Weighted Lorentz Spaces and Hardy's Inequalities

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We are going to review the recent developments on the theory of **weighted Lorentz spaces** and **Hardy's inequalities**. We will see how functional properties of the spaces (like normability) are related to boundedness of the Hardy operator, and the study of these inequalities give rise to several classes of weight functions.

The following is a general scheme of the main topics of this talk:

NORMABILITY FUNCTION SPACES	$=$	BOUNDEDNESS MAXIMAL OPERATOR
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INEQUALITIES FOR WEIGHTS	$=$	BOUNDEDNESS HARDY OPERATOR

Hardy's Inequalities

POWER WEIGHTS

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^{p-r-1}\,dx\right)^{1/p} \le C\left(\int_0^\infty f^p(x)x^{p-r-1}\,dx\right)^{1/p}$$

In terms of the derivative, this inequality has a dual version:

$$\left(\int_0^\infty |F(x)|^p x^{-r-1}\,dx\right)^{1/p} \le C\left(\int_0^\infty |F'(x)|^p x^{p-r-1}\,dx\right)^{1/p}$$

which also admits an extension to several variables for suitable domains and measures:

$$\|F\|_p \le C\|\nabla F\|_p$$

These kind of results are also related to Poincaré's inequality and Sobolev's embeddings.

Hardy's Inequalities

POWER WEIGHTS

$$\boxed{\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^{p-r-1}\,dx\right)^{1/p} \le C\left(\int_0^\infty f^p(x)x^{p-r-1}\,dx\right)^{1/p}}$$

GENERAL WEIGHTS: Muckenhoupt, Maz'ya, Talenti, etc.

$$\boxed{\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p w(x)\,dx\right)^{1/p} \le C\left(\int_0^\infty f^p(x)w(x)\,dx\right)^{1/p}}$$

Hardy Operator: $\qquad Sf(x) = \dfrac{1}{x}\displaystyle\int_0^x f(y)\,dy$

$$\boxed{\left(\int_0^\infty \big(Sf(x)\big)^p w(x)\,dx\right)^{1/p} \le C\left(\int_0^\infty f^p(x)w(x)\,dx\right)^{1/p}}$$

$$\boxed{S : L^p(w) \longrightarrow L^p(w)}$$

Motivated by the Real Interpolation Theory and the so called Rearrangement Invariant spaces, it turns out that one needs to consider these inequalities only for decreasing functions.

$$\boxed{S : L^p_{\mathrm{dec}}(w) \longrightarrow L^p(w)}$$

To understand this argument, we need to define the nonincreasing rearrangement of a function:

Nonincreasing Rearrangement

The nonincreasing rearrangement of a function f is:

$$\boxed{f^*(t) = \inf\{s;\ \overbrace{|\{x; |f(x)| > s\}|}^{\lambda_f(s)} \le t\}}$$

Observe that, in fact, f^* is a decreasing function. Another way to define f^* is by means of the **Layer-Cake formula**: given a set E we consider $E^* = (0, |E|) \subset \mathbb{R}^+$. Since

$$f(x) = \int_0^\infty \chi_{\{y: f(y) > s\}}(x)\, ds$$

then

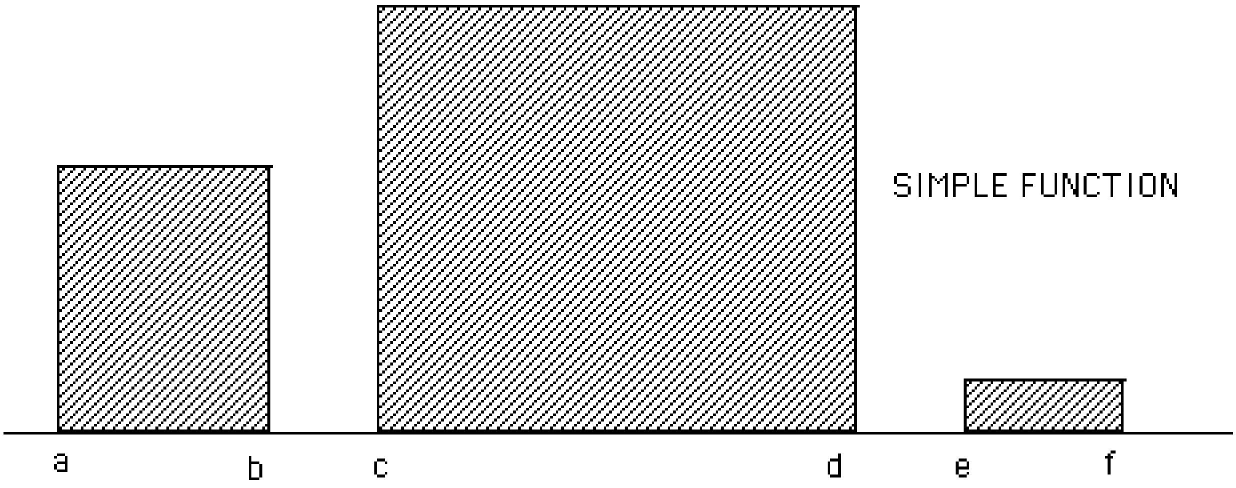
$$f^*(t) = \int_0^\infty \chi_{\{y: f(y) > s\}^*}(t)\, ds.$$

