The Vasicek Model

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The Vasicek Model

The Vasicek model (Vasicek 1978) is one of the earliest no-arbitrage interest rate models.
It

- is based upon the idea of *mean reverting* interest rates
- gives an explicit formula for the (zero coupon) yield curve
- gives explicit formulaes for derivatives such as bond options
- can be used to create an interest rate tree

AR(1) Processes

An *autoregressive process of order 1* (AR(1)) is a *stochastic process* of the form

$$Y_t = a + bY_{t-1} + \epsilon_t$$

where a and b are constants, ϵ_t is a random error term, typically

$$\epsilon_t \sim N(0, \sigma^2)$$

Features of the AR(1)

- If $b = 1$, the process is a *random walk*
- If $b = 0$ then Y_t is just *white noise*
- If $0 < b < 1$ the process is said to be *mean-reverting*.
- The parameter b measures the persistence in Y_t and $1 - b$ is called the *speed of mean-reversion*.
- Parameters a, b can be found by a regression of Y_t on Y_{t-1}.
- b literarily is the *autocorrelation* of Y_t since

$$b = \frac{\text{Cov}(Y_t, Y_{t-1})}{\text{Var}(Y_{t-1})} = \frac{\text{Cov}(Y_t, Y_{t-1})}{\text{Std}(Y_{t-1})\text{Std}(Y_t)} = \text{Corr}(Y_t, Y_{t-1})$$

since $\text{Std}(Y_{t-1}) = \text{Std}(Y_t)$.

Mean Reversion

- If a process is mean reverting, it tends to revert to a constant, long run mean
- The speed of mean reversion measures the average time it takes for a process to revert to its long run mean
- Mean reversion is reasonable for interest rates - random walk makes no sense because it is economically unreasonable to think that interest rates can "wander of to infinity" or become arbitrarily large.
- So what is the typical speed of mean reversion in interest rates? Weekly data gives 0.9961 for 3m T-bills. This is *extremely high* persistence and indicates that interest rates are *close to* random walk...

The Short Rate

Many term structure models, including Vasicek's, assume that the fundamental source of uncertainty in the economy is the *short rate*.

The short rate is the annualized rate of return on a very short term investment (lets assume daily).

The Vasicek Model

The Vasicek model simply assumes that the short rate, r_t follows an AR(1),

$$r_t = a + br_{t-1} + \epsilon_t . \qquad (1)$$

That is, Vasicek just assumes a purely *statistical model* for the evolution of interest rates.

Alternative Representations

The model is sometimes written,

$$\triangle r_t = \kappa(\theta - r_{t-1})\triangle t + \sigma \triangle z_t \tag{2}$$

where

$$\triangle r_t = r_t - r_{t-1} \tag{3}$$

and $\triangle z_t$ is a normally distributed error term.
Using the definition of $\triangle r_t$, and set $\triangle t = 1$, we have

$$r_t = \kappa\theta + (1 - \kappa) r_{t-1} + \sigma \triangle z_t \tag{4}$$

which is an AR(1) if $a = \kappa\theta$ and $b = 1 - \kappa$.

We can also set $\triangle t$ to something different from 1. In this case we assume that

$$\triangle z \sim N(0, \triangle t)$$

Differential Notation

Tuckman writes

$$dr_t = \kappa(\theta - r_t)dt + \sigma dw_t$$

This is a *stochastic differential equation* where dw is *Brownian motion*. You should interpret it to mean

$$\triangle r_t = \kappa(\theta - r_{t-1})\triangle t + \sigma \triangle z_t \tag{5}$$

In other words, we interpret dr_t to mean $\triangle r_t = r_t - r_{t-1}$. Therefore, any time you see a the differential notation in Tuckman, translate by equating to the *first difference*.

Vasicek Zero Coupon Curve

Let $B(t, T)$ denote the time t price of a zero coupon bond with maturity date T. It is

$$B(t,T) = \exp(-A(t,T)r_t + D(t,T)) \tag{6}$$

$$A(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa} \tag{7}$$

$$D(t,T) = \left(\theta - \frac{\sigma^2}{2\kappa}\right)[A(t,T)-(T-t)] - \frac{\sigma^2 A(t,T)^2}{4\kappa} \tag{8}$$

In other words, the Vasicek model gives us an exact expression for the value of a zero coupon. We can compare this to other models for zero coupon bond prices, such as the *cubic spline* model and *Nelson-Siegel*.

Properties of the Vasicek term structure

Lets take some example parameter values to see how the Vasicek term structure behaves. Reasonable values are

$$\kappa = (1 - 0.9961) \times 52 \approx 0.2,$$

$$\sigma = 0.0025 \times \sqrt{52} \approx 0.018,$$

(180 basis p/ year) and

$$\theta = 0.05$$

(the average risk free rate).

Effects of mean reversion

- The smaller the mean reversion in interest rates, the more we expect interest rates to remain close to their current levels.
- If we expect interest rates to change very slowly, long dated bonds tend to move in parallel to short.
- Thus, higher (lower) speed of mean reversion ($\kappa = 1 - b$) lead to a steeper (less steep) term structure because *the term structure depends on future expected short rates* (illustrated on next slide)

Vasicek Bond Options

Consider an option on a zero coupon. The underlying bond has maturity T and the option matures at T_0 (we assume $T_0 < T$ why?).

The value of a call option with strike K is

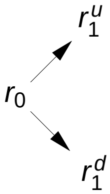
$$C = B(t,T)N(d_1) - KB(t,T_0)N(d_2) \tag{9}$$

where d_1 and d_2 have expressions that mimic those in the Black-Scholes model.

