Integrable Spin Chains and the AdS/CFT Correspondence

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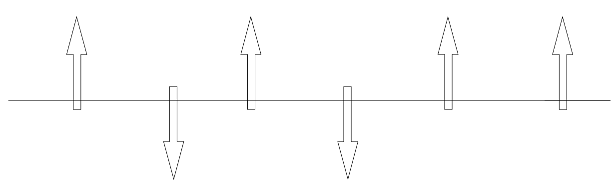
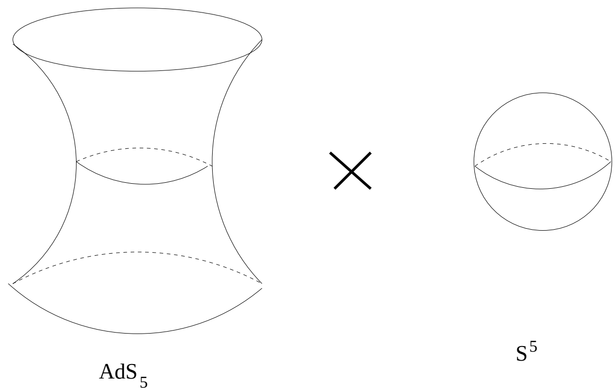
Geometry and Physics after 100 Years of Einstein's Relativity

Potsdam, April 5-8 2005



The AdS/CFT Correspondence

IIB Superstrings on $AdS_5 \times S^5$



tension: $\frac{1}{\alpha'}$ string coupling: g_s

$\mathcal{N} = 4$ Super Yang-Mills Theory

't Hooft coupling: $\lambda = N g^2_{\mathrm{YM}}$ inverse size of group: $\frac{1}{N}$

Maldacena's conjecture: These two theories are dual. [Maldacena '97]

$$\boxed{\frac{R^2}{\alpha'} = \sqrt{\lambda} \qquad 4\pi g_s = \frac{\lambda}{N}}$$

The symmetry $\mathfrak{psu}(2,2|4)$ matches! Problem: String theory is tractable if $\alpha' \to 0$, while Gauge theory is tractable if $\lambda \to 0$.

IIB Superstrings on $AdS_5 \times S^5$

A background preserving all superymmetries (very few do):

$$ds^2 = R^2 \left(ds^2_{\mathrm{AdS}_5} + ds^2_{\mathrm{S}_5} \right)$$

$$ds^2_{\mathrm{AdS}_5} = d\rho^2 - \cosh^2 \rho \, dt^2 + \sinh^2 \rho \, d\Omega_3 \qquad ds^2_{\mathrm{S}5} = d\theta^2 + \cos^2 \theta \, d\psi^2 + \sin^2 \theta \, d\tilde{\Omega}_3$$

Two-dimensional worldsheet with coordinates σ, τ embedded into this 10-dim. space: $X^m = X^m(\tau, \sigma), Y^m = Y^m(\tau, \sigma)$.

Bosonic action of the worldsheet string σ-model:

$$S_{\mathrm{b}} = \frac{R^2}{4\pi\alpha'} \int d\tau \, d\sigma \left[G^{(\mathrm{AdS}_5)}_{mn} \partial_a X^m \partial^a X^n + G^{(\mathrm{S}_5)}_{mn} \partial_a Y^m \partial^a Y^n \right]$$

Classically the full σ-model (fermions included) is integrable.

[Bena,Polchinski,Roiban '03]

The quantization of the model is currently ill-understood.
Thus even the spectrum of free strings $(g_s = 0)$ is unknown.

This situation is likely to change soon (more on this later).

$\mathcal{N}=4$ Super Yang-Mills Theory

Gauge group $SU(N)$, where N is the number of "colors".

Field content: gauge field A_μ, three complex scalars $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ (or: six real), fermions $\Psi^A_\alpha, \Psi^A_{\dot\alpha}$. All fields are $N \times N$ matrices.

