B** 120 in.^{3} **C** 240 in.^{3} **D** 360 in.^{3}

- About 2 steps

- Intermediate unknown (dimensions of larger carton if student does not recognize that one-half cubed equals one-eighth)

- Nonroutine

Solution: The problem uses the standard volume formula, but the application is not straightforward because the lengths of the sides are not known. The student must realize that plugging in one half of each of the unknown dimensions results in one-eighth the original volume.

NAEP Grade 8 (Hard: 13 percent correct)

Question: A cereal company packs its oatmeal into cylindrical containers. The height of each container is 10 inches and the radius of the bottom is 3 inches. What is the volume of the box to the <u>nearest </u>cubic inch? (The formula for the volume of a cylinder is $V = \pi r^2 h$.)

Answer:_____ cubic inches.

Did you use a calculator on this question? ___ Yes ___ No

- About 2 steps

- No intermediate unknown

- Routine

Solution: The problem requires a straightforward application of the formula for the volume of a cylinder, except that rounding the answer is required.

Discussion of Volume Problems: Given that Singapore's framework emphasizes volume more than U.S. frameworks do, it is not surprising that even an easy Singapore test item is richer, involves the most steps, and is less routine than those typically found in the United States. Although all the sample items assess an understanding of volume, the Singapore problem requires that students break the volume formula into area multiplied by height and separately compute the area. It also requires that students understand that the bases of the two tanks have different areas. Because the problem can be computed in several ways, students who are good at computation could simplify the arithmetic by computing appropriate individual ratios. The Texas grade 8 test item also uses the volume formula but requires students to know how to compare the volumes of two boxes when the dimensions are not known but information on the ratio of dimensions is known. The Ohio grade 6 and the NAEP grade 8 items require little more than procedural knowledge of formulas for the volume of a rectangular solid or cylinder. In summary, the Singapore problem requires analyzing the details of the problem to realize that the key to its solution it to compare the ratio of the areas of the two containers. The Texas problem is also challenging in that students must realize that the volume formula produces results in the form of the cube of the ratios, although this follows straight from the formula, unlike the Singapore volume problem. NAEP and the grade 6 Ohio assessment basically require only an application of the formula.

3. Mean/Average

Singapore Grade 6 (hard)

Question: Mary spilled some soya sauce on her Continual Assessment Results Slip, as shown below. Part of her English and Mathematics marks cannot be seen.

```
                    RESULTS SLIP


     English          8
     Mathematics      7
     Science          92
```

Her average score is 84 marks. What can be the largest difference between her English and Mathematics marks?

- About 5 steps

- Intermediate unknown (the total of the marks)

- Nonroutine

Solution: The problem requires an unusual representation of the formula for the mean involving three numbers, two of which have both unknown and known digits. The expression is simplified to find the sum of the two numbers with the unknown digits. Students then maximize the difference between numbers, given the known sum and the known first digits.

North Carolina Grade 6

Question: Jason had these scores on six math quizzes: 79, 80, 85, 70, 100, and 75. Sean's scores on the first five quizzes were 80, 83, 75, 95, and 76. What score does Sean need on his sixth quiz to give him the same average as Jason?

(A) 79   (B) 80   (C) 81   (D) 82

- About 3 steps

- No intermediate unknowns

- Routine

Solution: This problem requires more than a simple application of the formula for finding an average. Students must find the unknown number by understanding that the averages will be the same for two sets of six numbers only if the sums are identical.

New Jersey Grade 8

Question: Erin calculated the mean of 5 numbers to be 38. Then she found that she had made an error and had written 40 for one of the numbers when she should have written 30. What is the mean of the correct 5 numbers?

A. 28   B. 30   C. 36   D. 40

- About 3 steps

- No intermediate unknown

- Nonroutine

Solution: The problem requires a deep understanding of the formula for the mean because no individual numbers are given, only a prior average for five numbers and a change in one number. Students have to know that the changed number reduces the unknown numbers sum by the amount of change while reducing the mean by the changed sum divided by the number of items.

NAEP Grade 8 (hard: 19 percent correct)

Score	Number of Students
90	1
80	3
70	4
60	0
50	3

Question: The table above shows the scores of a group of 11 students on a history test. What is the average (mean) score of the group to the nearest whole number?

- About 8 steps

- No intermediate unknown

- Routine

Solution: Solving the problem requires only a straightforward application of the formula for the mean and a knowledge of how to round.

Discussion of Mean/Average Problems: Despite the fact that data and statistics are not emphasized in Singapore's curriculum in grades 1–6, comparing test items reveals significant differences in depth and difficulty between the U.S. and Singaporean assessments. The hard Singapore grade 6 problem has the most steps and involves computing an unknown as an intermediate step. The Singapore item also requires an understanding of how to solve a problem involving the mean when some numbers have both known and unknown digits; the nonroutine solution requires students to simplify the expression for the sum of the numbers with the unknown digits and then use the expression. The nonroutine New Jersey grade 8 problem, although less complicated than the Singapore problem, involves computing a new mean when the change in numbers is known but not the numbers themselves. North Carolina's grade 6 item also requires more than a simple application of the mean formula because it involves understanding how the sum of the numbers affects the mean and how to find the maximum difference. The grade 8 NAEP item requires

only a straightforward application of the mean formula and is the least complex of the four. Overall, the Singapore problem is by far the most analytically challenging with no clear easy path presented for a solution. New Jersey's grade 8 problem is also novel in its computation of a new mean on the basis of a change in numbers when the numbers themselves are unknown, but this problem is not nearly as open-ended as the Singapore problem. The NAEP grade 8 problem is a simple and straightforward formula application and is actually easier than North Carolina's grade 6 problem, which requires reasoning that two averages with the same number of items are equal if their sums are equal.

4. *Percent*

Singapore Grade 6 (hard)

Question: At Mrs. Ong's shop, there were two vases for sale at $630 each. She sold one of them at this price and earned 40 percent of what she paid for it. She sold the other vase later at a 20 percent discount. If the two vases had the same costs, how much did Mrs. Ong earn altogether?

- About 6 steps

- Intermediate unknowns (original prices)

- Nonroutine

Solution: The problem requires students to represent the expression for earnings as the sum of item price, less item cost. The cost of one vase can be solved from the information in the problem, which also gives the cost of the second vase. The price of the reduced item can also be solved from the information on the percent reductions. The information on prices and costs can then be entered into the earning's formula to obtain total earnings.

Ohio Grade 6

Question: Elliot wanted a 26-inch bike at Bill's Bike Shop; he looked at a regularly priced $120 bike marked ¼ off. In addition to the discount, the store was offering a $10 rebate. At Barney's Bike World, the same bike was advertised at 20% off the regular price of $100. From which store should Elliott buy his bike and why? Explain your answer or show your work.

- About 2 steps

- No intermediate unknown

- Routine

Solution: The fairly routine problem requires computing two different percent reductions in price and subtracting an additional $10 discount from one of the reductions. The problem is somewhat complicated by stating reductions first as fractions and then as percents.

North Carolina Grade 8

Question: The manufacturer's price for a tent is $28.50. A camping store that bought the tents sells them for $40.76. *Approximately* what was the percent of increase in the price of the tents?

(A) 43.0%   (B) 57.0%   (C) 75.4%   (D) 76.0%

- About 2 steps

- No intermediate unknown

- Routine

Solution: Students have to compute only the increase and divide by the base price.

NAEP Grade 8 (hard: 16 percent correct)

Question: If the price of a can of beans is raised from 50 cents to 60 cents, what is the percent increase in the price?

(A) 83.3%   (B) 20%   (C) 18.2%   (D) 16.7%   (E) 10%

- About 2 steps

- No intermediate unknowns

- Routine

Solution: The problem is the same as North Carolina's grade 8 problem, but with easier computations.

Discussion of Percent Problems: The Singapore percentage problem is a complicated price problem involving multiple steps and the calculation of unknowns. The problem necessitates that students understand the relationship between price and cost and that they be able to use that relationship to compute unknown costs. Students must then apply the notion of percentage reduction to compute another unknown price and calculate earnings, given prices and costs. Ohio's grade 6 problem is slightly more complicated than the other U.S. problems in that it states reductions as both fractions and percents and it adds another step, requiring students to compute a discount. Compared with the Singaporean complex multistep problem with unknowns, the Ohio problem is quite routine. The North Carolina grade 8 problem is a straightforward application of a formula. Finding the answer to the NAEP question about percentage change requires only a simple calculation of a formula for percentage and even simpler math than the North Carolina item, and although the arithmetic is simple, students are allowed to use calculators. In this problem set, only the Singapore grade 6 percent problem requires substantial strategic thinking. The information in the Singapore problem must be used in the right sequence to identify and calculate the key intermediate unknowns that need to be calculated, and then the results of this calculation are used to compute the unknowns in the final answer.

5. Spatial Sense (no grade 6 U.S. state problem available)

Singapore Grade 6 (Easy)

Question:

8   This figure shows a solid.

Which one of the following is a net of the solid?

(1)
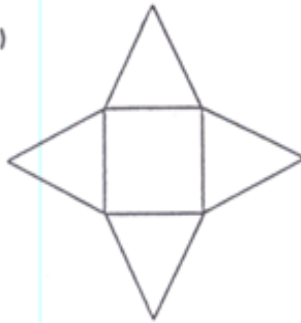
(2)
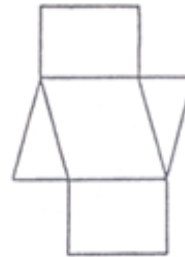
(3)
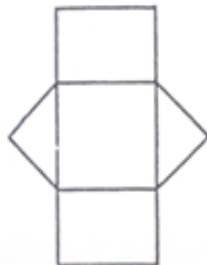
(4)
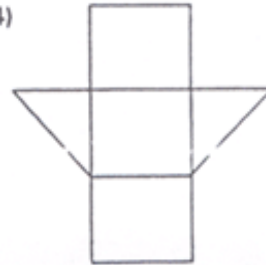
- About 2 steps

- No intermediate unknown

- Nonroutine

Solution: Although the problem requires only unfolding the three-dimensional figure and matching its shape to one of the offered choices, the figure is very unusually shaped. The correct answer choice shows the original figure flipped over.

Texas 8th Grade

Question: The picture below shows a toolbox with a black handle.
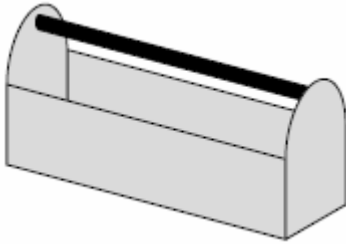
Which drawing best represents a top view of the toolbox?

F

H

G

J

- About 1 step

- No intermediate unknowns

- Routine

Solution: Getting the answer requires only that students imagine a simple perspective view of the top of a toolbox.

NAEP Grade 8 (Hard: 45 percent correct)

American Institutes for Research®
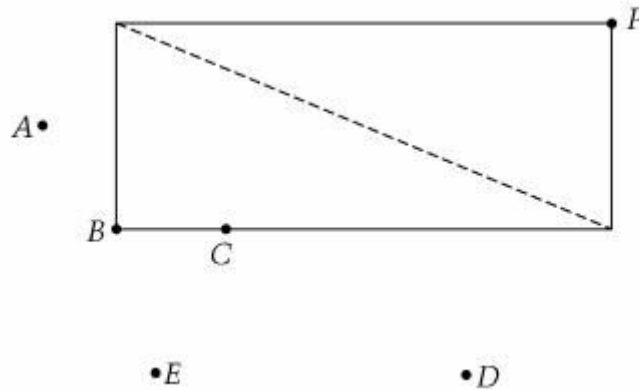
Question: When the rectangle above is folded along the dotted line, point *P* will touch which of the lettered points?

(A) A   (B) B   (C) C   (D) D   (E) E

- About 2 steps

- No intermediate unknowns

- Routine

Solution: Solving the problem requires students to visualize a fold along the diagonal of a simple rectangle and realize that only one point will match.

Discussion of Spatial Sense Problems: All three problems assess spatial sense. The Singapore problem tests students' ability to visualize a two-dimensional shape from its three-dimensional faces. The problem contains an unusual three-dimensional shape, and accurate visualization is required to translate the figure from three to two dimensions. The Texas grade 8 problem calls only for a simple identification of the top view of a toolbox. The NAEP problem, although more difficult than the Texas item in that a traditional rectangular figure must be folded along the diagonal to locate a new outer point, requires much less spatial visualization than Singapore's complicated figure. The lack of a single spatial problem among the Florida, Ohio, North Carolina, and Texas grade 6 sample items indicates that this topic does not receive much emphasis in the United States, a possible explanation for the particularly low U.S. ranking on geometry. A comparison of the three more difficult spatial problems reveals that only the Singapore problem requires the application of a spatial concept to a novel and unfamiliar situation.

6. Angles

Singapore Grade 6 (moderate and difficult)

Question: In the figure, ABCE and CDEF are rhombuses.

(a) Name the angle in the figure which is equal to angle CDE. (moderate)

(b) Find the size of the angle AEF. (difficult)
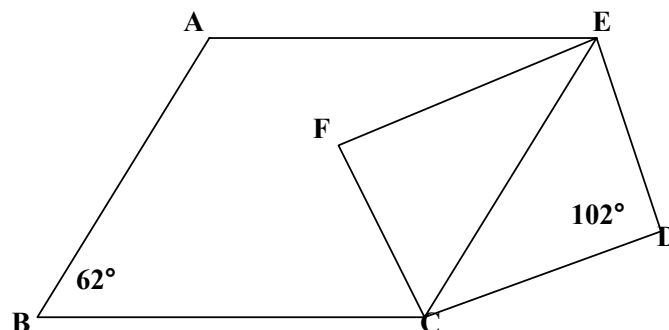
- About 5 steps

- Intermediate unknown (Angle FEC)

- Nonroutine

Solution: This is a difficult problem requiring the student to find unknown angles using the properties of rhombi and triangles. Part B has many steps and several intermediate unknowns. Its solution involves identifying the overlaid rhombuses and using the facts that opposite angles of a rhombus are equal, the sum of the angles is 360º, and the diagonals of a rhombus bisect the vertex angles. The intermediate steps include a lengthy reasoning chain. Students must see that ∠CFE is the same size as ∠CDE, then use the measure of ∠CDE to find the measure of ∠FED, divide ∠FED in half to find the measure of ∠FEC, then use ∠ABC to find the size of ∠AEC, and finally subtract out ∠FEC to get ∠AEF.

Texas Grade 6

Question: Find the measure, in degrees, of ∠2 in square *WXYZ*.
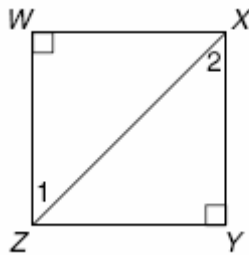
- About 1 step

- No intermediate unknown

- Routine

Solution: All students have to know is that a square has angles of 90 degrees and that the diagonal divides the angles in half.

North Carolina Grade 8

Question: ΔABC ≅ ΔDEF with a right angle at B and ∠C = 42º.

American Institutes for Research®
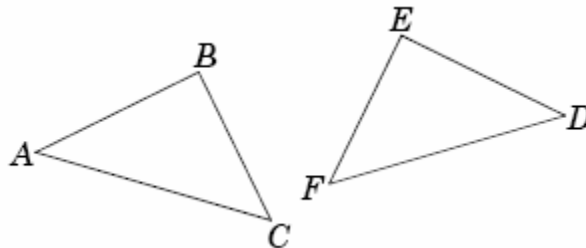
What is the measure of ∠D?

(A) 42º   (B) 48º   (C) 58º   (D) 90º

- About 2 steps

- Intermediate unknown (measure of Angle A)

- Routine

Solution: To find an intermediate unknown angle, this problem requires knowing that the sum of the angles of a triangle is 180 degrees and that a right angle is 90 degrees. Students have to appropriately match a similar triangle that is deceptive in that one triangle has to be flipped to see the similarity.

NAEP Grade 8: (33 percent correct)

Question: In the triangle, what is the degree measure of ∠ABC?
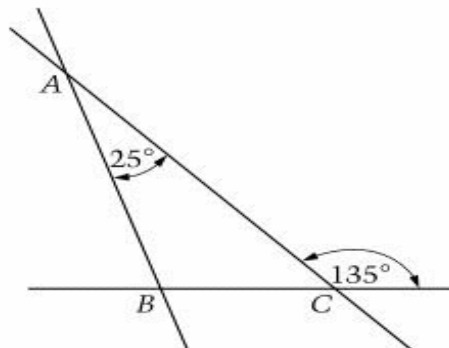
A) 45   B) 100   C) 110   D) 135   E) 160

- About 2 steps

- Intermediate unknown (measure of Angle ACB)

- Routine

Solution: Students solve the problem by using the knowledge that the angles around a straight line add up to 180 degrees to find an intermediate unknown and using the knowledge that the sum of the angles of a triangle is also 180 degrees to find the remaining angle.

Discussion of Angles Problems: All the problems involve finding an unknown angle. The Texas grade 6 item requires only a straightforward application of knowledge of the basic features of a square and its 90-degree angles. The North Carolina and NAEP grade 8 problems are similar in difficulty, each involving unknown angles in a triangle with an additional step for similar triangles or the sum of angles around a line. The Singapore question embeds conceptual knowledge about angles and about computing angles in two overlapping rhombuses. It further requires knowing the sum of the angles of a rhombus and the sum of the angles of a triangle and then using this information to

calculate unknown angles in the figure. Comparing the difficulty among the four problems shows that the Singapore problem-solving strategy is not at all obvious or straightforward. The solution to other three angle problems requires a simple application of angle formulas, although the NAEP problem does involve an intermediate unknown.

7. Rate or Distance

Singapore Grade 6 (difficult)

Question: Lee and Chan drove from Town P to Town Q. They started their journeys at different times. Lee drove at an average speed of 45 km/h and took 40 min. Chan drove at an average speed of 72 km/h and reached Town Q at the same time as Lee.

a) How far was Town P from Town Q?

b) How many minutes later than Lee did Chan start his journey?

- About 5 steps

- Intermediate unknown (how long Chan drove)

- Nonroutine

Discussion: The problem requires a thorough understanding of the concept of rate. Students must translate Lee's rate from minutes to hours (or vice versa) and use this information to calculate distance. Knowing the distance and the faster car's speed, students can calculate the time to travel the distance. The differences in times are then used to compute the second start time relative to the first.

North Carolina Grade 6

Question: The distance to the beach is fifty miles more than twice the distance to the mountains. If *d* represents the distance to the mountains, which expression represents the distance to the beach?

(A) 50*d*   (B) 2*d* + 50   (C) 50*d* + 2   (D) 100*d* + 50

- About 1 step

- No intermediate unknowns

- Routine

Solution: The problem requires only a simple expression for distance involving no calculations.

Florida Grade 8

Question: A car traveling at a certain speed will travel 87 feet per second. How many yards will the car travel in 90 seconds if it maintains that same speed?

(A) 87 yards   (B) 653 yards   (C) 2,610 yards   (D) 7,830 yards

- About 1 step

- No intermediate unknown

- Routine

Solution: A simple application of the rate, time, and distance formula is made slightly more complicated by the need to translate distance from feet to yards.

U.S. NAEP Grade 8 (12 percent correct)
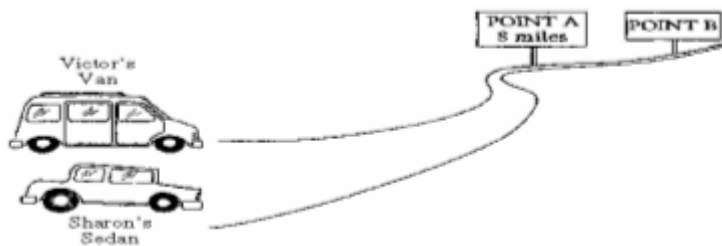
Question:

a) Victor's van travels at a rate of 8 miles every 10 minutes. Sharon's sedan travels at a rate of 20 miles every 25 minutes. If both cars start at the same time, will Sharon's sedan reach point A 8 miles away, before, at the same time, or after Victor's van? Explain your reasoning.

b) If both cars start at the same time, will Sharon's sedan reach point B (at a distance further down the road) before, at the same time, or after Victor's van? Explain your reasoning.

- About 2 steps

- No intermediate unknowns

- Routine

Solution: The NAEP problem involves a relatively straightforward computation of distance as the ratio of known distance to time. To obtain the answer to both parts of the question, students simply compute rates of speed of each car, which turn out to be identical. The NAEP question is open-ended, requiring students to explain their answers.

Discussion of Rate or Distance Problems: All but the grade 6 North Carolina distance problem involve using a formula: distance equals rate multiplied by speed. The NAEP problem involves a very simple computation of two rates (i.e., distance divided by time), showing that they are equal. The Florida problem is a simple application of the distance formula, but requires translating from feet into yards. The Singapore grade 6 problem involves a much more difficult test of students' understanding of the distance formula than the other three. It requires students to carefully represent the information in the problem and then reason through multiple steps and solve for an unknown. The Singapore problem requires a firm understanding of how to apply the formula relating rate, distance, and time to reason through the information in the problem, solve for the

required intermediate unknowns, and eventually derive answers to the distance and time questions. The other three problems basically involve only routine applications of the distance formula.

8. Data Analyses

Singapore Grade 6 (hard)

Question:

Number of children per family
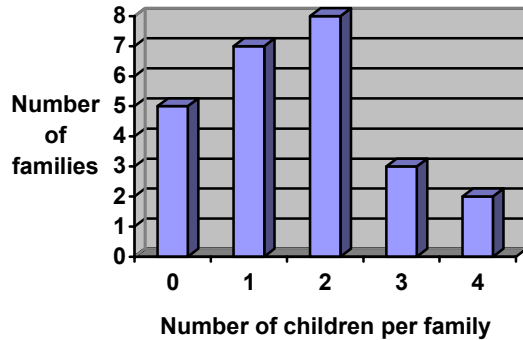
The graph shows the number of children per family in a housing estate.

(A)   What is the total number of children in the housing estate?

(B)   There are 25 families in the estate. What percentage of the families in the estate has fewer than 3 children?

- About 4 steps

- No intermediate unknowns

- Routine

Solution: The Singapore problem is a straightforward reading of a chart. The first answer requires simply that students multiply the number of children per family by the number of families and sum across different-sized families. The second problem is also a simple computation in which students determine what percentage of the total number of families have fewer than a given number of children.

Texas Grade 6

Question: Sam kept a list of the points made by the highest-scoring basketball players at school. He used the scores to make a stem-and-leaf plot, as shown.

Stem	Leaf
1	4 9
2	0 5 6 8 8

What is the median of this set of points?

(F) 14   (G) 23   (H) 25   (I) 28

- About 2 steps

- No intermediate unknowns

- Routine

Solution: Students must do a straightforward median computation, which is made more difficult by having to interpret a stem-and-leaf diagram.

North Carolina Grade 8

Question: Scoring leaders in the NBA for 1986—1995 had these averages: 30, 37, 35, 33, 34, 32, 30, 33, 30, 29.

Which box plot accurately illustrates these averages?

- About 4 steps
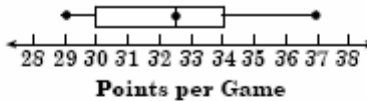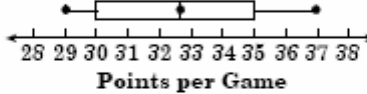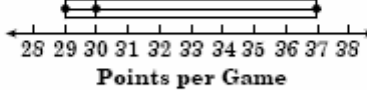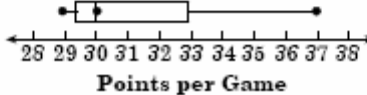
- No intermediate unknowns

- Routine

Solution: From a box-and-whisker diagram, students must compute all its parts: median, inter-quartile range, and range.

