THE UNIVERSITY OF CALGARY
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The Lighthouse Theorem – A Budget of Paradoxes

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Research Paper No. 848
July 1 2005

Calgary
Alberta
Canada

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Though Coxeter has gone,
Geometry lives on.

1. INTRODUCTION. The story began in 1993, when Dick Bumby, then Problems Editor of this MONTHLY, sent me the following problem to referee:

> Prove that every prime $p > 7$ of the form $3n + 1$ can be written as $p = \sqrt[6]{a^2 + 4762800b^2}$ for a unique choice of natural numbers a and b.

Paradox 1. The problem was deemed unsuitable for the MONTHLY, but here it is.

Fortunately the proposer, Joseph Goggins, had included the motivation for the problem. He was searching for integer-edged triangles whose Morley triangle was also integer-edged. The Morley triangle theorem is still not as well-known as it deserves to be. Its comparatively short history is outlined in Section **9**. Figure 1 shows its simplest form. The pairs of angle-trisectors of the triangle ABC meet at points which form an equilateral triangle PQR, regardless of the shape of the original triangle. Goggins found an infinity of triangles ABC with integer edge-lengths whose Morley triangle also had an integer edge-length.

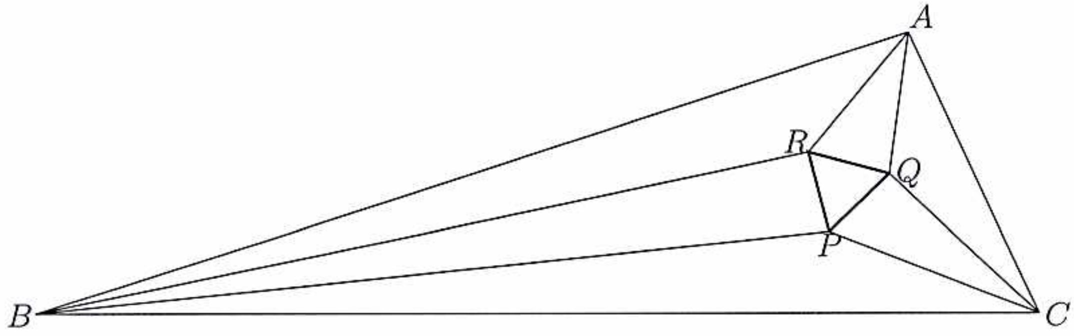

Figure 1: The simplest form of Morley's Theorem.

The combination of geometry and number theory is dear to my heart, and I set to work on the general problem. To trisect angles you need to solve cubic equations, which have three roots. Also, just as angle-bisectors come in pairs, so do (pairs of) trisectors come in triples. I soon discovered that there was not one, but as many as 18 Morley triangles. I excitedly rang Coxeter, but he claimed not to know that; nor did another geometer who was visiting him. Perhaps they didn't want to destroy my delight of discovery?

I rang John Conway, and he said, "Funny thing! I was just looking at that last weekend." So I dashed off a paper [10] and sent it to the MONTHLY. A while later the editor, John Ewing, said that he didn't normally intervene in the refereeing process, but the situation was unusual. One referee had said something like, "O.K., but there are references and other things missing." A second said, "Isn't this like a paper I refereed for you a few years ago?" Indeed it was! It had been written by the first referee, John Rigby, who hadn't resubmitted it.

Paradox 2. Rigby knew that there are 18 Morley triangles, and he knew that Morley knew, but few others seemed to know.

Our paper wasn't resubmitted, either.

Paradox 3. But here it is!

Goggins's real problem was answered by the following:

Theorem[5]. *If a rational-edged triangle has a rational Morley triangle, then either the original triangle is equilateral (and 6 of the 18 Morley triangles are rational—in fact, congruent to the original triangle), or it is Pythagorean and belongs to a one-parameter family (and just 2 of the 18 Morley triangles are rational), or it belongs to a two-parameter family of triangles (and all 18 Morley triangles are rational).*

The Pythagorean family has edge-lengths

$$2t(3-t^2)(1-3t^2), \quad (1-t^2)(1-14t^2+t^4), \quad (1+t^2)^3$$

for rational t, while the more general solution has edge-lengths

$$x_i(x_i^2-1)(x_i^2-9)/(x_i^2+3)^3 \qquad (i=1,2,3)$$

where $\quad 3(x_1+x_2+x_3)=x_1x_2x_3.$

We omitted to mention an observation of Goggins that the lengths of the trisectors are also rational. In fact the reader may like to reconstruct the Conway-Doyle 7-piece jigsaw proof of Morley's theorem by cutting out an equilateral triangle of edge 1001, three triangles with edge-length triples (1001,1716,1859), (1001,9464,9555), (1001,2695,2464) and three with edge-length triples (9555,2695,12005), (2464,1716,3740), (1859,9464,10985). If you want to be sure that they fit together, use the cosine law to calculate the angles.

Paradox 4. Although [5] is a numbertheoretic result, published in *Acta Arithmetica*, it was reviewed in *MR* under the heading Geometry.

But our purpose here is to discuss:

The Lighthouse Theorem. *Two sets of n lines at equal angular distances, one set through each of the points B, C, intersect in n^2 points that are the vertices of n regular n-gons. The circumcircles of the n-gons each pass through B and C.*

That is, they form a **coaxal system** with BC as **radical axis.** In the exceptional case that the sets of lines are parallel, one n-gon is at infinity.

Paradox 5. Although the theorem is quite striking and an immediate consequence of the following well-known propositions of Euclid, it doesn't seem to be in the literature:

III.20. The angle at the centre of a circle is twice that at the circumference.

III.21. Angles in the same segment are equal.

III.22. Opposite angles of a cyclic quadrilateral are supplementary.

The first of these also gives: the angle in a semicircle is a right angle; and the last leads to: exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. And, in the limiting case, we have: the angle between tangent and chord is equal to the angle in the alternate segment. For example, in Figure 3, the chord PU makes two supplementary angles with the tangent at P; these are equal in size to $\angle PVU$ and $\angle PBU$, one in each of the segments into which PU partitions the circle.

Informal demonstration of the Lighthouse Theorem for $n = 3$.

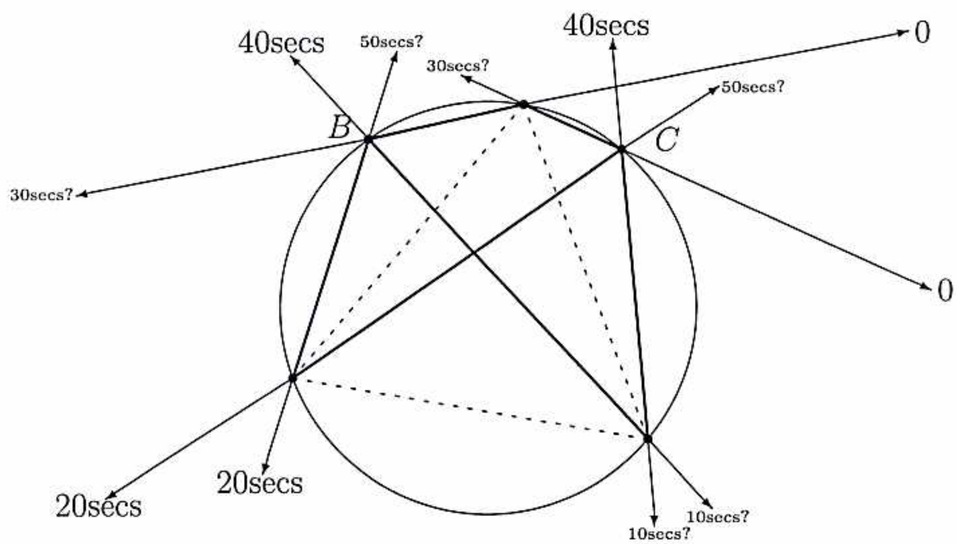

Figure 2: Part of the Lighthouse Theorem for $n = 3$.

In Figure 2, each of two lighthouses at B and C has one doubly-infinite beam, and each rotates with a uniform angular velocity of one revolution per minute. It's night-time. I take photographs every 20 seconds and superimpose them. The locus of the point of intersection of the beams is a circle (Euclid III.21). The point traces out the circle with uniform angular velocity twice that of the lighthouses (Euclid III.20). At 20-second intervals the beams will be equally inclined at angles that are multiples of $\pi/3$, and the points on the circle will be at angular distances $2\pi/3$. In other words,

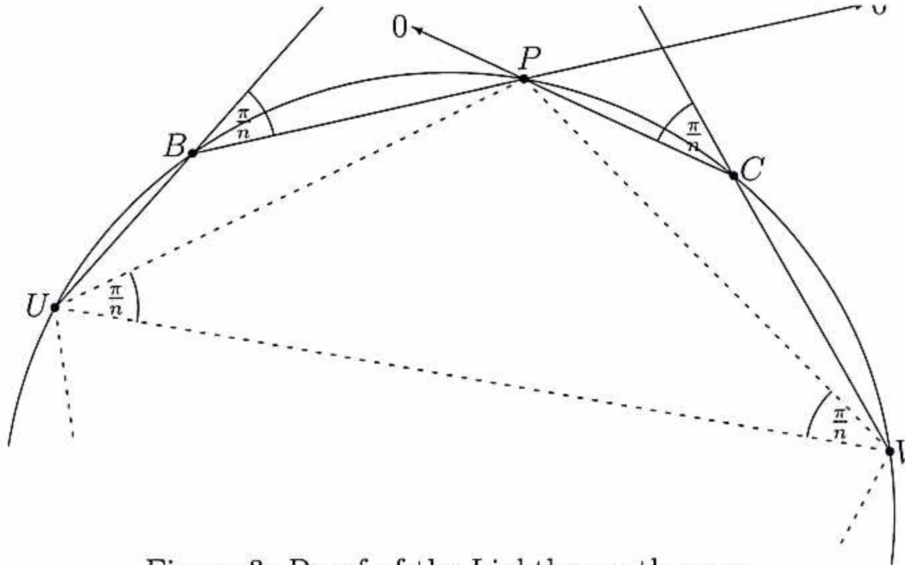

Figure 3: Proof of the Lighthouse theorem.

Proof of the Lighthouse Lemma.

I write β and γ for the magnitudes of angles CBP and BCP, respectively, and refer to them as the **phases** of the two lighthouses. Number the beams from each lighthouse from 0 to $n-1$ cyclically towards the baseline BC, so that they are counted in opposite senses. In Figure 4, BPY and UBX are beams 0 and $n-1$ from lighthouse B, while CPX and VCY are beams 0 and $n-1$ from lighthouse C.

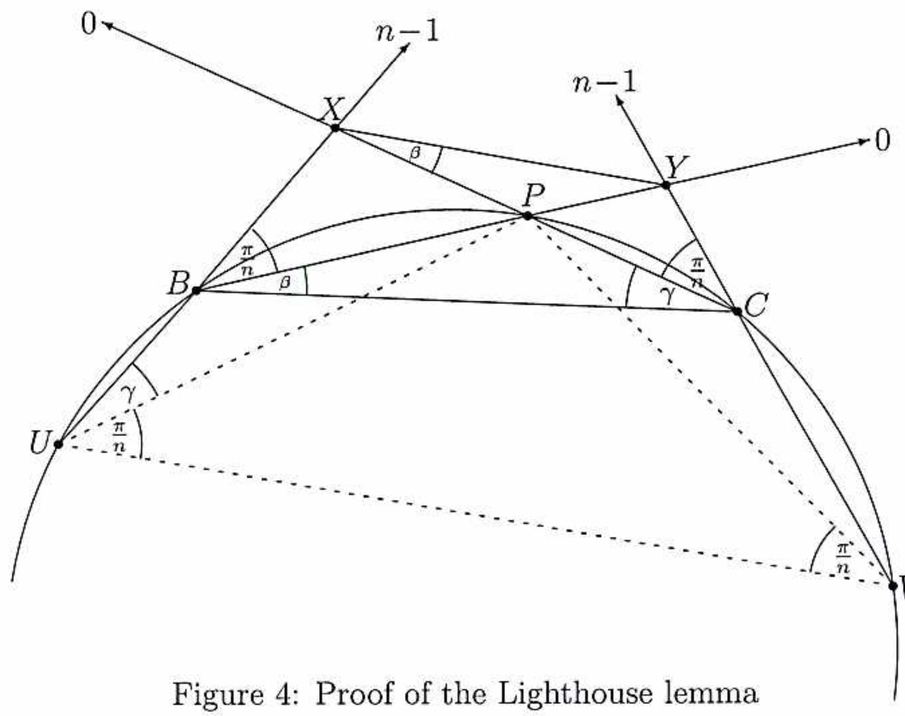

Figure 4: Proof of the Lighthouse lemma

Now $\angle XBY = \pi/n = \angle XCY$, so that $BCYX$ is cyclic. Then, by Euclid III.21, $\angle YXC = \angle YBC = \beta$, implying that XY makes an angle $\gamma - \beta$ with BC. Now the difference between angles BUV and XBC is $(\pi/n + \gamma) - (\pi/n + \beta) = \gamma - \beta$, making XY parallel to UV (and also parallel to the tangent at P to the circle $UBPCV$). Any edge of any n-gon thus makes an angle with any other edge of any n-gon that is a multiple of π/n. ■

Observe that all edges of all n-gons make angles with BC that differ from $\gamma - \beta$ only by multiples of π/n. Our proofs still hold if β or γ (or both) are changed by a multiple of π/n, although the pictures look different.

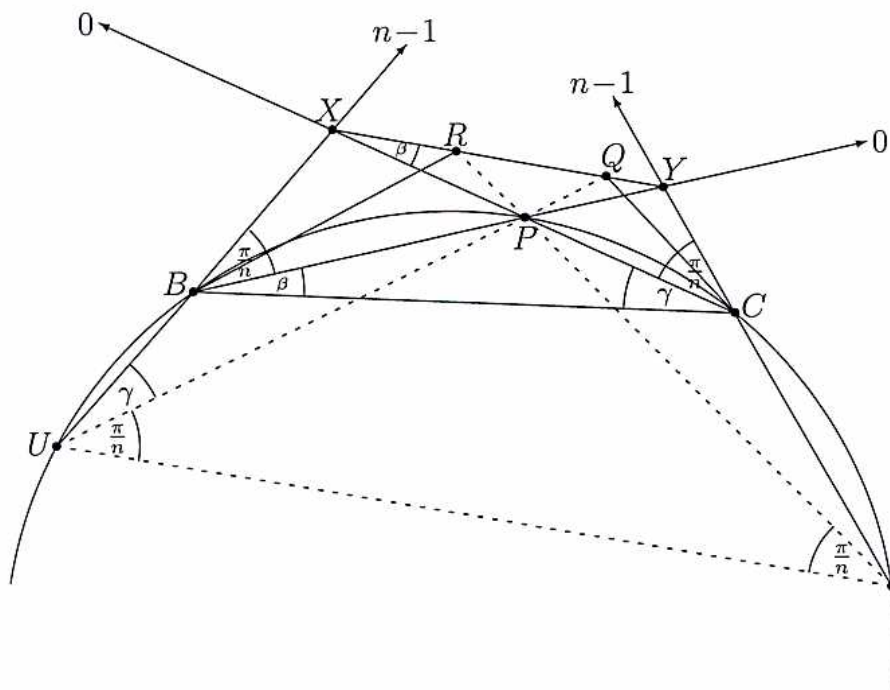

Figure 5: Proof of the Lighthouse Duplication Theorem

Proof of the Lighthouse Duplication Theorem. The notation of Figure 5 is as in Figure 4, and the edges UP and VP intersect the edge XY at Q and R, respectively. From the Lighthouse Lemma, $\angle QRP = \pi/n$ and is therefore equal to $\angle PBX$. Hence $RXBP$ is cyclic and $\angle RBP = \angle RXP = \angle YXC = \angle YBC = \beta$. Accordingly, R lies on a beam through B of phase 2β. Similarly, Q lies on a beam through C of phase 2γ. ■

Note. To avoid repetition when I prove Morley's theorem, it is convenient to observe that $QYCP$ and $BCYX$ are also cyclic; that $\angle PRQ = \angle PQR = \pi/n$; that $\angle BPX = \angle CPY = \beta + \gamma$; that $\angle BPR = \angle CPQ = \pi/n + \gamma$; and that $\angle BRP = \angle CQP = (n{-}1)\pi/n - \beta - \gamma$. This last angle may be written $(n{-}2)\pi/n + \alpha$, where $\alpha + \beta + \gamma = \pi/n$.