Technically (Feynman diagrams!) very similar to the gauge theories describing nature (QCD, Standard Model).

Physics different: $\beta = 0 \Rightarrow$ Theory is (super)conformal, with symmetry $\mathfrak{psu}(2,2|4)$. This places strong constraints on the structure of correlation functions. E.g. two-point functions:

$$\boxed{\langle \mathcal{O}_n(x)\, \mathcal{O}_m(0) \rangle = \tfrac{\delta_{nm}}{x^{2\Delta_n}}\,.}$$

Here Δ_n is the anomalous conformal (or scaling) dimension of the quantum operator $\mathcal{O}_n$. "Good" operators are complicated composites of the elementary fields (mixing problem), and depend on N and the gauge coupling $\lambda = N g^2_{\mathrm{YM}}$:

$$\mathcal{O} = \mathrm{Tr}\left(\mathcal{X}\mathcal{Y}\mathcal{Z}\mathcal{F}_{\mu\nu}\Psi^A_\alpha(\mathcal{D}_\mu\mathcal{Z})\ldots\right)\mathrm{Tr}\left(\ldots\ldots\right)\,\ldots + \ldots$$

Resolving the mixing problem leads to the spectral problem of $\mathcal{N}=4$: One should diagonalize the dilatation operator D :

$$\boxed{D \cdot \mathcal{O}_n = \Delta_n\, \mathcal{O}_n\,.}$$

String Predictions for $\mathcal{N} = 4$

A key proposal for AdS/CFT: [Maldacena '97; Gubser,Klebanov, Polyakov; Witten '98]

$$\boxed{\begin{array}{ccc} \text{string energy} & \leftrightarrow & \text{conformal dimension} \\ E & = & \Delta \end{array}}$$

Recent dramatic progress in certain limits involving states with large angular momentum J_1, J_2, J_3 on the five-sphere S^5. E.g.:

$$\boxed{\mathrm{Tr}\, \mathcal{X}^{J_1} \mathcal{Z}^{J_3}}$$

I BMN limit *"plane wave limit"* [Berenstein,Maldacena,Nastase '02]
$J_3 \gg 1, \quad J_1 = 2, 3, \ldots$

II Near-BMN limit [Parnachev,Ryzhov'02; Callan,Lee,McLoughlin,Schwarz,Swanson,Wu '03,04]
Like I, but focusing on $1/J_3$ corrections.

III Frolov-Tseytlin limit *"spinning strings limit"* [Frolov,Tseytlin '03]
$J_3 \gg 1, \quad J_1 \gg 1.$

In all cases an effective, analytic expansion parameter emerges:

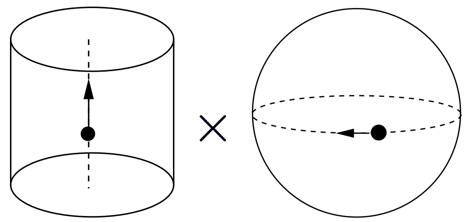
$$\boxed{\lambda' = \frac{\lambda}{L^2}} \qquad L = J_1 + J_3$$

II: [Callan et.al.'03] III: [Serban,MS '04]

I appears to work beautifully, $\mathrm{II}, \mathrm{III}$ have subtle problems ...

The BMN Limit

In the BMN limit one focuses on the geometry seen by a "tiny" string moving along a great circle on S^5:



It effectively sees a plane wave background with metric

$$ds^2 = -4\, dx^+ dx^- - \mu^2\, (x^i)^2 (dx^+)^2 + (dx^i)^2$$

The worldsheet theory becomes free, albeit massive: [Metsaev '01]

$$S_{\mathrm{b}} = \int d\tau d\sigma (\tfrac{1}{2}\, \partial_a x^i\, \partial^a x^i - \tfrac{\mu^2}{2}\, x^i\, x^i)$$