NAEP Grade 8 (Hard: 2 percent correct)

METRO RAIL COMPANY	
Month	Daily Ridership
October	14,000
November	14,100
December	14,100
January	14,200
February	14,300
March	14,600

Question: The data in the table above has been correctly represented by both graphs shown below.
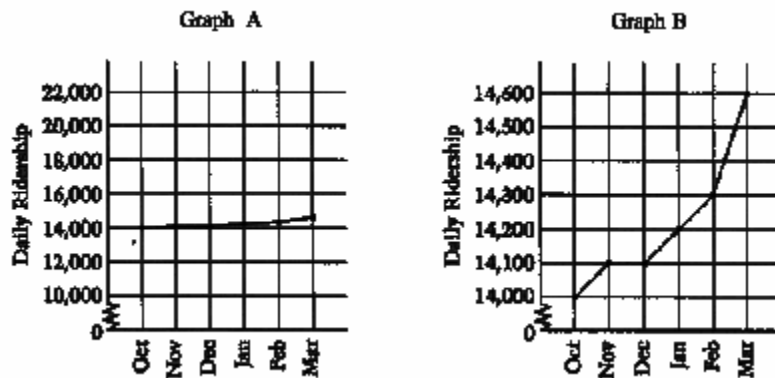
(A)   Which graph would be best to help convince others that the Metro Rail Company made a lot more money from ticket sales in March than in October? Explain your reason for making this selection.

(B)   Why might people who thought that there was little difference between October and March ticket sales consider the graph you chose to be misleading?

- About 1 step

- No intermediate unknowns

- Routine

Solution: The problem requires a fairly straightforward interpretation of data and line graphs of the same data in which different ranges on the vertical axis change the slope and the look of the graph.

American Institutes for Research®

Discussion of Data Analysis Problems: None of the four problems requires more than a routine solution. Although Singapore considers its problem hard, it requires only knowing different ways to read a bar chart. The high difficulty level for this problem is evidence that Singaporean students do not practice the interpretation of real-world evidence to the same degree that they practice more theoretical mathematics. The Texas problem requires students to use data in a stem-and-leaf diagram to compute a median. The North Carolina problem includes a more complicated box chart, but it involves no interpretation. The NAEP problem requires interpretation, but it is quite simple and no calculations are involved. Comparing the applied data questions, we find all four examples to be relatively straightforward analyses of the information in the charts. The Singapore problem does require extensive arithmetic, which must be done without the aid of a calculator. Singapore assesses upper primary-grade students' arithmetic skills not by using separate arithmetic problems but by imbedding arithmetic in more advanced questions.

9. Algebra and Functions

Singapore Grade 6 (moderate)

Question: Jim uses 1-cm cubes to build a solid. He starts with a square base using 100 cubes. The second layer of 81 cubes and the third layer of 64 cubes are also arranged in the form of squares. The figure shows the three layers of cubes.
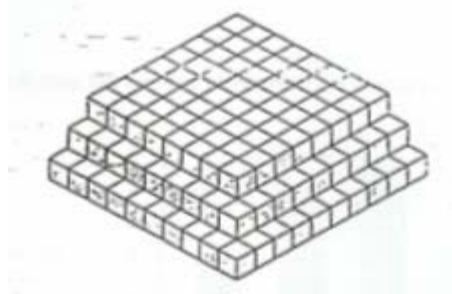
Jim continues building the solid in the same way.

a) How many cubes will there be in the next layer?

b) If the topmost layer consists of 9 cubes, how many layers does the solid have?

- About 5 steps

- Intermediate unknowns (identifying the pattern)

- Nonroutine

Solution: This nonroutine geometric pattern problem requires students to realize that the number of cubes equals the square of the number on one side. The problem is also made more complicated by giving students information about the number of cubes in the topmost layer so that they must determine the number of the layer and then use that information to find the total number of layers.

Texas Grade 6

Question: What is the rule to find the value of a term in the sequence below?

Sequence	
Position, n	Value of term
1	1
2	4
3	7
4	10
5	13
n	?

(F) $n + 3$ (G) $3n-2$ (H) $3n$ (J) $n-2$

- About 1 step

- No intermediate unknown

- Routine

Solution: Students must identify a simple sequence by plugging it into different formulas.

New Jersey Grade 8

Question: The first four pentagonal numbers are indicated below. The numbers are calculated by counting the dots. Based upon the pattern indicated, what is the **sixth** pentagonal number?

1   5   12   22
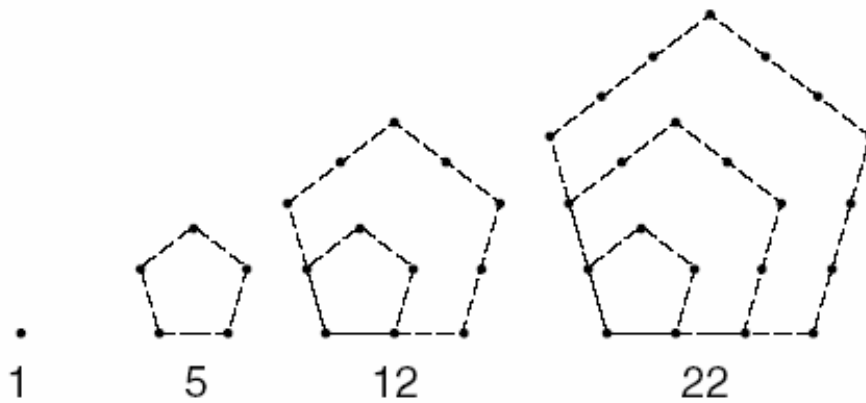
(A) 35   (B) 51   (C) 70   (D) 92

- About 4 steps

- Intermediate unknown (the underlying pattern)

- Non-routine

Solution: This is an unusual problem in which a pattern of dots follows the progression of a geometric figure. Students have to write out the dots in each figure and recognize that the number of dots (i.e., line segments) increases by three for each new pentagon added. Students also have to project two pentagons forward. So the total number of dots in the sixth figure is $1 + 4 + 7 + 10 + 13 + 16 = 51$.

NAEP Grade 8 (hard: 5 percent correct)

Question:

A	B
2	5
4	9
6	13
8	17

14	?

If the pattern shown in the table were continued, what number would appear in the box at the bottom of column B next to 14?

(A) 19   (B) 21   (C) 23   (D) 25   (E) 29

- About 2 steps

- No intermediate unknown

- Routine

Solution: The sequence is very simple. It requires students double the number and add 1.

Discussion of Algebra and Functions Problems: These problems contain algebraic problems that require identifying a pattern in a sequence, but there are significant differences among them. Although the grade 8 NAEP item was designated difficult, it involves only recognizing a simple pattern of doubling a number and adding 1. It is on par with the Texas grade 6 problem, which involved a more unusual sequence, although the Texas problem is multiple choice and the NAEP problem is short answer. The New Jersey grade 8 problem contains an unusual and complicated pattern built around successively larger pentagons. The moderately difficult Singapore problem again involves complicated nonroutine solution steps similar to those in the New Jersey problem. Students must recognize that the pattern involves the area of a square, that the length of one side is the square root of the number of cubes, and that the successive layers are one cube less in length.

Looking across the nine questions as a group is instructive; Exhibit 5–4 summarizes findings on item difficulty across all the problem types. The grade 6 items from state tests, as expected, required the fewest steps to solve, and none had an intermediate unknown or required a nonroutine solution. To our surprise, the grade 8 NAEP problems were not much more difficult, rarely requiring solving for an intermediate unknown or a nonroutine solution strategy. Despite the relative simplicity of NAEP, the pass rates on the NAEP problems are shockingly low. Two-thirds of the problems could be answered by fewer than one in four test-takers. The grade 8 state problems were more difficult than the NAEP problems in terms of the number of steps required, intermediate unknowns, and nonroutine solutions, and the Singapore problems were much more difficult than the state's grade 8 items.

Exhibit 5–5. Indicators of Difficulty of Harder Assessment Items: Grade 6 Singapore, Grades 6 and 8 States, and Grade 8 NAEP

	# Steps for Solution	Intermediate Unknown Variable (%)	Nonroutine Solution[*] (%)
Singapore	5	78	89
Grade 6 State	2	0	0
Grade 8 State	3	44	56
Grade 8 NAEP	2	22	0
[*]Involves a solution that is other than simple recall, a straightforward application of a formula, or a recognition of a familiar concept or figure			

The most difficult items on an assessment are a measure of expectations for what the better mathematics students should know. On this basis, Singapore has much higher expectations for the mathematics problems that its excellent students in mathematics should be able to solve than does the United States.

CONCLUSIONS

Singapore administers high-stakes assessments used primarily for pupil placement at grades 4, 6, and 10. The grades 6 and 10 assessments, which are uniform across the country, also use value-added measures of how well schools performed in relation to expected gains to reward schools for performance. These measures are adjusted to reflect the characteristics of each school's students. The items on the Singapore assessments are more likely to require deeper problem-solving skills than items on the U.S. NAEP or state assessments. Items on Singapore's tests are more likely require preparing constructed responses, solving for intermediate unknowns, and developing nonroutine solutions than are U.S. test items.

Compared with the difficult and moderately difficult problems on grade 6 PSLE, the U.S. grade 6 problems were quite easy. Surprisingly, the items on the grade 8 NAEP that were designated as hard were also relatively simple. The grade 8 hard problems from the state tests were more difficult, but still easier than the items from Singapore's grade 6 test. Clearly, the United States needs to make its tests more rigorous if it is to match Singapore's impressive performance on international assessments. To improve performance and make U.S. assessments more like those administered in Singapore, the following reforms of the U.S. assessment system should be considered:

1. States should increase the depth and rigor of assessment items on state mathematics tests, especially in grade 6. The grade 6 problems we examined were routine, usually requiring only one or two steps. The grade 8 problems, although more challenging, were still less difficult than items on Singapore's grade 6 test in terms of the number of solution steps required, intermediate unknowns, and non-routine solutions. To strengthen their assessments, the states should do the following:

   - Expand the range of items on state assessments to include more items that measure advanced levels of mathematics proficiency. NCLB requires that assessments measure advanced levels of student performance, but the focus of assessments has been on problems that differentiate students at the proficient level, the level required for school improvement determinations. States need to include problems assessing advanced levels of difficulty. Note that adaptive assessments, although beyond the scope of this study, are a strategy for extending test difficulty while holding down assessment burden (Wainer, 2000).

   - Create an online assessment bank with rigorously validated problems categorized by difficulty. This assessment bank could also include items from high-performing countries outside the United States so that test designers can gauge the difficulty of their assessments against world-class standards. It makes no sense for each state to develop its items independently. The Council of Chief State School Officers (2004) is already developing a test bank for science items. With federal support, such as the inclusion of NAEP items, the test bank would save money and improve item quality. A move toward a common item pool would also enable states to compare the relative difficulty of their tests because states would know the difficulty levels of the items on their respective assessments.

2. NAEP should review and strengthen its mathematics items. When we conducted this study, we expected to find that NAEP, the premier U.S. assessment, would contain grade 8 mathematics items more difficult than those on state assessments that are intended mainly to identify schools in need of improvement. However, we did not find NAEP's grade 8 items to be particularly difficult. In fact, NAEP's difficult items are easier than the difficult problems on Singapore's grade 6 test. The low percent correct on the difficult NAEP assessment items that we examined was particularly disturbing, given that generally these problems appeared to be less difficult than items on the state assessments aimed at the same grade. Either the state assessments establish very low pass criteria or the students perform differently on the high-stakes state assessments than on NAEP. Because our study is exploratory, we recommend a set of validation studies:

   - Adding NAEP items to high-stakes state assessments would validate current NAEP assessment results. The pass rates on relatively straightforward NAEP items that have been classified as difficult are so low in some cases that one wonders how these students are able to pass state examinations when the state questions we sampled appeared somewhat more difficult. Finding out whether the NAEP responses are valid is important not only to test developers but also to educators and policymakers who use the NAEP results as a gauge of the progress of U.S. mathematics education.

- Allowing students from grades 6 and 8 from Singapore to take the grade 8 NAEP would set world-class standards for advanced levels of mathematics performance. At the start of this study, the director of the Singapore assessment system offered to allow Singaporean sixth and eighth graders to take the NAEP. Singaporean students are the best in the world, and it would be instructive to see how students who achieve at world-class standards would fare on the NAEP.

- Students in grades 6 and 8 from the United States could also take the Singaporean PSLE. While having the world's best take NAEP would be a measure of NAEP's difficulty and of U.S. students' performance against world-class standards, having Singapore grade 6 students take the NAEP grade 8 assessment would be a measure of how well NAEP actually assesses performance.

3. The United States should consider using Singapore's high-quality assessment system as a model for NCLB assessment and accountability provisions. We should consider limiting high-stakes testing to key grades, develop value-added measures of school improvement, and develop a national assessment in mathematics to achieve greater test uniformity. These changes are fundamental, and there will be strong differences of opinion over whether it is desirable to alter NCLB so soon after its passage and over whether the idea of a national assessment is a good one. We have identified the following reforms because the purpose of our study is to examine what the U.S. system could do to model features that are important to Singapore's successful system:

- We should consider adopting the Singapore approach of administering high-stakes tests only at critical grades levels. Under this approach, U.S. schools could retain annual end-of-year tests, but high-stakes assessments for purposes of determining school improvement or pupil accountability would only be administered at critical grades. In Singapore, high-stakes assessments are given only at grades 4, 6, and 10. The United States might want to use grades 4, 8, or 12, as NAEP currently does. Spacing high-stakes assessments would be a visible move to address educators' concerns about excessive high-stakes assessments. Retaining annual grade-by-grade assessments supports continuous improvement, but for purposes of school accountability, it may be sufficient to have high-stakes assessments only at strategic transition points of student progress.

- We should consider adopting value-added measures to gauge school performance. The United States could use Singapore's value-added methodology, which rewards schools for greater than predicted student growth based on the entering test scores of their students, as a model. Value-added accountability, which takes into account the entering abilities of a school's students, is superior to the traditional U.S. methodology that holds schools solely accountable for outcomes.

- We should consider developing and implementing, within the framework of NCLB, a national test at key transition grades to ensure test comparability across states. Singapore assesses nationally at the end of primary and of secondary school. In the United States, discrepancies between states' pass rates on NAEP and pass rates on their own assessments are large. This discrepancy is a threat to the credibility of the NCLB assessment system. Requiring states to participate in NAEP, as NCLB currently does, is only an imperfect way of addressing the need for a common

national assessment yardstick.. The development of a voluntary national test has been proposed in the past, but the U.S. Congress has explicitly prohibited federally sponsored national testing (U.S. Congress, 2000b). However, current experience under NCLB, which depends on strong outcomes-based accountability, is not consistent with the large state-by-state assessment variability. There now exists reasonably strong agreement about what mathematics content should be included in assessments, even though there is less agreement on how to teach that content. This general agreement may now advance the idea of a national test as the practical solution to an otherwise intractable situation. Ultimately, a uniform assessment at critical student transition points may be necessary to also replicate the student results achieved using Singapore's uniformly equitable assessment system.

CHAPTER 6. TEACHERS OF MATHEMATICS

If the intended curriculum is specified by frameworks, the available curriculum by textbooks, and the valued curriculum by assessments, then teachers determine the realized curriculum. One recent study of school determinants of student outcomes concluded that teacher quality "is likely the most important schooling factor, with a much larger impact than other commonly measured attributes such as class size" (Goldhaber, 2004). The NCLB requirement that only highly qualified teachers deliver instruction offers a way to assess and develop teachers who understand mathematics content and understand how to teach that content for mathematics understanding. Meeting this requirement will not be easy in the United States.

Liping Ma's (1999) influential study of elementary mathematics teachers in China and the United States, *Knowing and Teaching Elementary Mathematics,* focuses on teachers' understanding of mathematics and their ability to teach others to understand mathematics. In discussing how a good teacher might teach division of fractions (e.g., 1 ¾ ÷ ½), she describes a knowledgeable Chinese teacher:

> My teacher helped us understand the relationship between division by fractions and division by positive integers–division remains the inverse of multiplication, but meanings of division by fractions extend meanings of whole-number division: the measurement model (findings how many halves there are in 1¾) and the partitive model (finding a number such that half of it is 1 ¾).

Ma found that most of the Chinese teachers she studied viewed this problem in the same way, and we believe that most Singaporean teachers would also. Ma, however, found that "many teachers in the United States failed to show this understanding."

U.S. teachers themselves do not share Ma's assessment of their mathematical abilities. On TIMSS-1999 surveys, for example, the proportion of U.S. teachers who said that their confidence in their preparation to teach grade 8 mathematics was high was the second highest among 35 countries. Teachers in Singapore, in contrast, ranked themselves 19th, substantially lower than U.S. teachers.

To learn more about how Singaporean and U.S. teachers really compare in quality, we looked at their knowledge and at the stages of their preparation. Teachers' cognitive ability, content knowledge, and professional training are important in teacher quality and student achievement (Whitehurst, 2002). We also looked at pre-service preparation in mathematics (Ma, 1999), and certification of teachers through exams (Goldhaber and Brewer, 2000). We looked at the following indicators of teacher qualifications and development:

- How do the mathematical abilities of entrants into the primary school teaching profession compare with those of other college graduates?

- How much formal mathematics coursework do prospective teachers receive in their pre-service coursework?

- Do teacher-licensing policies require prospective teachers to pass rigorous mathematics examinations?

- Do new teachers receive support from induction programs?

- What kinds of professional development do teachers receive to continually improve their mathematics skills, and how much do they get?

MATHEMATICS ABILITY OF ENTRANTS INTO TEACHER PREPARATION

Given what we have already seen of Singapore's centrally organized education system, it is not surprising that Singapore also has a centralized system for teacher preparation designed to recruit good candidates from the beginning of the teacher development process. Singapore has only one teacher education institution, the National Institute of Education (NIE). Candidates for entrance into NIE are rigorously screened, and financial incentives are offered to encourage better students to enter the profession (Ministry of Education, Singapore, 2004). Prospective teachers must also apply to the Ministry of Education, which has established stringent, uniform selection criteria. Applicants are assessed on the "totality of their academic and non-academic achievements," and, unless exempted, they must pass an Entrance Proficiency Test (EPT) *before* they enter formal pre-service education. Only applicants with excellent scores on their secondary school exit examination are exempted from the EPT (Ministry of Education, 2000c). The Ministry's goal is to recruit students from the top third of each grade cohort (Ministry of Education, Singapore, 2001b). In the case of mathematics, the top third of Singaporean students score, on average, in the top 10th percentile of students in the 38 countries tested on the TIMSS-R.

To encourage highly qualified students to enter teaching, the Ministry pays students who are accepted into the NIE a stipend. NIE students receive a regular salary, as well as full tuition and fees, from the Ministry. To further attract the best-qualified students, the amount of compensation increases for better students. For example, precollege students who are enrolled in the most advanced pre-university program and pass rigorous A level exams receive greater compensation than teacher candidates with academically less selective backgrounds (Ministry of Education, Singapore, 2004).

The U.S. approach to teacher education is much less centralized than Singapore's. More than 1,500 separate higher education institutions offer degree programs in education (Wang, Coleman, Coley, and Phelps, 2003). These institutions typically have no special requirements for students entering teacher preparation programs beyond those they require from all students.

However, when students are matriculating, many education schools require students to take the PRAXIS I. This is a skills assessment that students take early in their college program "to measure reading, writing, and mathematics skills." The PRAXIS I is usually given during the early part of a student's enrollment in education school, so it might be considered a screening exam, similar in purpose to the entrance exam Singapore gives its students prior to acceptance to education school.

Education Training Service (ETS), although a world leader in the testing field, does not publish independently established norms to indicate the level of difficulty of the PRAXIS I (ETS, 2004). There is no way to tell, for example, whether one might expect that the mathematics questions could be solved by a typical grade 8 student or a typical grade 10 student. In the absence of published PRAXIS test norms, we did our own informal evaluation of the difficulty of PRAXIS mathematics questions by using sample questions from PRAXIS I that ETS has posted on its Web site. Although these sample questions may not capture the full range of questions on the PRAXIS, they are what ETS offers to prospective test takers who want information about the kinds of questions they might expect to see on the examination. ETS has posted 10 mathematics questions from the PRAXIS I Academic Skills Assessment for prospective teachers. All 10 questions are shown in Exhibit 6–1.

Exhibit 6–1. Sample Questions on the PRAXIS I Content Assessment in Mathematics (http://ftp.ets.org/pub/tandl/0730.pdf)

1. Which of the following is equal to a quarter of a million?

    (A) 40,000   (B) 250,000   (C) 2,500,000   (D) 1/4,000,000   (E) 4/1,000,000

2. Which of the following fractions is least?

    (A) 11/10   (B) 99/100   (C) 25/24   (D) 3/2   (E) 501/500

3. Which of the sales commissions shown below is greatest?

    (A) 1% of $1,000   (B) 10% of $200   (C) 12.5% of $100   (D) 15% of $100   (E) 25% of $40

4. For a certain board game, two dice are thrown to determine the number of spaces to move. One player throws the two dice and the same number comes up on each of the dice. What is the probability that the sum of the two numbers is 9?

    (A) 0   (B) 1/6   (C) 2/9   (D) 1/2   (E) 1/5

5. If $P/5 = Q$, then $P/10 =$

    (A) 10Q   (B) 2Q   (C) Q/2   (D) Q/10   (E) Q/20

Car Model	Frequency
K	7
X	9
W	7
J	8

6. The chart above gives information about the distribution of four compact-car models in a company parking lot. Which of the following figures best represents the given data?

(A)   (B)   (C)

(D)   (E)

x	y
0	5
2	11
6	23
7	26
10	35

7. Which of the following formulas expresses the relationship between *x* and *y* in the table above?

(A) y = x+ 5   (B) y= x + 6   (C) y = 3x+ 5   (D) y = 4x – 1   (E) y = 4x – 5

WIND-CHILL CHART

Temp. (F)				Wind Speed (m.p.h.)				
	5	10	15	20	25	30	35	40
50°	48	40	36	32	30	28	27	26
40°	37	28	22	18	16	13	11	10
30°	27	16	9	4	0	–2	–4	–6
20°	16	4	–5	–10	–15	–18	–20	–21
10°	6	–9	–18	–25	–29	–33	–35	–37
0°	–5	–21	–36	–39	–44	–48	–49	–53
–10°	–15	–33	–45	–53	–59	–63	–67	–69
–20°	–26	–46	–58	–67	–74	–79	–82	–85
–30°	–36	–58	–72	–82	–88	–94	–98	–100
–40°	–47	–70	–85	–96	–104	–109	–113	–116
–50°	–57	–83	–99	–110	–118	–125	–129	–132

8. The temperature today is 10°F, but it feels as cold as it did last week when the temperature was – 10°F and the wind speed was 10 miles per hour. According to the chart above, what is the wind speed today?

(A) 10 m.p.h.   (B) 15 m.p.h.   (C) 20 m.p.h.   (D) 25 m.p.h.   (E) 30 m.p.h.
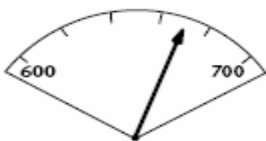
9. On the scale above, the arrow most likely indicates

(A) 630½   (B) 635   (C) 660 ½   (D) 670   (E) 685

10. Some values of x are less than 100. Which of the following is NOT consistent with the sentence above?

(A) 5 is not a value of x.   (B) 95 is a value of x.   (C) Some values of x are greater than 100.

(D) All values of x are less than 100.   (E) No numbers less than 100 are values of x

Even though the PRAXIS I samples might not represent the full range of item difficulty, they do reveal several interesting, and distressing, patterns:

- All the items are multiple choice, as are all items on the PRAXIS I exam. Sixty-nine percent of Singapore's grade 6 assessment, in contrast, consists of short- or extended-answer questions, as are 35 percent of NAEP questions. Only two of the five states whose student assessment we examined rely solely on multiple-choice assessments. We expect our students to answer more complicated questions than our prospective teachers.