The Lighthouse Duplication Theorem has many ramifications. The edges of the n-gons form n families of $\binom{n}{2}$ parallel lines. Two families intersect in $\binom{n}{2}^2$ points, so that the complete configuration contains $\binom{n}{2}^3$ points, though here some points (for example, the vertices of the n-gons) are counted by multiplicity. We now know that several of these points lie on additional beams through the lighthouses, which in turn generate new n-gons (or at least $(n/2)$-gons if n is even) whose edges intersect again, and so on indefinitely. Also, as we shall see in the proof of Morley's theorem, some sets of points lie on new lines, beams from additional lighthouses.

3. MORLEY'S THEOREM. For $n = 3$ the Law of Small Numbers [23] intervenes, because $\binom{n}{2}$ is no larger than n, and the process is in some sense closed, and we have

The Morley Miracle. *The 9 edges of the equilateral triangles of the Lighthouse Theorem for $n = 3$ are the Morley lines of a triangle.*

Paradox 9. We have a proof of the complete Morley theorem, without even having a triangle to start with.

In fact the 9 edges of the 3 triangles from one pair of lighthouses form 27 equilateral triangles of which 18 are genuine Morley triangles. The other 9 comprise 3 sets of what Conway has called the **Guy Faux triangles**, one set from each of 3 pairs of lighthouses situated at B & C, C & A, and A & B, where A is a mystery point, yet to be determined. A good way to count the Morley triangles is to notice that they are each formed from two sides of a GF-triangle together with a third side taken from a different GF-triangle: $3 \times 3 \times 2 = 18$.

Yet another proof of Morley's theorem. I will in fact prove more: what was probably known to Morley, and certainly to Rigby [39], (see also [40]).

Theorem. *The circumcircles of the 9 GF-triangles meet in threes, not only at the vertices of the triangle ABC, but also at 9 other points.*

More precisely, any choice of two GF-circles (i.e., the circumcircles of the GF-triangles), one from each of two of the three families through $B\&C$, $C\&A$, $A\&B$, pass through two pairs of the six Morley points on one of the 9 Morley lines. These, together with the circle from the third family that passes through the remaining two points on the Morley line, concur in an additional point. We have a configuration of 27+9+3 points, 9 circles and 9 lines, each circle passing through 3+3+2 points and each line through 6+0+0 points (see Figure 9).

Nothing that Euclid couldn't have done. In Figure 6 the labels are as in earlier figures, but in the particular case $n = 3$, so that PQR is a typical potential Morley triangle. The point A is defined by $\angle RBA = \beta$ and $\angle RAB = \pi/3 - (\beta + \gamma) = \alpha$, say. [Warning: it's best not to connect the edge CA, lest one assume things that, while true, have not yet been established.]

Draw the circle BRA and let PR and QR cut it again at S and T respectively. From triangle ABR, $\angle ARB = \pi - \alpha - \beta$. From the Note following the proof of the

Duplication Theorem, we obtain $\angle BRP = \pi/3 + \alpha$, so that

$$\angle QRA = 2\pi - \pi/3 - (\pi/3 + \alpha) - (\pi - \alpha - \beta) = \pi/3 + \beta.$$

Hence RST is a GF-triangle generated by lighthouses at A and B with phases α and β. By the Duplication Theorem, $\angle RAQ = \alpha$ and $\angle RQA = \pi - \alpha - (\pi/3 + \beta) = \pi/3 + \gamma$.

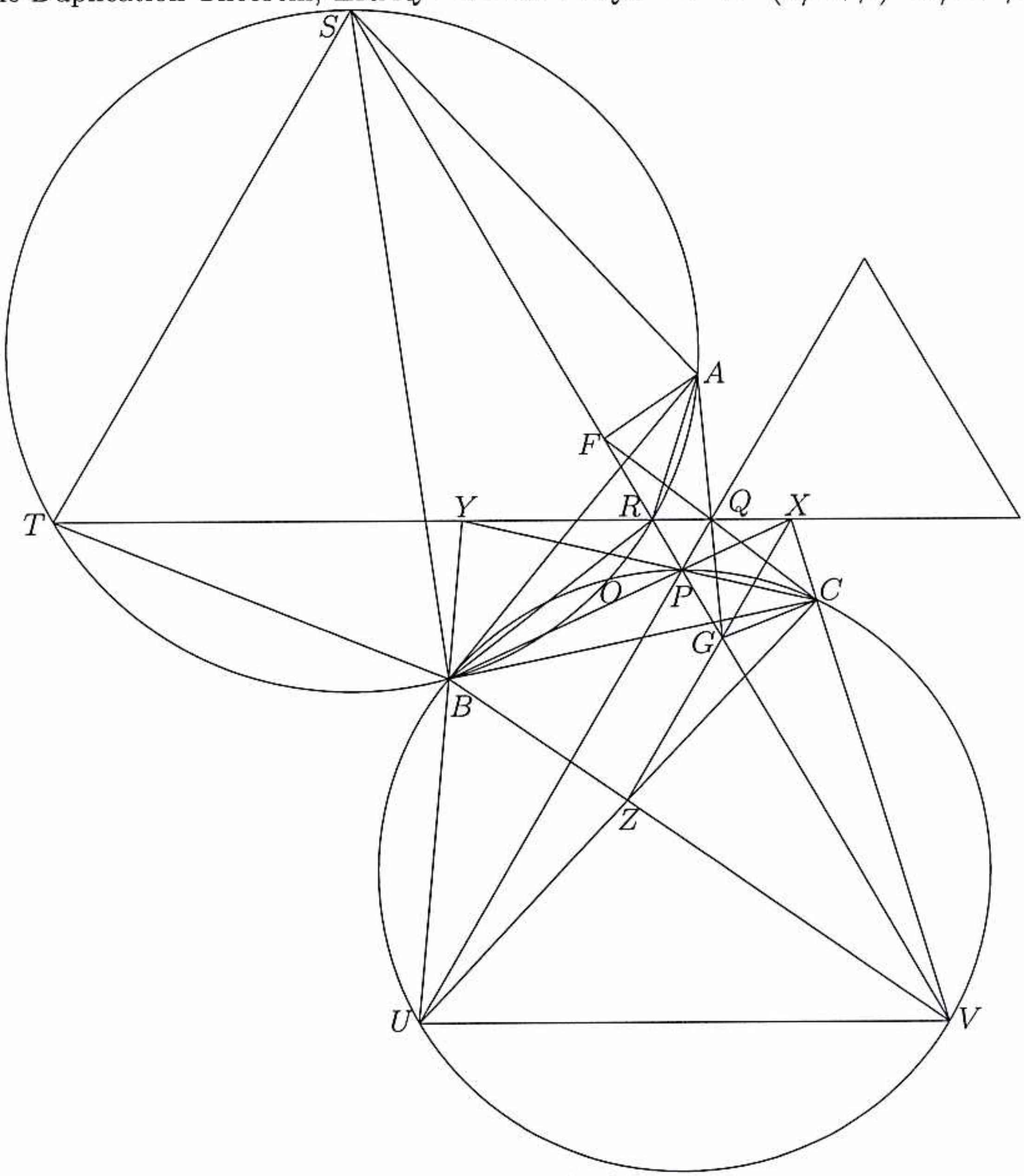

Figure 6: Proof of Morley's theorem

Let CQ meet PR in F. The Note also asserts that $\angle CQP = \pi/3 + \alpha$, whence $\angle RQF = \pi - \pi/3 - (\pi/3 + \alpha) = \pi/3 - \alpha$ and $\angle RFQ = \alpha = \angle RAQ$ which shows that $RQAF$ is cyclic. It follows that $\angle CFA = \angle QFA = \angle QRA = \pi/3 + \beta$ and $\angle FAQ = \pi/3$.

Let XZ, an edge of the GF-triangle XYZ, meet PR in G. Since XZ is parallel to PU, $GXQP$ has angles $\pi/3$ and $2\pi/3$ and so is cyclic. Moreover, C lies on the same circle, because the Note gives $\angle CQP = \pi/3 + \alpha$ which is also the measure of $\angle CXP = \angle CXB = \angle CYB$. So $\angle GQP = \angle GXQ = \pi/3 - \gamma$ and since $\angle RQA = \pi/3 + \gamma$ we see that A, Q and G are collinear and AF and AG are beams through A inclined at $\pi/3$. Also $\angle GCF = \angle GCQ = \angle GXQ = \pi/3$ so that CF and CG are beams through C inclined at $\pi/3$. We thus see that $AFGC$ is cyclic, $\angle CAG = \angle CFG = \alpha$, and $\angle ACF = \angle AGF = \angle QGP = \angle QXP = \gamma$, implying that FG is an edge of a GF-triangle, EFG say, generated by lighthouses at C and A with phases γ and α, respectively.

Finally, let the GF-circles ARB and BPC meet again at O. Then $\angle AOB = \angle ARB = \pi - \alpha - \beta$ and $\angle BOC = \angle BPC = \pi - \beta - \gamma$. This means that

$$\angle COA = 2\pi - (\pi - \alpha - \beta) - (\pi - \beta - \gamma) = \pi/3 + \beta = \angle CFA = \angle CGA$$

and that the GF-circles BPC, C**GF**A, ARB concur at O. ∎

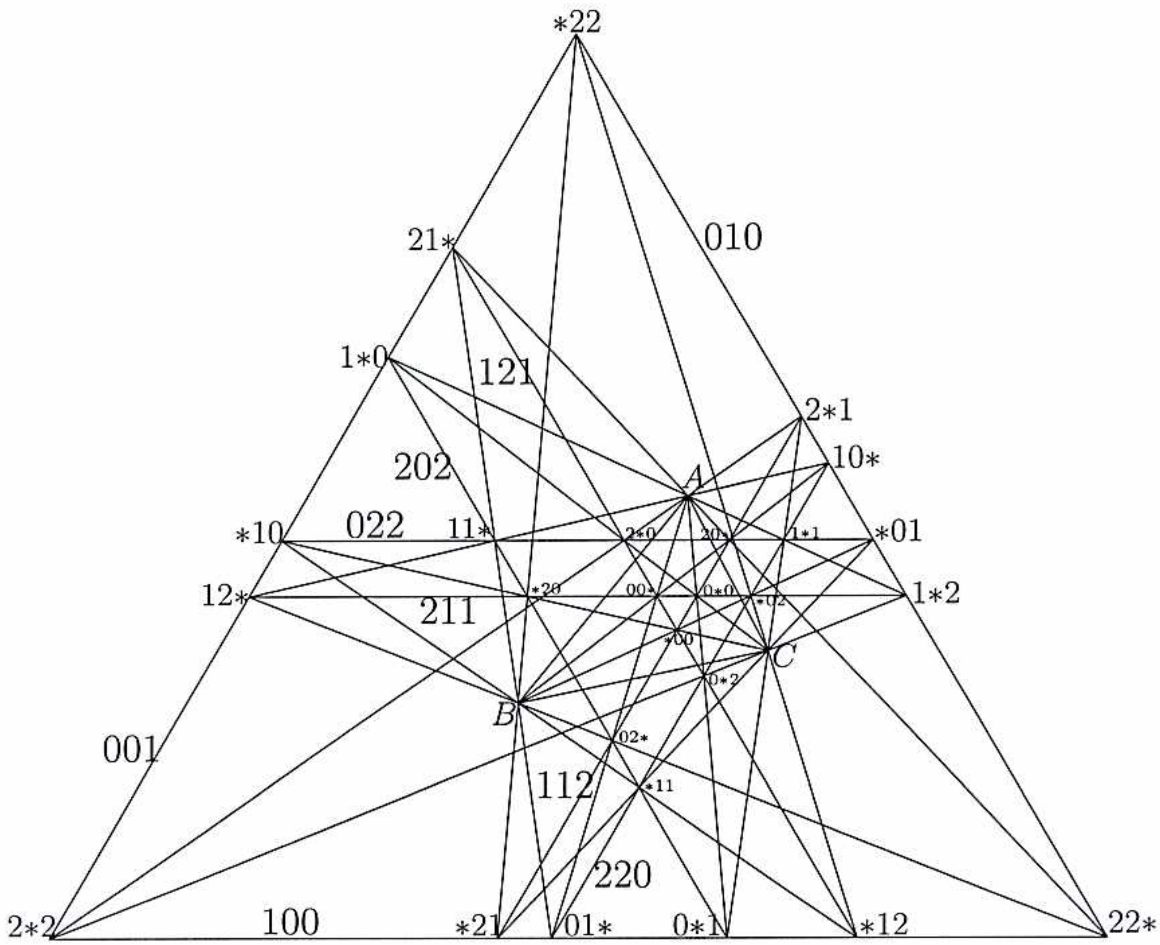

Figure 7: The eighteen Morley triangles

The point O is associated with the Morley line $VGPRFS$ in the sense that the three circles through O intersect the line in the pairs of points $\{P, V\}$; $\{G, F\}$; $\{R, S\}$.

Figure 7 shows the complete configuration of 18 trisectors, 27 Morley points, 9 Morley lines, 18 Morley triangles and 9 GF-triangles. The figure is labelled according to Conway's scheme, in which the 27 Morley points each receive a 3-digit label comprising two numbers 0, 1 or 2 and a star. The (1st, 2nd, 3rd) position of the star indicates the vertex (A, B, C) of the triangle that is *not* responsible for the point. For example, $1{*}2$ is the intersection of beam 1 from a lighthouse at A with beam 2 from a lighthouse at C. Remember that the beams from the (A, C)-lighthouse pair are numbered starting from 0 for the internal trisector proximal to the edge CA and continuing across CA with the beams counted in opposite senses.