So quantization may be performed in a "textbook fashion", the spectrum is found explicitly, and yields predictions for conformal dimensions in $\mathcal{N}=4$ gauge theory. E.g. $(\lambda' = \lambda/J^2$ with $J \gg 1)$

$$\boxed{\mathrm{Tr}\, \mathcal{X}^2 \mathcal{Z}^J + \ldots \qquad \Rightarrow \qquad \Delta_n = J + 2\sqrt{1+\lambda'\, n^2}\, .}$$

This is an all-loop prediction! It requires "BMN scaling". Highly non-trivial, unproven property of gauge theory. It is known to fail in closely related models. [Serban,MS '04; Fischbacher,Klose,Plefka '04]

Integrable Spin Chains and AdS/CFT

In the remainder of this talk I will present a lot of evidence that both sides of the correspondence can be described by an integrable long-range spin chain.

I suggest that, for $N = \infty$, in gauge theory we may identify the dilatation operator with a long-range spin chain Hamiltonian.

I also suggest that free strings on $AdS_5 \times S^5$ are described by a spin chain. It is currently not known how to derive the spin chain from the quantum σ-model, but I will present compelling "spectroscopic evidence".

So what is a spin chain, and what does it mean for it to be integrable?

Quantum Spin Chains

The Heisenberg XXX spin $\frac{1}{2}$ chain: [Heisenberg '28]

State space: $\ldots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots$

Pauli matrix

Hamiltonian: $H = \sum_{\ell=1}^{L} \frac{1}{2}(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1})$.

Spectral problem: $H \cdot \Psi = E\,\Psi$.

This involves diagonalizing a $2^L \times 2^L$ matrix! Easy for small L, direct approach hopeless for large L !

Permutation Operator

Rewrite Hamiltonian: $\boxed{H = \sum_{\ell=1}^{L} (1 - P_{\ell,\ell+1})\,.}$

Ferromagnetic ground state: $H \cdot |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle = 0$.

Excitations $\downarrow$ (magnons): $|\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\rangle\,.$

Clearly H conserves the number of $\uparrow$ and $\downarrow$ spins.

Action:

$$\boxed{H \cdot |\ldots\uparrow\downarrow\uparrow\ldots\rangle = 2\,|\ldots\uparrow\downarrow\uparrow\ldots\rangle - |\ldots\downarrow\uparrow\uparrow\ldots\rangle - |\ldots\uparrow\uparrow\downarrow\ldots\rangle}$$

Magnons are Particles

$\uparrow = \mathcal{Z}$-particle $=$ hole $\qquad \downarrow = \mathcal{X}$-particle $=$ magnon

$|\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\rangle = |\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{Z}\mathcal{Z}\mathcal{Z}\rangle = |\ldots\mathcal{X}\ldots\mathcal{X}\ldots\rangle$.

One-magnon States:

$$|\Psi\rangle = \sum_{1\leq\ell\leq L} \Psi(\ell)\, |\ldots\mathcal{Z}\overset{\overset{\ell}{\downarrow}}{\mathcal{X}}\mathcal{Z}\ldots\rangle\,,$$

Schrödinger equation $H\cdot|\Psi\rangle = E\,|\Psi\rangle$:

$$E\,\Psi(\ell) = 2\,\Psi(\ell) - \Psi(\ell-1) - \Psi(\ell+1)\,.$$

Fourier transforming, magnons start to “move”,

$$\Psi(\ell) = e^{i\,p\,\ell}\,,$$

with the dispersion law $E = 2 - e^{i\,p} - e^{-i\,p}$, i.e.

$$\boxed{E = 4\sin^2\tfrac{p}{2}\,.}$$

This solves the one-magnon problem. What about many?