- Nine of the 10 items require only a one-step solution. Only question 8 requires two steps.

- Only one problem, question 8, requires the test taker to solve for an intermediate unknown, the –33 wind-chill.

- The sample questions require little more than an understanding of the definition of a concept or a mechanical application of the concept to a routine question. Question 2, for example, simply requires knowing that a fraction whose numerator is smaller than its denominator is less than 1 and that such a fraction has a smaller value than a fraction whose numerator is greater than its denominator. Question 6 requires understanding how to read a bar graph. Question 7 requires only that the test taker plug numbers into a formula. To solve question 9, one needs only an understanding of intervals on a dial.

Given that the U.S. teacher preparation system does not rigorously screen education majors, as does Singapore's, proportionately fewer people with high mathematical ability enter education school. Exhibit 6–2 shows the SAT mathematics scores of prospective teachers, identified as those who took the PRAXIS II exam, the most common teacher licensure test required by states, with the SAT scores of students in other areas of study (Gitomer, Latham, and Ziomek, 1999). Students who take the PRAXIS II exam usually do so to meet state certification requirements. These students have made it through their education school coursework and are likely to go into teaching once certified.
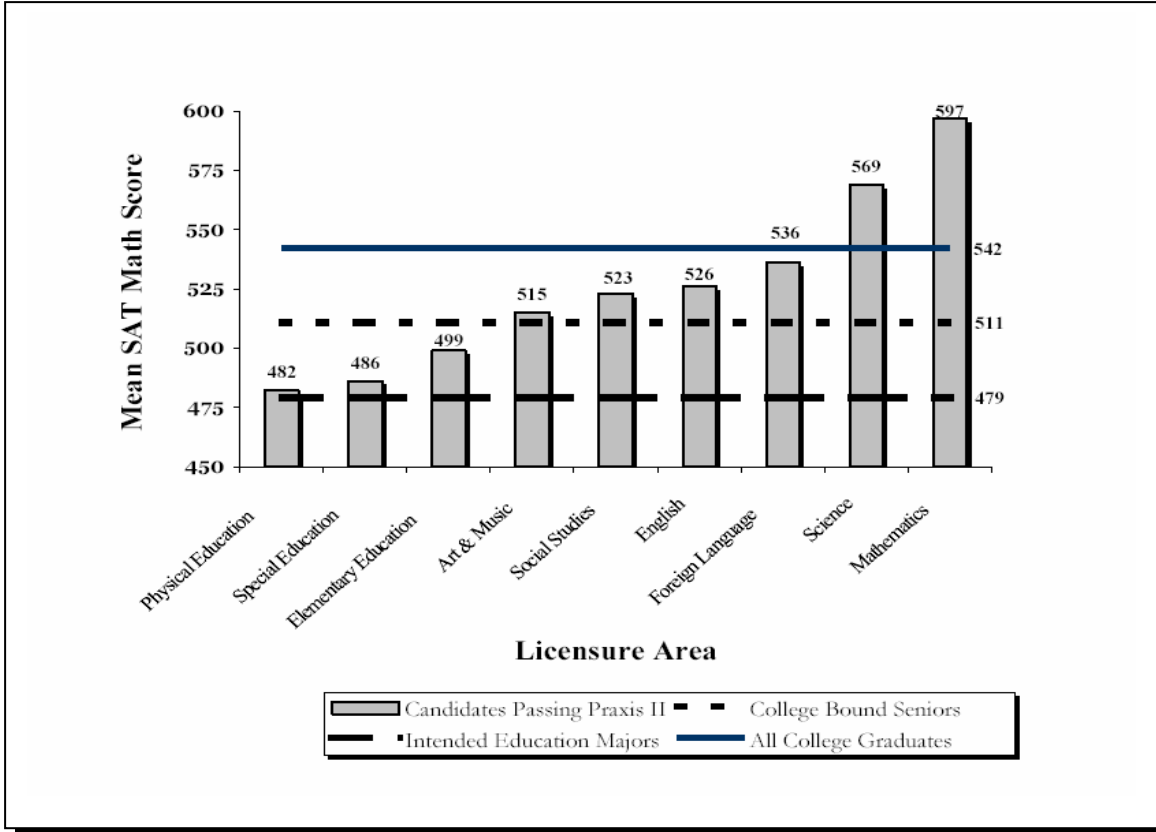
As Exhibit 6–2 shows, candidates seeking an elementary school license score about 30 points below the average college-bound senior on the SAT. About two-thirds of a high school cohort receive some postsecondary education, so elementary education majors appear to be only slightly above the U.S. average student. Because U.S. students, on average, score below the average of most industrialized nations on TIMSS, and U.S. teacher candidates are below average among all U.S. students, it is safe to say that prospective teachers' knowledge of core mathematics is much below that of Singaporean education school enrollees.[10]

---

[10] Although there is no direct way to relate SAT mathematics scores to national norms for a whole cohort, including those who do not attend college, a rough estimate can be obtained if several assumptions are accepted. About two-thirds of an age group now attend college. If we assume that all the college attendees score better than noncollege attendees and that SAT testtakers are similar to the college-going population, we can roughly estimate that the average SAT mathematics score of college graduates is better than about two-thirds of the cohort of all students. The prospective elementary teachers' score of about 30 points below the overall SAT average equals about one-third of a standard deviation on the SAT scores, or equivalent to about 10 percentiles among SAT test takers. This would place the elementary education majors in about the mid-50s range within the total cohort, or only slightly above the average U.S. student on a test such as TIMSS.

Exhibit 6–2. Comparison of SAT Mathematics College Entrance Scores of Prospective U.S. Teachers Taking the PRAXIS II Licensure Exam Compared With All College Graduates

Mean SAT Math Score

Licensure Area	Score
Physical Education	482
Special Education	486
Elementary Education	499
Art & Music	515
Social Studies	523
English	526
Foreign Language	536
Science	569
Mathematics	597

Reference lines: All College Graduates = 542; College Bound Seniors = 511; Intended Education Majors = 479

Legend: Candidates Passing Praxis II — College Bound Seniors — Intended Education Majors — All College Graduates

PRE-SERVICE TRAINING

Given that U.S. teacher candidates come to pre-service education with weaker initial mathematics knowledge than Singaporean teacher candidates, U.S. teacher preparation courses could be an important opportunity for prospective primary school teachers to catch up to their Singaporean counterparts in mathematics preparation. This is often, however, not what happens, even though U.S. teacher candidates typically take more college courses than do candidates in Singapore.

Primary school teachers in Singapore typically have considerably less college education than their U.S. counterparts. Two of three Singaporean primary teachers have earned only a two-year college certificate, usually a general education certificate from the National Institute of Education (NIE). Of course, their sound pre-college mathematics preparation and passage of a rigorous screening exam mean that these teachers know the mathematics taught at primary school very well before they ever attend college. Most secondary school teachers have at least a four-year college degree, consistent with the greater academic demands of secondary-level teaching, and they are required to have advanced academic training in the subject they teach (Ministry of Education, Singapore, 2003a).

Students who intend to teach primary school prepare for the General Education Diploma at the NIE. At the NIE, they take two years of pre-service course work designed to help them teach the national primary school curriculum, which includes mathematics. Pre-service coursework falls into six main areas of study (Singapore National Institute of Education, 2003):

- *Education Studies* courses (four courses in technology, teaching and learning theory, individual differences) teach candidates the key concepts and principles of education necessary for effective teaching and reflective practice.

- *Curriculum Studies* courses (four English and four mathematics courses) give trainee teachers the skills to teach their subjects.

- *Curriculum Content* courses (four English courses, two math courses, and two in one of the following: art, music, social studies, and science) reinforce subject content.

- *Academic Subjects* courses (two courses in advanced academic content including biology, English, geography, history, mathematics, or physics with chemistry) enhance training in one academic subject.

- *Practicum* courses (seven courses) give new teachers practice in basic teaching skills as they plan and deliver classroom lessons under the guidance and supervision of teachers and NIE supervisors.

- *Language Enrichment and Academic Discourse Skills* courses (2 courses) improve how candidates use oral and written language in teaching.

Singaporean primary teacher candidates take about six mathematics courses, about 12 semester hours, during their two-year preparation at the NIE. The focus of teacher preparation is pedagogical content—how to teach specific subjects (Shulman, 1986). Excluding practicum, three-fourths (18 of 24) of teacher preparation courses are in content area studies: curriculum studies, curriculum content, and academic subjects.

These three content areas expose prospective primary school teachers to the mathematics concepts and pedagogy they will need to teach the mathematical content in Singapore's primary syllabus. Curriculum studies coursework, for example, covers the teaching of basic mathematical ideas in all the major topics in the Singapore mathematical framework, including whole numbers, fractions and decimals, algebra, average, rate and speed, percent, ratio and proportion, money, measure and mensuration, geometry, graphical representation, and statistics (Singapore National Institute of Education, 2003).

Along with mathematical content, prospective teachers are exposed to mathematical pedagogy in the use of the abacus; in drill and practice; in heuristics for problem solving; in mathematics assessment, both the construction of tests and alternative assessment methods; in the teaching of low-achieving pupils; and in analyzing pupil mathematical errors and misconceptions. In addition, the academic subject area allows teachers to develop more advanced understanding in one discipline of their choosing, including mathematics. Overall, Singapore's teacher preparation program is tightly linked to preparing teachers to teach the content in Singapore's frameworks.

Prospective elementary school teachers in the United States, in contrast, take four years of college and often an additional year in an education school. What they study in education school varies depending on what college or university they attend. The National Council for Accreditation of Teacher Education (NCATE), the primary national accreditation body for education schools, has developed standards for what elementary education students should study in mathematics and other subjects. Although NCATE accreditation is not mandatory, NCATE has accredited 575 schools of education, and another 100 are in the pre-accreditation process (NCATE, 2004). NCATE's mission

> is accountability and improvement in teacher preparation. The NCATE accreditation process establishes rigorous standards for teacher education programs, holds accredited institutions accountable for meeting these standards, and encourages unaccredited schools to demonstrate the quality of their programs by working for and achieving professional accreditation. (NCATE, 2004)

NCATE has partnerships with 48 states, including all our sample states except New Jersey, to coordinate and conduct joint reviews of schools of education.

With respect to subject matter content in mathematics, NCATE developed its intended mathematics content and teaching standards to align with content and teacher standards from NCTM (2004a). NCATE's standards for the preparation of elementary school teachers in mathematics integrate subject area content preparation and preparation in teaching methods and assessment:

> **STANDARD 2.3. Mathematics—Candidates know, understand, and use the major concepts, procedures, and reasoning processes of mathematics that define number systems and number sense, geometry, measurement, statistics and probability, and algebra in order to foster student understanding and use of patterns, quantities, and spatial relationships that can represent phenomena, solve problems, and manage data;**

> *Supporting explanation*

> Candidates are able to teach elementary students to explore, conjecture, and reason logically using various methods of proof; to solve non-routine problems; to communicate about and through mathematics by writing and orally using everyday language and mathematical language, including symbols; to represent mathematical situations and relationships; and to connect ideas within mathematics and between mathematics and other intellectual activity. They help students understand and use measurement systems (including time, money, temperature, two and three dimensional objects using non-standard and standard customary and metric units); explore pre-numeration concepts, whole numbers, fractions, decimals, percents and their relationships; apply the four basic operations (addition, subtraction, multiplication, and division) with symbols and variables to solve problems and to model, explain, and develop computational algorithms; use geometric concepts and relationships to describe and model mathematical ideas and real-world constructs; as well as formulate questions, and collect, organize, represent, analyze, and interpret data by use of tables, graphs, and charts. They also help elementary students identify and apply number sequences and proportional reasoning, predict outcomes and conduct experiments to test predictions in real-world situations; compute fluently; make estimations and check the reasonableness of results; select and use appropriate problem-solving tools, including mental arithmetic, pencil-and-paper computation, a variety of manipulatives and visual materials, calculators, computers, electronic information resources, and a variety of other appropriate technologies

to support the learning of mathematics. Candidates know and are able to help students understand the history of mathematics and contributions of diverse cultures to that history. They know what mathematical preconceptions, misconceptions, and error patterns to look for in elementary student work as a basis to improve understanding and construct appropriate learning experiences and assessments. (NCATE, 2003)

To meet these standards, teachers would certainly have to know mathematics well. They would have to understand the NCTM process standards and be able to reason logically and use various methods of proof to solve nonroutine problems. They would have to be able to use measurement systems on multidimensional objects and with nonstandard units. They would have to know basic arithmetic operations, but also be able to model computational algorithms. They would have to be able to use geometric concepts to model mathematical ideas, apply number sequences and proportional reasoning, predict outcomes through conducting real-world experiments, and so on.

The standards for teachers' mathematics preparation are extremely ambitious, but what is less clear is how NCATE expects teachers to obtain this knowledge through their coursework. Because NCATE does not establish target levels for formal coursework in mathematics preparation, we cannot determine this expectation. Indeed, these standards do not provide a model teacher education program or suggest how teacher education schools should ascertain whether they are successfully producing mathematically well-prepared teachers.

The Conference Board of the Mathematical Societies, an organization representing 16 professional societies involved in the mathematical sciences, does recommend that prospective U.S. teachers take at least 9 semester hours of "fundamental ideas in mathematics" (Conference Board, 2000). These fundamental ideas include the core mathematical topics in the five NCTM strands. However, most U.S. elementary school teachers who teach mathematics are education majors, not mathematics majors, so they actually take fewer mathematics courses than the typical college graduate. On average, education majors take only three-quarter the credit hours in mathematics of a typical college graduate—6.3 credit hours compared with 8.3 credit hours (NCES, 2002).

Given that the average elementary school teacher takes fewer mathematics courses than the average U.S. college graduate, we wondered how U.S. primary teachers' mathematics course preparation compares with that of their counterparts in Singapore. No large-scale representative survey has been conducted on mathematics coursework for elementary education majors, so we were limited to examining undergraduate and graduate coursework from a few selected NCATE institutions.

We sought to pick a school of education at random from a list of NCATE-approved education schools in each of the seven selected states we discussed in prior chapters. We excluded Ohio because an Ohio elementary education degree covers only grades K–4, and we excluded Texas because we could not find any Texas schools that posted detailed course requirements online. Exhibit 6–3 provides information about the five education schools we examined. Each school requires only one elementary mathematics course for its elementary education degree, and this course is a teaching methods class. The University of Maryland is noteworthy in that it also requires 8 hours of general mathematics and recommends 4 hours of statistics in addition to education school coursework. With that one exception, these NCATE-approved education schools display a fundamental gap between the extensive mathematical knowledge that the challenging NCATE standards say that they require for accreditation of elementary education programs and the limited exposure to mathematics that accredited programs actually provide to prospective elementary school teachers.

Exhibit 6–3. Mathematics Course Work Required by Education Schools for Preparation of Teachers of Elementary Education, By Sampled Institution

Institution	Total Program Units (Excluding Field Work)	Mathematics Course Work (Required or Recommended)	Units
California State University, Bakersfield (graduate)	28	• Curriculum and Instruction: Mathematics	3
University of North Florida (undergraduate)	54	• Mathematics Methods	4
University of Maryland, Baltimore County (undergraduate)	33 plus general education courses	• Teaching Mathematics in the Elementary School (8 hours of general mathematics required plus 4 of statistics recommended)	4
William Paterson University, New Jersey (undergraduate)	20	• Learning and Assessment in Mathematics	2
University of North Carolina (Chapel Hill)	44	• Teaching Mathematics in the Elementary Grades	3
Source: Web sites of the institutions in August 2004.			

We also examined the coursework required by a prestigious Master of Education program for elementary school teachers offered by Columbia University Teachers College, which *U.S. News and World Report* ranked as one of the top education schools in the country in 2004. We found it to provide no more rigorous mathematics instruction than the NCATE-accredited undergraduate programs. The Teachers College program starts in a different place, assuming that its students have already earned an undergraduate degree, although not necessarily in education. Its course list illustrates the degree to which a major graduate program recognizes the possibility that its elementary education students have a deficit in their undergraduate mathematics education. Both New York state and Teachers College require prospective teachers to complete at least 6 hours of preparation in mathematics (Columbia University Teachers College, 2004a), roughly the national average for elementary teachers and below the 8.3 hours a typical college graduate takes. In addition, the Teachers College program requires prospective elementary education teachers to take the following coursework, in addition to classroom teaching (Columbia University Teachers College, 2004b):

- Two courses in *Curriculum Development in Elementary Education* with an emphasis on the interrelationship of instructional fields and team curriculum development

- One course in *Group Process Strategies, a* laboratory course that develops teaching skills, strategies based group process, cooperative learning, synectics, role-play, and concept development

- One course in *Teaching and Learning in the Multicultural Classroom.*

- One course in *Child Development* that examines risk and resilience in early development from birth to 8 years

- One course in philosophical, historical, and sociological *Educational Foundations*

- Two courses in *Methods of Teaching Reading*

- One course in *Methods of Teaching Math*

- One course in *Methods of Teaching Science*

- One course in *Special Education Methods*

- One course in *Health Education Methods*

If elementary mathematics teachers complete this program, they study methods of teaching mathematics in only 1 of 12 courses, excluding practicum, taken over a two-year period, no more than would be required if they taught health. What becomes clear through looking at both the Teachers College course list and the courses required by NCATE-accredited undergraduate programs is that most prospective elementary school teachers are expected to devote very little of their preparation time to mathematics content or mathematics pedagogy.

TEACHER CERTIFICATION

Teacher certification requirements are another way that the education system can ensure that teachers have received adequate preparation. Once they have completed their pre-service training, primary teachers in Singapore and the United States follow different certification procedures. Because Singapore screens teachers before they attend the NIE, completion of the NIE course work and course examinations automatically certifies candidates and allows them to enter the teaching profession.

Certification is more complicated in the United States. States set licensing requirements for initial certification in different ways, generally using credit hours in particular subjects and a teacher licensing examination. Because we have already examined coursework preparation for the elementary school teacher, we focus here on the mathematics portion of teacher licensing examinations. Although no single examination is used for teacher licensing in all states, we have chosen to look at the PRAXIS II series, which is used by 35 of the 43 states that require a teacher licensing examination for certification (ETS, 2004). The PRAXIS II tests prospective teacher knowledge of subject matter and of pedagogical knowledge.

As Appendix B6–1 shows, states differ as to which PRAXIS II tests they use to assess prospective teachers' mathematics knowledge. Each of the four PRAXIS II exams for prospective elementary teachers has a different content and pedagogy emphasis. The released items available on the ETS Web site for these PRAXIS II exams are limited in number, but the differences in test focus and difficulty level of the exams are quite apparent from these items.

The PRAXIS II Curriculum, Instruction, and Assessment exam (10111) is the most commonly used PRAXIS II, required by 18 states. According to ETS (2004a), "Many questions pose particular problems that teachers might face in the classroom, and many are based on authentic examples of student work." The exam's emphasis is on the assessment of pedagogical knowledge within content areas. The sample question in Exhibit 6–4 requires test takers to discover that the common source of the student's mistake in subtracting fractions is to subtract numerators and denominators from each other, instead of first transforming them to a common denominator (i.e., answer C). This assessment does not test teachers' mathematics skills in any comprehensive way.

Exhibit 6–4. PRAXIS II: Elementary Education: Curriculum, Instruction, and Assessment (10011) (http://ftp.ets.org/pub/tandl/0011.pdf)

$$\frac{4}{16} - \frac{1}{8} = \frac{3}{8} \qquad \frac{5}{9} - \frac{1}{2} = \frac{4}{7} \qquad \frac{7}{16} - \frac{1}{5} = \frac{6}{11}$$

5. The examples above are representative of a student's work. If the error pattern indicated in these examples continues, the student's answer to the problem $9/11 - 1/7$ will most likely be

(A) $\frac{10}{4}$

(B) $\frac{8}{7}$

(C) $\frac{8}{4}$

(D) $\frac{9}{3}$

The PRAXIS II Elementary Education: Content Knowledge Assessment, used by 17 states, is the most frequently used PRAXIS II assessment of teachers' knowledge of mathematics. Five sample questions are shown in Exhibit 6–5.

Exhibit 6–5. PRAXIS II: Sample Mathematics Questions on PRAXIS II Elementary Education: Content Knowledge (10014). Calculators Permitted: (http://ftp.ets.org/pub/tandl/0014.pdf)

Question 1: Riding on a school bus are 20 students in 9th grade, 10 in 10th grade, 9 in 11th grade, and 7 in 12th grade. Approximately what percent of the students on the bus are in 9th grade?

(A) 23% (B)43% (C)46% (D)76%

Question 2: Which of the following is equal to 8^4?

(A) 4,032   (B) 4,064   (C) 4,096   (D) 4,128
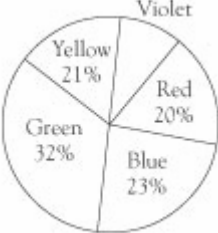
Question 3: In the formula $x = 10y$, if y is positive and the value of y is multiplied by 2, then the value of x is:

(A) divided by 10   (B) multiplied by 10   (C) halved   (D) doubled

Question 4:

Exhibit 6–5. PRAXIS II: Sample Mathematics Questions on PRAXIS II Elementary Education: Content Knowledge (10014). Calculators Permitted: (http://ftp.ets.org/pub/tandl/0014.pdf) (Continued)

The area of the shaded region is

(A) 30   (B) 52   (C) 64   (D) 116

Question 5:

The circle graph above represents the percent of colored gems in a collection. If the collection has a total of 50 gems, how many gems are violet?

(A) 2   (B) 3   (C) 4   (D) 5

These five PRAXIS II Content Knowledge (10014) questions are straightforward to solve. The problems have only one or two steps and require only routine applications of definitions and formulas. Only question 5 involves an intermediate unknown, which the teacher can find through a simple calculation of the unknown percentage value of the violet gems. Despite the simple arithmetic used in these problems, prospective teachers are allowed to use calculators.

The third ETS licensing exam for elementary teachers that includes mathematics questions focus is the PRAXIS II Multiple Subjects Assessment for Teachers: Content Knowledge (10140). ETS (2004c) says this exam is "designed to measure knowledge and higher-order thinking skills of prospective elementary school teachers." No state currently requires prospective elementary teachers to take this exam, although its use is permitted to meet licensing requirements in Alaska. Sample problems from this exam appear in Exhibit 6–6.

Exhibit 6–6. PRAXIS II: Sample Mathematics Questions on Multiple Subjects Assessment for Teachers (MSAT): Content Knowledge (10140) (Calculators Permitted): http://ftp.ets.org/pub/tandl/0140.pdf

1. If today is Thursday, what day of the week will it be 120 days from today?

   (A) Saturday   (B) Friday   (C) Wednesday   (D) Tuesday

2. Which of the following figures could be folded along the dotted lines to form a rectangular solid with no overlap? (Figures are drawn to scale.)
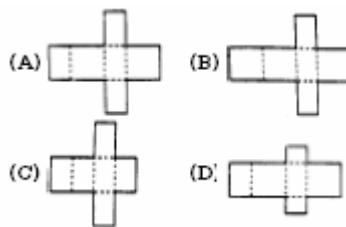
3. Each side of square S is 18 inches long, and the area of S is at least 3 times the area of a rectangle that is 9 inches wide. What is the greatest possible length, in inches, of the rectangle?

   (A) 9   (B) 12   (C) 18   (D) 27

4. When the values of a and b are interchanged, the value of which of the following expressions remains the same?

   (A) $a - b$ (B) $a^2 - b^2$ (C) $a^2 + b^2$ (D) $a(a + b^2)$

5. Ms. Pérez owns two properties in different towns. The first property is assessed at $125,000 and has a tax rate of $6.35 per thousand dollars. The second property has the same assessed value but is taxed at the rate of $0.00587 per dollar. Which of the following represents the total tax due on these two properties?

