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Four functions and sixteen Eisenstein series arising from Ramanujan's ${}_1\psi_1$ function

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In this paper we present sixteen series and show how they arise out of Ramanujan's ${}_1\psi_1$ function. First, we consider four special cases of Ramanujan's ${}_1\psi_1$ summation formula, which we shall call f_0, f_1, f_2, and f_3. Then we obtain another twelve functions by applying three different cases to f_0, f_1, f_2, and f_3. From these sixteen functions we obtain sixteen Eisenstein series and their corresponding q-series expansions. Several of the q-series expansions can be found in (1), (3), (7)-(12). Glaisher (5) and Zucker (13),(14) presented the complete set of sixteen series. The significant point is that four functions and sixteen Eisentein series all originate from one source, namely Ramanujan's ${}_1\psi_1$ function.

1 Introduction

Ramanujan gave the famous formula (1.1), now called the Ramanujan ${}_1\psi_1$ summation formula ((11), Chapter 16, Entry 17). G. H. Hardy ((6), p. 222) described it as "a remarkable formula with many parameters". There are a number of proofs for the equation (1.1), for example (2), (3), and (15). Ramanujan's ${}_1\psi_1$ summation formula is

$$\begin{aligned}
&\prod_{k=1}^{\infty} \frac{\left(1+zq^{2k-1}\right)\left(1+q^{2k-1}/z\right)\left(1-q^{2k}\right)\left(1-\alpha\beta q^{2k}\right)}{\left(1+\alpha zq^{2k-1}\right)\left(1+\beta q^{2k-1}/z\right)\left(1-\alpha q^{2k}\right)\left(1-\beta q^{2k}\right)} \\
= \;& 1+\left\{\frac{1-\alpha}{1-\beta q^2} qz + \frac{1-\beta}{1-\alpha q^2}\frac{q}{z}\right\} \\
& + \left\{\frac{(1-\alpha)\left(q^2-\alpha\right)}{\left(1-\beta q^2\right)\left(1-\beta q^4\right)}(qz)^2 + \frac{(1-\beta)\left(q^2-\beta\right)}{\left(1-\alpha q^2\right)\left(1-\alpha q^4\right)}\left(\frac{q}{z}\right)^2\right\} \\
& + \left\{\frac{(1-\alpha)\left(q^2-\alpha\right)\left(q^4-\alpha\right)}{\left(1-\beta q^2\right)\left(1-\beta q^4\right)\left(1-\beta q^6\right)}(qz)^3\right. \\
& \left. + \frac{(1-\beta)\left(q^2-\beta\right)\left(q^4-\beta\right)}{\left(1-\alpha q^2\right)\left(1-\alpha q^4\right)\left(1-\alpha q^6\right)}\left(\frac{q}{z}\right)^3\right\} + \ldots,
\end{aligned} \tag{1.1}$$

where $|\beta q| < |z| < 1/|\alpha q|$ and $q = e^{i\pi\tau}, Im(\tau) > 0$, and so $|q| < 1$. The Jordan-Kronecker function is a special case of the series on the right hand side of (1.1) and is defined as follows.

((15), p. 37) Let

$$F(a,b) = \sum_{n=-\infty}^{\infty} \frac{b^n}{1-aq^{2n}}, \tag{1.2}$$

where $|q^2| < |b| < 1$ and $a \neq q^{2k}$, $k = 0, \pm 1, \pm 2, ...$ Taking $\alpha = 1/a$, $\beta = a, z = -ab/q$, and divide by $1 - a$ in (1.1) gives

$$F(a,b) = \prod_{n=1}^{\infty} \frac{\left(1 - abq^{2n-2}\right)\left(1 - \frac{q^{2n}}{ab}\right)\left(1 - q^{2n}\right)^2}{\left(1 - aq^{2n-2}\right)\left(1 - \frac{q^{2n}}{a}\right)\left(1 - bq^{2n-2}\right)\left(1 - \frac{q^{2n}}{b}\right)}. \tag{1.3}$$