Magnon Scattering: The Bethe Ansatz

Two-magnon states:

$$|\Psi\rangle = \sum_{1\le\ell_1<\ell_2\le L} \Psi(\ell_1,\ell_2)\; |\ldots\mathcal{Z}\overset{\overset{\ell_1}{\downarrow}}{\mathcal{X}}\mathcal{Z}\ldots\mathcal{Z}\overset{\overset{\ell_2}{\downarrow}}{\mathcal{X}}\mathcal{Z}\ldots\rangle\,.$$

First guess nearly works, with $(K=2)$:

$$\Psi(\ell_1,\ell_2) = e^{i\,p_1\,\ell_1 + i\,p_2\,\ell_2}\,, \qquad \boxed{E = \sum_{k=1}^{K} 4\sin^2\frac{p}{2}\,,}$$

except when the magnons collide at $\ell_2 = \ell_1 + 1$, where:

$$E\,\Psi(\ell_1,\ell_2) = 2\,\Psi(\ell_1,\ell_2) - \Psi(\ell_1-1,\ell_2) - \Psi(\ell_1,\ell_2+1)\,.$$

Problem fixed by Bethe's ansatz: [Bethe '31]

$$\boxed{\Psi(\ell_1,\ell_2) = e^{i\,p_1\,\ell_1 + i\,p_2\,\ell_2} + S(p_2,p_1)\,e^{i\,p_2\,\ell_1 + i\,p_1\,\ell_2}\,.}$$

Elementary algebra gives:

$$S(p_1,p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_1} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} + 1}\,.$$

We can think of $S(p_1,p_2)$ as an S-matrix.

Bethe's Equations and Factorized Scattering

Periodic boundary conditions $\Psi(\ell_1,\ell_2)=\Psi(\ell_2,\ell_1+L)$ yield the spectrum through Bethe's equations:

$$e^{ip_1L}=S(p_1,p_2)\quad\text{and}\quad e^{ip_2L}=S(p_2,p_1)\,.$$

A little extra work allows to solve the K-magnon problem in the same fashion $(k=1,\ldots,K)$:

$$e^{ip_kL}=\prod_{\substack{j=1\\j\neq k}}^{K}S(p_k,p_j)\,.$$

The S-matrix is factorized, and the spin chain is integrable. The (hidden) reason is the existence of L conserved charges:

$$[Q_r,Q_s]=0\,.$$

Here $Q_1=\sum p_k$ = momentum and $Q_2=H$ = energy.
It is useful to change variables to rapidities $u_k=\frac{1}{2}\cot\frac{p_k}{2}$:

$$\boxed{\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L=\prod_{\substack{j=1\\j\neq k}}^{K}\frac{u_k-u_j+i}{u_k-u_j-i}\,,\qquad E=\sum_{k=1}^{K}\frac{1}{u_k^2+\frac{1}{4}}\,.}$$

Spin Chains and $\mathcal{N} = 4$ Gauge Theory

Now recall the dilatation operator:

$$\boxed{D \cdot \mathcal{O} = \Delta\, \mathcal{O}\,.}$$

The dimensions Δ may be computed in perturbation theory, case-by-case. It is much more efficient to directly study D:

$$\boxed{D = \sum_{l=1}^{\infty} g^{2l}\, D_{2l} \qquad \text{where} \qquad g^2 = \frac{\lambda}{8\pi^2}\,.}$$

This way we can disentangle the mixing problem from the Feynman diagram computation. At one-loop, one e.g. finds

[Minahan,Zarembo; Beisert,Kristjansen,Plefka,MS, '02]

$$\mathcal{O} = \mathrm{Tr}\, \mathcal{X}^K \mathcal{Z}^{L-K}\; : \qquad D = L + g^2\; \mathrm{Tr}\, [\mathcal{Z}, \mathcal{X}][\frac{\delta}{\delta \mathcal{X}}, \frac{\delta}{\delta \mathcal{Z}}] + \ldots$$

Let us see how D_2 acts; e.g. for $K = 2$:

$$\mathcal{O}_p = \mathrm{Tr}\, \mathcal{X} \mathcal{Z}^p \mathcal{X} \mathcal{Z}^{L-2-p} \Rightarrow D_2 \cdot \mathcal{O}_p = 2\; (2\, \mathcal{O}_p - \mathcal{O}_{p-1} - \mathcal{O}_{p+1})$$

+ double trace operators

The XXX spin chain! $\boxed{D_2 = H = \sum_{\ell=1}^{L} (1 - P_{\ell,\ell+1})\,.}$

[Minahan,Zarembo '02]

Curiosity, or Tip of the Iceberg?