Figure 8 is a schematic diagram of the 9 Morley lines and 18 Morley points.

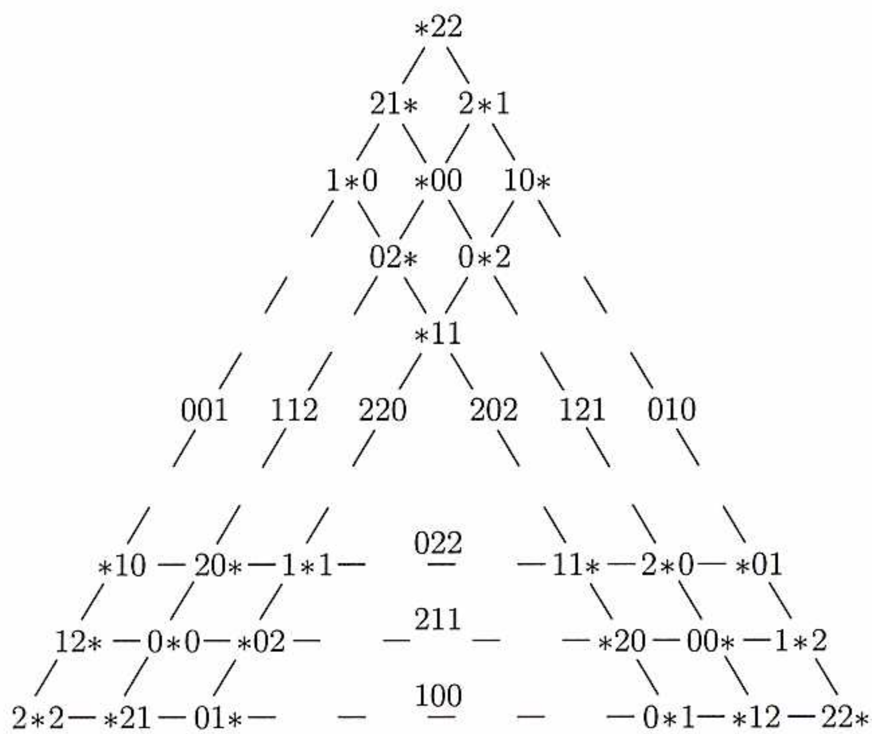

Figure 8: Schematic diagram of Morley points, lines and triangles, and GF-triangles

The six points on any Morley line come two from each of the three pairs of lighthouses, with two point-labels with a star in the first position, two with it in the second, and two with it in the third. On any Morley line only two of the digits 0,1,2 occur in any one of the three positions. The line-label (large in Figure 7) consists of the three digits which do *not* occur in the three positions. Each line-label has a repeated digit and a different digit, and the sum of the three digits is congruent to 1 modulo 3. For example, the points labelled $*12$, $0*2$, $*00$, $00*$, $2*0$, $21*$, lie on the Morley line with label 121.

The GF-triangles are found by choosing any vertex, say 0∗1 in the bottom row of Figure 8, and then taking the only other two points on the Morley lines through that vertex that have stars in the same position (here the B position): 2∗2 and 1∗0. The other 18 triangles are genuine Morley triangles whose vertices have their stars in the three different positions.

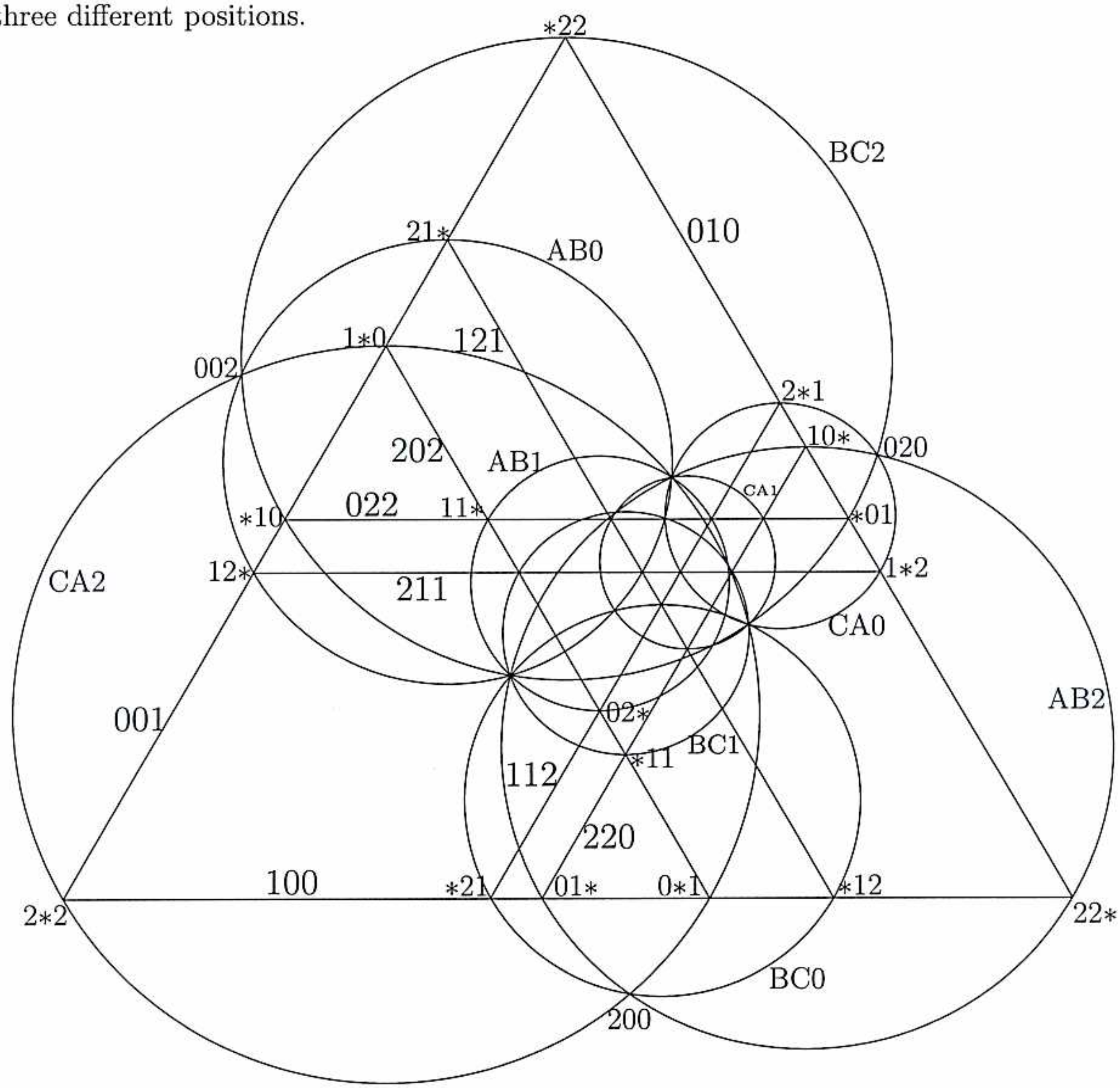

Figure 9: Nine Morley lines and nine GF-circles

Figure 9 shows the 9 Morley lines and 9 GF-circles, with more detail shown in Figure 10. There are 27 Morley points, 9 points of concurrence of triads of GF-circles, and 3 vertices of the original triangle. Each circle passes through 3+3+2 of these points. Each of the 9 Morley lines passes through 6 Morley points. The label for each of the 9 points of concurrence of triads of GF-circles is a three-digit number that differs from the label of the corresponding Morley line in only one digit: the digit that differs from the other two is changed to the third possibility, so that the sum of the digits of the label of an associated point is congruent to 2 modulo 3, as follows:

Morley line label	100 010 001	211 121 112	022 202 220
associated point label	200 020 002	011 101 110	122 212 221

We may include the circumcircle of triangle ABC in each of the coaxal systems of GF-circles. The three lines of centres of the systems are the perpendicular bisectors of the edges of ABC and they concur at its circumcentre.

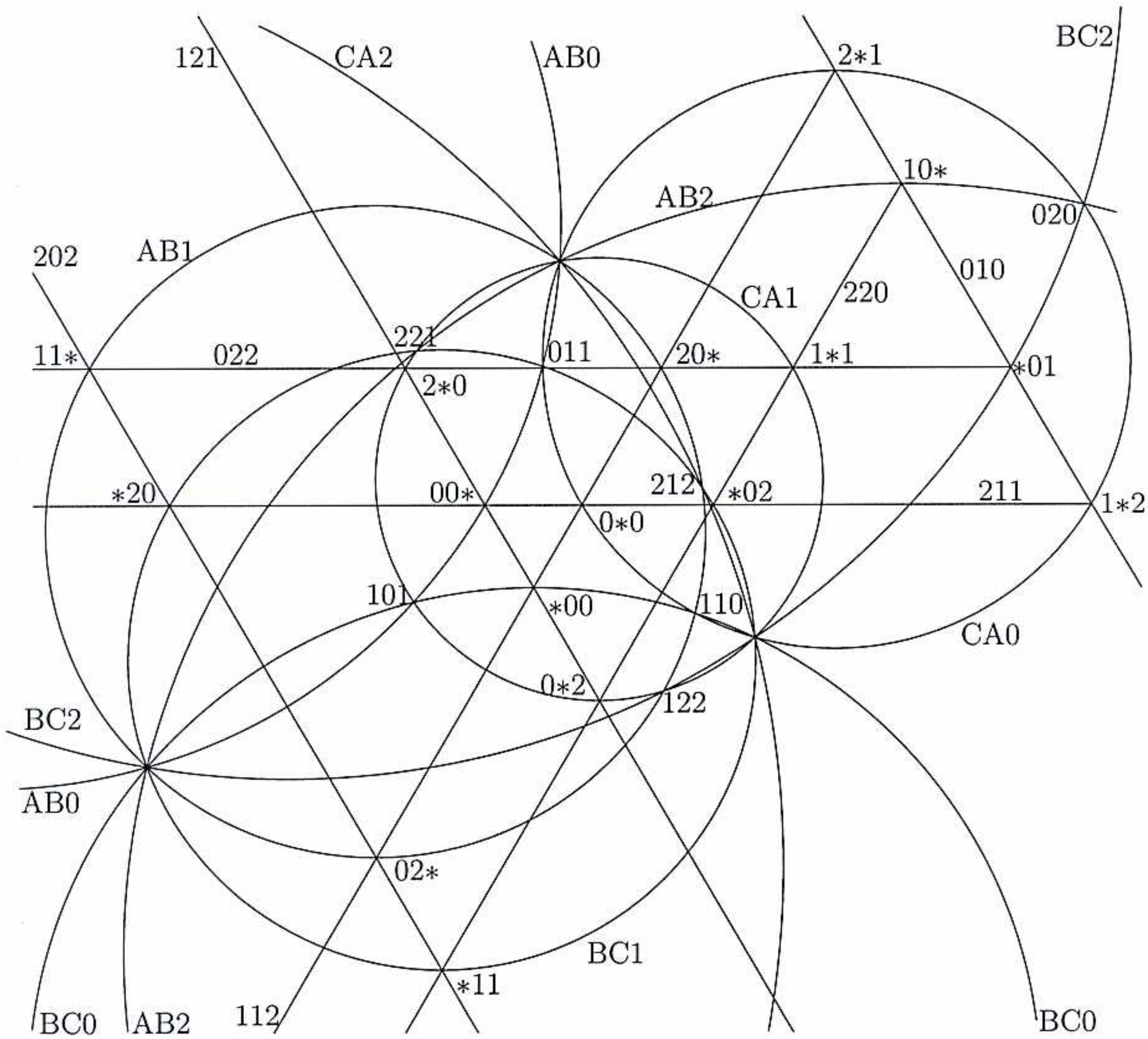

Figure 10: Enlargement of part of Figure 9

If we omit the Morley lines and Morley points from Figure 9 we are left with a pleasing configuration of 9 circles and 12 points with 5 points on each circle, 6 circles through each of 3 points, and 3 circles through each of the other 9 points.

Here are two tables that facilitate the location of all the points, lines and circles.

GF-circle	Morley points	Morley lines	associated points
BC0	∗00 ∗21 ∗12	100 121 112	200 101 110
BC1	∗11 ∗02 ∗20	211 202 220	011 212 221
BC2	∗22 ∗10 ∗01	022 010 001	122 020 002
CA0	2∗1 0∗0 1∗2	211 010 112	011 020 110
CA1	0∗2 1∗1 2∗0	022 121 220	122 101 221
CA2	1∗0 2∗2 0∗1	100 202 001	200 212 002
AB0	21∗ 12∗ 00∗	211 121 001	011 101 002
AB1	02∗ 20∗ 11∗	022 202 112	122 212 110
AB2	10∗ 01∗ 22∗	100 010 220	200 020 221

Morley line	GF-circles	Morley points	associated point
211	BC1 CA0 AB0	∗20 ∗02 0∗0 1∗2 12∗ 00∗	011
121	BC0 CA1 AB0	∗12 ∗00 2∗0 0∗2 00∗ 21∗	101
112	BC0 CA0 AB1	∗00 ∗21 2∗1 0∗0 02∗ 20∗	110
022	BC2 CA1 AB1	∗10 ∗01 1∗1 2∗0 20∗ 11∗	122
202	BC1 CA2 AB1	∗20 ∗11 1∗0 0∗1 11∗ 02∗	212
220	BC1 CA1 AB2	∗11 ∗02 0∗2 1∗1 01∗ 10∗	221
100	BC0 CA2 AB2	∗21 ∗12 2∗2 0∗1 01∗ 22∗	200
010	BC2 CA0 AB2	∗01 ∗22 2∗1 1∗2 22∗ 10∗	020
001	BC2 CA2 AB0	∗22 ∗10 1∗0 2∗2 12∗ 21∗	002

Conway's extraversion. The best insight into the Morley configuration is probably provided by Conway's "extraversion". An "A-flip" of a triangle, for example, replaces angles A, B, C with $-A$, $\pi - B$, $\pi - C$, respectively. It takes us from THE Morley triangle, 000, with vertices ∗00, 0∗0, 00∗, to triangle 011 with vertices ∗11, 0∗1, 01∗. Repeated use of A-, B- and C-flips generates the toroidal map of Figure 11, where similarly labelled Morley triangles are to be identified. There are 9 hexagonal regions, bounded by $9 \times 6/2 = 27$ edges, meeting 3 at each of the 18 Morley triangles.

The Morley Group. The least obvious of the five abstract groups of order 18 is the semidirect product of $C_3 \times C_3$ with C_2. One rarely meets such things in the heat of battle, but here it is: a set of relations is $x^2 = y^3 = z^3 = 1$, $yz = zy$, $yxy = x = zxz$, and one way to realize it is with $y = AB$, $z = AC$, $x = ABA$. Let's give it a name!