2 Sixteen series

Let

$$f_0(\theta) = \frac{1}{2}\cot\frac{\theta}{2} + 2\sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \sin m\theta = \frac{1}{i} F\left(e^v, e^{i\theta}\right)\left[v^0\right], \tag{2.4}$$

$$f_1(\theta) = \frac{1}{2}\cot\frac{\theta}{2} - 2\sum_{m=1}^{\infty} \frac{q^{2m}}{1 + q^{2m}} \sin m\theta = \frac{1}{i} F\left(e^{i\pi}, e^{i\theta}\right), \tag{2.5}$$

$$f_2(\theta) = \frac{1}{2}\csc\frac{\theta}{2} + 2\sum_{m=0}^{\infty} \frac{q^{2m+1}}{1 - q^{2m+1}} \sin\left(m + \frac{1}{2}\right)\theta = \frac{e^{\frac{i\theta}{2}}}{i} F\left(e^{i\pi\tau}, e^{i\theta}\right), \tag{2.6}$$

$$f_3(\theta) = \frac{1}{2}\csc\frac{\theta}{2} - 2\sum_{m=0}^{\infty} \frac{q^{2m+1}}{1 + q^{2m+1}} \sin\left(m + \frac{1}{2}\right)\theta = \frac{e^{\frac{i\theta}{2}}}{i} F\left(e^{i\pi + i\pi\tau}, e^{i\theta}\right). \tag{2.7}$$

Here $F\left(e^v, e^{i\theta}\right)\left[v^0\right]$ means let $a = e^v$ in (1.2), then expand in powers of v and extract the constant term. Equations $(2.5) - (2.7)$ are doubly periodic and are called elliptic functions; equation (2.4) is not doubly periodic and is called the zeta function. These series all converge for $-Im\,(2\pi\tau) < Im\theta < Im\,(2\pi\tau)$. Replacing θ with $\theta + \pi, \theta + \pi\tau$, and $\theta + \pi + \pi\tau$ in $(2.4) - (2.7)$, respectively, and simplifying, gives

$$\begin{aligned}
f_0(\theta + \pi) &= -\frac{1}{2}\tan\frac{\theta}{2} + 2\sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{1 - q^{2m}} \sin m\theta, \\
f_0(\theta + \pi\tau) &= \frac{1}{i}\left[\frac{1}{2} - 2\sum_{m=1}^{\infty} \frac{q^m}{1 - q^{2m}} \cos m\theta\right], \\
f_0(\theta + \pi + \pi\tau) &= \frac{1}{i}\left[\frac{1}{2} - 2\sum_{m=1}^{\infty} \frac{(-1)^m q^m}{1 - q^{2m}} \cos m\theta\right], \\
f_1(\theta + \pi) &= -\frac{1}{2}\tan\frac{\theta}{2} - 2\sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{1 + q^{2m}} \sin m\theta, \\
f_1(\theta + \pi\tau) &= \frac{1}{i}\left[\frac{1}{2} + 2\sum_{m=1}^{\infty} \frac{q^m}{1 + q^{2m}} \cos m\theta\right], \\
f_1(\theta + \pi + \pi\tau) &= \frac{1}{i}\left[\frac{1}{2} + 2\sum_{m=1}^{\infty} \frac{(-1)^m q^m}{1 + q^{2m}} \cos m\theta\right],
\end{aligned}$$

$$\begin{aligned}
f_2(\theta+\pi) &= \frac{1}{2}\sec\frac{\theta}{2} + 2\sum_{m=0}^{\infty}\frac{(-1)^m q^{2m+1}}{1-q^{2m+1}}\cos\left(m+\frac{1}{2}\right)\theta, \\
f_2(\theta+\pi\tau) &= 2\sum_{m=0}^{\infty}\frac{q^{m+\frac{1}{2}}}{1-q^{2m+1}}\sin\left(m+\frac{1}{2}\right)\theta, \\
f_2(\theta+\pi+\pi\tau) &= 2\sum_{m=0}^{\infty}\frac{(-1)^m q^{m+\frac{1}{2}}}{1-q^{2m+1}}\cos\left(m+\frac{1}{2}\right)\theta, \\
f_3(\theta+\pi) &= \frac{1}{2}\sec\frac{\theta}{2} - 2\sum_{m=0}^{\infty}\frac{(-1)^m q^{2m+1}}{1+q^{2m+1}}\cos\left(m+\frac{1}{2}\right)\theta, \\
f_3(\theta+\pi\tau) &= -2i\sum_{m=0}^{\infty}\frac{q^{m+\frac{1}{2}}}{1+q^{2m+1}}\cos\left(m+\frac{1}{2}\right)\theta, \\
f_3(\theta+\pi+\pi\tau) &= 2i\sum_{m=0}^{\infty}\frac{(-1)^m q^{m+\frac{1}{2}}}{1+q^{2m+1}}\sin\left(m+\frac{1}{2}\right)\theta.
\end{aligned} \tag{2.8}$$