This was the special case of planar one-loop anomalous dimensions for operators made from the partons $\mathcal{Z}$, $\mathcal{X}$.
This led to an integrable spin chain.
Is this just a curiosity, or something deeper?

- What about all other operators in $\mathcal{N} = 4$?
- What about higher loops?
- And what about the conjectured dual string description?

Scattering with Fermions and Derivatives

Let us consider two further interesting examples:

- $\mathcal{O} = \mathrm{Tr}\,\mathcal{U}^K \mathcal{Z}^{L-K}$, $\mathfrak{u}(1|1)$ symmetry. $\mathcal{U}$ is a "gaugino."

  Two-magnon states:

  $$|\Psi\rangle = \sum_{1\le \ell_1<\ell_2\le L} \Psi(\ell_1,\ell_2)\ |\ldots\mathcal{Z}\overset{\overset{\ell_1}{\downarrow}}{\mathcal{U}}\mathcal{Z}\ldots\mathcal{Z}\overset{\overset{\ell_2}{\downarrow}}{\mathcal{U}}\mathcal{Z}\ldots\rangle\,.$$

  Making Bethe's ansatz, one now finds

  $$\Psi(\ell_1,\ell_2) = e^{i\,p_1\,\ell_1+i\,p_2\,\ell_2} - e^{i\,p_2\,\ell_1+i\,p_1\,\ell_2}\,.$$

  This means that the fermion $\mathcal{U}$ is free at one loop, and the S-matrix is (XY model!) [Callan,Heckmann,McLoughlin,Swanson '04; MS '04]

  $$\boxed{S_{\mathfrak{u}(1|1)} = -1\,.}$$

- $\mathcal{O} = \mathrm{Tr}\,\mathcal{D}^K \mathcal{Z}^{L}$, $\mathfrak{sl}(2)$ symmetry. $\mathcal{D}$ is a covariant derivative.

  Two-magnon states:

  $$|\Psi\rangle = \sum_{1\le \ell_1\le\ell_2\le L} \Psi(\ell_1,\ell_2)\ |\ldots\mathcal{Z}(\overset{\overset{\ell_1}{\downarrow}}{\mathcal{D}}\mathcal{Z})\ldots\mathcal{Z}(\overset{\overset{\ell_2}{\downarrow}}{\mathcal{D}}\mathcal{Z})\ldots\rangle\,.$$

  Here one finds for the S-matrix "almost" the same result as for $\mathfrak{su}(2)$ (XXX Heisenberg magnet with spin $= -\frac{1}{2}$!) [Beisert '03]

  $$\boxed{S_{\mathfrak{sl}(2)} = S^{-1}_{\mathfrak{su}(2)}\,.}$$

The $\mathcal{N}=4$ Integrable Super Spin Chain

The partons $\mathcal{Z}$, $\mathcal{X}$ form a closed $\mathfrak{su}(2)$ subsector of $\mathcal{N}=4$.

All six scalars at one loop form $\mathfrak{so}(6)$ spin chain. [Minahan,Zarembo '02]

Integrable spin chain techniques for anomalous dimensions had already been applied in QCD. [Lipatov '97; Braun, Derkachov, Manashov '98]

Now recall that the full superconformal symmetry is $\mathfrak{psu}(2,2|4)$.