Building a Vasicek Tree

- We will try to build an interest rate tree which is consistent with the assumed short rate dynamics in the Vasicek model.
- We will build a monthly model.
- Assume the following parameters: $\kappa = 0.025$ (*k* in Tuckman), $\sigma = 0.0126$, and $\theta = 0.15339$.
- Lets assume that the current short rate is 5.121% and also that the probability of going up/down in the first month is $\frac{1}{2}$.

Lets try to build a tree by setting $\triangle z_t = +-1$ with probability $\frac{1}{2}$. Our tree will look something like



where

$$r_1^u = r_0 + \kappa(\theta - r_0)/12 + \frac{\sigma}{\sqrt{12}} = 5.506\%$$

and

$$r_1^d = r_0 + \kappa(\theta - r_0)/12 - \frac{\sigma}{\sqrt{12}} = 4.7786\%$$

We should verify that

$$E(\triangle r) = \kappa(\theta - r_0)/12$$

and

$$\text{Std}(\triangle r) = \frac{\sigma}{\sqrt{12}}$$

which follows trivially here.

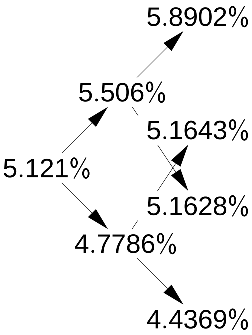
Next, lets try the same for the the next period in the tree....
If in the down-state ($r_1 = r_1^d = 4.7786\%$) at t=1, we get that the up move should equal

$$r_2^{ud} = r_1^d + \kappa(\theta - r_1^d)/12 + \sigma/\sqrt{12} = 5.1643\%$$

and likewise, if in the up-state, we get that the down move should be

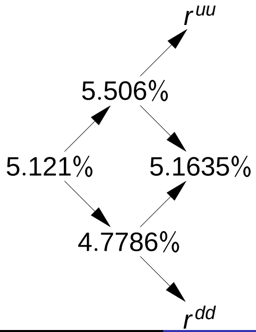
$$r_2^{du} = r_1^u + \kappa(\theta - r_1^u)/12 + \sigma/\sqrt{12} = 5.1628\%$$

In other words, our interest rate tree is



Designing a recombining tree

We need a recombining three such to avoid too many branches when we add time-steps.
Lets consider relaxing the 50/50 probability assumption. If we do, we also need to adjust the rate in the up-up and down-down to maintain the correct mean and variance of the future interest rate.
The tree is



where we assign a probabilities p and q of an up move conditional upon being in either state.
The number 5.1635 on the middle branch is the expected value of the interest rate after two periods.
The expected value after one period is

$$5.121 + 0.025(15.339 - 5.121)/12 = 5.1423$$

the expected value of the interest rate in period 2 is

$$5.1423 + 0.025(15.339 - 5.1423)/12 = 5.1635$$

We now solve for r^{uu}, r^{dd}, p and q using the mean and variances in the following way. Consider p and r^{uu}:
The expected future value given an up move in period 1 is

$$5.506 + 0.025(15.339 - 5.506)/12 = 5.5265$$

This means that our binomial tree must give

$$p \times r^{uu} + (1 - p)5.1635 = 5.5265$$

Secondly, we know that the standard deviation is $0.0126/\sqrt{12} = 0.003637$ per month so we should have

$$p \times (r^{uu} - 5.5265\%)^2 + (1-p) \times (5.1635\% - 5.5265\%)^2 = 0.3637\%$$

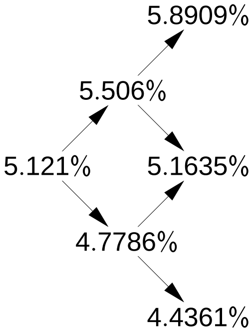
We now have two equations in two unknowns and we find

$$p = 0.499$$

and

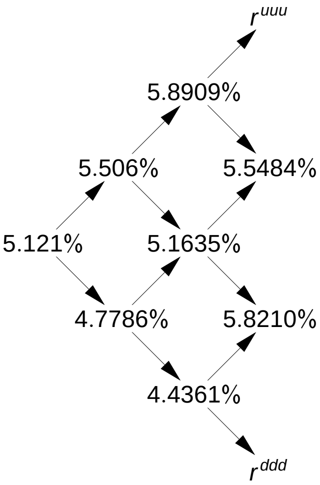
$$r^{uu} = 5.8909\%.$$

We solve for q and r^{dd} using the same technique. We get $q = 0.5011$ and $r^{dd} = 4.4361\%$, giving the recombining tree



Next...

... we find the nodes in month 4. The method is entirely analogous to what is we did at time 3 (see Tuckman p. 238).


r^{uuu}
5.8909%
5.506%
5.5484%
5.121%
5.1635%
4.7786%
5.8210%
4.4361%
r^{ddd}

Here the two middle branches are assigned $\frac{1}{2}$ probabilities while the move from 5.8909 and up is assigned probability p and the 4.4361% to 5.8210% is assigned probability q.

We now solve for r^{uuu}, p and r^{ddd}, q in as at time $t = 3$.

Concluding remarks

- Even with the Vasicek model, we need to solve non-linear equations to construct the tree
- More complicated models typically imply that we need to solve even more complicated equations.
- Shortcomings of the Vasicek model:
  - Interest rate can become negative
  - Difficult to make the model fit the yield curve exactly (spot rate function is not flexible enough)
  - Constant local volatility is not realistic, but it is easy to generalize...
- Numerous generalizations are possible... stay put