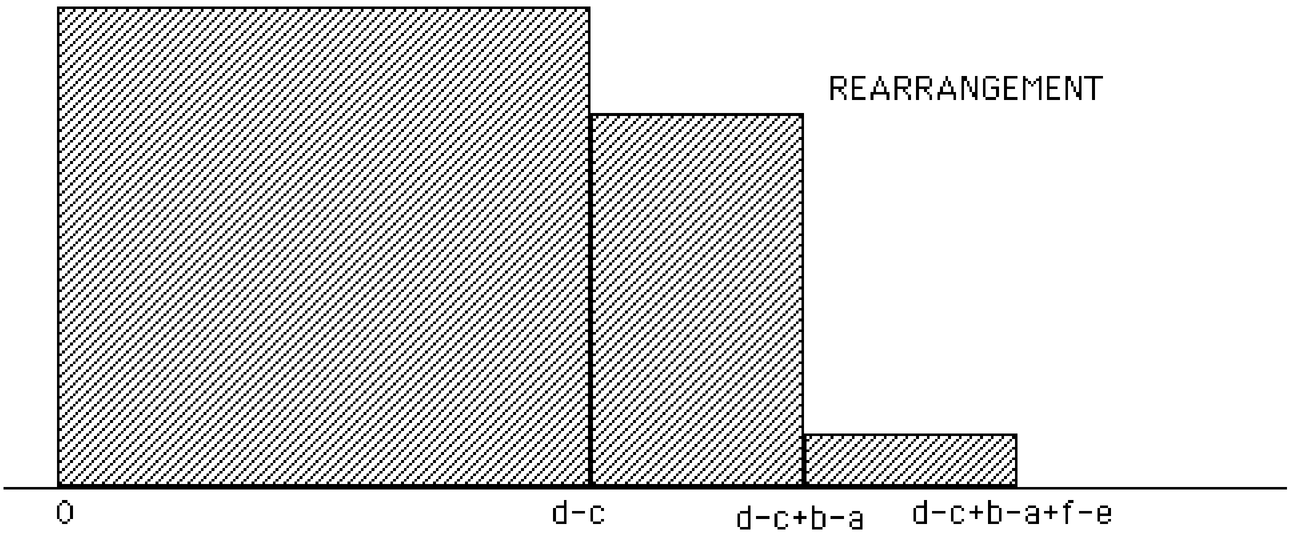
Hence, we can recover a function by means of its level sets: $\{f > t\}$. The spaces where the norm only depends on the measure of these sets are, essentially, the **Rearrangement Invariant spaces**. For example the Lebesgue L^p spaces:

$$\|f\|_{L^p} = \left(\int_0^\infty (f^*(t))^p\, dt \right)^{1/p}.$$

Other examples are the Lorentz spaces $L^{p,q}$ and the Lorentz-Zygmund spaces $L^p(\log L)^\alpha$.

For simple functions, it has a very geometric interpretation:





Weighted Lorentz Spaces (G.G. Lorentz, 1951)

We define the weighted Lorentz space $\Lambda^p(w)$ as

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty (f^*(x))^p w(x)\, dx \right)^{1/p} < +\infty.$$

Similarly, we define the **weak-type** version:

$$\|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} f^*(t) W^{1/p}(t) < +\infty,$$

with $W(t) = \int_0^t w(x)\, dx$.