   (A) 6.35(125) + 0.00587(125)   (B) (6.35 + 0.00587) (250,000)

   (C) 6.35(125,000) + 5.87(125,000)   (D) 6.35(125) + 0.00587(125,000)

6. The numbers 1 to 20, inclusive, are individually written on 20 cards, and the cards are then placed in a hat. One card is drawn at random. What is the probability that the number on the card chosen is not a multiple of 3?

   (A) 7/10   (B)1/2   (C)3/10   (D) 1/20

      The sample problems in Exhibit 6–6 illustrate ETS's intent that this exam measure teacher knowledge of higher order skills in mathematics. Question 1 is a nonroutine arithmetic problem that requires test takers to realize that they must use the remainder of a division problem to obtain the answer. Question 2 requires test takers to understand the need to visualize a rectangular figure in which the faces have to have common edges of the same length. Question 3 involves solving for an intermediate unknown area of a square and then remembering to reduce the answer by one-third.

Question 4 requires test takers to use variables and the commutative properties of addition. Question 5 is an arithmetic problem that requires interpreting place values imbedded in a real-word taxation problem and identifying a correct numerical expression. In problem 6, an otherwise straightforward probability problem is complicated by requiring test takers to realize that they have to find the probability of all numbers, not multiples of 3, between 1 and 20. All these questions require reasoning skills.

The remaining PRAXIS II content-related assessment is the Elementary Education: Content Area Exercises test (20012), which is designed "to measure how well prospective teachers of students in the elementary grades can respond to extended exercises that require thoughtful, written responses" (ETS, 2004d). There were no released items in mathematics. This is not a test of mathematical knowledge but of the test taker's ability to analyze student work in an open-ended question context.

Of all the PRAXIS I and II exams, the questions in the PRAXIS II: MSAT (Exhibit 6–6) were the most demanding in mathematical reasoning and in requiring nonroutine solutions. The questions on this exam were more mathematically demanding than the harder items we examined on grade 8 NAEP and on the state grade 8 assessments, but they still do not involve the multistep reasoning required by some of Singapore grade 6 problems. However, the major weakness with the MSAT is not in the test, which is far more difficult than the other PRAXIS tests for elementary teachers. The weakness is that no states require it for their teacher licensure. The tests the states do require are no harder than the problems on states' grade 6 exams.

INDUCTION SUPPORT FOR NEW TEACHERS

Bridging the gap between theory and practice is a challenge for new teachers. Compared with experienced teachers, novice teachers "differ in the array of examples and strategies they can use to explain difficult concepts to students, in the range of strategies they can employ for engaging students who are at different performance levels, and in the degree of fluency and automaticity with which they employ the strategies they know" (Elmore, 2002). One means that school systems can use to address the difficulties faced by new teachers is to provide induction programs. Such programs provide new teachers with special support in their first year or years of teaching, a period during which many teachers leave the teaching profession, by making the initial period of entry into the classroom smoother and easier.

Singapore provides all new teachers with induction support during their first year (Ministry of Education, Singapore, 1999). Each new teacher is given a 20 percent reduction in his or her teaching load. This released time allows new teachers to observe more experienced teachers and receive on-the-job training. The Ministry also requires that the principal or vice-principal manage each school's induction program and that heads of departments oversee each mentoring arrangement. In addition, an online teachers' network gives new teachers ready access to assistance from experts.

Only slightly more than half (56 percent) of new U.S teachers participate in formal teacher induction programs in their first year of teaching (Choy, Chen, and Ross, 1998). Although 38 states have formal induction programs, the coverage and quality of the programs vary considerably. Programs differ in duration, intensity, and support for mentors. In Rhode Island and Massachusetts, for example, only about 15 percent of new teachers participate in locally determined induction program (Hirsch, Koppich, and Knapp, 2001). A recent review of U.S. induction programs found that "although many states have induction policies, the overall support for new teachers in the United

States is fragmented due to wide variation in legislation, policy, and type of support available" (Wang, Coleman, Coley, and Phelps, 2003).

The early years of teaching can be extraordinarily challenging, and many potentially good teachers drop out of the profession early. The early years are also when solid teaching habits are formed. Singapore recognizes how crucial the early years of teaching are by providing strong universal induction support for their teachers, whereas the U.S. system does so only in uneven, sporadic ways.

PROFESSIONAL DEVELOPMENT

New teachers are not the only ones who require assistance. Experienced teachers also need to continually keep their skills up to date. They may need to supplement inadequate pre-service preparation, update their knowledge of curriculum content and research on effective pedagogy, learn to understand new technologies and incorporate them into teaching, or need special help on particular aspects of teaching. Although we found many examples of U.S. school systems that provide excellent professional development, we found that Singapore does a much more systematic, thorough, and high-quality job of providing teachers with professional development than the United States does.

Singapore's Ministry of Education conceives of effective professional training as an integrated, planned program that draws together a variety of different kinds of professional learning experiences to support teachers' continual improvement:

> By training, we mean planned, not incidental, activities of learning. Such activities may take a variety of forms such as regular workshops, seminars, courses and conferences. They may include self-paced learning at home through the Internet, CD ROMs and other take-home packages that are made available to meet the diverse learning needs to learn at our own pace. They may also include discussion sessions both within and outside the school. On the job training is an effective way to learn and these include peer supervision, mentoring, action learning etc.

> Training should be part and parcel of improving work capability and not something unconnected and done in isolation. The Ministry is therefore asking that every teacher have a training roadmap worked out at the beginning of each year through discussion between the teacher and his supervisor. The roadmap will take into account the teacher's existing knowledge and ability, and identify what new areas of knowledge and improvement will benefit the teacher." (Ministry of Education, 1998)

Singapore's approach to continual professional development has four essential elements (Hean, 2000; Ministry of Education, Singapore, 1999):

- *An annual target of 100 hours* of professional development for each teacher, including compensated leave time for training.

- *A modular approach to teacher training* that allows teachers to upgrade their skills in different topics to different depths. NIE structures programs and courses in building blocks at basic, intermediate, or advanced levels to match teachers' current knowledge and skills.

- *Introduction of online learning* to supplement face-to-face instruction. NIE has established a long-term goal of providing approximately half of all professional training online.

- *Formal recognition and accreditation* of teachers for courses taken, thus linking professional training to advanced degrees and higher salaries.

To support nontraditional training, the Singapore Ministry of Education has developed the Professional Development Leave Scheme, or PDL, which provides a month and a half of half-paid leave for every year that a teacher serves. Teachers accumulate half-pay leave, which they may begin to use after six years of service. Teachers most often use the PDL to pursue post-graduate studies.

To further diversify learning, the PDL also covers private sector attachments or community development projects. The MOE provides full pay for up to four weeks if teachers embark on their attachments during school vacation. The MOE facilitates attachments by building up a network of contacts in the business and community sectors. Teachers use a dedicated Web site that provides information and advice for Singapore teachers on outside attachments. Participating companies include Citibank, Creative Technology, IBM, and Shell. Some teachers use PDL to learn new skills such as strategic planning or communication through e-newsletters. Others learn specific content knowledge; for example, some teachers worked with a bioengineering company. Some teachers benefit from learning more about the skills that companies want to see from students. The experience has energized many teachers, and some participants establish communities of teacher learners to share their experiences and continue learning from other teachers (Ministry of Education, Singapore, 2004a).

Professional development in the United States tends to be far less integrated and enriching for new and experienced teachers than it is in Singapore. Although isolated local examples of professional training are effectively integrated with mathematics and support continual teacher development, the overall picture of teacher improvement opportunities is not positive (Elmore, Burney, and Deanna, 1997; Sykes and Burian-Fitzgerald, 2004).

An evaluation of the Eisenhower program, the federal government's major means of supporting professional development in mathematics, found a direct connection between the intensity and duration of professional development and the likelihood of change in teacher practice. However, most teachers in the Eisenhower-supported program did not receive the extended professional development opportunities required to produce change.

> [T]he average time span of a professional development activity was less than a week; the average number of contact hours was 25, and half of the teachers were in activities that lasted 15 hours or less; most activities did not have collective participation or a major emphasis on content; and most activities had limited coherence and a small number of active learning opportunities. (Birman, Desimone, Porter, and Garet, 2000).

The evaluation concludes that "nationwide, the typical professional development experience was not of high quality." Most professional development activities amount to no more than a day spent in a specific content area (Parsad, Lewis, and Farris, 2001), and teachers do not believe that the professional development they receive is relevant.

Given how little time U.S. teachers spend on professional development and how little the professional development they do receive relates to their daily work, it is not surprising that teachers

do not value professional development as much as other forms of advice. One study shows that 72 percent of teachers rate advice from other teachers on how to improve their teaching as very valuable, whereas 37 percent find professional development valuable (Feistritzer, 1999).

Of course, some U.S. states and school systems are doing a good job of providing teachers with professional development that is well thought out, relevant to classroom activities, and provided in a systematic way. Connecticut, for example, has created a comprehensive "professional model" for training teachers (Sykes and Burian-Fitzgerald, 2004). This comprehensive approach to teacher quality includes an entry test for those wishing to enroll in a teacher education program, a subject matter examination taken during education school, graduation from an NCATE-approved program, a mentor or support team in the first year of teaching, completion of a subject matter portfolio at the end of the second year of teaching, and continuing professional education to renew the teaching license. Connecticut backs up this program with high salaries for teachers.

Singapore's professional development program offers sustained learning opportunities through a modularized approach that adjusts to teachers' learning needs and is integrated into a continual learning process that includes experiential training experiences in nonschool settings. In contrast, professional training in U.S. school systems focuses on short-term workshops that fit into teachers' released time. Evaluations suggest that these experiences are not likely to improve teachers' performance in mathematics; the teachers themselves agree and do not seem to value this training very much.

CONCLUSIONS

Singapore's elementary mathematics teachers, like other elements of Singapore's mathematics instruction system, are superior in overall quality to their U.S. counterparts. Singapore's system holds teachers in high esteem and backs up that esteem with a commitment to provide students with an able, well-prepared teaching force. Rigorous entrance exams sort out the most qualified teacher candidates, drawn from a pool of students who are already among the best in the world in mathematics, before they are accepted into the teacher education program. The government pays teachers while they earn their teaching credentials, and primary school teacher candidates undertake a program of study that is organized around content in mathematics and other subject areas. First-year teachers receive a reduced course load and are guided by an expert teacher mentor. Ongoing professional training needs are addressed through an individualized development plan, and teachers' skills are improved and kept current by 100 hours of annual professional training.

The United States faces different challenges than does Singapore when it comes to developing a strong cadre of mathematically able elementary education teachers. Students who enter U.S. elementary education preparation programs are typically not strong in mathematics, and this initial deficit is compounded at every stage of the preparation and development process. We found serious weaknesses in each part of the U.S. system, compared with the Singapore system. Given the strengths of the Singaporean system, the United States should consider the following ways to strengthen the quality of the mathematics preparation and development that elementary school teachers in the United States receive:

1. NCATE and the education schools it accredits should consider fully implementing the mathematics portion of the NCATE Elementary Standards. The standards are mathematically rigorous, but the coursework that schools offer is not. Implementing the Conference Board of the Mathematical Sciences recommendation that students take a

minimum of 9 hours of formal mathematics coursework would be a major first step toward meeting NCATE's Elementary Standards.

2. ETS should consider strengthening the mathematics content in questions on the PRAXIS I and II taken by prospective elementary teachers:

   - ETS should conduct and publish a norming study identifying at which grade level the content of the PRAXIS I and II mathematics questions is set. The published questions are substantially less rigorous than questions found on Singapore's grade 6 PSLE exam items, and they are less rigorous than the sample questions we found from state assessments administered to eighth graders.

   - ETS should consider comparing PRAXIS mathematics items to items from the entrance exam that Singapore administers to its teachers. It would also be informative to test a representative sample of prospective U.S. teachers with test items from Singapore's examination, in order to measure their performance against that of teachers in a world-class mathematics system. ETS should consider revising its PRAXIS I and II mathematical questions on the basis of these findings.

3. States should consider shifting their licensing requirements for elementary teachers to require the PRAXIS II Multiple Subjects Assessment for Teachers: Content Knowledge (10140). This appears to be a much more challenging measure of the higher-order thinking skills in mathematics of prospective elementary school teachers than the typically required PRAXIS II Content Knowledge (10014), which involves routine or two-step mechanical solutions to problems. Given that many states use the PRAXIS assessments to demonstrate that new teachers meet NCLB requirements for teachers to be highly qualified, the shift to a more challenging assessment would better fulfill the intent of this legislation.

4. States should consider reviewing current policies and practices for ensuring that teachers are qualified in mathematics to see whether a comprehensive approach such as Singapore's is needed. States should consider replicating the practices of reform states with comprehensive "model professional" systems. Connecticut (Sykes and Burian-Fitzgerald, 2004) has such a system, which includes an entry test for those wishing to enroll in a teacher education program, a subject matter examination taken during education school, graduation from an NCATE-approved program, a mentor or support team in the first year of teaching, completion of a subject matter portfolio at the end of the second year of teaching, and continuing professional education to renew the teaching license, all backed up by high teacher salaries.

5. School systems should consider reviewing their current professional development policies and practices for supporting highly qualified teachers to determine whether they, like Singapore, provide ongoing and sustained professional development opportunities tied to content delivery.

6. School systems should consider having teachers specialize in mathematics and other content areas from the earliest grades, a practice that is the norm in the Chinese education system that Liping Ma describes, a system that produces teachers with in-depth understanding of mathematics (1999). In Chinese classrooms, students as young as first

graders move from room to room to be taught by teachers who specialize in teaching mathematics and other subjects. U.S. elementary school teachers have to know enough to teach all subjects. The detailed requirements of the NCATE standards for the elementary mathematics teacher are no more detailed than the requirements for English language arts, science, social studies, the arts, and health education. This is a lot of content, probably too much, to learn. The idea of early specialization is not novel. Although the United States does not formally practice specialization in the early grades, the common practice of team teaching in elementary schools often results in specialization, because different members of the team concentrate on specific content areas, including mathematics.

CHAPTER 7. EXPLORATORY ANALYSES OF U.S. SINGAPORE PILOT SITES

Of all the elements of Singapore's successful mathematics system, its textbooks are the easiest to transfer to U.S. schools. Several U.S. school districts are using Singaporean mathematics textbooks in an attempt to emulate Singapore's international success in mathematics. We looked at four of these districts: the Baltimore City Public School System; the Montgomery County Public Schools, in the Maryland suburbs of Washington, DC; the North Middlesex Regional School District in Massachusetts; and Paterson Public School No. 2 in Paterson, New Jersey. These four pilot sites introduced Singapore mathematics textbooks as a replacement for their regular U.S. mathematics textbook. We examined student outcomes and program experiences in each pilot site to determine whether Singapore's textbooks worked well in a U.S. environment.

The challenges arising during the process of introducing a textbook designed for a different mathematics system also helped us understand the need for systemic change in U.S. mathematics programs. Implementation problems were routinely caused by the lack of alignment between the Singaporean books and the mathematics frameworks and assessments in the states where the pilots occurred; problems also arose when the pilots were implemented in only a few of the district's schools. However, the missing system element acting most severely as a drag on the success of the pilots proved to be teachers who lacked the educational preparation needed to teach the content in Singapore's books. All the pilot sites, to varying degrees, encountered problems with teachers who lacked the educational preparation needed to teach the content in Singapore's demanding books.

This exploratory evaluation was an opportunity to study the use of the Singapore textbooks in diverse U.S. school settings and to help us gain insight into whether teachers saw the same advantages that our comparative textbook analyses in chapter 3 suggest. The study also helped us identify challenges that teachers face in using the Singapore textbooks and the conditions under which the textbooks can be effectively used. The sites yield information about whether students' scores improve when the Singapore textbooks are introduced. Information from the pilots will also be helpful in guiding conditions and measures for more rigorous, confirmatory studies (Gilbert, Mosteller, and Tukey, 1976; Cronbach and Associates, 1985).

We reviewed staff experiences and student results from the pilot sites, focusing on three questions:

- What are the general magnitude and pattern of outcome improvements in the Singapore pilots in relation to a comparison group?

- What advantages and challenges do teachers perceive in using the Singapore mathematics textbooks?

- What are the drawbacks of introducing the textbooks in the absence of other components of Singapore's system?

METHODOLOGY

The first step in selecting the pilot sites for this study was to identify all the districts and schools that were using the Singapore mathematics textbooks. There is no master list of these districts, so we used several sources to find appropriate sites: distributors of the Singapore textbooks, newspaper stories about Singapore mathematics, Web searches, and conversations with U.S. experts involved in Singapore mathematics.

More than 80 school systems were identified as using Singapore mathematics textbooks, but many were unsuitable for inclusion in this study. Many sites used the Singapore textbooks only as supplemental materials. In other sites, the Singapore texts were not used schoolwide, making the assessment results useful only when they were broken down classroom by classroom. In addition, a number of users were private schools, which generally do not participate in state assessments, making it impossible for us to include their outcome data. In addition to these criteria, statistical analyses required that the sites met the following criteria:

- Had a Singapore mathematics program in place for at least two full school years

- Measured gain scores for at least a two-year period on either the same students or a repeated cross-section of students at a particular grade

- Could compare treatment gains with gains made by a comparison group of nontreatment schools or test publishers' norms

We identified four school systems that met these criteria: the Baltimore City Public School System; the Montgomery County Public Schools, in the Maryland suburbs of Washington, DC; the North Middlesex Regional School District in Massachusetts; and Paterson Public School No. 2 in Paterson, New Jersey. The sites had about 2,000 Singapore program participants ranging from first through eighth graders. The Montgomery County site accounted for about three-fourths of the participants.

Because financial resources to launch new data collections were unavailable, we relied on information already available from the pilot sites. All the sites had test score data from their annual assessments in school years 2000–2002. In three of the four sites, net progress of the Singapore pilot students was measured by comparing their improvements in mathematics results on state mathematics assessments with changes in the scores of a comparison group. The comparison group composition varied, depending on data availability; in some cases, schools similar to pilot schools were used. In other cases, we used the district or state average. The fourth site was a special districtwide program that independently administered its own standardized test. In this site, the students' beginning assessment scores on a national norm-referenced test provided a baseline, and improvements from this baseline were used as a measure of progress. In addition to student outcomes, teacher reactions to the new curricula were obtained from surveys given in two sites. We also visited with participants at all the sites.

In an exploratory study of this kind, important limitations arise from relying on existing site data. One limitation is that the school districts use nonrandom processes to select treatment and comparison schools, which introduces the possibility of selection bias from differences between the self-selected group of pilot schools and the comparison group. A second limitation is that the pilot sites may not be representative of the general school population. Notably, no large urban district

participated in the pilots, although the pilot schools do include a wide range of school poverty levels. An unusual third limitation was the lack of documentation on the extend that providers of professional development to these pilots were themselves familiar with, and trained in, the Singapore curriculum and pedagogy which is still new here in the U.S.

Fourth, although teacher survey data describing teachers' opinions about the advantages and disadvantages of the Singapore mathematics program were available from two sites, these data cannot substitute for a common, comprehensive questionnaire. Teachers' reactions from these two sites provided only a limited range of experience, and the questions used omitted important issues, such as individual teachers' mathematics preparation.

Bias in the study can also occur in looking at any newly implemented program. Improvements in outcomes may be understated for relatively new programs because implementation tends to improve as it is strengthened over three or more years (Berends, Chun, Schuyler, Stockly, and Briggs, 2001). However, new programs may also generate better outcomes because of heightened levels of teacher attention and interest. Greater levels of professional training associated with initial implementation of the textbooks in several of the pilot sites may, in its own right, generate better results, independent of the effects of the Singapore textbooks.

These possible sources of error place an extra burden on us to demonstrate large differences in results to improve the likelihood that observed differences in outcomes are valid and mathematically important. Clearly, the present results should be considered only suggestive and warrant much more rigorous and widespread replication.

Despite the limitations in relying on existing data, we were able to gather important information about the nature of program implementation and its outcomes. These results provide the first independent, empirical look at the outcomes from using the Singapore mathematics textbooks in different U.S. classroom situations. This initial information will be useful to the many U.S. school systems that having heard about Singapore's impressive TIMSS results are deciding whether to introduce the Singapore mathematics textbooks in their own schools.

STUDENT OUTCOMES

The outcomes for each of the four sites are summarized in Exhibit 7–1. In general, a change of two-tenths of a standard deviation, or roughly 6 percentile points, is thought to be educationally significant (Rosenthal and Rosnow, 1984). Because we used existing data from sites with unknown collection procedures and where population characteristics may be changing, we have elected not to present standard errors for our estimates because they may convey a degree of precision that does not actually exist.

The **Montgomery County Public Schools (MCPS)** are located in a suburban community with mixed income families bordering Washington, DC. MCPS began using Singapore textbooks in the 2001 school year in four treatment schools, covering grades 1 through 5, that volunteered for participation in the program. The MCPS identified four schools with characteristics similar to those of the pilot schools to serve as comparison schools in reporting its change scores.

Exhibit 7–1. Outcomes of Singapore Mathematics Pilot Sites in Relation to Comparison Group, 2000–2002[1]

Singapore Pilot Site (1)	Comparison Group (2)	Treatment Group (3)	Assessment (4)	Outcomes Measured As (5)	Change in outcomes		Treatment Minus Comparison (8)
					Treatment Group (6)	Comparison Group (7)	
Montgomery County Public Schools, Maryland (Pilots: 1,593 pilot scores in grades 2 and 4)[2]	Four similar schools in district	Grades 2 & 4 schools (greater professional training)	Natl. test	Percentiles: Gr. 2 Math	9	3	6
				Percentiles: Gr. 4 Math	0	2	–2
				Percentiles: Gr. 2 Math Computation	20	12	8
				Percentiles: Gr. 4 Math Computation	9	3	6
		Grades 2 & 4 schools (lesser professional training)	Natl. test	Percentiles: Gr. 2 Math	0	3	–3
				Percentiles: Gr. 4 Math	-4	2	–6
				Percentiles: Gr. 2 Math Computation	5	12	–7
				Percentiles: Gr. 4 Math Computation	5	3	2
North Middlesex Regional School System, Mass. (22)[2]	District schools not in Singapore math	Grade 4	State test	Percent Advanced	32	1	31
				Percent Proficient or Advanced	7	1	6
Patterson #2, New Jersey (63)[2]	District mean	Grade 4	State test	Percent Advanced	1	8	–7
				Percent Proficient or Advanced	9	10	–1
Baltimore City Public Schools, Ingenuity Project Baltimore, Maryland (84 in grade 6; 96 grade 7)[2].	National percentile norms	Grade 6	Natl. test	Percent At Least 97th Percentile	17	[3]	17
				Percent at Least 76th Percentile	5	[3]	5
		Grade 7	Natl. test	Percent at Least 97th Percentile	4	[3]	4
				Percent at Least 76th Percentile	7	[3]	7

[1]We have omitted standard errors of estimates to emphasize the exploratory nature of the findings and because of the many unknown design factors in the sites that may influence standard error estimates.
[2]Numbers are the size of the students in the beginning treatment group.
[3]Comparison group score is the pretest score of the treatment group so that there is no change in the comparison group.

Teachers in the four MCPS pilot schools were provided with summer and school year professional development in Singapore mathematics. The training started in the summer prior to implementation, when teachers involved in the Singapore mathematics pilot received in-service training in the use of the curriculum materials. The training sessions introduced teachers to the Singapore mathematics textbooks and helped them plan initial lessons. Teachers in pilot schools also received four after-school training sessions lasting from 4 to 7 p.m. and facilitated by MCPS mathematics curriculum staff. The sessions modeled the teaching of key topics, such as fractions, and gave teachers strategies for helping students approach the complicated multistep word problems in the Singapore textbooks. These sessions were supplemented with school-based mini-training sessions, conducted during regular school hours, that covered topics specific to grade-level teams (Gross and Merchlinsky, 2002).