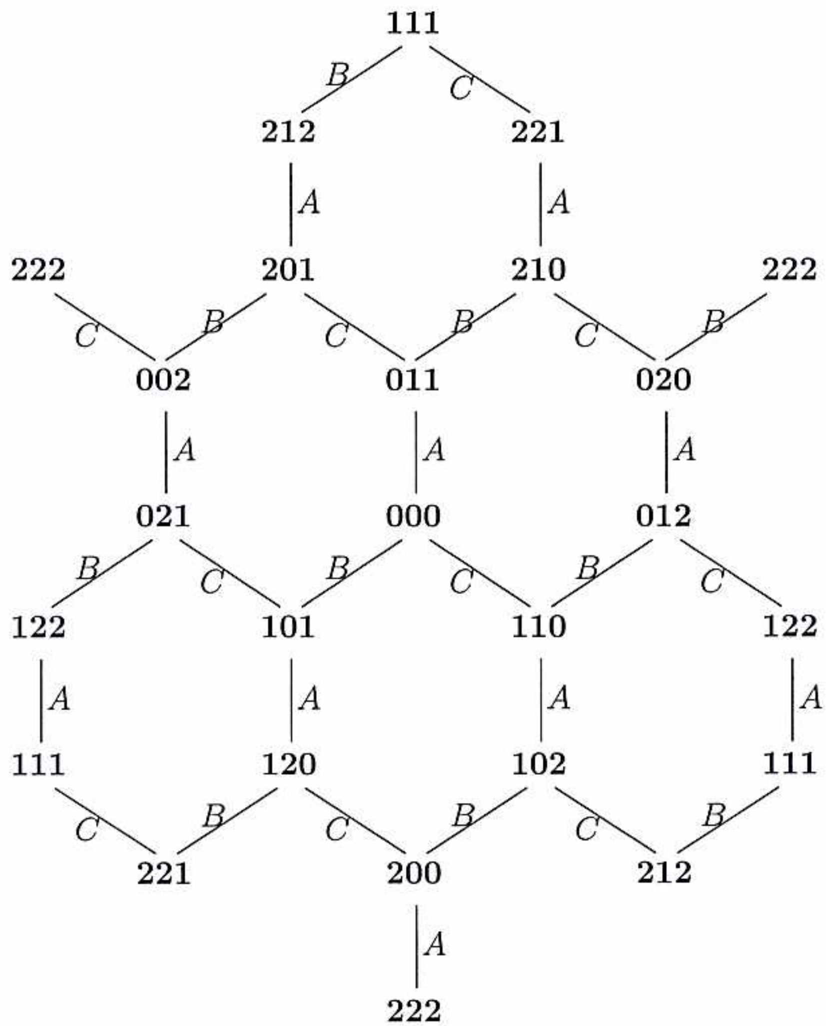

Figure 11: The 18 Morley triangles related by extraversion

Where are the Morley centres? There is some debate as to what constitutes a "centre" of a triangle, but for the equilateral triangle, one point, and hence at least one Morley centre (namely 000) is beyond doubt; but 222 and 111 should also be considered. Label the centre of a Morley triangle with the 3-digit number that gives the vertices on replacing the digits in turn with a star. For example, 120 is the centre of the Morley triangle with vertices *20, 1*0, 12*. In Figure 12 the Morley lines are dotted and the 18 Morley centres form a "crate", which, when extended by the 9 GF-centres (smaller unlabelled dots) appears to consist of 27 cuboids, 18 of which have 5 Morley vertices and 3 GF-vertices, and the other 9 have 6 Morley vertices and 2 GF-vertices. There appear to be 108 body-diagonals that bisect each other in fours at 27 points that form a half-sized "crate". Nine of them are collinear in threes (the dashed lines in Figure 12).

Paradox 10. Figure 12 looks three-dimensional.

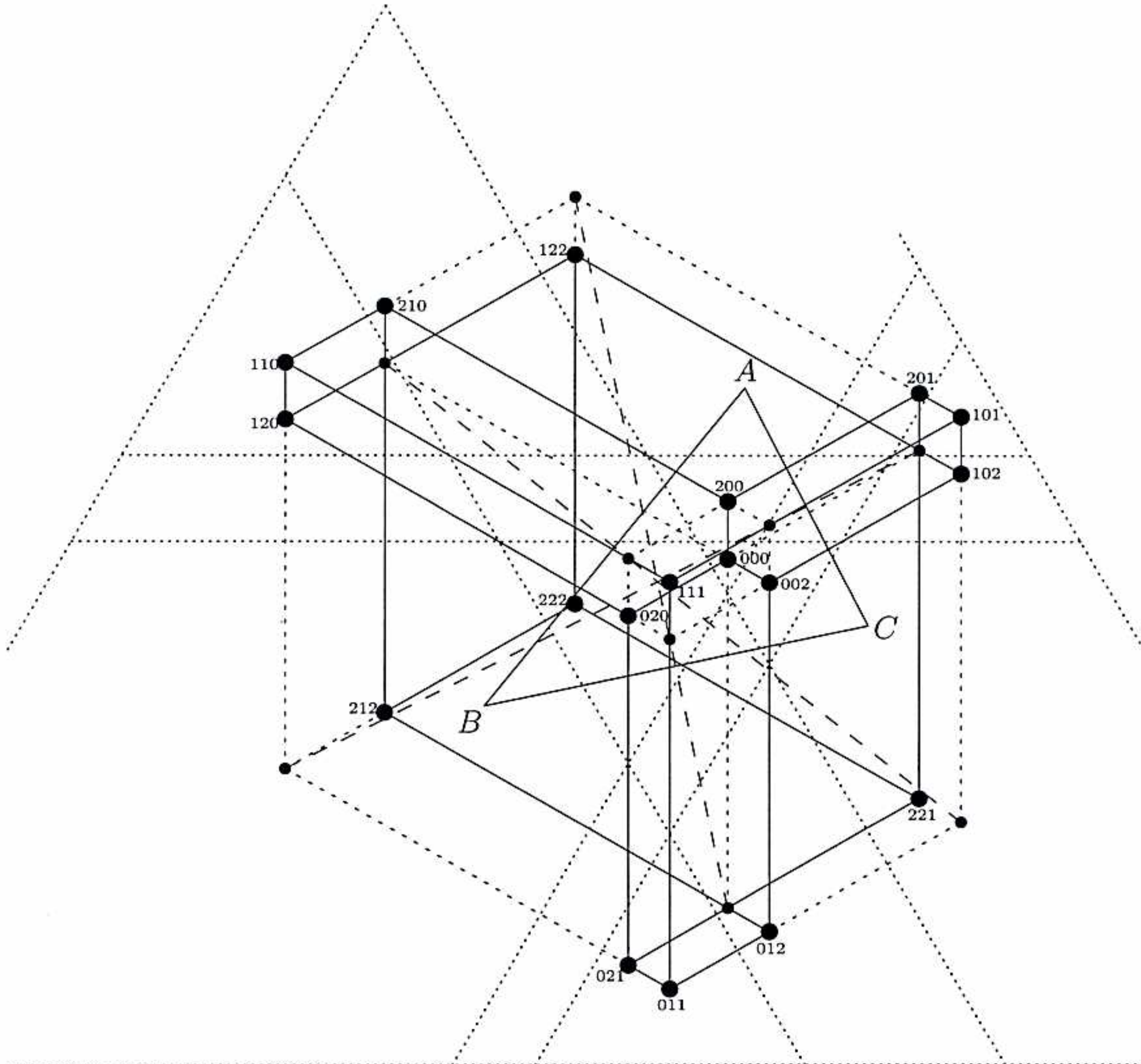

Figure 12: Eighteen Morley centres and nine GF-centres

4. THE LIGHTHOUSE THEOREM FOR $n = 2$.

For $n = 2$, $\binom{n}{2} < n$, so we don't get the ramifications that occur for larger n, but even so, the Lighthouse Theorem still has a lot to tell us. Paradox 6 no longer arises, while Paradoxes 7 and 8 are not so apparent, because we expect a rectangular hyperbola and a circle to intersect in four points. In Figure 13 two pairs of perpendicular lines through lighthouses at E and F intersect at the vertices of two 2-gons, say AH and BC, segments that are at an angle $\pi/2$ with each other. In other words:

Theorem. *The altitudes of a triangle* (ABC) *concur* (*at the orthocentre* H).

Paradox 11. Another triangle theorem without having a triangle.

More symmetrically, each of the 4 points A, B, C, H is the orthocentre of the triangle formed by the other three. This is a special case of the theorem that any conic through the intersections of two rectangular hyperbolas is a rectangular hyperbola.

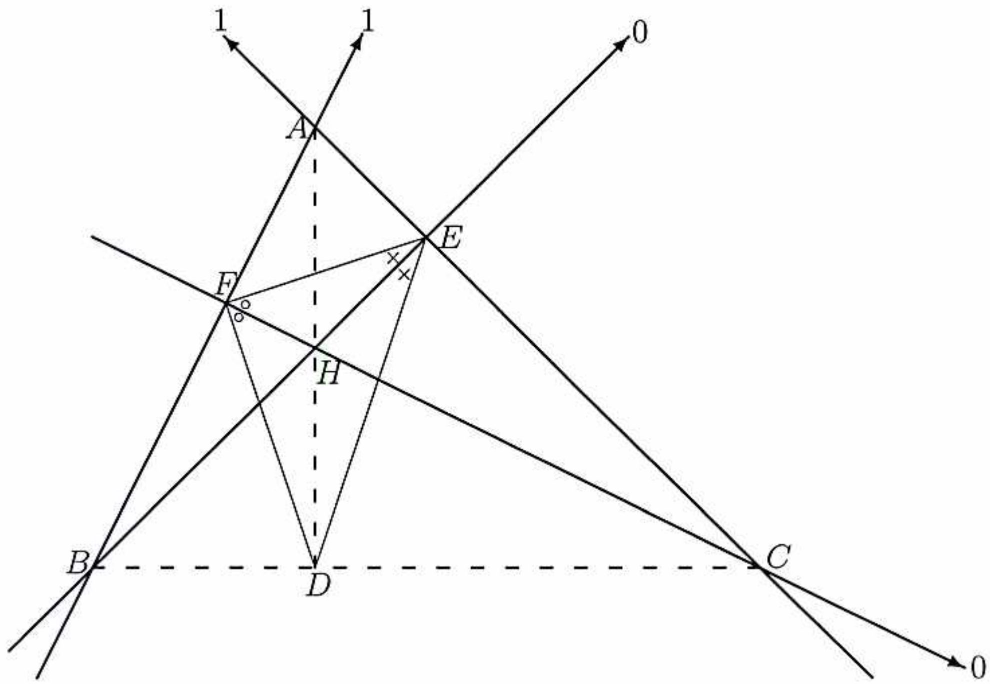

Figure 13: The altitudes of a triangle concur: also the angle-bisectors

But more: if the two 2-gons meet at D, then the Lighthouse Duplication Theorem tells us that $\angle DEH = \angle HEF$ and $\angle DFH = \angle HFE$ (i.e., EH & AC and FH & AB are the pairs of angle-bisectors at vertices E and F of triangle DEF). The angle-bisectors of triangle DEF concur in four points, the incentre H and the excentres A, B, C, *provided* we know that DH & BC are the angle-bisectors of angle D. But $\angle BDF = \angle BHF$ ($BDHF$ cyclic) $= \angle EHC$ (vertically opposite) $= \angle EDC$ ($EHDC$ cyclic). ■

Figure 13 contains six cyclic quadrangles and three rectangular hyperbolas!

Paradox 12. A third triangle theorem without a triangle.

Note that the theorems are quite general. Any triangle DEF is obtained from lighthouses at E and F whose beams are phased to pass simultaneously through D.

5. THE THRICE SIXTEEN THEOREM AND OTHER BONUSES

Figure 14 shows a cyclic quadrangle 0123. The incentres of triangles 123, 023, 013, 012 are the points that are respectively labelled 00, 11, 22, 33. The excentres are labelled with the other twelve 2-digit base-4 numbers. The midpoints of the segments joining these in-(& ex-)centres are labelled, somewhat arbitrarily, as follows:

the point	04	05	06	14	15	16	24	25	26	34	35	36
is mid of	10-13	10-12	22-23	00-03	00-02	21-20	01-02	11-13	00-01	11-12	01-13	02-03
& mid of	20-23	31-33	32-33	30-33	21-23	31-30	31-32	30-32	10-11	21-22	20-22	12-13

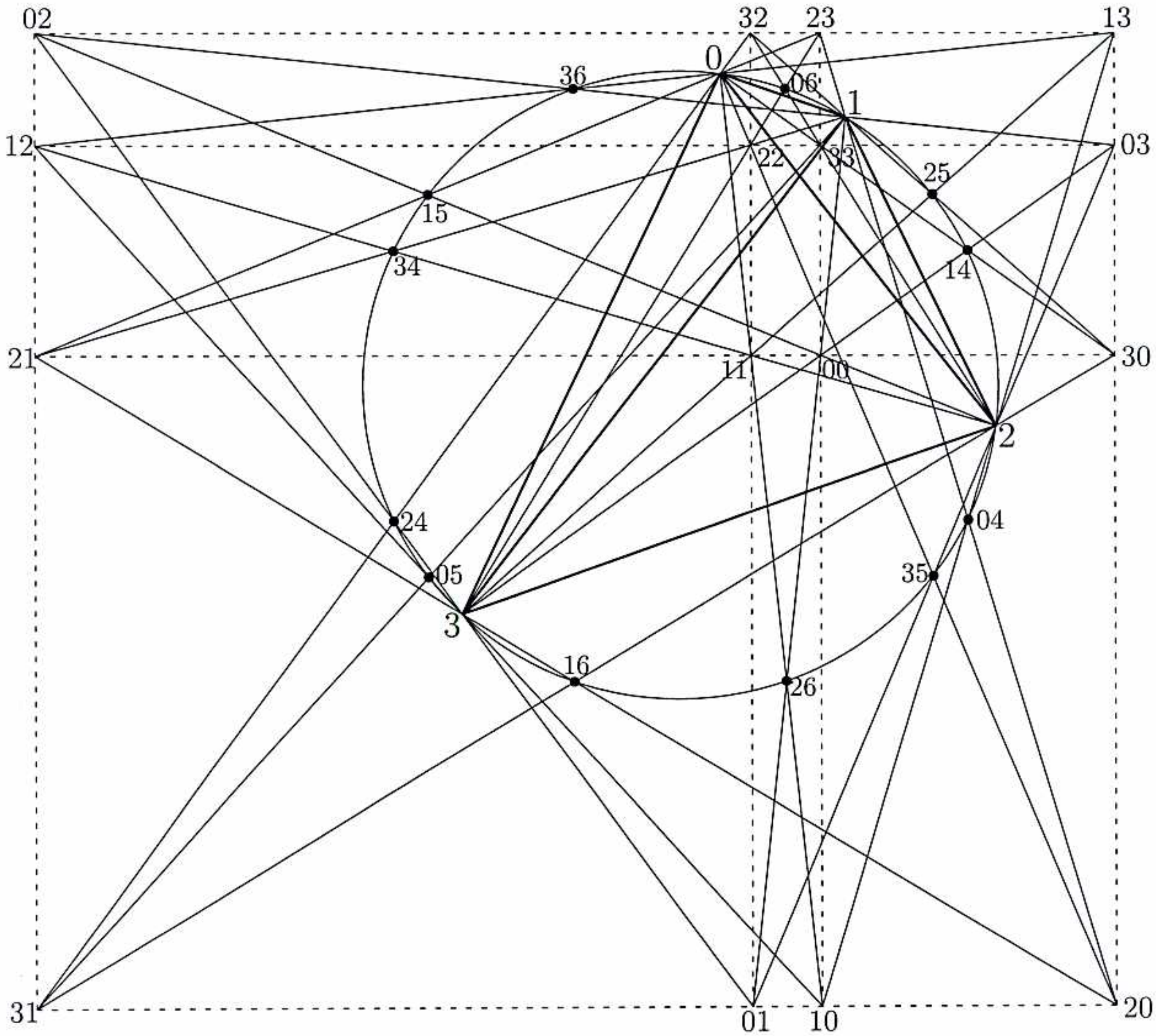

Figure 14: The Thrice Sixteen Theorem

The Thrice Sixteen Theorem states that the 4×4 incentres of the four triangles 123, 023, 013, 012 lie in fours in a rectangular array on two perpendicular sets of 4 parallel lines. Secondly, the 4×6 midpoints of segments joining these centres coincide in twelve pairs, one pair (the dots in Figure 14) at each end of six diameters of the **16-point circle** 0123. Thirdly, this circle is the **9-point circle** for each of the 4×4 triangles ab ac ad, where a and $\{b, c, d\} \in \{0, 1, 2, 3\}$. Each of the 16 centres is the orthocentre of one of these triangles, implying that the 16 circumcentres are the reflexions of the 16 incentres in the centre of the 16-point circle. Note that the four centres in any of the eight rows and columns come one each from the four concyclic triangles, so that the 2-digit labels form a pair of orthogonal latin squares.