We need to consider a variant of these spaces:

$$\|f\|_{\Gamma^p(w)} = \left(\int_0^\infty (f^{**}(x))^p w(x)\, dx \right)^{1/p} < +\infty,$$

where $f^{**}(t) = Sf^*(t) = \frac{1}{t} \int_0^t f^*(s)\, ds$ (**Maximal Function**).

$$\|f\|_{\Gamma^{p,\infty}(w)} = \sup_{t>0} f^{**}(t) W^{1/p}(t) < +\infty.$$

Observe that, for example, $\Gamma^1(1) = \{0\}$ and $\Gamma^{1,\infty}(1) = L^1$.

REMARKS

- If $w = 1$, $\Lambda^p(w) = L^p$ and if $w(t) = t^{p/q-1}$, $\Lambda^p(w) = L^{q,p}$.

- $\Lambda^p(w) \subset \Lambda^{p,\infty}(w)$ and $\Gamma^p(w) \subset \Gamma^{p,\infty}(w)$.

- Since $\boxed{f^* \leq f^{**}}$, $\Gamma^p(w) \subset \Lambda^p(w)$ and $\Gamma^{p,\infty}(w) \subset \Lambda^{p,\infty}(w)$.

- $\| \cdot \|_{\Lambda^p(w)}$ is a norm $\Leftrightarrow$ w is decreasing and $p \geq 1$ (**Lorentz**). One of the **main questions** on the theory of Lorentz spaces is to characterize when the space itself is a Banach space (which may happen even if w is not a decreasing function).

- $\| \cdot \|_{\Lambda^p(w)}$ is a quasi-norm $\Leftrightarrow$ $W(2t) \leq CW(t)$, i.e. $W \in \Delta_2$. (**Carro-Soria**).

- Since $\boxed{(f+g)^{**} \leq f^{**} + g^{**}}$, $\Gamma^p(w)$ is a Banach space ($p \geq 1$) and similarly for $\Gamma^{p,\infty}(w)$ ($p > 0$):

  This is proved by showing that

$$f^{**}(t) = \sup_{|E|=t} \frac{1}{|E|} \int_E f(x)\, dx.$$

  In both cases $\| \cdot \|_{\Gamma^p(w)}$ and also $\| \cdot \|_{\Gamma^{p,\infty}(w)}$ satisfy Minkowski's inequality and hence they are always a norm.

Hardy's Inequalities (continued)

We return to the study of Hardy's inequality for nonincreasing functions:

$$\boxed{S : L^p_{\text{dec}}(w) \longrightarrow L^p(w)}$$

It is known that if we consider the **Hardy-Littlewood maximal function**

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy$$

then $(Mf)^* \approx f^{**}$ (**Riesz, Wiener, Herz, Bennett and Sharpley**), and hence:

$$\boxed{M : \Lambda^p(w) \to \Lambda^p(w) \Leftrightarrow S : L^p_{\text{dec}}(w) \to L^p(w)}$$

$$\boxed{w \in B_p : \qquad r^p \int_r^\infty \frac{w(t)}{t^p} dt \leq C \int_0^r w(t) dt}$$

Ariño-Muckenhoupt (1990)

$$\boxed{M : \Lambda^p(w) \to \Lambda^p(w) \Leftrightarrow S : L^p_{\text{dec}}(w) \to L^p(w) \Leftrightarrow w \in B_p}$$

Neugebauer (1991) If $p > 1$:

$$\boxed{M : \Lambda^p(w) \to \Lambda^{p,\infty}(w) \Leftrightarrow S : L^p_{\text{dec}}(w) \to L^{p,\infty}(w) \Leftrightarrow w \in B_p}$$

$$w \in R_p : \qquad \frac{W(t)}{t^p} \le C \frac{W(s)}{s^p}, \quad s \le t$$

Carro-García del Amo-Soria (1993, 1996) If $p \le 1$:

$$M : \Lambda^p(w) \to \Lambda^{p,\infty}(w) \Leftrightarrow S : L^p_{\mathrm{dec}}(w) \to L^{p,\infty}(w) \Leftrightarrow w \in R_p$$

Soria (1998)

$$M : \Lambda^{p,\infty}(w) \to \Lambda^{p,\infty}(w) \Leftrightarrow S : L^{p,\infty}_{\mathrm{dec}}(w) \to L^{p,\infty}(w) \Leftrightarrow w \in B_p$$

???		BOUNDEDNESS MAXIMAL OPERATOR
		‖
INEQUALITIES FOR WEIGHTS	═══	BOUNDEDNESS HARDY OPERATOR

Embeddings and Normability

Question: When is a Lorentz space a Banach space?