3 Sixteen Eisenstein series

The method used in this section is similar to the one used by Glaisher (5) with the exception that our notation is simpler. We can rewrite the results of (2.4) – (2.7) in terms of trigonometic functions, namely the cotangent and the cosecant as follows

$$\begin{aligned}
\frac{1}{2}\sum_{n=-\infty}^{\infty}\cot\left(\frac{\theta}{2}+n\pi\tau\right) &= \frac{1}{2}\cot\frac{\theta}{2} + 2\sum_{m=1}^{\infty}\frac{q^{2m}}{1-q^{2m}}\sin m\theta = f_0(\theta), \\
\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\cot\left(\frac{\theta}{2}+n\pi\tau\right) &= \frac{1}{2}\cot\frac{\theta}{2} - 2\sum_{m=1}^{\infty}\frac{q^{2m}}{1+q^{2m}}\sin m\theta = f_1(\theta), \\
\frac{1}{2}\sum_{n=-\infty}^{\infty}\csc\left(\frac{\theta}{2}+n\pi\tau\right) &= \frac{1}{2}\csc\frac{\theta}{2} + 2\sum_{m=0}^{\infty}\frac{q^{2m+1}}{1-q^{2m+1}}\sin\left(m+\frac{1}{2}\right)\theta = f_2(\theta), \\
\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\csc\left(\frac{\theta}{2}+n\pi\tau\right) &= \frac{1}{2}\csc\frac{\theta}{2} - 2\sum_{m=0}^{\infty}\frac{q^{2m+1}}{1+q^{2m+1}}\sin\left(m+\frac{1}{2}\right)\theta = f_3(\theta).
\end{aligned} \tag{3.9}$$

The Eisenstein series are defined by ((4), `p.376`),

$$E_{2k}(\tau) = \sum_{(m,n)\neq(0,0)}\frac{1}{(m+n\tau)^{2k}},$$

where $k = 2, 3, ...$ Expanding the left hand side into partial fractions, then expanding both sides of the expression (3.9) in ascending powers of θ, and equating the coefficients of θ^{2t-1}, we find

$$\begin{aligned}
\sum_{(m,n)\neq(0,0)} \frac{1}{(2m+2n\tau)^{2t}} &= (-1)^{t-1}\left[\frac{B_{2t}}{2t} - 2\sum_{m=1}^{\infty}\frac{m^{2t-1}q^{2m}}{1-q^{2m}}\right]\frac{\pi^{2t}}{(2t-1)!},\\
\sum_{(m,n)\neq(0,0)} \frac{(-1)^n}{(2m+2n\tau)^{2t}} &= (-1)^{t-1}\left[\frac{B_{2t}}{2t} + 2\sum_{m=1}^{\infty}\frac{m^{2t-1}q^{2m}}{1+q^{2m}}\right]\frac{\pi^{2t}}{(2t-1)!},\\
\sum_{(m,n)\neq(0,0)} \frac{(-1)^m}{(2m+2n\tau)^{2t}} &= \frac{(-1)^t}{2^{2t-1}}\left[\frac{\left(2^{2t-1}-1\right)B_{2t}}{2t} + 2\sum_{m=0}^{\infty}\frac{(2m+1)^{2t-1}q^{2m+1}}{1-q^{2m+1}}\right]\frac{\pi^{2t}}{(2t-1)!},\\
\sum_{(m,n)\neq(0,0)} \frac{(-1)^{m+n}}{(2m+2n\tau)^{2t}} &= \frac{(-1)^t}{2^{2t-1}}\left[\frac{\left(2^{2t-1}-1\right)B_{2t}}{2t} - 2\sum_{m=0}^{\infty}\frac{(2m+1)^{2t-1}q^{2m+1}}{1+q^{2m+1}}\right]\frac{\pi^{2t}}{(2t-1)!}.
\end{aligned}$$