Exercise for the reader. Prove the Thrice Sixteen Theorem. Hint: use Euclid III.21 and the Lighthouse Theorem for $n = 2$ with pairs of lighthouses at each of the six pairs of points (0,1), (2,3), (0,2), (3,1), (0,3), (1,2) to show that the join 00-11 is perpendicular to 22-32 and many other perpendicularities. Use the Lighthouse Lemma to prove that 00-11 is parallel to 23-32 and many other parallelisms. Prove also that each of the twelve sets of six points shown in the following table are concyclic.

sets of concyclic points						centre
0	3	20	10	23	13	04
0	2	31	12	33	10	05
0	1	33	23	32	22	06
1	2	03	33	00	30	14
1	3	23	02	21	00	15
1	0	30	20	31	21	16
2	1	01	31	02	32	24
2	0	32	13	30	11	25
2	3	11	00	10	01	26
3	0	21	11	22	12	34
3	1	22	01	20	03	35
3	2	12	03	13	02	36

There is a fourth manifestation of 16 if we combine these 12 circles with the 16-point circle counted with multiplicity four as the circumcircle of the triangles 123, 230, 301, 012. In Figure 14 each of the sextuples (1,2,3,04,05,06), (2,3,0,14,15,16), (3,0,1,24,25,26), (0,1,2,34,35,36) lies on the 16-point circle. In Figure 15 the incentre grid is dashed; the lighthouse beams are dotted; and the 16-point circle is drawn with a thick line.

Further manifestations of 16 come from the 4 sets of 4 orthocentric points $a0, a1, a2, a3$ ($a =$ 0,1,2,3). Each of the 16 incentres ab is the orthocentre of the triangle $a\bar{b}$, where $\bar{b}$ denotes the 3-element complement of b in the set $\{0, 1, 2, 3\}$. These 16 triangles have a common 9-point circle, namely the 16-point circle, whose centre is N, say. If O_{ab} is the circumcentre of triangle $a\bar{b}$, then, since N is the midpoint of the segment of the Euler line that joins the orthocentre to the circumcentre, the 16 circumcentres form a 4×4 grid that is congruent to the grid of orthocentres, being its reflexion in N. Moreover the 16 circumcircles are congruent, since their radii are each twice that of the 9-point radius. The 16 centroids form a similar grid, with one-third the linear dimensions. We have not included this in Figure 16, which shows the orthocentre grid dashed and the circumcentre grid solid, together with the 9-point centre N and the 16 congruent circumcircles.

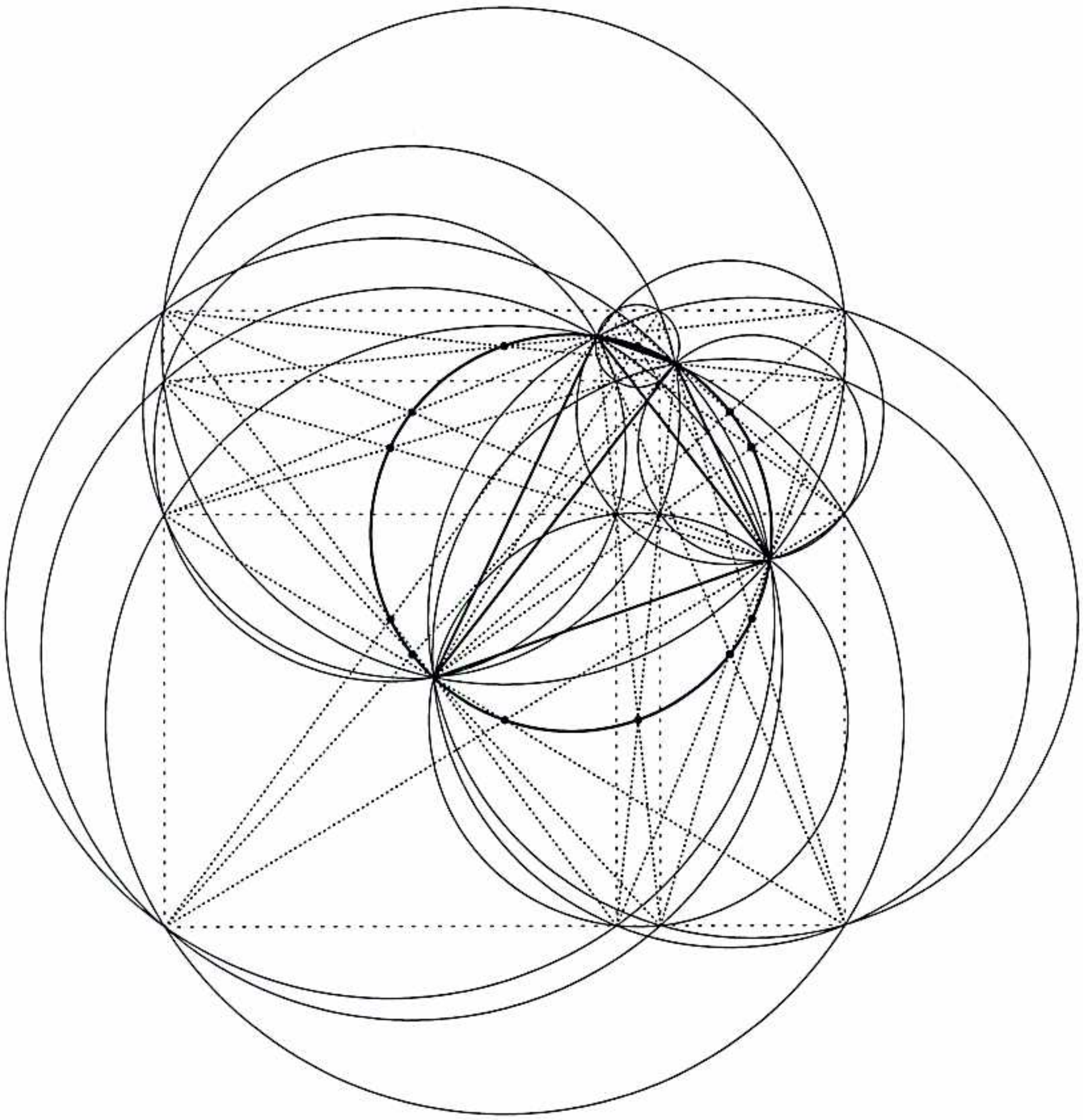

Figure 15: Twelve circles and four times one other

Take five! (Further exercise for the reader). Draw a figure with five concyclic points, forming $\binom{5}{3} = 10$ triangles with five 4×4 grids of incentres coinciding as 40 pairs.

There are 72 Morley triangles! If we repeat our proof of Morley's theorem with the lighthouses at phases $\frac{\pi}{6} - \gamma$ and $\frac{\pi}{6} - \beta$ in place of β and γ, then we get 18 Morley triangles for a triangle with base angles $3(\frac{\pi}{6} - \gamma) = \frac{\pi}{2} - C$ and $\frac{\pi}{2} - B$. This is the triangle BHC, where H is the orthocentre of ABC. The edges of these Morley triangles make angles with BC which differ only by multiples of $\frac{\pi}{3}$ from $(\frac{\pi}{6} - \gamma) - (\frac{\pi}{6} - \beta) = \gamma - \beta$ and so are parallel to the edges of the Morley triangles of ABC. Wth 18 more from each of the triangles CHA and AHB we have a grand total of 72. Moreover, in the rational case [5] all 72 have rational edges! Note that although BC, CA, AB are rational, AH, BH, CH are not: these, as well as the altitudes and the area, are rational multiples of $\sqrt{3}$.

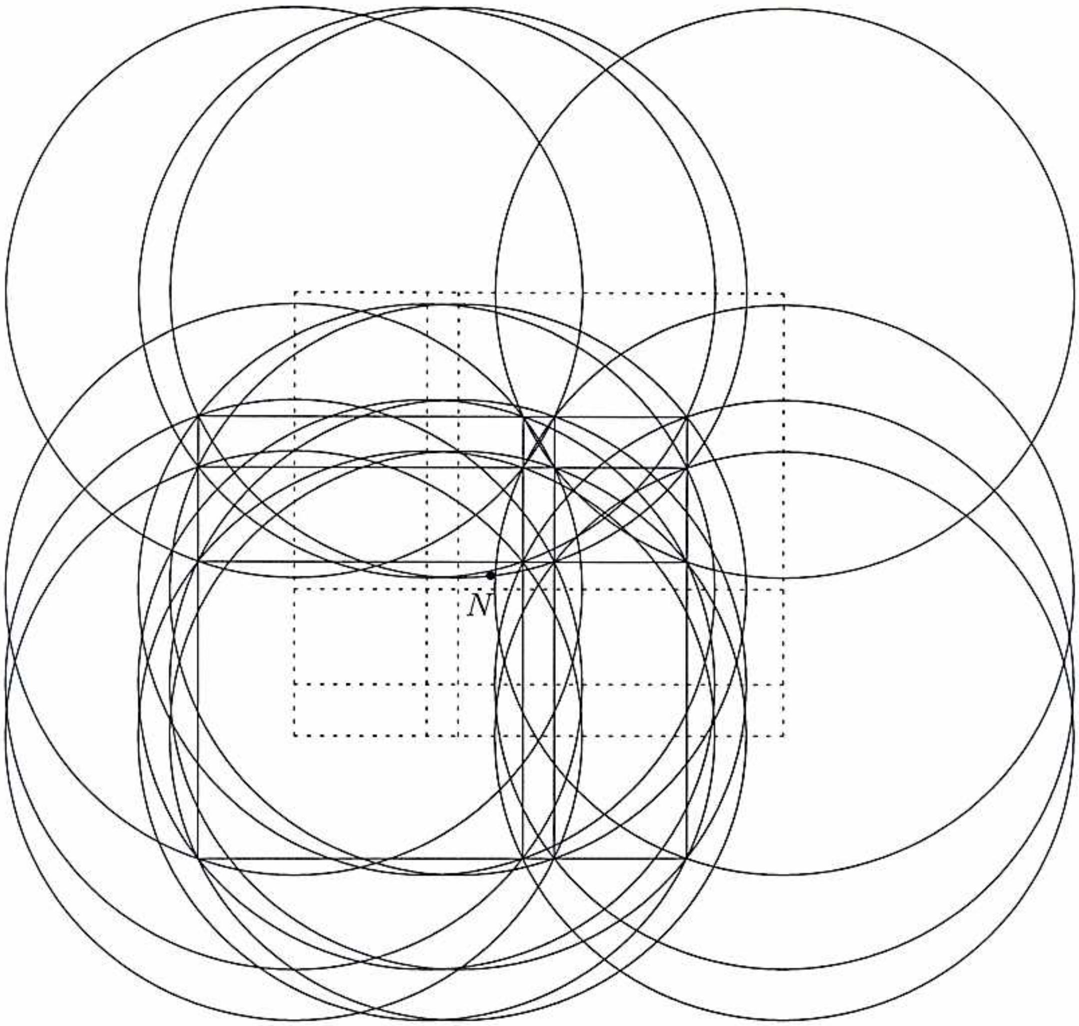

Figure 16: Sixteen orthocentres, circumcentres and circumcircles

Buy three: get two free!

Theorem. *If a triangle is inscribed in a rectangular hyperbola, then its orthocentre also lies on the hyperbola, at the opposite end of the diameter of the hyperbola through the fourth point of intersection of the hyperbola with the circumcircle of the triangle.*

I prefer synthetic proofs, but this analytic one is so attractive that I can't resist:

Proof. By choice of axes and scale, we may take the rectangular hyperbola to have equation $xy = 1$. Let the triangle have vertices $(t, 1/t)$, $(u, 1/u)$, $(v, 1/v)$. The line through the first two of these has slope $-1/tu$. The line through the third point, with perpendicular slope, has equation $y - 1/v = tu(x - v)$, which intersects the hyperbola again in $(-1/tuv, -tuv)$. By symmetry, we see that the altitudes of the triangle concur at this point of the hyperbola. If the equation to the circumcircle of the triangle is $x^2 + y^2 + 2gx + 2fy + c = 0$, then the x-coordinates of its points of intersection with the hyperbola are given by $x^4 + 2gx^3 + cx^2 + 2fx + 1 = 0$. The product of the roots is 1, so the fourth point has x-coordinate $1/tuv$, at the opposite end of the diameter of the hyperbola through the orthocentre of the triangle. ■

In case there was ever any doubt about Paradox 8, here is a list of the 27 Morley points. They lie in threes on 9 GF-circles which pass through 2 of the vertices A, B, C. They also lie in threes on 9 rectangular hyperbolas which pass through 2 of A, B, C. The following table shows what lies on what. Rows of 3 points are on a GF-circle through the 2 vertices listed at their head. Columns of 3 points are on a GF-hyperbola through these vertices.