Equivalently: When can we find a norm which is comparable to the functional defining the Lorentz space? As we shall see, this is **most of the times** equivalent to saying that we can take the maximal function f^{**} in place of f^*; i.e., when $\Lambda = \Gamma$.

Ariño–Muckenhoupt, Sawyer (1990)

$$p > 1 : \Lambda^p(w) \text{ normable} \Leftrightarrow \Lambda^p(w) = \Gamma^p(w) \Leftrightarrow w \in B_p$$

Carro-García del Amo-Soria (1996)

$$\Lambda^1(w) \text{ normable} \Leftrightarrow \Lambda^1(w) \subset \Gamma^{1,\infty}(w) \Leftrightarrow w \in R_1$$

Soria (1998)

$$\Lambda^{p,\infty}(w) \text{ normable} \Leftrightarrow \Lambda^{p,\infty}(w) = \Gamma^{p,\infty}(w) \Leftrightarrow w \in B_p$$

We observe that for the space $\Lambda^1(w)$ we do not have a description of the equivalent norm in terms of f^{**}, contrary to the other cases (i.e., in case of normability, **is $\Lambda^1(w)$ some kind of Γ space?**).

The case $\Lambda^1(w)$

From now on we assume that $\Lambda^1(w)$ is normable (i.e., $w \in R_1$), and without loss of generality, we can also suppose that w is a decreasing weight. We observe that we have the embeddings

$$\boxed{\Gamma^1(w) \subset \Lambda^1(w) \subset \Gamma^{1,\infty}(w)}$$

and that the endpoints are Γ-spaces. When is it the case that either $\Lambda^1(w) = \Gamma^1(w)$ or $\Lambda^1(w) = \Gamma^{1,\infty}(w)$?

- Equality $\Lambda^1(w) = \Gamma^1(w)$ is equivalent to $w \in B_1$.
- On the other hand, $\Lambda^1(w) = \Gamma^{1,\infty}(w)$ if and only if one of the following conditions holds:
  (1) $0 = w(\infty) < w(0) < \infty$ and $w \in L^1$,
  (2) $0 < w(\infty) \leq w(0) < \infty$ (i.e., $w \approx 1$).
  Carro-Pick-Soria-Stepanov (2001).

But, if we consider the weight $w(t) = (1 - \log t)\chi_{(0,1)}(t)$, then

$$\{0\} \neq \Gamma^1(w) \subsetneq \Lambda^1(w) \subsetneq \Gamma^{1,\infty}(w).$$

$$\boxed{\text{Is there any } \Gamma \text{ space between } \Gamma^1(w) \text{ and } \Gamma^{1,\infty}(w) \text{ which is } \Lambda^1(w)\text{?}}$$

First Try

[J. Martín, Soria (2002)]

There is a natural scale of Γ spaces between $\Gamma^1(w)$ and $\Gamma^{1,\infty}(w)$:

$$\Gamma^{1,q}(w) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^+; \|f\|_{\Gamma^{1,q}(w)} < \infty\},$$

where

$$\|f\|_{\Gamma^{1,q}(w)} = \left(\int_0^\infty (f^{**}(t))^q (W(t))^{q-1} w(t)\, dt \right)^{1/q}.$$

It is easy to show that, if $1 \leq p \leq q \leq \infty$:

$$\boxed{\Gamma^1(w) \subset \Gamma^{1,p}(w) \subset \Gamma^{1,q}(w) \subset \Gamma^{1,\infty}(w)}$$

We want to know if there is any hope to finding $1 < p < \infty$ for which $\Lambda^1(w) = \Gamma^{1,q}(w)$. We can now prove the following:

$$\boxed{\Lambda^1(w) \subset \Gamma^{1,q}(w),\ 1 \leq q < \infty \text{ if and only if } \Lambda^1(w) = \Gamma^1(w)}$$

Hence the answer to $\Lambda^1(w) = \Gamma^{1,q}(w)$ is **NO** for a general weight.