Staff in two of the MCPS schools took extensive advantage of professional development opportunities to help implement Singapore mathematics; staff from the two others did not. These two groups of pilot schools differed in the observed quality of how they implemented the Singapore mathematics textbooks and in their results. A detailed case study of the professional development conducted at the most successful of the four MCSP is included in Appendix C.

As Exhibit 7–1 shows, the MCPS pilots yielded uneven results (gross and Merchlinsky, 2002; Merchlinsky and Wolanin, 2003). The first four shaded rows display the grades 2 and 4 net results for the Singapore pilot schools that received the greater amount of professional training. The improvement in outcomes for the treatment schools (column 6) was larger than the improvement in outcomes for the comparison schools (column 7) in three of four grades. The remaining four shaded rows display the results for the Singapore pilot schools that received less training. These Singapore pilots were out-performed by the controls in three of four comparisons. These findings may reflect the greater demands the Singapore textbooks place on teachers to teach students to understand mathematical concepts compared with traditional U.S. textbooks that place more emphasis on the mechanics.

The North Middlesex Regional School District serves the towns of Ashby, Pepperell, and Townsend in north-central Massachusetts, near the New Hampshire border. Approximately 5,000 students are enrolled in the district's four elementary schools, two middle schools, and one high school. The Singapore mathematics pilot sites included one school at grade 4, two at grade 6, and one at grade 8.

North Middlesex reported two-year changes in students' mathematics results when the pilot sites were compared with schools in the district that did not use the Singapore texts (Champigny, 2003). The proportion of students in the Singapore mathematics pilot who scored at the advanced level on assessments rose by 32 percentage points, from 13 percent to 45 percent. In contrast, the proportion of students in the control group who achieved advanced levels rose by only 1 percentage point. Thus, the school achieved a net improvement of 31 percent (column 8). The proportion of students who scored either proficient or advanced in the Singapore pilots also rose but by a smaller amount, from 65 percent to 72 percent, or 7 percentage points. The controls improved by 1 percentage point, for a net gain of 6 percentage points.

In response to the strong performance, especially in improving the proportion of students achieving at the advanced level, of the Singapore mathematics pilot sites in North Middlesex, the Massachusetts state superintendent called a statewide meeting to encourage other school systems to

use the Singapore's mathematics program. Massachusetts is also supporting the North Middlesex school system in its use of Singapore's mathematics textbooks to further assess the district's success.

The Paterson Public Schools, located in urban Paterson, New Jersey, were taken over by the state because of consistently poor test scores. Paterson Public School No. 2 (PS2), the district's Singapore mathematics pilot site, serves students that start out at risk; 98 percent are eligible for free or reduced-price lunch. Sixty percent are from homes in which the first language is not English. Forty-seven percent transferred into or out of the school in a single year, more than four times the state average. The high number of transfers limits students' meaningful exposure to the Singapore mathematics program.

As Exhibit 7–1 shows, we used PS2's mathematics assessment results on the grade 4 New Jersey state assessment (New Jersey school report card, 2002), comparing them with the district's grade 4 average for 2000 and 2002. Overall, no evidence shows that using the Singapore mathematics textbooks resulted in net positive effects on the state's grade 4 assessment scores. In fact, the percentage of students in the Singapore mathematics pilot who scored at the advanced level changed little, whereas the districtwide average showed some improvement. PS2 did improve the percentage of students at the advanced plus proficient level, but no more than the district as a whole did.

PS2 illustrates two problems with introducing the Singapore textbooks in isolation. First, PS2 is a very low-income school with an annual turnover in its student body of around 40 percent. Therefore, a large number of the students taking the grade 4 New Jersey mathematics assessment were new to the Singapore program in grade 4, which is not a fair test of the effectiveness of the mathematics textbook. As long as the Singapore textbook is used in only a single school with a highly mobile population, the presence of a substantial fraction of students who have not had prior exposure to the textbook will remain a problem. The second problem is that PS2 used two different grade 4 textbooks because of the lack of correspondence between the topics in the Singapore grade 4 textbook and the content on the grade 4 state assessments. The most serious discrepancies occurred in the areas of statistics, which Singapore treats without real-world applications, and probability, which Singapore does not cover at all until the middle grades. Not surprisingly, the PS2 teachers found having to use two textbooks with two different styles of presentation disorientating.

The Baltimore City Public Schools, with support from the nonprofit Abell Foundation, began the **Ingenuity Project**, a multiyear program designed to give selected Baltimore City Public School students identified as gifted a rigorous and enriched program. Admitted students enter the program in the first or sixth grades and are expected to have excellent attendance, with no unexcused absences. Students need to attain an 80 average or above in all course work to remain in the program. If at any time during the year Ingenuity students do not show consistent effort to meet standards, they and their parents are asked to meet with the school and Ingenuity staff to assess the students' difficulties and determine the necessary steps for improving academic performance. Students who cannot meet the demands of the project cannot re-enter the program the next year.

Johns Hopkins University administered the Sequential Tests of Educational Progress (STEP) norms for fall and spring to Ingenuity Project students. All tests were administered "in level" and were scored with grade-appropriate national norms. Because this test was administered only to the Ingenuity Project students, and not to the entire school population of the Baltimore City Public Schools, we used national norms on the STEP as our control, rather than other schools in the district.

Ingenuity Project students started out as above-average performers and their achievement improved with participation. Ingenuity Project students in grade 6 exhibited improvements in average scores and in the percentage of students achieving in the highest quartile. Particularly impressive was the 17 percent gain in the percent of grade 6 Singapore mathematics students who placed in the top three percentiles. Ingenuity project students in grade 7 maintained their gains and continued to improve in terms of increasing the proportion of students achieving the top three percentile and upper quartile groups. The improvement in Ingenuity students' results comes from an intervention that also integrates Singapore mathematics into a special program for selected students and teachers. The Singapore mathematics results come from taking all these factors together.

Overall, the pilot results show that under favorable conditions, Singapore mathematics textbooks can produce significant boosts in achievement, but introducing textbooks alone is insufficient to achieve improvement. The two pilot sites where the use of the Singaporean textbooks resulted in substantial two-year improvements in school outcomes, especially in moving students to advanced levels of performance, were sites with relatively small, stable populations. These may be ideal situations for introducing the Singapore textbooks. The site that serves a highly mobile population, where topics in the Singapore textbook do not match up well with topics on the state assessments and where no net test-score improvement occurred, may be the type of site where introducing a challenging mathematics textbook across all grades may be too much to ask for. The Montgomery County results for this large mixed economic system were also interesting because improvements in the Singapore pilot schools were correlated with the intensity of the schools' participation in professional training, suggesting that teacher acceptance and commitment to the new Singapore mathematics program may be key to its success. Collectively, these results suggest that the systemic context in which the Singapore textbooks are introduced has to be given careful consideration and addressed through broader based reforms to ensure that appropriate training and alignment with state frameworks and assessments in place.

IMPLEMENTATION OF THE SINGAPORE PILOTS

Two of the pilot districts, Montgomery County and North Middlesex, conducted formal surveys of teachers in pilot schools. The Montgomery County Public Schools evaluation staff also conducted site reviews. The perceptions of the educators involved in the Singapore mathematics pilot sites provide information on firsthand experience with the perceived strengths and weaknesses of the Singaporean textbooks and the challenges attendant on introducing the books into U.S. schools.

The experiences of the pilot site teachers at these two sites bear out our observations in chapter 4 about the key features of the Singaporean textbooks. Our comparisons suggest that teachers in the United States will find that the Singapore textbooks offer a richer treatment of mathematical content and a deeper level of conceptual development than they are accustomed to in U.S. textbooks. The spiraling of content found in the Singapore texts addresses a topic at one grade, assumes that it is mastered, and addresses it at a higher level in successive grades. They will also find mathematics problems that are more challenging and graphical aids that provide more useful problem-solving assistance than in traditional U.S. textbooks. However, our observations also suggest that U.S. teachers will also find the lack of real-world data analyses and the omission of probability in the Singaporean texts troubling.

We were also concerned that the typical U.S. teacher's lack of mathematics preparation compared with what a typical Singapore teacher receives may cause U.S. teachers difficulties in

teaching the more rigorous Singapore content. In addition, many U.S. teachers have been given no special training for addressing the special needs of students who are slower mathematically but for whom Singapore provides a special slower curriculum and extra mathematical assistance.

The **Montgomery County Public Schools** surveyed its teachers in the Singapore pilot schools to learn their perceptions of the new textbooks (Gross and Merchlinksy, 2002). Overall, the teachers liked the new textbooks, and 75 percent indicated that they felt MCPS was moving in the right direction by introducing them.

The empirical evidence from surveys completed by MCPS teachers and from the classroom observations pointed to classroom changes consistent with our expectations about the differences between the Singapore textbooks and the traditional U.S. textbooks (Gross and Merchlinsky, 2002):

- 61 percent of the Singapore pilot teachers said that they focus on topics in *greater depth* because of the Singapore curriculum; another 14 percent attributed their focus on greater depth to the combination of the training and curriculum.

- 57 percent of the Singapore pilot teachers said that they focus on a *topic longer* because of the Singapore curriculum; another 14 percent attributed their focus for longer periods of time on the combination of the training and curriculum.

- Classroom observation showed that teachers and their students in Singapore mathematics classrooms spent 22 percent of classroom time using pictorial representations, compared with 12 percent in control schools.

The MCPS teachers found the Singapore textbooks much less appealing when it came to instruction involving real-world activities. Only one in five strongly agreed that the Singaporean curriculum resulted in engagement in inquiry-oriented activities; about the same proportion agreed that the Singaporean texts provide applications of mathematics in a variety of contexts (MCPS, 2002). This is in keeping with our finding that the Singaporean books do not emphasize applied mathematics as much as U.S textbooks.

Implementing the new mathematics approaches embedded in the Singapore textbook was, as we pointed out in the empirical findings for Montgomery County, a challenge for teachers. Eighty-two percent of the MCPS teachers surveyed found that using the Singapore textbooks as their primary mathematics textbooks was very challenging. The comment of one MCPS teacher is representative of their reactions, both positive and negative.

> One of the principles I like about Singapore Mathematics is the idea of teaching to mastery. Mastery seems to be emphasized. That is really important, especially early on to develop a good number sense. That seems to be better than my understanding of ISM [MCPS current mathematics program]: teach it and if they don't get it they will get a chance again next year. In principle you teach to mastery in Singapore Math, but I am still feeling pressure to move on at a faster pace and not get the mastery Singapore Mathematics requires before moving on.

> I have difficulty knowing when to move on. For example, how long do I stay on addition before moving onto subtraction? I am feeling guilty about not completing the workbook. I can't wait for all students to catch up. (Gross and Merchlinksy, 2002)

The new more rigorous textbooks also brought to the surface the weakness in the elementary school teachers' mathematics preparation. During a focus group, one MCPS teacher said, *"Having to explain Singapore mathematics has made me understand that I never really understood the mathematics I was teaching."* This is a shocking admission to make before one's peers, but what was more shocking was that most of the other focus group participants felt the same way. Given these challenges, it is not surprising that classroom observations showed, as seen in Exhibit 7–2, that 53 percent of the pilot site teachers were moderately or fully successful in implementing the Singaporean textbook in the schools if they had availed themselves of the added training opportunities, but only 30 percent were successful if they had not. A detailed case study of the professional development conducted at the most successful of the four MCSP is included in Appendix C.

Exhibit 7–2. Independent Observations on the Extent of Implementation of Instructional Strategies in Singapore Mathematics Pilots in the Montgomery County Public Schools

Extent of Observed Implementation	Greater Professional Training	Lesser Professional Training	All Pilot Schools
None observed	6%	33%	18%
Struggling with implementation	41%	37%	38%
Moderately successful	31%	23%	29%
Highly successful	22%	7%	16%
Source: Gross and Merchlinsky (2002)			

The **North Middlesex Regional School District** reported on focus group responses, staff experiences, and the opinions of mathematics experts working with the district. Like the Montgomery County teachers, the Middlesex staff liked the depth and rigor of the mathematical content in the Singaporean textbooks; the clear presentations of concepts; the thoughtful, multistep problems; and the emphasis higher-level thinking. Staff also saw the early introduction of algebraic concepts as an advantage. This is surprising because Singapore does not introduce algebra until grade 6.

The North Middlesex staff also identified a number of challenges the Singapore textbooks posed for them. They noted the books' lack of correspondence with state frameworks; the problems faced by students who had not received adequate exposure to the textbooks in prior grades; and the difficulties associated with differences between Singaporean and U.S. English.

North Middlesex teachers were also asked which professional training topics they felt would be helpful in implementing the new curriculum (Champigny, 2003). They wanted a better understanding of strategies they could use to balance the state's mathematics framework with Singapore's. The teachers also wanted experts to observe them using the Singapore methods. Third, teachers wanted more time to meet with other teachers and talk about instructional strategies for teaching Singapore mathematics. And finally, the teachers would like greater access to resources that

contain mathematical problems appropriate for use with the Singapore textbooks because all the nontextbook resources they had were geared to their U.S. mathematics curriculum.

Exhibit 7–3. Strengths and Weaknesses of the Singapore Mathematics Curriculum Compared With Traditional Mathematics Curriculum, Based on Experience of North Middlesex Singapore Pilot Staff

Strengths	Challenges
• Greater depth/less breadth	• Some students have never had Singapore before and some of the strategies are new to them.
• Fewer topics covered each year; greater emphasis on mastery	• Written communication skills are not stressed
• Topics are revisited throughout the curriculum, but not retaught	• Matching to the State Framework; Probability strand is not covered
• Higher level of mathematical content	• Language differences including metric units*
• Higher level thinking	• Few hands-on activities are provided for grades 7 and 8.1
• Texts clearly present concepts using pictures, numbers and words	• Not enough practice within some of the units, so supplemental problems are necessary.
• Multistep, multiconcept, and multistrand problems.	
• Challenging word problems empower students to become critical thinkers	
• Algebraic concepts introduced at an earlier age	

*Refers to the older Singapore Third Edition of the *Primary Mathematics books.* A newer U.S. edition is now available with American examples.
Sources: Bisk, Condike, Hogan and Little (2002) and Champigny (2003)

In general, findings from teacher surveys and classroom observations in both Middlesex and Montgomery County about the use of the Singapore textbooks are consistent with our findings. Like us, the teachers noticed that the Singapore textbooks offer a deeper treatment of mathematical topics that returns to a topic only to teach it with more depth. The teachers also liked the books' visual explanations that concretely explained abstract concepts and the numerous multistep problems, differences we also noted between Singapore and U.S. textbooks.

The survey and focus group responses also identified serious challenges in bringing these new textbooks and methods into the U.S. classrooms. Teachers in the United States will need specially tailored professional development to successfully use the Singapore textbooks. The difficulties of students with weak mathematics preparation or those who have not been exposed to Singapore's curriculum in prior grades have to be addressed, as do the lack of real-world examples in the books, the lack of alignment between the Singapore textbooks and state frameworks, and the unfamiliar language and phrases used in the textbooks.

CONCLUSIONS

The Singapore mathematics textbooks produced uneven improvements in the pilot sites. In some sites, notably the smaller sites with stable enrollments and those that enrolled gifted students, students made remarkable progress. But the unevenness in the scores and the educators'

implementation concerns suggest that, for effective implementation, the textbooks should be introduced with considerable care. Because the curriculum is not repeated in the Singapore textbooks in the same way it is in U.S. textbooks, teachers of students in upper elementary grades, students transferring into a school from a non-Singapore textbook school, and students who may be weaker in mathematics face special challenges in using the Singapore textbooks.

Teachers face several problems, some in dealing with the new pedagogy and others stemming from their weak mathematical preparation. The curriculum presented in the Singaporean textbooks is quite different from that seen in U.S. mathematical textbooks, and unlike the Singapore teachers, U.S teachers have not been prepared using that curriculum. Participation in professional development tailored to implementing the content of the Singapore curriculum and textbooks is essential (Rosenbaum Foundation, 2001).

On the basis of our findings from the pilot sites, we recommend that the following actions be considered in districts and schools that plan to introduce Singaporean mathematics textbooks:

1. Districts should be prepared to provide teachers with extensive professional development opportunities to help them use Singapore mathematics textbooks effectively. Teachers need opportunities to receive formal training and opportunities to observe and meet with other teachers who use Singapore mathematics textbooks to discuss challenges and teaching strategies. Districts should also foster teachers' commitment to take such extra training. Although all the teachers in the Montgomery County pilots could have availed themselves of the extra training opportunities, only some did, and those who did had better results. Professional training requires a commitment of time, and teachers need to be encouraged and recognized for their in-service efforts.

2. The ideal time for districts to begin using Singapore books is in kindergarten and first grade because the textbooks are built around a spiral curriculum that builds new content on top of previously taught mathematics. The expectation is that once students are exposed to a topic, they will be able to build on the content learned and apply this content to more advanced mathematical problems. Continuing instruction of students who have used the Singapore curriculum from the beginning is far less difficult that arranging individualized or whole-class catch-up for students new to the program. If resources permit, the use of the Singapore books should also extend over a substantial geographic area. In this way, mobile U.S. students will have a better chance of transferring into and out of schools teaching the same material. In the same way, if "feeder schools" for a middle or high school have been taught the same material, student performance in the middle and high schools should also benefit.

3. Districts should offer weaker students more time and extra help in mathematics. They should identify weaker students beginning in grade 1 and provide them with extra mathematics. Singapore schools provide extra training in small groups, often after school and by a specially trained teacher; mathematically weaker U.S. students would benefit from the same type of high-quality extra assistance.

4. Districts should make sure that any the misalignment between the Singapore textbook and state mathematics frameworks and assessments is addressed in advance. Teachers in the pilot sites found that, at some grades, the Singapore mathematics textbooks do not include all the topics that their states' curriculum framework and assessments include,

statistics being the topic most frequently cited. Before adopting the Singapore mathematics textbooks, districts should conduct a curriculum alignment analysis and identify supplementary mathematics material, preferably from other Singaporean textbooks, to fill any gaps.

5. Districts should use the current U.S. edition of the Singapore textbooks. Teachers in the pilot sites noted that the older, third edition Singapore textbooks contain terms and examples unfamiliar to U.S. students. This U.S. edition, otherwise unchanged from the older version, uses the English measurement system, U.S. currency, and American examples and terminology.

Findings from this exploratory study are just that, exploratory and only broadly suggestive of what might happen if Singaporean textbooks were to be more widely introduced into U.S. schools. We recommend that more rigorous, experimental tests of the Singapore mathematics curriculum and textbooks be undertaken. The federal government's evaluation office within the Institute of Education Sciences, as part of its experimental test of promising interventions, should consider conducting a rigorous multiyear impact evaluation of the use of Singapore textbooks. Such an experimental study could more adequately control for potential selection differences among schools and students exposed to the Singapore textbooks, and those exposed to traditional mathematics textbooks. The impact evaluation should last four years or longer so that a cohort of students exposed to the textbooks can be followed over a sufficient number of grades to demonstrate whether using the Singapore textbooks leads to deeper understanding of mathematics among U.S. students. Since professional development is an important and customary part of any new program implementation, the teachers in study schools will of necessity need professional development specifically tailored to either the Singaporean or the different, control program and materials.

CHAPTER 8. NEXT STEPS

ACHIEVING U.S. MATHEMATICS REFORM

Singapore has a world-class mathematics system in which all elements focus on students mastering concepts in order to solve challenging mathematics problems. Singapore's well-prepared teachers understand and are able to teach the challenging mathematics presented in their problem-rich textbooks. Singapore's students work hard to prepare for their high-stakes assessments, and schools are rewarded for value-added improvements on uniform national assessments. All these elements—teaching, textbooks, and assessments—are aligned to Singapore's mathematically logical national framework. For the most part, the U.S. system does not match up well with Singapore's; its frameworks, textbooks, assessments, and teacher preparation and training are generally weaker and less rigorous. To close its performance gap with Singapore, the United States would do well to consider adopting reforms that would align its mathematics system more competitively for world-class success.

The U.S. mathematics instructional system has several strengths to build on. It is largely based on the NCTM process standards (e.g., reasoning, representation and communication), which are consistent with the need to start building student skills through tackling complicated real-world problems and data. The NCTM and state frameworks also place a greater emphasis on applied mathematics in their data analysis and probability strand than Singapore does. The U.S. nontraditional textbook used in this study offers good real-world problem sets, although it is weaker in conceptual development of mathematics topics than Singapore's textbook. Given the widespread criticisms of U.S. mathematics as a "mile wide and an inch deep," we were also pleasantly surprised to discover that three of our seven states had frameworks not unlike Singapore's. Although changes to improve the U.S. mathematics system should build on these strengths, the Singapore mathematics system's more rational and coherent design and the quality of its key elements should be considered as a model for U.S. reform.

The experiences of several of the U.S. pilot sites that introduced the Singapore mathematics textbooks without the other aspects of the Singapore system also illustrate the challenges that teachers faced when only one piece of the Singapore system was replicated. Some pilot schools and sites coped successfully with these challenges, but others had great difficulty. Professional training improved the odds of success, as did serving a stable population of students who were reasonably able at mathematics. These findings indicate that U.S. school systems that are considering adopting Singaporean textbooks should enlist teacher support for the reorientation necessary to teach its demanding content.

Confirming the Exploratory Findings

Ultimately, the U.S. system has to comprehensively attack the root causes of its students' mediocre mathematics performance to achieve widespread improvement. The reform considerations presented in the prior chapters offer a set of strategies for narrowing the differences between the U.S. mathematics system and Singapore's system. As these reforms are considered, it is important for reformers to keep in mind that they were derived from exploratory analyses. The analyses were rich in the depth of their descriptions, but they were limited in the scope of comparisons and warrant

validation over broader samples. We begin our concluding discussion of reforms by identifying additional studies that would validate and extend our findings.

1. **Frameworks:** Additional studies should extend the comparisons of Singapore, NCTM, and state standards to develop 50-state framework comparisons on topic coverage and outcomes, grade by grade. In addition it would be helpful to extend international comparisons to other Asian nations that score well on TIMSS, including Korea, Japan, and Taiwan, and to compare these with Singaporean and U.S. frameworks.

2. **Textbooks:** We noted that the Singapore mathematics textbook have features that might benefit special education and limited English-speaking students. In part this is due to their straightforward and clear presentation of concepts, and in part because they rely less on words and more on a concrete, pictorial approach to abstract concepts. Since help for these difficult to reach populations would be a separate gain, experimental studies to evaluate the effectiveness of the Singapore mathematics textbooks for these populations are suggested.

    We examined the overall structure, lessons, and problem sets of a traditional U.S. mathematics textbook and a U.S. nontraditional mathematics textbook. Analyses of additional major U.S. texts would be informative and a useful start toward developing a consumers' report for mathematics textbooks.

3. **Assessments:** The comparison of Singapore's grade 6 PSLE examination with the grade 8 NAEP and the grade 6 and grade 8 state assessments was based on small samples of released items. This small sample was nevertheless sufficiently telling to suggest that the agencies that administer NAEP and states tests would be well advised to conduct similar comparisons, using a full test item set. They can then respond to the challenge posed by PSLE by providing similarly rigorous assessments. Another suggestion is to consider assessing the comparative difficulty of tests by having students from each country take the other country's test. For example, the grade 8 NAEP could be administered to a sample of Singapore grade 6 students to assess how challenging NAEP would be for students from a world-class education system who are two grades below the NAEP testing grade. Similarly, the grade 6 Singapore assessment could be administered to a sample of U.S. students in grades 6 and 8.

4. **Teachers:** ETS should examine the mathematics items on the PRAXIS I and II tests to determine item difficulty and should publicly state the grade levels of the mathematical content on each examination. ETS might also consider conducting studies that formally compare the PRAXIS I test, a content test, with the entrance tests used in Singapore to select students for Singapore's National Institute of Education. Currently no state requires prospective elementary teachers to pass the PRAXIS II (10140) test, an alternative that poses more challenging mathematics problems, consistent with teachers' demonstrating higher-order thinking skills. ETS should consider conducting studies comparing prospective teachers' scores and pass rates on the PRAXIS II and PRAXIS I tests. An examination of the mathematics preparation of teachers who have graduated from a large number of U.S. education schools would also be informative.