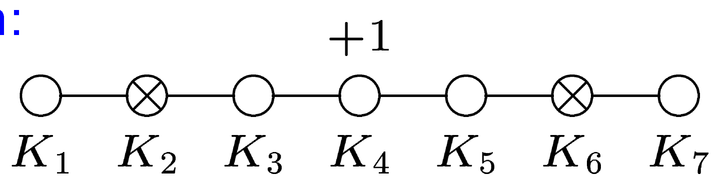
$\Rightarrow$ Full set of one-loop Bethe equations: [Beisert,MS '03]

$$\left(\frac{u_k^{(p)}+\frac{i}{2}V_p}{u_k^{(p)}-\frac{i}{2}V_p}\right)^L=\prod_{q=1}^{7}\prod_{\substack{j=1\\(p,k)\neq(q,j)}}^{K_q}\frac{u_k^{(p)}-u_j^{(q)}+\frac{i}{2}M_{pq}}{u_k^{(p)}-u_j^{(q)}-\frac{i}{2}M_{pq}}$$

M_{pq} - Cartan matrix, V_p - Dynkin labels of spin irrep:

$$M_{pq}=\left(\begin{array}{c|c|ccc|c|c} -2 & +1 & & & & & \\ \hline +1 & 0 & -1 & & & & \\ \hline & -1 & +2 & -1 & & & \\ & & -1 & +2 & -1 & & \\ & & & -1 & +2 & -1 & \\ \hline & & & & -1 & 0 & +1 \\ \hline & & & & & +1 & -2 \end{array}\right),\quad V_p=\left(\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \\ 1 \\ 0 \\ \hline 0 \\ \hline 0 \end{array}\right)$$

Dynkin diagram:



Higher Loop Integrability

It is possible to construct the dilatation operator to higher loop order. In the $\mathfrak{su}(2)$ sector one finds to three loops, with

$D = \sum_{\ell=1}^{L} \left(1 + g^2\,\mathcal{H}_2 + g^4\,\mathcal{H}_4 + g^6\,\mathcal{H}_6 + \ldots\right)\,,$ [Beisert,Kristjansen,MS '03]

$$\begin{aligned}
\mathcal{H}_2 &= \frac{1}{2}\,(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1})\,,\\
\mathcal{H}_4 &= -(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1}) + \frac{1}{4}\,(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+2})\\
\mathcal{H}_6 &= \frac{15}{4}\,(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1}) - \frac{3}{2}\,(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+2}) + \frac{1}{4}\,(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+3})\\
&\quad -\frac{1}{8}\,(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+3})(1 - \vec{\sigma}_{\ell+1} \cdot \vec{\sigma}_{\ell+2})\\
&\quad +\frac{1}{8}\,(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+2})(1 - \vec{\sigma}_{\ell+1} \cdot \vec{\sigma}_{\ell+3})
\end{aligned}$$

We see that the spin chain becomes increasingly long range.
Notice the presence of four-spin terms!
Truncating the Hamiltonian breaks integrability, however the model is perturbatively integrable in the sense that

$$\boxed{[Q_r(g), Q_s(g)] = 0\,.}$$

We conjectured that the model is non-perturbatively integrable.

The 3-loop result may be proven algebraically and, impressively, extended to the maximally compact subsector $\mathfrak{su}(2|3)$. [Beisert '03]

S-Matrix Extraction (Gauge Theory)

In order exploit the integrability we need to find the S-matrix. The usual Bethe ansatz does not work anymore. However, asymptotically (i.e. the particles are "far" apart) we still expect

$$\Psi(\ell_1,\ell_2) \sim e^{i\,p_1\,\ell_1+i\,p_2\,\ell_2} + S(p_2,p_1)\,e^{i\,p_2\,\ell_1+i\,p_1\,\ell_2} \text{ for } \ell_1 \ll \ell_2\,.$$

The 3-loop $\mathfrak{su}(2)$ chain may be embedded into the integrable long-range Inozemtsev spin chain. Knowledge of the exact wavefunctions yields a 3-loop Bethe ansatz. [Serban,MS '04]
However, the embedding is inconsistent with BMN scaling.