B C	C A	A B
$*00$ $*12$ $*21$	$0{*}0$ $1{*}2$ $2{*}1$	$00*$ $12*$ $21*$
$*11$ $*20$ $*02$	$1{*}1$ $2{*}0$ $0{*}2$	$11*$ $20*$ $02*$
$*22$ $*01$ $*10$	$2{*}2$ $0{*}1$ $1{*}0$	$22*$ $01*$ $10*$

In fact each 3-by-3 array may be thought of as an affine geometry with 9 points. Two sets of 3 parallel "lines" are the rows and columns, i.e. the GF-circles and the GF-hyperbolas. The other two sets of 3 parallel lines are the broken diagonals and correspond to sets of points on beams through B, C; or C, A; or A, B.

6. SO LITTLE DONE—SO MUCH TO DO.

Nine rectangular hyperbolas, each through 5 points, $\binom{5}{3}$ triangles inscribed in each. Where are the 90 orthocentres? Where do the circumcircles meet the hyperbolas again? Where are the circumcentres? The nine-point circles? The Euler lines? The 'buy 3, get 2 free' theorem gives us a good start.

For simplicity we look only at the three hyperbolas associated with the lighthouses at B and C. Their centres are all at the midpoint, M say, of BC. We simplify the labels of the Morley points by omitting the star and reading the two digits as a ternary number. For example, $*21$ is 7. Figure 17 (see also Figure 18) shows the nine points whose star is in the first ('A') position as $0, 1, 2, \ldots, 8$.

We confine our attention to the nine triangles BCa where $0 \le a \le 8$. Each is inscribed in hyperbola 0 or 1 or 2 according as $a \in \{0, 8, 4\}$ or $\{5, 1, 6\}$ or $\{7, 3, 2\}$. Their nine orthocentres a' lie respectively on these hyperbolas, and the nine sets $\{B, C, a, a'\}$ are each orthocentric:

$$aa' \perp BC, \quad aB \perp a'C, \quad aC \perp a'B$$

and the circumcircles of the triangles Caa' Baa', BCa' are the reflexions of the circumcircle of BCa in its edges Ca, aB, BC, and so each have the same circumradius, twice that of the radius of the common nine-point circle of the four triangles. Each of the nine points a'' is at the opposite end of the diameter through a of the appropriate GF-circle $BC057$, $BC813$, $BC462$, so that they form regular hexagons with the original points a inscribed in the GF-circles and are part of a manifestation of the Lighthouse Theorem for $n = 6$. Each is also at the opposite end of the diameter through a' of the appropriate hyperbola $BC084$, $BC516$, $BC732$. These nine diameters concur at M.

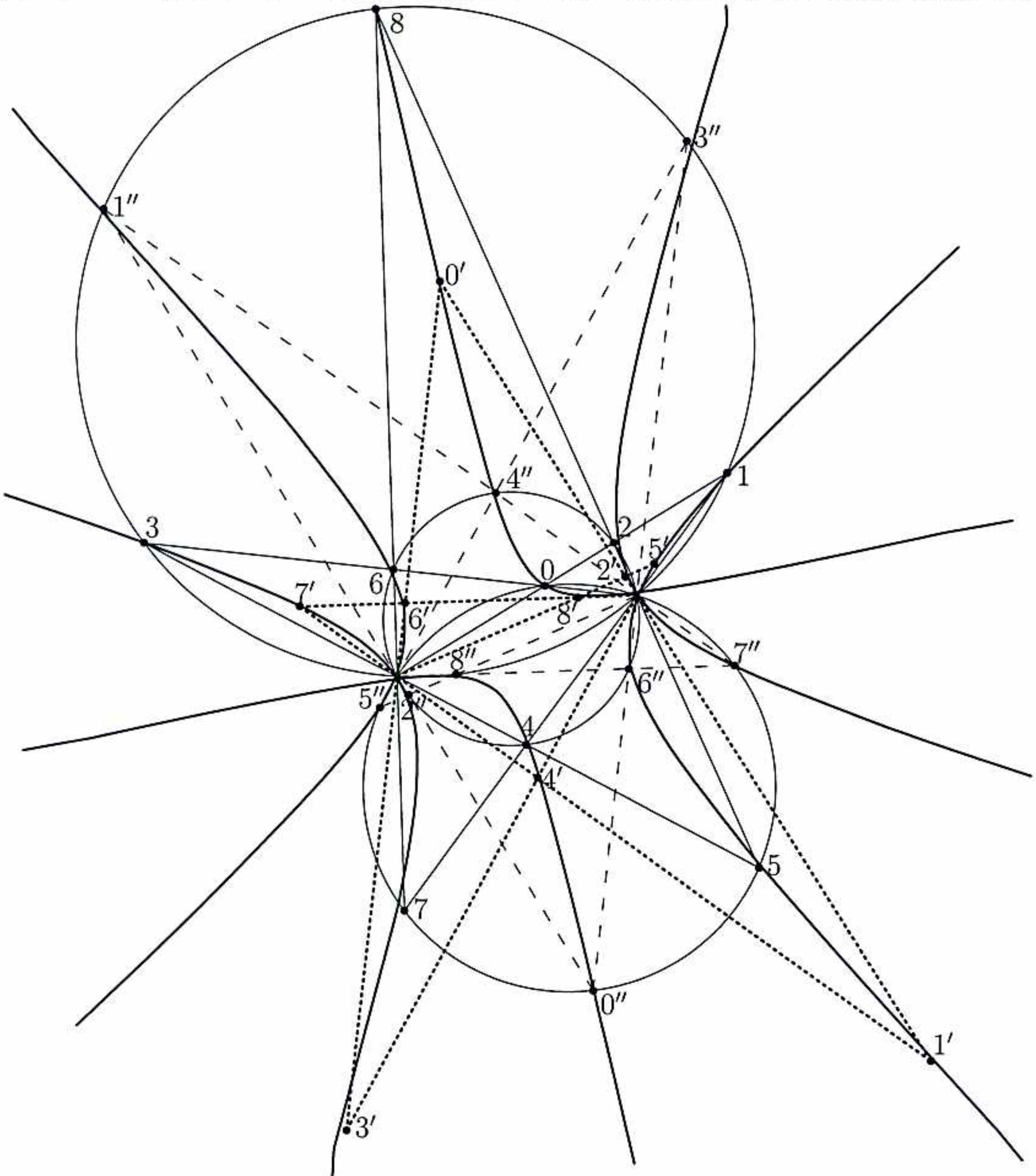

Figure 17: The three GF-hyperbolas through B and C

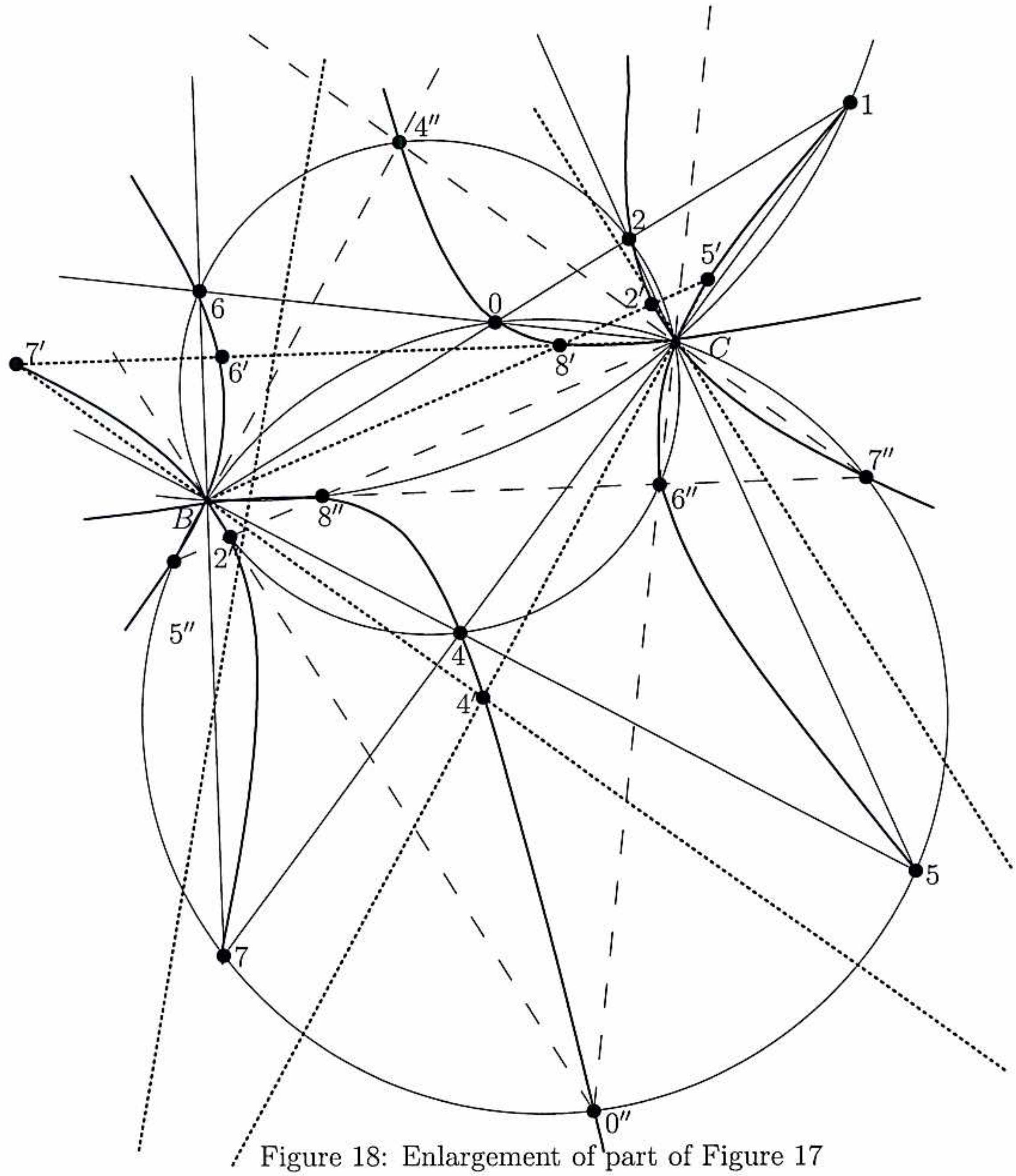

Figure 18: Enlargement of part of Figure 17

The nine points a''' are not shown in Figures 17 or 18, though Figure 19 contains three specimens. They may be variously described as the second intersection of aa' with the appropriate GF-circle or as the reflexion of a' in BC, i.e., as being generated by lighthouses whose beams are the reflexions in BC of the original beams through B and C or as the reflexion of a'' in the perpendicular bisector of BC.

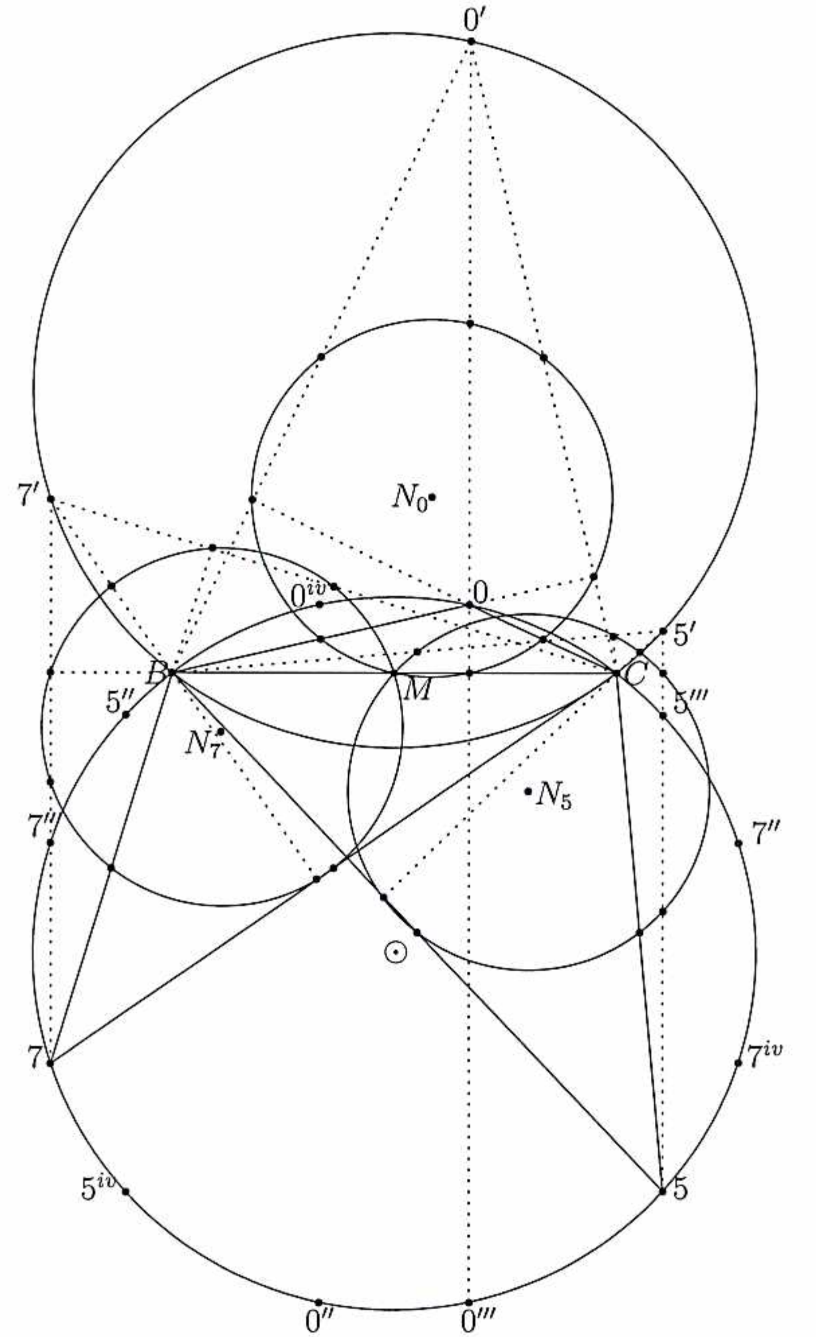

Figure 19: Further fun from Figure 17

The Four Nines Theorem. *The four sets of nine points,* $\{a\}$, $\{a'\}$, $\{a''\}$, $\{a'''\}$ *are the congruent configurations*

0	1	2	$0'$	$3'$	$6'$	$0''$	$1''$	$2''$	$0'''$	$3'''$	$6'''$
3	4	5	$1'$	$4'$	$7'$	$3''$	$4''$	$5''$	$1'''$	$4'''$	$7'''$
6	7	8	$2'$	$5'$	$8'$	$6''$	$7''$	$8''$	$2'''$	$5'''$	$8'''$

whose rows are points on beams through B *and columns points on beams through* C.

In Figure 19 some of the relations between these configurations can be seen as

1. $00'0'''$, $55'5'''$, $77'7'''$ are collinear and perpendicular to BC, with $00'$, $55'$, $77'$ equal in length.

2. $00''$, $55''$, $77''$ are diameters of the GF-circle 057.

3. $0'0''$, $5'5''$, $7'7''$ each have M as midpoint.

4. $0''0'''$, $5''5'''$, $7''7'''$ are parallel to BC.