5. **Textbook Pilots:** We should validate the exploratory findings of the effects of the Singapore textbooks on student outcomes in the U.S. pilot schools with a rigorous,

scientific study. Using different mathematical textbooks in the same school would be confusing and increase the possibility of spillover effects, so an experimental design should randomly assign the Singapore and traditional mathematics textbooks to schools rather than classrooms. Teachers in Singapore pilot schools should have access to training to prepare them to teach the content of the Singapore textbooks. However, since added training is likely to confound the treatment in the experimental design, the control schools should also receive extra professional training. Because the Singapore curriculum is not repeated across grades, the pilots should track student cohorts beginning in the first grade. If practical, they should cover schools in a broad geographic area within a district. This would increase the likelihood that students transferring into and out of schools within the district continue their exposure to Singapore mathematics. Textbook experiments should also aim to cover different district contexts, including stable smaller communities that had the most successful pilot results. When highly mobile lower income communities are included, such as those in which the introduction of the textbooks did not produce student gains in the exploratory study, a system should be put in place for incoming student evaluation and provision for tutoring to bring these students up to grade. In fact, studies could also experimentally assess the effectiveness of replicating Singapore's system of tutoring lower performing students using Singapore textbooks.

Periodically repeating the studies that compare frameworks, textbooks, assessments, and teachers would be desirable and would create a set of key indicators about the status of the U.S. mathematics delivery system.

Reform Strategies to Consider

Although more information about the needs and effects of reform is always desirable, the consequences of reform are necessarily uncertain. Some reform options that inspire considerable confidence and that do not radically depart from the present system might move forward immediately. Others that bring about more radical change may have to await further study. Some options presented in prior chapters are extensions of reforms already under way and should be relatively easy to implement. Others require modest or significant changes to the current U.S. system and require more discussion. Still others represent fundamental departures from the U.S. system and would require extensive debate and perhaps pilot testing before being widely enacted. This section rearranges the reform options by the degree of change involved.

Tinkering Options: Improve or extend reforms that are already under way. This initial set of options builds on efforts already under way and is politically easier to achieve than the other options.

- **Frameworks:** NCTM and states that currently use grade-band standards should consider shifting to grade levels, as an increasing number of U.S. states have already done. Grade bands fail to specify intended content grade by grade and are no longer adequate when students are assessed on their mathematics performance grade by grade.

  NCTM and the states should also consider strengthening their process standards (representation, reasoning, and communication) by improving their integration with content standards. The College Board is already developing a framework that links NCTM process standards with the five NCTM content areas.

- **Textbooks:** Publishers should consider building on the characteristics of nontraditional textbooks, which offer rich problem-based learning examples, by adding content that strengthens the development of the conceptual understanding of mathematical topics.

- **Teachers:** NCATE and its member schools should consider strengthening the pre-service preparation of teachers by ensuring that education students actually learn the mathematics set out in NCATE's challenging standards for the preparation of elementary teachers. Requiring prospective teachers to take at least 9 hours of formal mathematics, as proposed by the Conference Board on the Mathematical Sciences (2001), would be a good second step. A good first step would be scholarships for those prospective education majors who do well on a entrance exams, similar to Singapore's, for acceptance into schools of education.

  The federal government should also consider ensuring that current teachers of mathematics meet the NCLB requirements for highly qualified teachers. An independent evaluation from the Education Commission of the States suggests that the implementation of the highly qualified teacher provision has been weaker than for other major NCLB provisions. School systems should support federal NCLB efforts by requiring teachers of mathematics who are not currently highly qualified to focus their professional development on content-based coursework in mathematics.

- **Assessments:** States should consider achieving greater uniformity in the difficulty of their mathematics tests by developing a common national item bank of mathematics test questions. Such a mathematics item bank could be modeled after the one the Council of Chief State School Officers has developed for science. Items would be systematically validated for their topic relevance, test reliability, and difficulty.

This set of provisions covers all four reform areas, but because the provisions work within existing reform efforts, they fail to achieve some of the more fundamental changes required. The following sets of recommendations, more difficult to accomplish, begin efforts to move toward such fundamental change.

Leveraging Options: Use market leverage to bring about improvements. One way to strengthen mathematics instruction is to take advantage of the responsiveness of the decentralized U.S education system to market forces.

- **Frameworks:** States should consider reducing the number of mathematics topics they expect to be covered at each grade level, making the curriculum deeper.

- **Textbooks:** States and school systems should consider stopping the purchase of mathematically weak textbooks and replacing them with textbooks that have strong content development. To inform prospective textbook buyers about the quality of mathematics texts, public interest organizations could develop and maintain consumer reports that rate the quality of mathematics textbooks content and pedagogy. While the AAAS textbook rating system for middle school mathematics textbooks is a good start at developing such a system, NCTM or a similar professional organization should be funded to establish criteria and implement a rating system for use by developers and potential purchasers of textbooks.

- **Assessments:** Similarly, professional and other public interest organizations such as ACHIEVE should consider undertaking the rating of the quality and rigor of state assessments.

- **Teachers:** School systems should consider approving only providers of professional development that offer high-quality, content-rich professional training aligned with state and local mathematical standards.

- **Teachers:** States should consider requiring more rigorous forms of PRAXIS tests.

By affecting demand, such reforms should be straightforward to implement and could have strong and immediate impacts.

Program Strengthening Options: Stay within the current U.S. education structure, but strengthen the mathematical depth and rigor of its components. These recommendations require significant reform and may take longer to bring about. In addition, obtaining political consensus for them may be more difficult than for strategies that build on current reform policies or market processes.

- **Textbooks:** Textbook publishers should consider reducing the number of topics and lessons in their textbooks to give more textbook space for developing a rich mathematical treatment of each topic. Publishers could also improve their textbooks by cutting down on the number of illustrations and text largely unrelated to helping students learn mathematics. In this manner, U.S. textbooks would more closely model the logical topic organization, rich problem-based approach, and varied pictorial representations of mathematics concepts found in Singaporean textbooks.

- **Teachers:** School systems could consider having primary-school teachers, who are currently generalists, specialize as early as first grade. Current primary-grade teachers have to be experts in mathematics, reading, science, and social studies, which is not a plausible strategy. Liping Ma (1999) highlights the expertise of Chinese teachers in teaching mathematics for understanding, but she also notes that some Chinese primary-grade teachers specialize. In the United States, the Learning First Alliance recommends that only mathematics specialists teach mathematics in grades 5 and above. Alternatively, schools can easily and conveniently arrange for teachers to pair up, with the teacher who is knowledgeable and comfortable with mathematics teaching both their mathematics classes in exchange for, say, science and/or social studies.

- **Assessments:** States should consider increasing the rigor of state assessments by incorporating more multistep and nonroutine mathematics problems to assess advanced levels of mathematical understanding. Currently test are used mainly to identify schools in need of improvement, asking questions primarily to enable schools to differentiate low-performing students from other students. We found that the current approach leads to grade 6 questions that are several levels lower in difficulty than those on Singapore's grade 6 PSLE. A testing system that focuses primarily on identifying poor performers may achieve equity but not excellence. To improve the feasibility of extending the testing range, states should consider introducing computerized or online adaptive assessments as rapidly as possible. By adjusting the questions to the mathematical ability of the test taker, adaptive testing allows the inclusion of more questions in the upper range of mathematical

difficulty without increasing testing time. Such a test has the additional benefit of being able to identify, and encourage, highly proficient students.

These options round out reforms that might occur within the present structure of the U.S. delivery system, but they still leave the United States with a system that is fragmented and does not adequately reward schools for their contribution to student mathematical learning.

Systemic Reform Options: These reforms would require fundamental changes in the federal role in mathematics education. We identify them because they appear vitally important to Singapore's successful education system and thus would be advantageous reforms for the United States to consider and debate.

- **Standards:** The United States should consider creating national mathematics standards that define a common core of mathematics skills and concepts at each grade**.** States would be able to supplement this core to meet their particular priorities. Singapore officials point to their national framework as the structural feature that distinguishes their mathematical system. It is hard to imagine how the United States can solve the problems of unfocused textbooks or teachers who lack training in mathematical content until we reach a consensus on a common core of mathematics content for each grade. Congress has already taken the unprecedented action of recommending a phonics-based approach to teaching reading, so a national mathematics framework may be politically feasible.

- **Assessments:** The United States should consider developing national mathematics assessments at key grades to ensure that all students, regardless of where they live, take equitable and challenging assessments. Singapore's high-stakes national assessments are given only at critical transition grades. In contrast, NCLB requires that states test every year in grades 3–8 and once in high school, producing large state differences in outcomes compared with NAEP's uniform test results. NAEP's grades 4, 8, and 12 tests have been suggested as a basis for building a national assessment (Finn and Hess, 2004). The development of voluntary national tests in mathematics and reading has been previously attempted, but not carried through. However, strong NCLB accountability provisions make at least a voluntary uniform test more politically acceptable now than in the past.

Common assessments would also make possible the introduction of value-added performance measures, like Singapore's, into NCLB's accountability system. Value-added measures identify the school's contribution to outcome growth, something that current NCLB measures do not accomplish. The current method fails to adjust for entry-level differences among pupils' abilities and penalizes schools that start with lower-performing pupils.

COMMITMENT TO REFORM TO ACHIEVE NCLB GOALS

We began our analyses of the Singapore mathematics system by pointing out that Singapore has not always been as successful as it is now, as measured by its students' performance on international assessments. Singapore has become a world-class system by devoting more than a decade to developing and carefully implementing major reforms to strengthen all aspects of its mathematics system. Our analyses found a Singapore system in which components are aligned to produce students who understand and can apply mathematics concepts and procedures to solve a range of varied, nonroutine mathematics problems. The Singapore system features a logical and balanced mathematics framework; textbooks rich with multistep problems and pictorial illustrations

of abstract concepts; challenging assessments; and teachers who know mathematics and are prepared to teach it well.

The features that make the Singapore system strong have direct applicability to weaknesses we identified in the current U.S. mathematics program. Our students never go much beyond learning the mechanics of applying definitions and formulas to routine, simple, one-step problems. Because the U.S. system is large and diverse, we probably cannot reform our mathematics system by using Singapore's unitary approach. Instead, reform must come about through a state, local, and national partnership. The U.S. reform effort should also capitalize on its strengths in applied mathematics and its emphasis on 21st century thinking skills, but what the United States needs overall are the sound features of the Singapore mathematics system. Replicating them in the United States will require the same sustained commitment to developing a quality mathematics system that Singapore gave to its reform efforts. Students in the United States deserve at least that much if they are to have the quality mathematics opportunities envisioned in No Child Left Behind, opportunities that Singapore provides to its students.

APPENDICES

APPENDIX A. REFERENCES

American Association for the Advancement of Science AAAS (2000). Middle grades mathematics textbooks: a benchmarks-based evaluation. A study included in Project. 2061. Washington, DC.

American Federation of Teachers (2001). *Making standards matter 2001*.Washington, D.C.:AFT. Available at: http://www.aft.org/pubs-reports/downloads/teachers/msm2001.pdf.

Berends, M., Chun, J., Schuyler, G,. Stockly, S., and Briggs, R.. (2001). *Challenges of conflicting school reforms: effects of new American schools in a High-Poverty District.* Rand Corp: Santa Monica, Calif.

Birman, B., Desimone, L., Porter, A. C., and Garet, M. (2000). Designing professional development that works. *Educational Leadership 57*(8), 28-33.

Bisk, R., Condike, A.M., Hogan, R.. and Little, S.A. (2002). *Singapore mathematics information session.* Massachusetts Department of Education. Presented at December 6, 2002 session in Malden, MA.

Bransford, J.D., Brown, A.L., & Cocking, R.R., eds. (2000). *How people learn: brain, mind, experience, and school.* Committee on Developments in the Science of Learning Research and Educational Practice, Commission on Behavioral and Social Sciences and Education, National Research Council. Washington, DC. Available at: http://www.nap.edu.

California State Department of Education (1999). *Mathematics Content Standards for California Public Schools Kindergarten Through Grade Twelve*. Sacramento, Ca. ISBN 0-8011-1457-8.

Champigny, H. (2003). *North Middlesex Regional School District Singapore Mathematics Program: A District-scale Study of Program Effectiveness*. Presentation before January 2003 "An Invitation to a Singapore Mathematics Information Session" held by Massachusetts Department of Education." Worcester, Massachusetts.

Choy, S.P., Chen, X., Ross, M. (1998). *Toward better teaching: professional development in 1993-94 (*NCES 98-230). Washington, DC: U.S. Department of Education, National Center for Education Statistics.

College Board on line http://www.collegeboard.com/prod_downloads/about/news_info/cbsenior/yr2002/pdf/threeA.pdf.

Colvin, R. L. (February 23, 1997). *Why tiny Singapore is at top of the class*. Los Angeles Times.

Columbia University Teachers College (2004a). *Elementary/childhood education-pre-service in the department of curriculum & teaching. program description.* Available at: http://www.tc.edu/academic/C&T/Child-EdPre/detail.asp?id=Program+Description.

Columbia University Teachers College (2004b). E*lementary/childhood education-pre-service in the department of curriculum & teaching: master of arts.* Available at: http://www.tc.edu/academic/C&T/Child-EdPre/detail.asp?id=Master+of+Arts.

Conference Board of the Mathematical Sciences (2000). *The mathematical education of teachers.* In issues in mathematics education series. V.11. Washington, DC. Available at http//www.ams.org.

Council of Chief State School Officers (1995). State Curriculum Frameworks in Mathematics and Science: How Are They Changing across the States? Washington, D.C.: Council of Chief State School Officers.

Council of Chief State School Officers (2004). *The SCASS Exchange.* June, 2004. Available online at: http://www.ccsso.org/content/PDFs/EXJune04.pdf.

Cronbach, L., Ambron, S., Dornbusch, S., Hess, R., Hornik, R., Phillips, D., Walker, D., and Weiner, S. (1985). *Toward reform of program evaluation.* San Francisco, Calif: Jossey-Bass Inc.

Devlin, K. (2000). Finding your inner mathematician. *The Chronicle of Higher Education, 46*, B5.

Doolittle, P.E. (2001). *Multimedia Learning: Empirical Results and Practical Applications*. Available online: http://edpsychserver.ed.vt.edu/workshops/edtech/pdf/multimedia.pdf

Dossey, J.A. (1997). *Essential skills in mathematics: A comparative analysis of American and Japanese assessments of eighth-graders.* (NCES97-885). Lois Peak and Dawn Nelson project officers. Washington, DC: National Center for Education Statistics.

Driscoll, D.P. (2003). September 25 Memorandum form Commissioner on *Massachusetts Curriculum Frameworks.*

Edmonds, R. (1979). Effective schools for the urban poor. *Educational Leadership*, *37,* 15–27.

Education Commission of the States. ECS (2004). *ECS report to the nation: state implementation of the no child left behind act.* Denver, Col. Available online at www.ecs.org/NCLB.

Education Testing Service ETS (2004). *The PRAXIS series.* Available online at: http://www.ets.org/PRAXIS/.

Education Testing Service ETS (2004a). *The PRAXIS series. elementary education: curriculum, instruction, and assessment (10011)t.* Available online at: http://ftp.ets.org/pub/tandl/0011.pdf.

Education Testing Service ETS (2004b). *The PRAXIS series. elementary education: content knowledge (10014*)Available online at: http://ftp.ets.org/pub/tandl/0014.pdf.

Education Testing Service ETS (2004c). *The PRAXIS series. elementary education(10140):* Available online at: http://ftp.ets.org/pub/tandl/0140.pdf .

Education Testing Service ETS (2004d). *The PRAXIS series. elementary education content area exercises.* Available online at: http://ftp.ets.org/pub/tandl/0012.pdf.

Education Week (January 2003). *Quality Counts.* Available at http://www.edweek.org/sreports/QC03/.

Elmore, R. (2002). *Bridging the gap between standards and achievement: the imperative for professional development in education.* Washington, DC: Albert Shanker Institute.

Elmore, R.F. and Burney, D. (1997). *Investing in Teacher Learning: Staff Development and Instructional Improvement in Community School District #2, New York City.* New York, NY: National Commission on Teaching and America's Future & Consortium for Policy Research in Education.

Everyday Learning Corporation (2001). *Everyday Math.* Chicago, Illinois.

Feistritzer, C.E. (1999). *Professional Development Opportunities for Teachers* Testimony before the Senate Committee on Health, Education, Labor and Pensions, June 22, 1999.

Ferguson, R. F. (1991). Paying for public education: New evidence on how and why money matters. *Harvard Journal on Legislation, 28*, 465-498.

Ferrini-Mundy, Joan, and Thomas Schram, eds. (1997). *The recognizing and recording reform in mathematics education project: insights, issues, and implications. Journal for Research in Mathematics Education* Monograph No. 8. Reston, Va.: National Council of Teachers of Mathematics.

Finn, C.E. and Hess, F.M.(2004). On leaving no child behind. In *Public Interest* Fall 200. pp. 35-56.

Florida State Department of Education (2004). *Sunshine State Standards*. Available online at: http://www.firn.edu/doe/menu/sss.htm

Florida Comprehensive Assessment Test 9FCAT0 eighth grade mathematics assessment sample items available online at: http://www.firn.edu/doe/sas/fcat/pdf/fc8mib3a.pdf.

Gilbert, J., Mosteller, F., and Tukey, J. Steady social progress requires quantitative evaluation to be searching. In C.C. Abt (Ed.), *The evaluation of social programs,* Beverly ills, Calif.: Sage, 1976.

Gitomer, D., Latham, A., and Ziomek, R (1999). *The academic quality of prospective teachers: the impact of admissions and licensure testing."* Princeton, NJ: Educational Testing Service.

Goldhaber, D. and Brewer, D. (2000). Does teacher certification matter? High school certification status and student achievement. *Educational Evaluation and Policy Analysis, 22*: 129-145.

Goldhaber, d. (2004). "Why do we license teachers?" in *A qualified teacher in every classroom.* Cambridge, Mass: Harvard Education Press.

Gonzales, P., Guzman J.C., Partelow, L, et. al., (2004). Highlights From the Trends in International Mathematics and Science Study (TIMSS) 2003. U.S. Department of Education, National Center for Education Statistics. Washington, DC; U.S. Government Printing Office

Grissmer, D.W., Flanagan, A., Kawata, J., & Williamson, S. (2000).*Improving student achievement: what state NAEP test scores tell us.* RAND: Santa Monica.

Gross, S. and Merchlinsky, S. (2002). *Evaluation of the Singapore Math Pilot Program: Year I Report of Findings.* Montgomery County Public Schools: Office of Shared Accountability: MD.

Harris, C.A., Miller, S.P., & Mercer, C.D. (1995). Teaching initial multiplication skills to students with disabilities in general education classrooms. *Learning Disabilities Research & Practice, 10*, 180–195.

Hean, T.C. (2000). Speech by rear admiral Teo Chee Hean, Singapore minister for education and second minister for defense at the NIE teachers investiture ceremony. January 11, 2000.

Hiebert, J., Gallimore, R, Garnier, H., Bogard Givvin, K., Hollingsworth, H., Jacobs, J., Chui, A., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek, W., Manaster, C., Gonzales, P., and Stigler, J., (2003). *Highlights from the TIMSS 1999 video study of eighth-grade mathematics teaching.* Washington, DC: NCES.# 2003011

Hirsch, E., Koppich, J.E., Knapp, M.S. (2001). *Revisiting What States Are Doing to Improve the Quality of Teaching: An Update on Patterns and Trends*. Washington State Center For the Study of Teaching Policy. Huat, J. and Huat, L. (2001). *A handbook for mathematics teachers in primary schools.* Times Media Private Limited. Singapore.

Kendall, J. S. (2001). *A technical guide for revising or developing standards and benchmarks*. Aurora, CO: Mid-continent for Education and Learning.

Kennedy, L.M., & Tipps, S. (1998). *Guiding children's learning of mathematics,* (8th ed.). Belmont, CA: Wadsworth.

The Ingenuity Project. Available (2003). http://www.ingenuityproject.org.

Loveless T. (1999): "The Tracking Wars: State Reform Meets School Policy." Washington, DC: Brookings Institution.

Loveless, T. (2003). Trends in math achievement: the importance of basic skills. Presented in *Papers and Presentations, Mathematics and Science Initiative*. U.S. Department of Education**:** http://www.ed.gov/inits/mathscience/loveless.html**.**

Loveless, T**.** (2003). Do students have too much homework in *Brown Center Report on American Education*. Washington, DC: Brookings Institution.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Lawrence Erlbaum Associates Inc., Mahwah, NJ.

Maccini, P. & Gagnon, J.C. (2000). Best practices for teaching mathematics to secondary students with special needs. *Focus on Exceptional Children, 32*, 1–21.

Malzahn, K. (2002) *Status of Elementary School Mathematics Teaching*. Chapel Hill, NC: Horizon Research, Inc. Available: http://www.horizon-research.com.

Maryland State Department of Education (2000). *Maryland State Content Standards for Math. A*vailable: http://mdk12.org/mspp/standards/math/index.html.

Mathematical Association of America, Committee of the Mathematical Education of Teachers. A Call for Change: Recommendations for the Mathematical Preparation of Teachers of

Mathematics, edited by James R. C. Leitzel. Washington, D.C.: Mathematical Association of America, 1991.

Mayer, R.E. and Moreno, R. ( ). *A Cognitive Theory of Multimedia Learning: Implications for Design Principle. Available at:* http://www.eng.auburn.edu/csse/research/research_groups/vi3rg/ws/mayer.rtf

MariMari (2003) online at: http://www.marimari.com/content/singapore/general_info/people/people.html

McMillen, M.S., Henke, R., & McGrath, D.J. & Cohen, B.A. (2002). Qualifications of the public school teacher workforce: prevalence of out-of-filed teaching 1987-88 to 1999-2000. National Center for Education Statistics. NCES 2002-603

McKnight, C.C., Crosswhite, F.J., Dossey, J.A., Kifer, E., Swafford, J.O., Travers, K.T., & Cooney, T.J., (1987). *The underachieving curriculum: Assessing U.S. school mathematics from an international perspective.* Champaign, IL: Stripes Publishing.

Medrich, E and Griffith, J (1992). *International Mathematics and Science Assessment: What Have We Learned?* Tables B.16 and B.17. National Center for Education Statistics, U.S. Department of Education.

Mercer, C.D., Jordan, L., & Miller, S.P. (1996). Constructivistic math instruction for diverse learners. *Learning Disabilities Research and Practice, 11,* 147–156.

Merrill, David (2003). First principles of Instruction. *Submitted for publication to "Edicatopnal Technology Research & Development."* Available at http://www.id2.usu.edu/Papers/5FirstPrinciples.PDF.

Merchlinsky, S. & Wolanin, N. (2003). *Evaluation of the Singapore math pilot program Year 2 report of findings.* Montgomery County Public Schools, Office of Shared Accountability: Md.

Ministry of Education (MOE), Singapore (1997). *Primary mathematics. Published by Curriculum Development Institute of Singapore.*

Ministry of Education, Singapore (1998). "To lead by example." *Contact*. January: No. 12.

Ministry of Education (MOE) Singapore (1999). *A summary report from the steering committee on the review of the teacher training system.* Available at http://www1.moe.edu.sg/press/1999/pr170399.htm.

Ministry of Education (MOE), Singapore (2000a). *Primary Education. Available: http://www1.moe.edu.sg/primary.htm#what%20are%20the%20various%20stages%20in%20 primary%20education.*

Ministry of Education (MOE), Singapore (2000b). *Singapore number one in mathematics again in the Third International Mathematics and Science Study 1999 [TIMSS-R].*Singapore Ministry of Education Press Release, December 2000: Author. Available at: http://www1.moe.edu.sg/press/2000/pr06122000.htm.