Second Try

We observe that $\Gamma^{1,q}(w) = \Gamma^q(W_q)$, where $W_q(t) = W^{q-1}(t)w(t)$, and we have already shown that for this weight W_q things do not work. We check now with general weights:

$$\boxed{\Lambda^1(w) = \Gamma^q(v)}$$

This time the answer is ... **NO** again:

If $1 < q < \infty$:

$$\boxed{\Lambda^1(w) = \Gamma^q(v) \Leftrightarrow \Lambda^1(w) = \Gamma^{1,\infty}(w),\ w(\infty) = 0 \text{ and } w \in L^1}$$

But we have left open the case $q = 1$ ...

Third Try

Finally we are able to solve the problem of identifying $\Lambda^1(w)$ as a Γ space **in all cases**:

Theorem

If w is a decreasing weight, then:

(i) $w(\infty) = 0$ *if and only if there exists a weight v such that*

$$\boxed{\Lambda^1(w) = \Gamma^1(v)}$$

(ii) $w(\infty) > 0$ *and* $w \in L^\infty$ *(i.e.,* $w \approx 1$*) if and only if*

$$\boxed{\Lambda^1(w) = L^1}$$

(iii) $w(\infty) > 0$ *and* $w \notin L^\infty$ *if and only if there exists a weight v such that*

$$\boxed{\Lambda^1(w) = \Gamma^1(v) \cap L^1}$$

$L^1 \nsubseteq \Gamma^1(v)$, *and* $\Gamma^1(v) \nsubseteq L^1$.

A good thing about the previous result is that v can be made explicit. For example, if we recall the case $w(t) = (1 - \log t)\chi_{(0,1)}(t)$, for which

$$\{0\} \neq \Gamma^1(w) \subsetneq \Lambda^1(w) \subsetneq \Gamma^{1,\infty}(w).$$

then $\Lambda^1(w) = \Gamma^1(v)$, where

$$\boxed{v(t) = \begin{cases} \dfrac{9}{4}, & 0 < t \leq \dfrac{1}{4} \\ \dfrac{1 - 7t^2 + 6\log 4t}{4t^2}, & \dfrac{1}{4} < t \leq \dfrac{1}{2} \\ \dfrac{-1 + t^2 - 6\log t}{4t^2}, & \dfrac{1}{2} < t \leq 1 \\ 0, & 1 < t \end{cases}}$$

Idea of the Proof

We first show that $\Lambda^1(w) = \Gamma^1(v)$ is equivalent to

$$\frac{1}{r}\int_0^r w(s)\,ds \approx \frac{1}{r}\int_0^r v(s)\,ds + \int_r^\infty \frac{v(s)}{s}\,ds = S(S^*v)(r),$$

where S^* is the adjoint of S:

$$S^*f(r) = \int_r^\infty \frac{f(s)}{s}\,ds.$$

If we now assume that $w(\infty) = \rho > 0$, since $S(S^*v)$ is a decreasing function, the following limit exists:

$$\lim_{r\to\infty} \frac{1}{r}\int_0^r v(s)\,ds = \alpha \geq 0.$$

Since $S(S^*v) = S^*(Sv)$, then $\alpha = 0$. But, on the other hand

$$\rho = \lim_{r\to\infty} \frac{1}{r}\int_0^r w(s)\,ds \leq C \lim_{r\to\infty} \frac{1}{r}\int_0^r v(s)\,ds,$$

which gives a contradiction since $\alpha \geq \rho/C > 0$.

The converse is easy.

(ii) is trivial.

To show (iii), we observe that $\Lambda^1(w) \subset L^1$ is equivalent to $w(\infty) > 0$ and hence, if we define $u(t) = w(t) - w(\infty)$, by (i) we can find v such that $\Lambda^1(u) = \Gamma^1(v)$. Hence $\Lambda^1(w) = \Gamma^1(v) \cap L^1$. If $L^1 \subset \Gamma^1(v)$ then $\Lambda^1(w) = L^1$ which contradicts the fact that $w \notin L^\infty$. Similarly, if $\Gamma^1(v) \subset L^1$ then

$$t \leq C\left(V(t) + t\int_t^\infty \frac{v(s)}{s}\,ds\right),$$

and hence $v(\infty) > 0$ and $\Gamma^1(v) = \{0\}$. The rest is easy. $\square$