A novel type of chain appears to exist which exhibits BMN scaling. The Hamiltonian was constructed to 5 loops, and a Bethe ansatz conjectured. The S-matrix reads [Beisert,Dippel,MS '04]

$$S_{\mathfrak{su}(2)}(p_k,p_j) = \frac{\varphi_k-\varphi_j+i}{\varphi_k-\varphi_j-i}\,, \qquad E = \sum_{k=1}^{K} q_2(p_k)\,, \quad \text{with}$$

$$\varphi(p_k) = \tfrac{1}{2}\cot\left(\tfrac{p_k}{2}\right)\sqrt{1+8\,g^2\,\sin^2\left(\tfrac{p_k}{2}\right)}\,, \quad q_2(p_k) = \frac{1}{g^2}\left(\sqrt{1+8\,g^2\,\sin^2\tfrac{p_k}{2}}-1\right).$$

Similarily a 3-loop S-matrix may be extracted for $\mathfrak{u}(1|1)$ (but not for $\mathfrak{sl}(2)$) from Beisert's Hamiltonian by applying a new technique, the perturbative asymptotic Bethe ansatz (PABA). [MS '04].

$$S_{\mathfrak{u}(1|1)} = -e^{i\,\theta(p_k,p_j)}\,, \qquad S_{\mathfrak{sl}(2)} = ?$$

A *Gedankenexperiment* for AdS Strings

Let us assume that string theory on $AdS_5 \times S^5$ is integrable, and that it is also described by some long-range spin chain.

Then a Bethe ansatz should exist

$$e^{ip_k L} = \prod_{\substack{j=1\\ j\neq k}}^{K} S(p_k, p_j)\,. \qquad E = \sum_{k=1}^{K} q_2(p_k)\,,$$

Let us, in addition, assume the same dispersion law as in gauge theory: $q_2(p_k) = \frac{1}{g^2}\left(\sqrt{1 + 8\,g^2\,\sin^2\frac{p_k}{2}} - 1\right)$.

Consider the large L limit. Then, with $S = e^{i\theta}$,

$$p_k = \frac{2\pi}{L}\,n_k + \frac{1}{L}\sum_{j=1}^{K} \theta\!\left(\frac{2\pi}{L}\,n_k, \frac{2\pi}{L}\,n_j\right) + \mathcal{O}\!\left(\frac{1}{L^3}\right).$$

Thus, integrability predicts for large L [MS '04]

$$E = \sum_{k=1}^{K} q_2\!\left(\frac{2\pi}{L}\,n_k\right) + \frac{1}{L}\sum_{k,j=1}^{K} q_2'\!\left(\frac{2\pi}{L}\,n_k\right)\theta\!\left(\frac{2\pi}{L}\,n_k, \frac{2\pi}{L}\,n_j\right) + \ldots$$

Let's check!

S-Matrix Extraction (String Theory)

Luckily the $\frac{1}{J}$ corrections $\delta\Delta$ to the spectrum of strings in the plane wave geometry (near-BMN limit) have been computed.

[Parnachev,Ryzhov'02; Callan,Lee,McLoughlin,Schwarz,Swanson,Wu '03,04]

$$\Delta = J + \sum_{k=1}^{K} \sqrt{1+\lambda' \, n_k^2} + \frac{\delta\Delta}{J} + \mathcal{O}(\frac{1}{J^2})\,.$$