Ministry of Education (MOE), Singapore (2000c). *Committee of supply debate FY2000 minister's second reply on schools.* Available at: http://www.moe.gov.sg/speeches/2000/sp13032000d.htm

Ministry of Education (MOE), Singapore (2001a). *Primary mathematics syllabus.* Curriculum Planning and Development Division. Available: http://www1.moe.edu.sg/syllabuses/

Ministry of Education (MOE), Singapore (2001b). *Stepping up teacher recruitment. August 13 press release.*

Ministry of Education (MOE), Singapore (2001c). Speech by Radm (NS) Ted Chee Hean, minister for education and second minister for defense at the NIE teachers investiture ceremony on 4 July , 2001 Available at: http://www1.moe.edu.sg/speeches/2001/sp04072001.htm.

Ministry of Education (MOE), Singapore (2002). *Primary mathematics* (third edition). Singapore: Federal Publications.

Ministry of Education(MOE), Singapore (2003a). *Education statistics digest 2002.* Singapore: Author. Available: http://www1.moe.edu.sg/esd/index.htm.

Ministry of Education (MOE), Singapore (2003b). *Mission Statement of the Singapore education service.* Singapore: Author. Available: http://www.moe.gov.sg/corporate/mission.htm.

Ministry of Education (MOE), Singapore (2003c). *Mathematics Department.* http://www.moe.edu.sg/schools/rulang/org_mathematics.htm#LSP.

Ministry of Education (MOE), Singapore (2003d). *Primary School Leaving Exam.* http://www.moe.edu.sg/exams/psle/psle.htm.

Ministry of Education (MOE), Singapore (2003e).*The next phase in education: innovation and enterprise.* Speech by Mr. Tharman Shanmugaratnam, acting minister of education, at the MOE work plan seminar at Ngee Ann Polytechnic, October 2, 2003.

Ministry of Education (MOE), Singapore (2004). *Admission criteria for teaching.* Available at: http://www.moe.gov.sg/teach/AdmissionCriteria.htm.

Ministry of Education (MOE), Singapore (2004a). Speech. Mr. Tharman Shanmugaratnam, acting minister for education, at the teachers' conference on the *"Teacher as eduprenuer: exploring new frontiers" June 8, 2004.*Available at: *http://www.moe.gov.sg/speeches/2004/sp20040608.htm*

Montgomery County Public Schools (2001). *Mathematics Curriculum Framework.* Available: http://www.mcps.k12.md.us/info/curriculum/framework.html.

Mullis, I., Martin, M., Gonzalez, E., Gregory, K., Garden, R., O'Connor, K., Chrostowski, S., &. Smith, T. (2000). *TIMSS 1999 International Mathematics Report: Findings from IEA's Repeat of the Third International Mathematics and Science Study at the Eighth Grade.* Boston, Mass: TIMSS and PIRLS International Study Center, International Study Center Lynch School of Education, Boston College.

Mullis, I., Martin, M., Gonzalez, E., & Chrostowski, S., (2004). *TIMSS 2003 International Mathematics Report.* Boston, Mass: TIMSS and PIRLS International Study Center, Lynch School of Education, Boston College.

National Academy of Sciences (1996). *National Science Education Standards.* Washington. DC: Center for Science, Mathematics, and Engineering Education.

National Research Council (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, and B. Findell (eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

National Assessment for Educational Progress NAEP (2004). Th*e state assessment.* Available at: http://nces.ed.gov/nationsreportcard/about/state.asp.

National Assessment of Educational Progress (2004a) *Data Tool of* released questions available at: http://nces.ed.gov/nationsreportcard/itmrls/.

National Center For Education Statistics (NCES, 2000). *Pursuing excellence: comparisons of international eighth-grade mathematics and science achievement from a U.S. perspective, 1995 and 1999*, Office of Educational Research and Improvement. U.S. Department of Education: NCES 2001–028.

National Center For Education Statistics (NCES, 2002). *Digest of Education Statistics 2001.* Table 311.Mean number of semester credits completed by bachelor's degree recipients, by major and course area: 1976, 1984, and 1992-93. Washington: U.S. Department of Education.

National Council of Teachers of Mathematics (2000). *Principals and standards for schools mathematics.* Reston, VA. Available: http://www.nctm.org/standards.

National Council for the Accreditation of Teacher Education (2003). *Elementary education standards.* Developed by its constituent member the Association for Childhood Education International (ACEI) Available at: http://www.ncate.org/standard/new%20program%20standards/Program%20Report%20Templates%20(2004)/ACEI/ACEI%20standards.doc.

National Council for the Accreditation of Teacher Education, NCATE (2004). *Quick facts about NCATE.* Available at: http://www.ncate.org/ncate/fact_sheet.htm.

National Council for the Accreditation of Teacher Education, NCATE (2004a). *NCATE, making a difference.* Available at: http://www.ncate.org/pubs/diff.pdf.

National Research Council. (2001) *Adding It Up: Helping Children Learn Mathematics*. J Kilpatrick, J Swafford, and B Findell (Eds.), Washington, DC: National Academy Press

New Jersey School Report Card 2002. Available: http://nj.evalsoft.com.

New Jersey Grade 8 Middle School Statewide Assessment online at: http://www.state.nj.us/njded/assessment/ms/sample/test_book_math.pdf.

Newmann, F.M., Smith, B., Allensworth, E., and Bryk, A. (2001). Instructional program coherence: What it is and why it should guide school improvement policy. *Educational Evaluation and Policy Analysis.* Winter 2001, vol.23, no4. 297-321.

Nisbett, Richard (2003). *The geography of thought: how Asians and westerners think differently … and why.* NY, NY: The Free Press.

North Carolina End-of-Grade test sample items available online for grades 6 and 8 at: http://www.ncpublicschools.org/accountability/testing/eog/.

Ohio Sixth Grade Proficiency Test sample items available online at: http://www.ode.state.oh.us/proficiency/samples/sixth_previous.asp.

Parsad, B., Lewis, L., and Farris, E., (2001). *Teacher Preparation and Professional Development*: NCES 2001–088. Washington, DC: U.S. Department of Education, National Center for Education Statistics.

Partnership for 21st Century Skills (2003). *Learning for the 21st century*. Washington, D.C. Available at http://www.21stcenturyskills.org.

Pedulla, J., Abrams, L., Madaus, G., Russell, M., and Miao, J. (2003) *The perceived effects of state mandated testing programs on teaching and learning: Findings from a national survey of teachers*. Boston: NBETTP. [On-line]. Available at http://www.bc.edu/research/nbetpp/reports.html

Raimi, R. A., & Braden, L.S. (1998). State Mathematics Standards: An Appraisal of Math Standards in Forty-six States, the District of Columbia, and Japan. Washington, D.C.: Thomas B. Fordham Foundation.

Rosenbaum Foundation (2001). *Singapore primary mathematics teacher's guide.* Chicago, Ill.

Rosenthal, R. and Rosnow, R. (1984). *Essentials of behavioral analysis: Methods and data analysis. New York:* McGraw-Hill.

Rosenstein, J.G., Caldwell, J.H., Crown, W.D., (1996). New Jersey mathematics curriculum framework. New Jersey Mathematics Coalition and New Jersey Department of Education. Available at http://www.state.nj.us/njded/frameworks/math/math.pdf.

Scott Foresman-Addison Wesley (2004). *Mathematics*. Pearson Education Inc.

Shanmugaratnam, T. (2003). October 2 speech by acting Minister of Education at the MOE workplan seminar. Ngee Ann Polytechnic University.

Schmidt, W., Houang, R., and Cogan, L. (2002). A coherent curriculum. *American Educator*: Summer 2002. pp.1-17.

Shulman, L.(1986). "Those who understand: knowledge growth in teaching," *Educational Researcher*: February 1986.

Singapore National Institute of Education (2003). *Degree and Diploma Programs.* Available at http://www.nie.edu.sg/html/itt_deg_dip.htm.

Smith, M.S., & O'Day, J. (1991). Systemic school reform. In S. H. Fuhrman, & B. Malen (Eds.), *The politics of curriculum and testing (Politics of Education Association Yearbook*, 1990) (pp. 233-267). London: Taylor & Francis.

Snipes, J., Doolittle, F., and Herlihy, C. (2002). *Foundations for success: case studies of how urban school systems improve achievement.* MDRC and Great City Schools.

Sykes, G and Burian-Fitzgerald, M. (2004). "Cultivating quality in teaching" in *A qualified teacher in every classroom.* Cambridge, Mass: Harvard Education Press.

Tan, Y.K. (1995) Eds. "Performance indicators for school management (PRISM)." In Ginsburg, A. and Owen, E. Eds. APEC performance measurement systems symposium. Washington, DC: Education Forum.

Teachers College, Columbia University (2003). Elementary/Childhood Education-Pre-service Master of Arts Degree Requirements. Available at: http://www.tc.edu/programs/Child-EdPre/ma.htm.

Texas Assessment of Knowledge and Skills (TAKS) sample items available online for 6th and 8th grade at: http://www.ncpublicschools.org/accountability/testing/eog/.

Texas Education Code, §28.002*, Chapter 111. Texas Essential Knowledge and Skills for Mathematics Subchapter A. Elementary.* Available at : http://www.tea.state.tx.us/rules/tac/chapter111/ch111a.pdf.

Times Media Private Limited. (2003). *Primary Mathematics.* Content prepared by Curriculum, Planning and Publishing Division, Ministry of Education, Singapore.

Tyson-Bernstein, H (1988). *A conspiracy of good intentions: America's textbook fiasco.* Washington, DC: Council for Basic Education.

Tyson, H., & Woodward, A. (1989). Why students aren't learning very much from textbooks. *Educational Leadership, 47:*14-17.

U.S. Congress (2002). *No child left behind act of 2001.* Public Law 107–110 107th Congress.

U.S. Congress (2002a). *P.L. 107-110 -- No Child Left Behind Act of 2001. SEC. 602. amendment to the national education statistics act of 1994. (a) national assessment of educational progress.*

U.S. Congress (2002b). *H.R.1 conference report highlights: fact sheet.* House Education and the Workforce Committee. October, 2002. Available online at: http://edworkforce.house.gov/issues/107th/education/nclb/accountfact.htm.

U.S. Department of Education (1999). *Promising results, continuing challenges: final report of the national assessment of title.* Washington: DC: Planning and Evaluation Service.

U.S. Department of Education (2002). July 24, 2002 letter in *Key policy letters signed by the education secretary or deputy secretary.* Available at: http://www.ed.gov/policy/landing.jhtml.

U.S. News and World Reports (2004). *America's best graduate schools 2005.* Available online at: http://www.usnews.com/usnews/edu/grad/rankings/edu/brief/edurank_brief.php.

Van de Walle, J.A. (2001). *Elementary and middle school mathematics: Teaching developmentally* (4th ed.). Addison Wesley Longman, Inc.

Viadero, D. (September 27, 2000) *U.S. schools importing Singaporean texts. Education Week, I*

Valverde, G. & Schmidt, W. (1997). Refocusing U.S. math and science education. *Issues in science and technology.* Winter, 1997-98.

Wainer, H. (Ed.) with Dorans, N.J., Eignor, D., Flaugher, R., Green, B.F., Mislevy, R.J., Steinberg, L., and Thissen, D. (2000). *Computer-adaptive testing: a primer.* Mahwah, NJ: Lawrence Erlbaum Associates.

Wang, A., Coleman, A.B., Coley, R.J., Phelps, R.P. (2003). *Preparing teachers around the world.* Princeton, NJ: Education Testing Service. Available at http://www.ets.org/research/pic/prepteach.pdf.

Whitehurst, G.J. (2002). *Research on teacher preparation and professional development.* Paper presented at the White House Conference on Preparing Tomorrow's Teachers, March 5, 2002. Available at http://www.ed.gov/inits/preparingteachersconference/index.html.

Whitehurst, G.J. (2003). *Research on mathematics education. paper presented at* Secretary's Summit on Mathematics, U.S. Department of Education.

Wu, H. (1999) Basic skills versus conceptual understanding a bogus dichotomy *in Mathematics Education* Fall: Volume 20, Number 4. pp.1-7.

Xerox Corporation (1989). *Leadership through quality processes and tools review: quality you can copy.* Multinational Customer & Service Education, Rochester, NY.

APPENDIX B. CHARTS REFERENCED IN CHAPTERS 3 AND 6

Exhibit B3–1. Classification of State Mathematics Frameworks by Grade-Specific and Grade-band Categories, 2003*

State	Type of Framework	State	Type of Framework
Alabama	Grade Bands	New Hampshire	Grade Specific (Draft)
Alaska	Grade Bands	New Jersey	Grade Specific
Arizona	Grade Bands	New Mexico	Grade Specific
Arkansas	Grade Specific	New York	Grade Bands
California	Grade Specific	North Carolina	Grade Specific
Colorado	Grade Bands	North Dakota	Grade Bands
Connecticut	Grade Bands	Ohio	Grade specific
Delaware	Grade Bands	Oklahoma	Grade Specific
District of Columbia	Grade Specific	Oregon	Grade Specific
		Pennsylvania	Grade Bands
Florida	Grade Bands		
Georgia	Grade Specific	Rhode Island	Grade Bands
Hawaii	Grade Bands	South Carolina	Grade Bands
Idaho	Grade specific	South Dakota	Grade Specific
Illinois	Doesn't begin until Grade 3 (match assessments)	Tennessee	Grade Specific
Indiana	Grade Specific	Texas	Grade Specific
Iowa	None	Utah	Grade Specific
Kansas	Grade Specific	Vermont	Grade Bands
Kentucky	Grade Bands	Virginia	Grade Specific
Louisiana	Grade Bands	Washington	Grade Specific
Maine	Grade Bands	West Virginia	Grade Specific
Maryland	Grade Specific	Wisconsin	Grade Bands
Massachusetts	Grade Bands	Wyoming	Grade Specific
Michigan	Grade Bands		
Minnesota	Grade Specific		
Mississippi	Grade Specific		
Missouri	Grade Bands		
Montana	Grade Bands		
Nebraska	Grade Bands		
Nevada	Grade Specific		
*Based on a review of 2003 review of State mathematics frameworks on the Web.			

Exhibit B3–2. NCTM Algebra Standard for Grades Pre-K–2

Standards	Expectations
Instructional programs from pre-kindergarten through grade 12 should enable all students to—	In pre-kindergarten through grade 2 all students should—
– Understand patterns, relations, and functions	– Sort, classify, and order objects by size, number, and other properties; – Recognize, describe, and extend patterns such as sequences of sounds and shapes or simple numeric patterns and translate from one representation to another; – Analyze how both repeating and growing patterns are generated.
– Represent and analyze mathematical situations and structures using algebraic symbols	– Illustrate general principles and properties of operations, such as commutativity, using specific numbers; – Use concrete, pictorial, and verbal representations to develop an understanding of invented and conventional symbolic notations.
– Use mathematical models to represent and understand quantitative relationships	– Model situations that involve the addition and subtraction of whole numbers, using objects, pictures, and symbols.
– Analyze change in various	– Describe qualitative change, such as a student's growing taller; – Describe quantitative change, such as a student's growing two inches in one year.
National Council of Teachers of Mathematics (2000)	

Exhibit B3–3. Singapore's Topic Matrix—Primary 1 to 4 and Primary 5 and 6 (Normal Track)

P1	P2	P3	P4	P5	P6
WHOLE NUMBERS					
1. Number notation and place values up to 100 2. Cardinal and ordinal numbers 3. Comparing and ordering 4. Addition and subtraction of numbers within 100 5. Multiplication of numbers whose product is not greater than 40 6. Division of numbers not greater than 20	1. Number notation and place values up to 1000 2. Addition and subtraction of numbers up to 3 digits 3. Multiplication and division within the 2, 3, 4, 5 and 10 times tables	1. Number notation and place values up to 10 000 2. Addition and subtraction of numbers up to 4 digits 3. Multiplication tables up to 10×10 4. Multiplication and division of numbers up to 3 digits by a 1-digit number 5. Odd and even numbers	1. Number notation and place values up to 100 000 2. Approximation and estimation 3. Factors and multiples 4. Multiplication of numbers • up to 4 digits by a 1-digit number • up to 3 digits by a 2-digit number 5. Division of numbers up to 4 digits by a 1-digit number and by 10	1. Number notation and place values up to 10 million 2. Approximation and estimation 3. Multiplication and division of numbers up to 4 digits by a 2-digit whole number 4. Order of operations	
MONEY, MEASURES & MENSURATION					
1. Measurement of • length • mass in non-standard units 2. Time (12-hour clock) • o'clock • half past 3. Money • dollars ($) and cents (¢) • addition and subtraction of money in dollars only or in cents only	1. Measurement of • length : meter, centimeter • mass : kilogram, gram • volume : liter • time : hour, minute 2. Addition and subtraction of • length • mass • volume 3. Addition and subtraction of money (in compound units)	1. Units of measure • length : kilometer, meter, centimeter • mass : kilogram, gram • time : hour, minute, second, day, week, month, year • area : square meter, square centimeter • volume : liter, milliliter 2. Addition and subtraction of length, mass, volume and time (in compound units) 3. Addition and subtraction of money (in compound units using decimal notation) 4. Perimeter of rectilinear figures 5. Area and perimeter of • a square • a rectangle	1. Multiplication and division of length, mass, volume and time (in compound units) 2. Multiplication and division of money (in compound units using decimal notation) 3. Units of measure of volume: cubic centimeter, cubic meter 4. Volume of • a cube and a cuboid • liquid 5. Area and perimeter of a square, a rectangle and their related figures	1. Conversion of units of measure involving decimals and fractions 2. Volume of a cube and a cuboid 3. Area of a triangle	1. Area and circumference of a circle 2. Area and perimeter of a figure related to square, rectangle, triangle and circle 3. Volume of • a solid made up of cubes and cuboids • liquid
STATISTICS					
1. Picture graphs • Constructing, reading and interpreting	1. Picture graphs with scales • Constructing, reading and interpreting • solving problems	1. Bar graphs • reading and interpreting • solving problems	1. Tables • constructing, reading and interpreting • solving problems 2. Bar graphs • constructing, reading and interpreting • solving problems	1. Line graphs • reading and interpreting • solving problems	1. Pie charts • reading and interpreting • solving problems

American Institutes for Research®

P1	P2	P3	P4	P5	P6
GEOMETRY					
1. Shapes • rectangle;• square; • circle;• triangle. 2. Patterns: complete patterns according to • shape;• size; color	1. Shapes☺ • semicircle • quarter circle 2. Patterns: complete patterns according to • shape • size • orientation • two of the above attributes 3. Lines, curves and surfaces	1. Concept of angles	1. Perpendicular and parallel lines 2. Angles in degrees 3. Symmetry 4. Geometrical figures • rectangle • square • parallelogram • rhombus • trapezium • triangle 5. Properties of • a square • a rectangle 6. 2-D representation of a 3-D solid • cube and cuboid • solid made up of unit cubes	1. Angles • angles on a straight line • angles at a point • vertically opposite angles 2. 8-point compass 3. Properties of • a parallelogram • a rhombus • a trapezium • a triangle 4. Geometrical construction: Draw a square, a rectangle, a parallelogram, a rhombus and a triangle from given dimensions 5. Tessellation	1. Angles in geometric figures 2. 2-D representation of a 3-D solid • prism • pyramid 3. Nets of • a cube • a cuboid • a prism • a pyramid
FRACTIONS					
	1. Equal parts of a whole 2. Idea of simple fractions 3. Comparing and ordering like fractions	1. Equivalent fractions 2. Comparing and ordering unlike fractions	1. Addition and subtraction • like fractions • related fractions 2. Product of a proper fraction and a whole number 3. Mixed numbers and improper fractions	1.Addition and subtraction of • mixed numbers • unlike fractions 2. Product of fractions 3. Concept of fraction as division 4. Division of a proper fraction by a whole number	
DECIMALS					
				1. Multiplication up to 2 decimal places by a 2-digit whole number 2. Multiplication and division up to 3 decimal places by tens, hundreds, thousands	1. Number notation and place values up to 3 decimal places 2. Comparing and ordering 3. Addition and subtraction up to 2 decimal places 4. Multiplication and division up to 2 decimal places by 1-digit whole number 5. Conversion between decimals and fractions 6. Approximation and estimation 10
AVERAGE/ RATE/SPEED					
				1. Average 2. Rate	1. Time (24-hour clock) 2. Speed
RATIO/ PROPORTION					
				1. Ratio	Ratio and direct proportion
PERCENTAGE					
				1. Concept of percentage 2. Percentage of a quantity	One quantity as a percentage of another
ALGEBRA					
					1. Algebraic expressions in one variable • simplification • evaluation 12
Source: Ministry of Education Singapore (2001)					

Exhibit B3–4. NCTM Mathematics Standards: Grade K–2

Instructional programs from pre-kindergarten through grade 12 should enable all students to:	In pre-kindergarten through grade 2 all students should:
Numbers and Operations	
<u>Understand numbers</u>, ways of representing numbers, relationships among numbers, and number systems	• count with understanding and recognize "how many" in sets of objects; • use multiple models to develop initial understandings of place value and the base-10 number system; • develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections; • develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers; • connect number words and numerals to the quantities they represent, using various physical models and representations; • understand and represent commonly used fractions, such as 1/4, 1/3, and 1/2.
<u>Understand meanings</u> of operations and how they relate to one another	• understand various meanings of addition and subtraction of whole numbers and the relationship between the two operations; • understand the effects of adding and subtracting whole numbers; • understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally.
<u>Compute fluently</u> and make reasonable estimates	• develop and use strategies for whole-number computations, with a focus on addition and subtraction; • develop fluency with basic number combinations for addition and subtraction; • use a variety of methods and tools to compute, including objects, mental computation, estimation, paper and pencil, and calculators.
Algebra	
<u>Understand patterns</u>, relations, and functions	• sort, classify, and order objects by size, number, and other properties; • recognize, describe, and extend patterns such as sequences of sounds and shapes or simple numeric patterns and translate from one representation to another; • analyze how both repeating and growing patterns are generated.
<u>Represent and analyze</u> mathematical situations and structures using algebraic symbols	• illustrate general principles and properties of operations, such as commutativity, using specific numbers; • use concrete, pictorial, and verbal representations to develop an understanding of invented and conventional symbolic notations
<u>Use mathematical models</u> to represent and understand quantitative relationships	• model situations that involve the addition and subtraction of whole numbers, using objects, pictures, and symbols.
<u>Analyze change</u> in various contexts	• describe qualitative change, such as a student's growing taller; • describe quantitative change, such as a student's growing two inches in one year.
Geometry	
<u>Analyze characteristics</u> and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships	• recognize, name, build, draw, compare, and sort two- and three-dimensional shapes; • describe attributes and parts of two- and three-dimensional shapes; • investigate and predict the results of putting together and taking apart two- and three-dimensional shapes.
<u>Specify locations</u> and describe spatial relationships using coordinate geometry and other representational systems	• describe, name, and interpret relative positions in space and apply ideas about relative position; • describe, name, and interpret direction and distance in navigating space and apply ideas about direction and distance; • find and name locations with simple relationships such as "near to" and in coordinate systems such as maps.
<u>Apply transformations</u> and use symmetry to analyze mathematical situations	• recognize and apply slides, flips, and turns; • recognize and create shapes that have symmetry.
<u>Use visualization</u>, spatial reasoning, and geometric modeling to solve problems	• create mental images of geometric shapes using spatial memory and spatial visualization; • recognize and represent shapes from different perspective; • relate ideas in geometry to ideas in number and measurement; • recognize geometric shapes and structures in the environment and specify their location.
Measurement	
<u>Understand measurable attributes</u> of objects and the units, systems, and processes of measurement	• recognize the attributes of length, volume, weight, area, and time; • compare and order objects according to these attributes; • understand how to measure using nonstandard and standard units; • select an appropriate unit and tool for the attribute being measured.
<u>Apply appropriate techniques</u>, tools, and formulas to determine measurements	• measure with multiple copies of units of the same size, such as paper clips laid end to end; • use repetition of a single unit to measure something larger than the unit, for instance, measuring the length of a room with a single meter stick; • use tools to measure; • develop common referents for measures to make comparisons and estimates.
Data Analysis and Probability	
<u>Formulate questions</u> that can be addressed with data and collect, organize, and display relevant data to answer them	• pose questions and gather data about themselves and their surroundings; • sort and classify objects according to their attributes and organize data about the objects; • represent data using concrete objects, pictures, and graphs.
<u>Select and use</u> appropriate statistical methods to analyze data	• describe parts of the data and the set of data as a whole to determine what the data show.
<u>Develop and evaluate</u> inferences and predictions that are based on data	• discuss events related to students' experiences as likely or unlikely.
Source: National Council of Teachers of Mathematics (2000)	

American Institutes for Research®

Exhibit B6–1. State's Use of Different PRAXIS II Exams For New Teacher Licensing

State	Elementary Education: Curriculum, Instruction, and Assessment 10111	Elementary Education: Content Knowledge 10014	Multiple Subject Assessment Content Knowledge 10140	Elementary Education: Content Area Exercises 20012
Total	18	16	1	8
Alabama				
Alaska	X	X	X	
Arkansas				
California				
Colorado		X		
Connecticut	X			
Delaware	X			
District of Columbia	X	X		
Georgia		X		X
Hawaii	X			X
Idaho		X		
Indiana	X			
Kansas	X			
Kentucky	X	X		
Louisiana		X		
Maine				
Maryland		X		X
Minnesota		X		
Mississippi	X			
Missouri	X			
Nebraska				
Nevada	X			X
New Hampshire				
New Jersey		X		
New Mexico				
North Carolina	X			X
North Dakota	X			
Ohio				
Oklahoma				
Oregon				
Pennsylvania	X			
Rhode Island		X		X
South Carolina	X			X
South Dakota		X		
Tennessee		X		
Texas				
Utah	X			X
Vermont		X		
Virginia		X		
Washington		X		
West Virginia	X			
Wisconsin				
Wyoming	X			

Source: Education Testing Service (2004). *PRAXIS State Requirements. Available online at http://www.ets.org/PRAXIS/prxstate.html*

APPENDIX C. PROFESSIONAL DEVELOPMENT CASE STUDY

IMPLEMENTING SINGAPORE MATH AT COLLEGE GARDENS ELEMENTARY SCHOOL MONTGOMERY COUNTY PUBLIC SCHOOLS, MARYLAND

Robyn Silbey, School-based Math Specialist

Four Montgomery County schools were selected to participate in a Singapore Math pilot in June, 1999. Our initial charge from the district was to use the books *on* grade level. We were instructed to begin on page 1 of each book and continue uninterrupted and skip no pages, through the end of the book. Although an occasional U. S. reproducible workbook page could be used, no U. S. textbook resources were permitted. Concepts not appearing in Singapore Math were not to be taught, with the exception of topics such as customary measurement and money.