Again there are ramifications and more sets of nine points. The astute reader would demand a set $\{a^{iv}\}$ of reflexions of $\{a''\}$ in BC or of $\{a'''\}$ in M, forming a 4-group of configurations with $\{a\}$, $\{a''\}$, $\{a'''\}$. The points 0^{iv}, 5^{iv} and 7^{iv} have crept into Figure 19 which confines itself to illustrating triangles $BC0$, $BC5$ and $BC7$, with common circumcentre $\odot$. The Euler lines $\odot N_0 0'$, $\odot N_5 5'$, $\odot N_7 7'$ are not drawn. The nine-point centres N_0, N_5, N_7 form an equilateral triangle. The nine-point circles are congruent, each passes through M and they intersect at angles $\pi/3$, forming a pleasing cloverleaf.

Only a tithe. I have considered one-third of the GF-hyperbolas and three-tenths of the triangles inscribed in each. The "buy three, get two free" theorem gives us 180 bonus points from the 90 triangles. What further coincidences, collinearities, concyclicities, are there among these points?

Bifaux and Skewfaux. When you state the Morley theorem, you must be careful to specify the intersections of the *proximal* trisectors. What if you make a mistake and use the *distal* (duplicated) beams? You get a new set of GF-triangles whose edges are the Morley lines of a triangle, $A'BC$ say, with base angles 6β, 6γ, so that the original A is an incentre of $A'BC$. Or you might get mixed up and choose one proximal and one distal trisector, and get the Morley lines for a triangle $A''BC$ or $A'''BC$, with just one base angle doubled — A'' and A''' being the intersections of BA with CA' and CA with BA'.

Turning Morley inside-out. We could describe the construction of A' as the intersection of the **distal treblers** at B and C, in contrast to the *proximal trisectors* that are used in Morley's theorem. Construct B' and C' similarly (in Figure 20, arcs at A, bullets at B, circles at C), let the **proximal treblers** meet at α', β', γ', and let BC, $B'C'$ meet at α, etc., as in Figure 20. We then have

Theorem [15]

1. AA', BB', CC' concur at the circumcentre of ABC.

2. $A\alpha'$, $B\beta'$, $C\gamma'$ concur at the orthocentre of ABC.

3. The triads (α,β,γ), (α,β',γ'), (α',β,γ'), (α',β',γ) are each collinear.

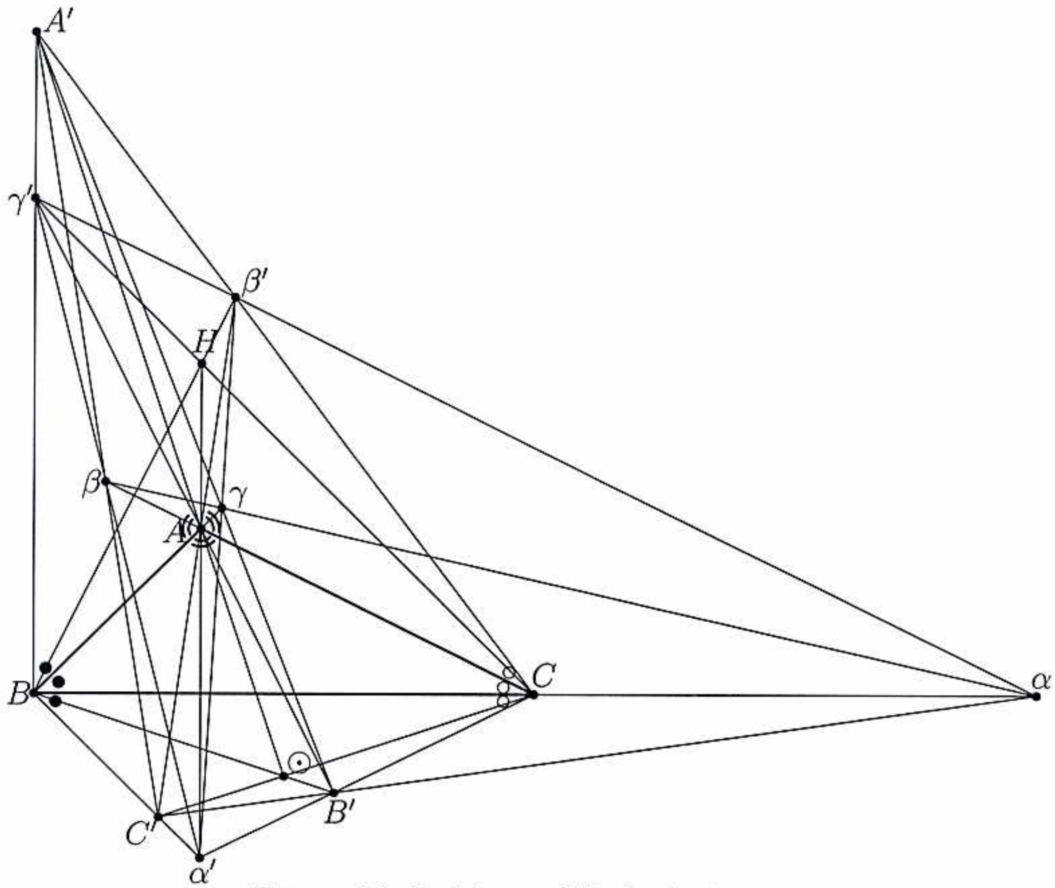

Figure 20: Inside-out Morley's theorem

Sketch of proof. 1. AB, AC are angle-bisectors of $\angle A'BC$, $\angle A'CB$, so that A is an incentre of $A'BC$ and AA' is a bisector of $\angle BA'C$ and makes angles $\frac{\pi}{2}-C$, $\frac{\pi}{2}-B$ with AB, AC and hence passes through $\odot$, the circumcentre of ABC. Similarly for BB', CC'.

2. α' is the reflexion of A in BC, so that $A\alpha'$ is perpendicular to BC.

3. $\odot$ is the perspector of triangles ABC, $A'B'C'$, so, by Desargues's theorem, $\alpha\beta\gamma$ are collinear. Similarly for triangles $A'BC$, $AB'C'$, etc. ■

Desargues distended. The last part of this proof reminds us that we may swap a pair of vertices of two triangles in Desargues's theorem and produce a new perspectrix. Since any of the 10 points may serve as perspector we have 40 perspectrices and a configuration of 25 points and 55 lines with 9 lines through each of the original 10 points and 6 lines through each of 15 new ones; four points on each of 15 new lines and three on each of the 10 old and the 30 new perspectrices.

Relabel Figure 20 with Alex Fink's beautiful labelling:

Fig.20 labels	$\odot$	A	B	C	A'	B'	C'	α	β	γ	α'	β'	γ'
Fink's labels	0	8	7	1	9	3	2	4	5	6	04	05	60

This shows the duality between the Desargues configuration and the Petersen graph (Figure 21), each of which has automorphism group S_5.

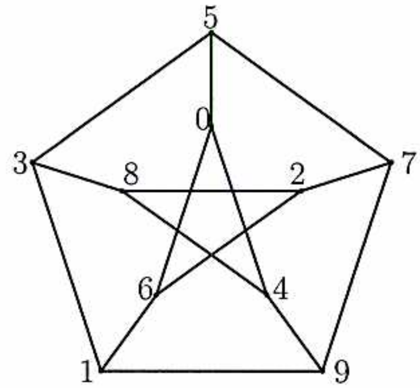

Figure 21: The Petersen graph

(a)	6	8	0	2	4
(b)	012	234	456	678	890
(c)	1	3	5	7	9
(d)	369	581	703	925	147
(e)	9023	1245	3467	5689	7801
(f)	61 57 48	83 79 60	05 91 82	27 13 04	49 35 26

Rows (a) & (c) of this table are perspectors in the Desargues configuration, and (b) & (d) are the corresponding perspectrices. Vertices (b) & (d) in the Petersen graph are adjacent to the corresponding vertices in rows (a) & (c). Row (e) shows the maximal independent sets of the Petersen graph; they comprise the $5 \cdot \binom{4}{2}$ pairs of points which have a common Desargues line. The other $\binom{10}{2} - 30$ pairs, which are independent in the Desargues configuration, are the 15 edges of the Petersen graph. These are given as five triples of pairs in row (f) of the table. The first pair is a copy of the entries in (a) & (c) and, in the Petersen graph, is the edge perpendicular to the 2nd & 3rd pairs. These pairs can also be read as two-digit labels of the 15 points that amplify the Desargues configuration. For example, read the middle triple as 'the joins of 9 & 1 and of 8 & 2 meet in the point whose 2-digit label is 05' etc. Then, with perspector 0,

triangles 871 & 932 have perspectrix 4 5 6
triangles 971 & 832 have perspectrix 4 05 60
triangles 831 & 972 have perspectrix 04 5 60
triangles 872 & 931 have perspectrix 04 05 6

which is a description of the relabelled Figure 20.

7. THE LIGHTHOUSE THEOREM WHEN $n \geq 4$

When $n > 3$, think of the n-gons as complete graphs on n vertices, two-dimensional representations of regular $(n-1)$-dimensional simplexes. There are n sets of $\binom{n}{2}$ parallel lines that intersect in $\binom{n}{2}^3$ points, though I now count points with multiplicity. The n^2 vertices are each counted $\binom{n-1}{2}$ times. If n is even the "diameters" concur, and for certain n there are other concurrencies, originally enumerated by G. Bol [3] and often rediscovered (see [38] for a good account).

Figures 22 and 23 are for $n = 4$. In Figure 22 the beams are dotted and the quadrangles solid. In Figure 23 the original beams are omitted, the edges of the quadrangles are dotted, and the "duplicated" beams are solid. I number the beams as before and will label the intersections of the lighthouse beams with the hexadecimal numbers 0, 1, ..., 9, a, b, ..., f=15 which should be thought of as two-digit quaternary numbers $\beth\gimel$, where $\beth$ and $\gimel$ are the numbers of the beams from B and C. Quadrangle q $(0 \leq q \leq 3)$ then has vertices whose quaternary digits add to q modulo 4:

$q = 0$ is 00=0 13=7 22=a 31=d; $q = 1$ is 01=1 10=4 23=b 32=e;
$q = 2$ is 02=2 11=5 20=8 33=f; $q = 3$ is 03=3 12=6 21=9 30=c.

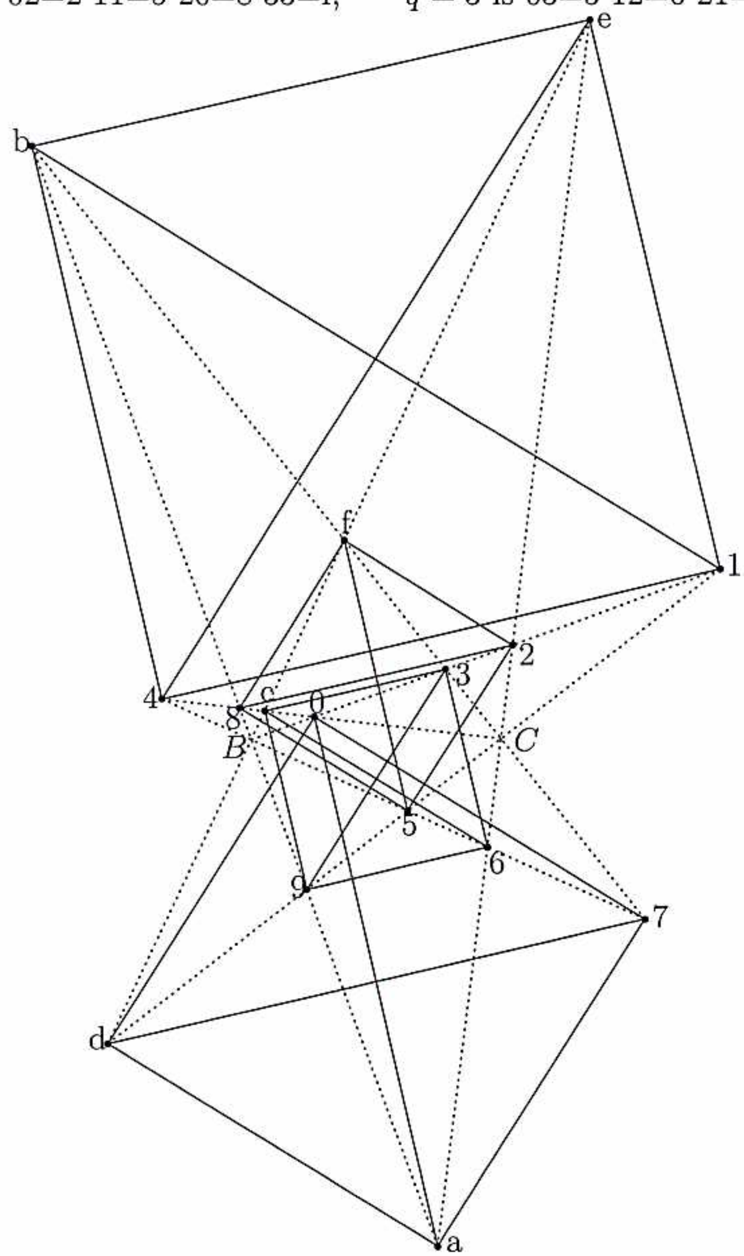

Figure 22: The four 4-gons formed from lighthouses with $n = 4$

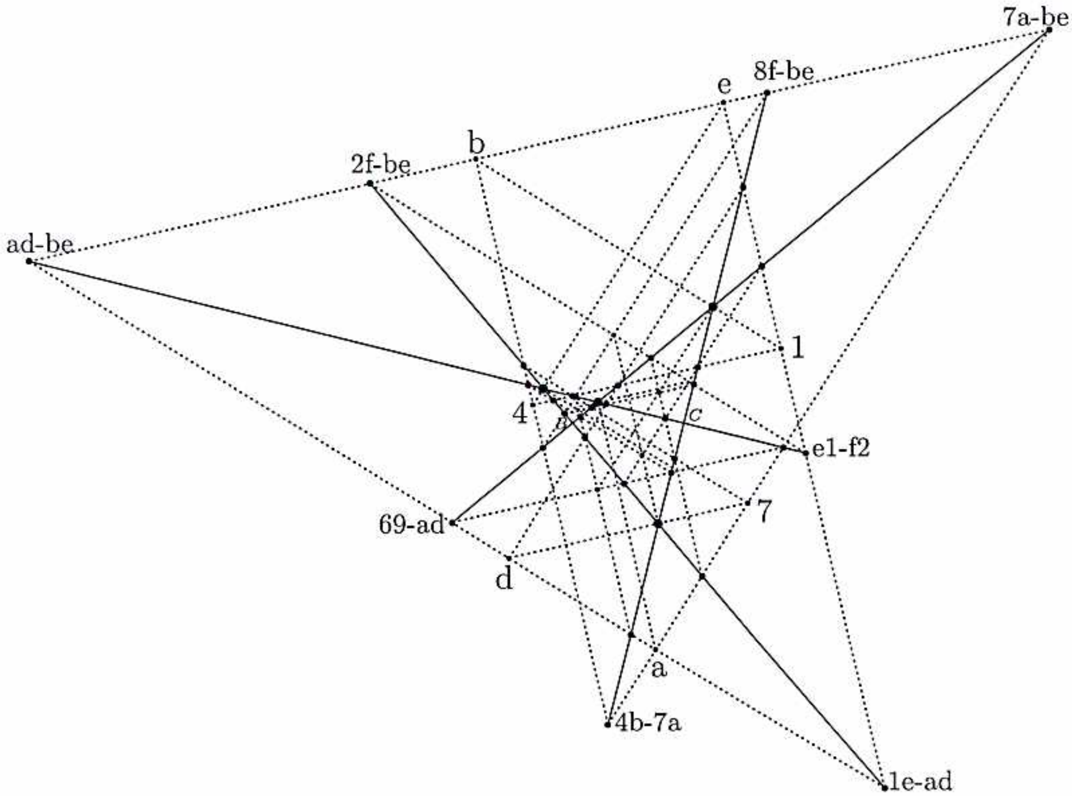

Figure 23: The duplicate beams through edge-intersections when $n = 4$

Edges of a quadrangle are denoted by concatenating the point-labels in these cyclic orders, starting at the "earliest" end. For instance, if $q = 2$ the edges of "length" 1 are 25, 58, 8f and f2, and the two of "length" 2 (the "diagonals") are 28 and 5f. Then the Lighthouse Duplication Theorem shows that the following columns of ten intersections lie on beams through B with phases 2β and $2\beta + \frac{\pi}{2}$ and through C with phases 2γ and $2\gamma + \frac{\pi}{2}$. The hexadecimal numbers have been converted back to base 4, to better show the pattern.