The result for $\delta\Delta$ in the multi-excitation case indeed exhibits the above form of a double sum! This allows to "extract" the S-matrix in the following sectors to $\mathcal{O}(\frac{1}{L})$: [Arutyunov, Frolov, MS; MS '04]

$$\begin{aligned} S_{\mathfrak{su}(2)} &= \frac{\varphi_k - \varphi_j + i}{\varphi_k - \varphi_j - i} \prod_{r=2}^{\infty} e^{2\, i\, \theta_{r,r+1}(p_k,p_j)}\,, \\ S_{\mathfrak{u}(1|1)} &\simeq -e^{-\, i\, \theta_{1,2}}\,, \\ S_{\mathfrak{sl}(2)} &\simeq \frac{\varphi_k - \varphi_j - i}{\varphi_k - \varphi_j + i} \prod_{r=1}^{\infty} e^{-2\, i\, \theta_{r,r+1}(p_k,p_j)}\,, \end{aligned}$$

where $\theta_{r,r+1}(p_k,p_j) = (\frac{1}{2} g^2)^r \, (q_{r,k}\, q_{r+1,j} - q_{r+1,k}\, q_{r,j})$. Notice the following interesting relation between the three S-matrices:

$$\boxed{S_{\mathfrak{sl}(2)} = S_{\mathfrak{u}(1|1)}\, S_{\mathfrak{su}(2)}^{-1}\, S_{\mathfrak{u}(1|1)}\,.}$$

Is the AdS/CFT Conjecture Wrong?

Note that the S-matrices we found, while similar, are different in gauge and string theory.

This (technically) explains the three-loop discrepancies between gauge and string theory found in the near-BMN and the Frolov-Tseytlin (spinning strings) limit.

Does this disprove the AdS/CFT conjecture? Not necessarily. The underlying spin chain structures are valid for different regimes of the coupling constant.

It is quite conceivable that the S-matrix gets "dressed" as the coupling strength increases.

It is very interesting that the dispersion relations appear to be unaffected when one goes from weak to strong coupling.

Right or wrong, I will finish by using the correspondence to learn something for the gauge theory from string theory.

An Application of the AdS/CFT Correspondence

Recently a many-year effort to compute three-loop anomalous dimensions of twist-two operators in QCD was completed.

[Moch,Vermaseren,Vogt '04]

Based on experience with "translating" scaling dimensions at one and two loops from QCD to $\mathcal{N} = 4$, a conjecture for the three-loop dimensions of these operators in $\mathcal{N} = 4$ was put forward. [Kotikov,Lipatov,Onishchenko,Velizhanin '04]

Twist-two operators are, in the present language, very short spin chains of length $L = 2$ in the $\mathfrak{sl}(2)$ sector:

$$\boxed{\operatorname{Tr} \mathcal{D}^K \mathcal{Z}^2}$$

Their space-time spin K can be arbitrary.

Can we check the above conjecture, using integrability? Let us apply the relation $S_{\mathfrak{sl}(2)} = S_{\mathfrak{u}(1|1)}\, S^{-1}_{\mathfrak{su}(2)}\, S_{\mathfrak{u}(1|1)}$ we found on the string side to gauge theory. This yields the Bethe ansatz

$$e^{ip_k L} = \prod_{j=1}^{K} \frac{\varphi_k - \varphi_j - i}{\varphi_k - \varphi_j + i}\, e^{2i\,\theta(p_k,p_j)}\,.$$

Solving it for twist two we reproduce the above conjecture! Note that the ansatz should work for any twist; recently it was successfully tested against field theory for twist three. [Eden '05]

Physics is One

This subject nicely demonstrates that what was true in 1905 remains true 100 years later: Good problems interconnect many subdisciplines. Here we have an interesting example, novel types of integrable long-range spin chains, which appear in subjects which naively look completely different:

- A theory containing quantum gravity such as string theory on $AdS_5 \times S^5$.
- A supersymmetric gauge theory such as $\mathcal{N} = 4$ Yang-Mills, which is closely related to the theory of the strong interactions (QCD).
- Condensed matter theory? Long-range systems have not been studied so much, and these new systems might find interesting applications in solid state theory.