In early July, a member of the central math office and I were asked to provide initial training for the pilot program. An initial two-day training was hastily put together for delivery within 10 days. We decided that a good place to start was for teachers to make and share their observations of how the new Singapore books compared with their U.S counterparts. It was noted that teacher's edition in the U.S. have a wraparound feature that relates each student page to instructional ideas for the teacher. Suggestions for pre-assessing, introducing, and teaching the student pages are provided. Discussion questions and follow up ideas are also available, as well as reproducible worksheets for differentiated instruction. All of these features are "wrapped around" annotated reductions of the student pages. In the Singapore books, however, its was noted that the teacher's edition is far less wedded to the student book. A graphic at the beginning of each instructional unit tells the number of periods or lessons, the accompanying workbook pages and some activities that are to be sprinkled throughout the unit. There are no specific directions or suggestions as to when the activities should be intertwined into the unit. Some activities relate directly to a lesson, while others seem to be "stand-alones" or games. In addition, activities include materials and visuals to which we have no access, so we are required to substitute with what we assume are comparable materials.

We distributed books to our participants and offered ideas for using the books. We allowed time for teachers to plan a unit. This initial training was received poorly. We were not experts in Singapore Math ourselves and were unable to answer most questions. In short, we did not adequately prepare teachers emotionally or academically to teach from these materials.

I returned to my school (working halftime, or 20 hours per week) in August, with the charge to train my building's teachers in Singapore Math. Our administrator wisely arranged grade level teams to have their "specials" (art, music, physical education) at the same time so that each grade level team had common planning time each week.

In broad-brush strokes, the professional development training we implemented had six components:

1. Team meetings with grade levels 1 through 5
2. Customized support
3. Modeling lessons in each classroom, grades 1 through 5

4. Half days for "lesson study," grades 1 through 5
5. After-school district-sponsored trainings
6. School-wide staff meetings with a focus on mathematics and program implementation

1. Team meetings

I met with each grade level team at least once every week for about 45 minutes. Weekly grade level meetings during the first year consisted of grasping the new approaches to topics in the student book and gleaning ideas for instruction from the teacher's guide. There are approximately 40 lessons in each Singapore year, compared with our approximately 170 days of instruction so we are forced to make decisions about how and when to spread the longer Singapore lesson across more than one day.

As in the US, Singapore books use the "Concrete—Pictorial—Abstract" approach. The Singapore books often show illustrations of models similar to base ten blocks, counters, and so forth. This provides a comfort level for teachers, and we used time at our weekly team meetings to discuss manipulatives and how they could best be used. We availed ourselves of the "Notes for Teachers," found prior to each major concept. They are often quite concise and required some elaboration and further discussion. Another focus was the "colour patch", which appears throughout the student book (see below for examples). To quote the preface in each student book, "The colour patch is used to invite active participation from the pupils and to facilitate oral discussion." Although we embraced this concept with open arms, it was difficult to implement in the short run. The first year, some teachers were too overwhelmed with learning and teaching Singapore Math to facilitate meaningful discourse. The colour patches were used, but not in conjunction with effective questioning strategies. In subsequent years, we strived the meet the following goal: ***Every class at every grade level is expected to routinely engage their students in active discourse.*** Part of our weekly meetings from the second year forward was scheduled to provide attention to regularly asking higher order thinking questions and integrating math terminology into classroom discussions. We wrote questions that supported the colour patch and discussed strategies for getting students to respond in small groups and with partners.

Two examples of colour patches (shown with [ ]) and the questions we discussed at weekly meetings are shown below.

6 tens ×3 = [ ] tens

- What are some reasons you think the answer is ___?

- What are some ways to represent this problem?

- What are some things you need to know in order to find the answer?

- How would the product change if 6 tens were multiplied by 7? If 6 hundreds were multiplied by 3?

- How could this problem be solved or checked using another operation?

90 – 17 = [ ] (computed mentally in grade 3, approximate process shown below)

- What are some reasons you think the answer is ___?

- What helps you decide how to decompose 90?

- What are some other ways to mentally calculate this difference?

- How would the process change if 17 were subtracted from 92? From 98?

- How can addition be used to check your answer?

```
  90 – 17
  /  \              20 – 17 = 3      70 + 3 = 73
 70  20
```

One stunning advantage to the Singapore texts is their emphasis on the connectedness of concepts. Subtraction is presented within the context of addition, and division is presented within the context of multiplication. In Liping Ma's book, "Knowing and Teaching Elementary Mathematics," she notes that, "the power of a concept depends on its relationship with other concepts." One of the many reasons that Singapore children excel internationally is their intertwined web of conceptual understanding, which deepens the understanding of each individual concept.

It was imperative for us to comprehend the scope of learning for the teachers at each grade level. Typically, when implementing a new program, quick learners and highly motivated teachers can grasp the "big ideas" and run with them fairly early on. This implementation was different. The approach to teaching Singapore Math is unlike the typical U.S. approach for most strands at every grade level. With each new unit and virtually every lesson within it, teachers must relearn the mathematics and think about how they will convey it to the children. Moreover, teachers need a thorough understanding of the Singapore concept development so they can create additional classroom examples and practice. For example, second grade students are asked to mentally compute 156 + 99. Teachers need to furnish additional examples, such as 275 + 198 or 495 + 321, and guide children to understand how to mentally find their sums. Teachers reported that they spent an average of 10-15 hours *per week* studying Singapore Math. Team meetings helped us address these issues.

2. Customized Support

Our classes were heterogeneously grouped, so we had a wide range of abilities in each class. Because I was working closely with the books at every grade level, I was in the best position to help teachers customize the program for their grade levels. For example, the fourth grade unit on measurement assumes that students are able to convert between metric units. Particularly in the first year of implementation, this is not the case. I provided teachers with a large part of the third grade measurement unit, which they taught prior to using their own grade level materials.

Some students demonstrate deep understanding of the grade level material. We strived to challenge them with enrichment and extension. Acceleration for grades 2 through 4 typically involved the corresponding unit from the next year's text. Grade 1 is different because the book handles numbers to 100 in separate units throughout the year. Grade 2 begins with an assumption of that knowledge, so grade 1 must be completed in its entirety before acceleration in grade 2 makes sense. We found that acceleration in grade 5 was not necessary. Upon closer examination and comparison with our district's standards, we found that the Singapore Math texts at grade 5 corresponded to indicators from grades 5 and 6, and even a handful from grade 7.

In addition to weekly team meetings, I was also "on call" for individual teachers seeking help and advice or needing to vent. Their needs were assorted. Some teachers resisted teaching from the books entirely and needed individual coaching. Other teachers felt they could grasp the processes for their own understanding, but not thoroughly enough to teach it effectively. Teachers wanted to know how to help strugglers and high flyers. I addressed each and every issue individually, unless I found that there was a pattern, at which point I devised a plan for a more global response.

The Grade 1 vignette that follows illustrates some of the mathematical and pedagogical issues teachers needed to understand and needed assistance with implementing.

Grade 1 vignette: Grade 1 appeared to move very slowly at the beginning, with a great deal of emphasis on the composition and decomposition of numbers. Teachers struggled with the idea of crawling through this, thinking their children already understood what "4 " means. Weeks later, the children were required to apply what they have learned as they find sums greater than 10. To add $9 + 4$, for example, students are expected to decompose 4 as 3 and 1. Then, they are to compose 10 from 1 and 9, and add the remaining 3 for a sum of 13.

```
 9  +  4
      /\      becomes   (9 + 1) + 3
     1   3                 10 + 3 = 13
```

Later in first grade, this initial exposure to the concept of regrouping is revisited as children are asked to find the sums of $38 + 7$ using decomposition and composition.

Near the end of first grade, students learn to add two-digit numbers without regrouping. All number sentences for addition and subtraction are presented horizontally. The children's thinking is shown in the book using arrow notation. The gist of the two-digit addition found near the end of first grade instruction is shown below. By this time, children have a thorough understanding of two-digit place value and are able to compute most addition mentally.

```
              + 30              + 2

54 + 32  ---------→  84 ---------→ 86
```

3. Modeling lessons in each classroom

I modeled what I believe to be "best practices" in 7-10 classrooms per week, or about twice per month for each teacher. Appointments were made with individual teachers during the weekly planning meetings. The twice-monthly schedule was eventually replaced with an as-needed basis. Teachers in the third year of implementation no longer needed much assistance, while newcomers needed a great deal of guidance. Consistently, the most frequently requested lessons involved decomposition and composition in the primary grades, problem solving using the "bar model" in the upper grades, and mental math for all grades.

My original implementation plan was to offer to teach a lesson, teach it, briefly discuss it with the teacher, and move to the next teacher. Results using this process were mixed at best. Teachers did not necessarily focus on the strategies and methods I wished for them to see. Moreover, in an effort to be compassionate, I allowed teachers to divide their attention between my work with their children and their other assorted duties—grading papers, reading memos, and so forth. In time, I created a more formalized process. I am convinced that this process was instrumental in improving the quality of mathematics teaching in our building.

Prior to each lesson, the teacher and I discussed the lesson I would be modeling. The teacher was provided with my Lesson Observation Form that s/he must complete during the lesson. The Lesson Observation Form focuses the teacher's attention on the parts of an effective lesson as well as the students' behavior during the lesson. After the lesson we debriefed using the Lesson Observation Form for talking points. We discussed the features of the lesson, its alignment with the student book, the students' reactions, and so forth. The general headings are shown below. The actual forms, along with "look fors" are attached at the end of this document.

Lesson Observation Form

Focus on Teaching
Pre-assess/Warm up/Preview
Goal-setting introduction
Body
  (Mathematics, questioning, grouping)
Close/Reflection

Focus on Learning
Evidence of engagement
Evidence of ongoing assessment
Evidence of differentiation

Next Steps
What would you like to try?

The Lesson Observation Form helped the teacher focus on the parts of the lesson that I felt were most important for him or her to see. The teacher must stay focused throughout the lesson in order to complete the form. Finally, the teacher was invited to think about implementing something that suits his or her style in the future. Using the Model, Coach, Apply process, the teacher and I team-taught when he or she felt comfortable.

Eventually, the teacher and I switched roles as the teacher modeled a lesson while I completed the form.

4. **"Lesson Study" professional half days**

During the first two years of the pilot, and about four times per year, a grade level team was released from classroom duties to participate in a three-hour staff development experience. The process for the half-day was an outgrowth of the in-class individual modeling outlined above. In this version, the team and I spent about 30 minutes prior to the lesson discussing what they will see, while I modeled a lesson in one of their classrooms. We discussed (a) the lesson, (b) its goals, (c) the "look fors," (d) evidence of student understanding, and (e) follow up activities. All grade level teachers observed the lesson. Each teacher completed the Lesson Observation Form. After the lesson, we reconvened and critically reviewed the lesson, using teachers' comments on the Lesson Observation Forms as a starting point. We discussed what the children learned, how they learned it, and how the lesson can be improved. With the exception of the teacher's class in which I modeled the lesson, teachers used the information gleaned from the experience to teach the same lesson the next day.

5. **After-school district-sponsored trainings**

After-school district-sponsored trainings were provided by the district's math office. Trainings were from 4:00 p.m. to 7:00 p.m.. The trainings had global themes that may related to content and/or process. Attendance was voluntary but encouraged. I attended all trainings. Each time, about half of the staff at my school attended as well. I believe the training was worthwhile because the specialists offered a distanced, global view of the Singapore Math curriculum. In addition, teachers from other pilot schools were able to compare experiences and exchange information.
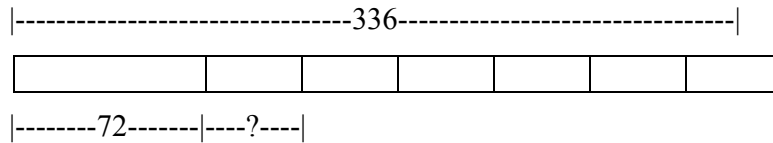
6. **School wide staff meetings**

I designed and presented school wide or *vertical* staff meetings throughout the pilot implementation period. We had about four staff meetings related to Singapore Math the first year and two meetings the second year. The focus of the meetings was on the program implementation and mathematics content. Implementation meetings were on topics such as parent education, rich discourse, and writing in math. Content meetings included work on concepts such as bar modeling.

Bar modeling presented the greatest challenge to our teachers. During the first year, teachers taught the bar model as an illustration to a worked out problem. In year 2, we all learned how the bar model can actually lead to a *solution plan* for the problem. To draw the bar model, children must deeply understand and comprehend what the problem is asking. They must identify and label the whole and its parts. As children attach values to each part of the bar model, they can identify whether the missing value represents a part or a whole. Because the model is drawn to a rough proportional scale, children can predict the relative magnitude of the solution.

Here is a sample problem and how the bar model leads to its solution.

American Institutes for Research®

Rosa has 336 shells. She keeps 72 shells and gives the rest to her friends. She shares the remaining shells evenly between 6 friends. How many shells does each friend receive?

```
       |-------------------------------336-------------------------------|
       _____
      |        |        |        |        |        |        |        |
       -----------------------------------------------------------------
       |--------72-------|----?----|
```

Note that the bar model identifies the whole, 336, and the part that is known, 72. It also shows that there are six equal unknown parts that reflect the difference between 72 and 336. After students draw the model, it is apparent to them that they must first find the difference between 336 and 72, and then divide the difference by 6.

$$\text{Solution: } (336 - 72) \div 6 = 44$$

Overall, teacher response to trainings and support was overwhelmingly appreciated. Because a staff position was dedicated to the implementation and success of this program, teachers knew they are able to seek and obtain the services and support outlined above. Teachers readily admitted when they were frustrated or puzzled, something they may not feel as comfortable reporting directly to their principals. Although I was unable to successfully resolve all questions and concerns, teachers knew they are not alone. I firmly believe that the building level support was the primary reason for the successful implementation of this program at College Gardens.

Lessons Learned

The Montgomery County Public Schools Singapore Math pilot was originally slated for a period of two years. An optional extension was offered, so College Gardens solicited input from the parents, teachers, staff, and children. It is unanimously agreed that we would continue. We repeated the extension process each spring, and College Gardens is currently in its fifth year of implementation (grade 5 implementation is eliminated in 2003-2004).

This was important because a two-year pilot is simply not enough time to judge the success of this program. The first year of implementation is largely spent becoming accustomed to the vastly different style, format, presentation, approach and philosophy of the program. Beginning with the second year, teachers become comfortable enough to begin the true professional development required to maximize the program's effectiveness.

I am proud of the work I did to facilitate the Singapore Math pilot in my school. Our test scores improve significantly over the course of the pilot, which pleased district leaders. More importantly, though, is the increased confidence and pleasure in doing math. Teachers, students, and parents demonstrated a newfound respect and enjoyment in mathematics.

I would happily facilitate the implementation of Singapore Math in another setting. However, this time, I would make the following adjustments:

A. Trainer of trainers

Our district's teachers were at an unfair disadvantage because specialists who knew mathematics, but who had no experience with Singapore Math, provided their initial exposure to the program. In a future Singapore Math pilot, individuals experienced in using Singapore Math will provide initial training. Initial training will take place over a period of five days. Training will include, but not necessarily be limited to the following:

a. A brief overview of the Singapore national standards so that participants have a backdrop for the program.

b. An exploration of the overarching teaching methods found in the Preface of every student book: (1) The notion of teaching using Concrete—Pictorial—Abstract and (2) the meaning and use of the colour patches.

c. The advantages of this program, briefly highlighting attributes such as the approach to number and operation, mental math, problem solving, and the tight connectedness of concepts.

d. A "book walk", in which teachers see the sequence of the instruction at their grade level and comprehend the importance of teaching in the prescribed sequence.

e. A close examination of the format of the student and teacher books. We will discuss specifically how the Teacher's Edition supports the student book.

f. A discussion of the format and purpose of the practice exercises, review exercises, and revisions.

g. Time with grade level teams to see the overview of the first unit of study in their books.

h. Time with grade level teams to actually plan and simulate instruction for the first few lessons of the first unit.

By the end of the week, participants will feel emotionally and academically prepared to embark on this exciting journey. For students, meaningful instruction will begin from Day One.

B. Program consistency

To begin with, trainers perceived that teachers were too overwhelmed with the task at hand to utilize grade level texts other than their own. The beauty and consistency of this program is most clearly seen from a broader view. The Singapore Math texts look virtually identical in grades 1 through 5. The major difference in appearance between books is the complexity of the problems as students' mathematical understandings deepen. This point needs to be made early on. Teachers' ability to effectively implement this program exponentially improves if they have an understanding of the whole, rather than just their specific part. For this reason, all

staff, not just impacted grade levels, would attend meetings and/or workshops that focus on content traces. Grade 5 teachers would learn how composition and decomposition begins in grade 1. Grade 1 teachers would learn how part-part-whole is used throughout the program in decimal operations, bar models and problem solving.

C. Materials

Our teachers were forced to guess about the manipulatives recommended for teaching lessons from the book. Audio-visual materials were cited on numerous occasions, to which we had no access. More materials and greater insight of the program will maximize the strength of this program.

D. Connection to U.S. National, State, and District Standards

Starting with the second year of implementation, teachers were required to prepare students for the state assessment. Consequently, Singapore math instruction is compacted and muddied because time must be spent on topics such as customary measurement, probability, money, and so forth. A discussion around the accountability of Singapore and U.S. standards must take place prior to future implementation.

CONCLUSION

Many factors contribute to the success of Singapore Math in Singapore. Teachers are highly regarded and respected professionals, on a par with attorneys and accountants here in the U.S. Education is a top priority, reinforced by families and the community at large. Parents are able to effectively assist their children in mathematics, as they have a deep understanding of their own. These conditions are simply not the same in the U.S. Regardless of duration, a U.S. pilot cannot exactly simulate the programming in Singapore.

College Gardens Elementary School experienced a high degree of success in the Singapore Math pilot. Our teachers armed themselves with deeper understanding of math. Parents rallied behind them, although they were unable to be of great academic assistance. And yet, I would caution anyone seeking to implement Singapore Math to be fully cognizant of the significant differences in the mathematical emphasis between Singapore and the U.S. Singapore Math is very strong in some areas, but sorely lacking in others. Elementary instruction in Singapore contains no probability. Data analysis and statistics are flat and literal. Money and measurement differ and we must teach our own systems to our children.

While *any* math program can benefit from the components we put into place, through the vehicle of Singapore Math, our teachers:

- learned to thoughtfully plan and analyze their daily instruction;

- focused on higher order thinking questions that challenged all students:

- provided meaningful instruction using a variety of strategies and methods; and

- used cooperative learning strategies and small group instruction.

Meanwhile, our students developed resourcefulness, independence and confidence. In short, teachers became effective facilitators of instruction. I believe that Singapore Math has powerful attributes that U.S. publishers and teachers may want to utilize in the future.

Lesson Observation Form

Focus on Teaching
Bridge/Pre-Assess for Readiness

Goal-Setting Introduction

Lesson Body

Closure/Preview (How will we use what we learned today in the future?)

Focus on Learning
What evidence was there that the students learned? (observation, discourse)

What was done to make sure every student was engaged?

Was the lesson differentiated? If so, how?

Next Steps
What would I like to try in my class based on the lesson?

Using the Lesson Observation Form

The Lesson Observation Form links to Model, Coach Apply

1. Keeps teacher actively engaged throughout the modeling process.
2. Provides talking points during coaching.
3. Places responsibility on teacher for the application of effective teaching strategies.

Pre-Modeling Conferences

1. Days prior to modeling
   a. Modeler collaborates with teacher on lesson content and format
2. Day of modeling
   a. Discuss "look fors"
   b. Focus on modeler's specific teacher behaviors

Focus on Teaching "Look Fors"

Bridge/Pre-Assess for Readiness What prior knowledge will students build upon?
Goal-Setting Introduction How will students know what they are expected to know and be able to do?
Lesson Body What tasks will help students deeply understand what is essential?
Closure/Preview What did students learn? How does it fit into the big picture? How will they build upon this knowledge in the future?

Focus on Learning "Look Fors"

What evidence was there that the students learned? Observation, discourse, written work
What was done to make sure every student was engaged? Group interactions, modeled thinking, Every Pupil Response, Responsibility accepted by students
How was the lesson differentiated? Enrichment, extensions, problem formulation, small group instruction

Post-Modeling Conference

Within 24-48 hours

1. Observation Form serves as basis for conversation
   a. What did you see?
   b. What questions do you have about what you saw?
   c. How can you apply what you saw to your own teaching style?
2. Open question-and-answer session
   a. Teacher chooses "Next Step" to try

At some point in the future

3. Process is reversed with these caveats:
   a. Same day pre-conference is optional
   b. Post-conference should be as soon as possible after the lesson
   c. Non-evaluative recording on Lesson Observation Form