2β	$2\beta + \frac{\pi}{2}$	2γ	$2\gamma + \frac{\pi}{2}$
00.13–30.03	00.13–10.23	00.13–01.10	00.13–03.12
01.10–31.00	01.10–11.20	02.11–03.12	01.10–02.11
02.11–32.01	02.11–12.21	10.23–11.20	10.23–13.22
03.12–33.02	03.12–13.22	12.21–13.22	11.20–12.21
10.23–20.33	20.33–30.03	20.33–21.30	20.33–23.32
11.20–21.30	21.30–31.00	22.31–23.32	21.30–22.31
12.21–22.31	22.31–32.01	30.03–31.00	30.03–33.02
13.22–23.32	23.32–33.02	32.01–33.02	31.00–32.01
00.22–02.20	30.12–32.10	00.22–02.20	03.21–01.23
03.21–01.23	11.33–13.31	30.12–32.10	11.33–13.31

The first eight rows contain intersections of sides of quadrangles, which each meet a pair of adjacent sides of the two adjacent quadrangles. Here "side" means the join of two adjacent vertices, and "adjacent" means "neighboring" in the cyclic order 0123. The last two rows contain (duplicates of) the four intersections of diagonals with the opposite diagonal of the opposite quadrangle: the set of four orthocentric points 0a-28, 39-1b, c6-e4, 5f-7d, (the larger dots in Figure 23). These constitute an imbedding of the Lighthouse Theorem for $n = 2$. Compare the labelling of Figures 12 and 23.

Fig.12 labels	E	F	H	A	B	C	D
Fig.23 labels	B	C	0a-28	5f-7d	39-1b	c6-e4	—
Relabel as	B	C	I	I_A	I_B	I_C	A

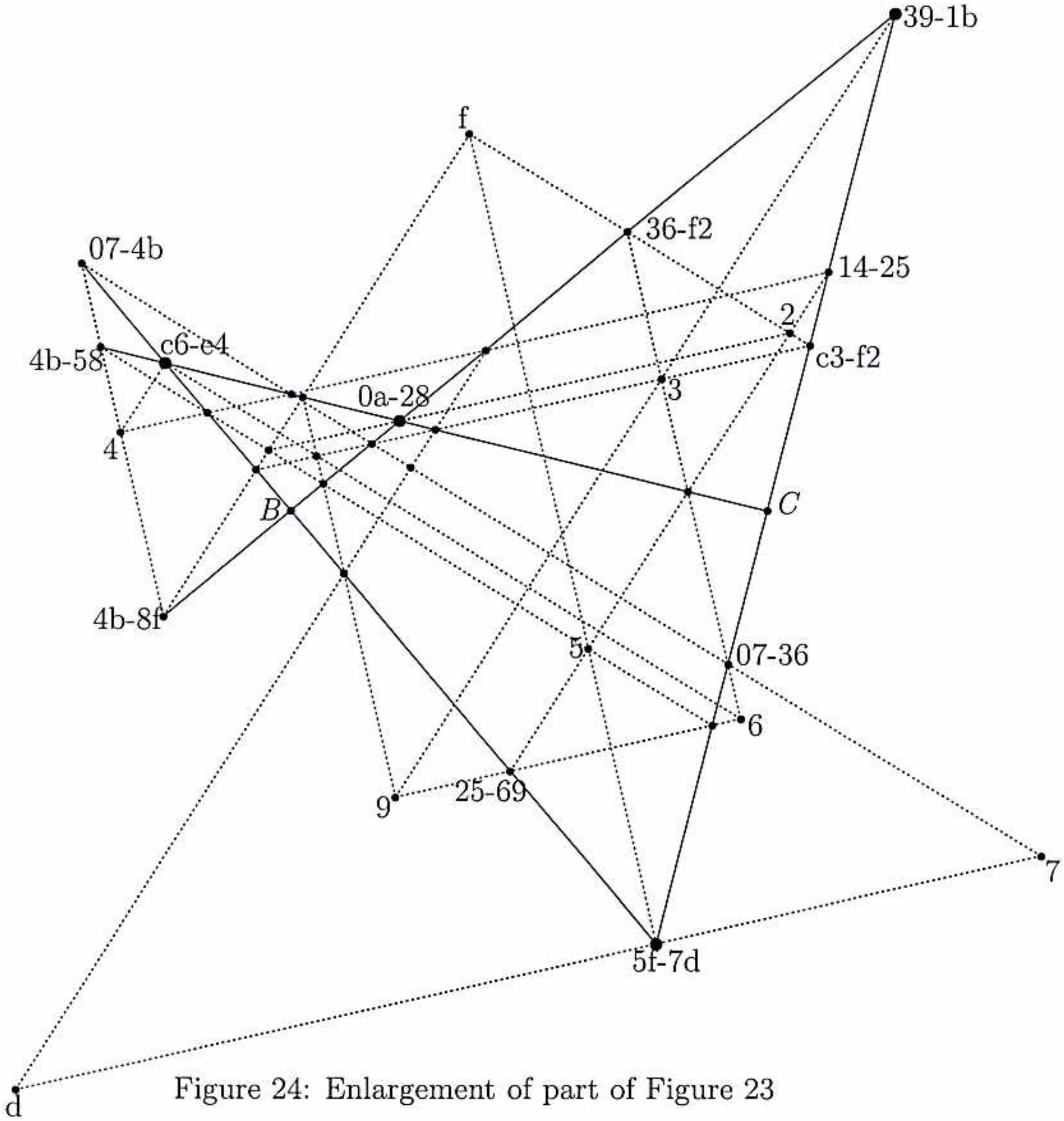

Figure 24: Enlargement of part of Figure 23

The label A, not shown in Figure 24, is for the intersection of the lines joining 5f-7d to 0a-28 and 39-1b to c6-e4. By the Lighthouse Duplication Theorem these perpendicular lines are the angle-bisectors of $\angle BAC$ and the beams BA and CA have phases 4β and 4γ.

The original beams are angle-bisectors of the four triangles BCI, where I is written collectively for the incentres of ABC. The original 16 points are the incentres of these triangles, given in the following table in the order 'incentre, excentre opposite I, excentre opposite B, excentre opposite C'.

Incentres of triangle BCI	(I = 0a-28)	are	0	a	2	8
Incentres of triangle BCI_A	(I_A = 5f-7d)	are	5	f	7	d
Incentres of triangle BCI_B	(I_B = 39-1b)	are	3	9	1	b
Incentres of triangle BCI_C	(I_C = c6-e4)	are	c	6	e	4

8. IS THERE A MALFATTI MIRACLE, TOO ?

The popular version of the Malfatti problem is to draw three circles, with each touching the two others and also touching two sides of a given triangle, say ABC. Steiner's construction begins by finding the point of contact of an incircle of triangle BIC with BC, where I is an incentre of ABC. In our case we would drop a perpendicular $0X$ from 0 (or from any of the other 15 points) onto BC. Then the two Malfatti circles that touch BC do so at points on either side of X at distance $d = r(1 + \tan\alpha)/2$, where r is the (appropriate) inradius of ABC and $\alpha + \beta + \gamma = \frac{\pi}{4}$. Is there any significance in the fact that W. E. Philip's proof of Morley's theorem [19, 25] and Steiner's construction for the Malfatti circles both start with the incircle whose centre is lit by beams of phases $B/n(= \beta)$ and $C/n(= \gamma)$?

We could set up 4-beam lighthouses at B and A with phases β and α, and two more at C and A with phases γ and α that would have angle-quadrisecting beams of triangle ABC and would give the 16 incentres of the four triangles BIA as well as those of the triangles CIA. These would guide us to the points of contact of the Malfatti circles with AB and with AC. Conway's "extraversion" [11] shows us that, as β and γ (and α, subject to the above condition) vary by multiples of $\frac{\pi}{4}$, we get no fewer than 32 triads of Malfatti circles (Steiner knew that there were that many). Note that a circle with centre X and radius d not only passes through the points of tangency of the two Malfatti circles with BC but also through their point of tangency with each other. More detail will be found in the forthcoming [6] and [11]. A parallel picture to Figure 11 has 16 hexagonal regions. Thanks again to Alex Fink, who identified the relevant group of order 32 as that with Hall-Senior (and Magma) number 34: the semidirect product of $C_4 \times C_4$ with C_2. It merits the name **Malfatti group**.

The 96 centres of the Malfatti circles lie 16 each on the four beams of the table between Figures 23 and 24, and 16 on each of the angle bisectors of BAC, but I haven't found companion lighthouses to locate the centres exactly. Are there houses that would shine light on the 32 radical centres of the triads, the points of concurrence of the internal common tangents to the three pairs of circles in a triad? There are always more questions than answers.

There are also four degenerate solutions to the Malfatti problem, consisting of an incircle taken with multiplicity three, and 24 semi-degenerate ones consisting of an repeated incircle (4 possibilities) and a third circle touching two sides (3 choices) and touching the incircle either proximally or distally. These solutions are in fact more relevant to Malfatti's original problem of maximizing the volume of three cylinders cut from a triangular prism. But that is another story [44].

The truth is out there. I leave $n = 5$ (which, with its $\binom{5}{2}^3 = 1000$ points, is particularly beautiful) and larger values of n to the reader. There's much, much more to be discovered. A glimpse of the richness for $n = 6$ is seen in Figures 17 to 19. Since 2 and 3 divide 6, we get three imbeddings of Lighthouse-2 and two imbeddings of Lighthouse-3. So there are two complete Morley configurations together with their interaction via $n = 2$ with its sets of orthocentric points. Are there further incidences among the 18 GF-circles?

9. HISTORY AND LITERATURE.

The Morley triangle theorem has an interesting history. It's a very Euclidean theorem, but it was discovered comparatively recently, by Frank Morley around the turn of the previous century, though he didn't publish anything about it until many years later [33, 34]. The first published proofs appeared in 1909: a trigonometric one by Satyanarayana [42] and a synthetic one by Naraniengar [35]. Naraniengar's proof, which hinges on a lemma, can be found in Coxeter & Greitzer [13]. It was rediscovered by Child [9] and by others. A direct proof seems to be elusive, though the one given here may qualify. There is an "inside-out" proof due to Bricard [7], expounded in [12]; it uses a lemma that is a corollary to the angle-bisector theorem. A similar proof is due to Bottema [4]. For other proofs, see [8, 21, 2]. There are few hints that there is more than one Morley triangle, but Honsberger [24, p. 98] asks the reader to show that Morley's theorem holds also in the case of the trisection of the exterior angles of a triangle. He gives a proof, for just one more Morley triangle, on pp. 163–164. This is quoted in [27], an article drawn to our attention by Coxeter, that is deserving of wider circulation.

There *is* a statement, rarely remembered or repeated, of the existence of 27 points lying six by six on nine (Morley) lines, at [25]. This is attributed to [19] where there are two proofs. The first is quite neat, attributed to W. E. Philip, and given in [25]. Figure 3 on [19, p.124] is very like our Figure 7. Glanville Taylor & Marr also give some attention to nonequilateral choices of three of the 27 Morley points. Johnson's next section, **§422**, gives the theorem that the incentres of the four triangles of a cyclic quadrangle form the vertices of a rectangle, and its generalization to the first sixteen of the Thrice Sixteen Theorem. This is attributed to Fuhrmann [17].

I hope that Rigby's paper [39] will eventually see the light of day. As he says there:

> ... I then consulted Morley's paper on the subject; this contains results that

rarely seem to be quoted, so I have included an account of aspects of the theorem that are apparently not generally known, especially the connection with cardioids inscribed in the triangle.

Honsberger [24] quotes Morley in this regard:

> If a variable cardioid touch the sides of a triangle the locus of its center, that is, the center of the circle on which the equal circles roll, is a set of 9 lines which are 3 by 3 parallel, the directions being those of the sides of an equilateral triangle. The meets of these lines correspond to double tangents; they are also the meets of certain trisectors of angles, internal and external, of the first triangle.

These are, of course, the 9 Morley lines of the triangle. Rigby generalizes this:
Theorem[39]. *The locus of the centres of epicycloids $C(m,n)$ touching a given triangle consists of $(n+m)^2$ lines (with an exception if the triangle is equilateral).*

Here the **epicycloid** $C(m,n)$ is the envelope of the line joining the points with polar coordinates $(1,m\theta)$ and $(1,n\theta)$, where $0 < m < n$ and $m \perp n$. The origin, O, is the **centre** of the epicycloid. E.g., $(m,n) = (1,2)$ is the **cardioid** and (1,3) is the **nephroid**. If m is allowed to be negative, we have a **hypocycloid**; for example $(-1,2)$ is a **deltoid** and $(-1,3)$ is an **astroid**. Better known are the connexions with the Steiner deltoid (the envelope of the Simson lines of the triangle) and the nine-point circle, their common tangents being parallel to the edges of the Morley triangles. See [12, 22, 32]. A referee has also pointed to [18, 29].
Query. If the lighthouses are rotating with different angular velocities, is there a connexion between the locus of the intersection of the beams with the epicycloids considered by Morley and Rigby?

The story began at about the time that I discovered a "lost notebook" of Conway, and we embarked on writing *The Triangle Book* [11]. Fortunately, in the interim, Steve Sigur has taken over, and, by the time you're reading this, the book will be available. Also, since the story began, news has been spreading, and many people, including D. J. Newman [36] and Alain Connes, have taken an interest in Morley's theorem. See `http://www.cut-the-knot.com/triangle/Morley`.

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