Similarly, we can get another set of twelve results by applying the same technique to (2.8) :

$$\begin{aligned}
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{((2m-1)+2n\tau)^{2t}} &= (-1)^{t-1}\left[\frac{\left(2^{2t}-1\right)B_{2t}}{2t} - 2\sum_{m=1}^{\infty}\frac{(-1)^m m^{2t-1}q^{2m}}{1-q^{2m}}\right]\frac{\pi^{2t}}{(2t-1)!},\\
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{((2m-1)+2n\tau)^{2t}} &= (-1)^{t-1}\left[\frac{\left(2^{2t}-1\right)B_{2t}}{2t} + 2\sum_{m=1}^{\infty}\frac{(-1)^m m^{2t-1}q^{2m}}{1+q^{2m}}\right]\frac{\pi^{2t}}{(2t-1)!},\\
\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{((2m-1)+2n\tau)^{2t+1}} &= \frac{(-1)^{t-1}}{2^{2t+1}}\left[E_{2t} - 4\sum_{m=1}^{\infty}\frac{(-1)^m (2m-1)^{2t} q^{2m-1}}{1-q^{2m-1}}\right]\frac{\pi^{2t+1}}{(2t)!},\\
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^{m+n}}{((2m-1)+2n\tau)^{2t+1}} &= \frac{(-1)^{t-1}}{2^{2t+1}}\left[E_{2t} + 4\sum_{m=1}^{\infty}\frac{(-1)^m (2m-1)^{2t} q^{2m-1}}{1+q^{2m-1}}\right]\frac{\pi^{2t+1}}{(2t)!};
\end{aligned}$$

$$\begin{aligned}
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(2m+(2n-1)\tau)^{2t}} &= \left[2(-1)^t\sum_{m=1}^{\infty}\frac{m^{2t-1}q^m}{1-q^{2m}}\right]\frac{\pi^{2t}}{(2t-1)!},\\
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2m+(2n-1)\tau)^{2t+1}} &= \left[2i(-1)^t\sum_{m=1}^{\infty}\frac{m^{2t}q^m}{1+q^{2m}}\right]\frac{\pi^{2t+1}}{(2t)!},\\
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^m}{(2m+(2n-1)\tau)^{2t}} &= \left[\frac{(-1)^t}{2^{2t-2}}\sum_{m=1}^{\infty}\frac{(2m-1)^{2t-1}q^{m-\frac{1}{2}}}{1-q^{2m-1}}\right]\frac{\pi^{2t}}{(2t-1)!},\\
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^{m+n}}{(2m+(2n-1)\tau)^{2t+1}} &= \left[\frac{i(-1)^t}{2^{2t-1}}\sum_{m=1}^{\infty}\frac{(2m-1)^{2t}q^{m-\frac{1}{2}}}{1+q^{2m-1}}\right]\frac{\pi^{2t+1}}{(2t)!};
\end{aligned}$$

$$\begin{aligned}
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{((2m-1)+(2n-1)\tau)^{2t}} &= \left[2(-1)^t\sum_{m=1}^{\infty}\frac{(-1)^m m^{2t-1}q^m}{1-q^{2m}}\right]\frac{\pi^{2t}}{(2t-1)!},\\
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{((2m-1)+(2n-1)\tau)^{2t+1}} &= \left[2i(-1)^t\sum_{m=1}^{\infty}\frac{(-1)^m m^{2t}q^m}{1+q^{2m}}\right]\frac{\pi^{2t+1}}{(2t)!},\\
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^m}{((2m-1)+(2n-1)\tau)^{2t+1}} &= \left[\frac{(-1)^{t-1}}{2^{2t-1}}\sum_{m=1}^{\infty}\frac{(-1)^m(2m-1)^{2t}q^{m-\frac{1}{2}}}{1-q^{2m-1}}\right]\frac{\pi^{2t+1}}{(2t)!},\\
\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^{m+n}}{((2m-1)+(2n-1)\tau)^{2t}} &= \left[\frac{i(-1)^t}{2^{2t-2}}\sum_{m=1}^{\infty}\frac{(-1)^m(2m-1)^{2t-1}q^{m-\frac{1}{2}}}{1+q^{2m-1}}\right]\frac{\pi^{2t}}{(2t-1)!}.
\end{aligned}$$

It is easy to see the sixteen series form one system. The numerator is either $1, (-1)^n, (-1)^m$, or $(-1)^{m+n}$; and the denominator contains either both even numbers, one odd and one even number, or both odd numbers. Zucker (13) mentioned without proof that the sixteen series can only be found in closed form for either even or odd, but never for both. We now have a better understanding of this from the way we derived the sixteen series.

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