Sister Celine's Methods, Theorems, and Demonstrations

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Hypergeometric Functions

First lets recall some well-known definitions for hypergeometric series. The pochhammer symbol Z(n) is defined by $(Z)_n = \prod_{m=0}^{n-1} (z + m)$. $\Sigma_n c_n$ is a hypergeometric series if $c_0 = 1$ and $(c_{n+1})/c_n = R(n)$, where R(n) is a rational function of n. R(n) can also be written as P(n)/Q(n) where P and Q are polynomials. Hypergeometric series are always infinite and very important. This can be seen by recognizing that many of the fundamental functions of analysis are hypergeometric functions: Binomial, Exponential, Logarithmic, and Trigonometric. That is they create a hypergeometric series. You can find out if a series is hypergeometric by using the following procedure.

1. Given a series $\Sigma_k t_k$. Shift the summation index k so that the sum starts at k = 0 with a nonzero term. Extract the term corresponding to k = 0 as a common factor so that the first term of the sum will be 1.
2. Simplify the ratio t_{k+1}/t_k to bring it into the form P(k)/Q(k), where P, Q are polynomials. (If you can't do this, the series is not hypergeometric.)

3. Completely factor the polynomials P and Q into linear factors, and write the term ratio in the form $$\frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x$$
   - The factor k+1 must be there, so if it is not then multiply by (k+1)/(k+1).
4. You have now identified the input series. It is the hypergeometric series in the form: $${}_pF_q\left[\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix}; x\right]$$ (Petkovsek, 36).

The following proof will show that $\Sigma_n c_n = c_0\ {}_pF_q$. $\quad \frac{c_{n+1}}{c_n} = \frac{(n+a_1)...(n+a_p)x}{(n+b_1)...(n+B_q)(n+1)}$

Given: For a hypergeometric series,

$${}_pF_q\left[\begin{matrix} a_1...a_p; x \\ b_1...b_q \end{matrix}\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n} \frac{x^n}{n!}$$

Let $\Sigma_n c_n$ be a hypergeometric series. Show $\Sigma_n c_n = c_0\ {}_pF_q$

Consider the terms c_n and recall that for a hypergeometric series, $c_{(n+1)}/c_n = R(n) = P(n)/Q(n)$ so that $c_{(n+1)} = c_n * P(n)/Q(n)$.

If n = 0: $\quad c_1 = c_0 * (P(0) / [Q(0)*1])$ or

$$c_1 = c_0 \frac{a_1 a_2 ... a_p}{b_1 b_2 ... b_q} x$$

Now let n=1: $\quad c_2 = c_1 * (P(1) / (Q(1)*2))$ or

$$c_2 = \frac{(1+a_1)(1+a_2)...(1+a_p)xc_1}{(1+b_1)(1+b_2)...(1+b_q)2}$$

Substitute for c_1

$$c_2 = \frac{a_1(1+a_1)a_2(1+a_2)...a_p(1+a_p)x^2 c_0}{b_1(1+b_1)b_2(1+b_2)...b_q(1+b_q)2}$$

Next n=2: $\quad c_3 = c_2 * (P(2) / (Q(2)*3))$ or

$$c_3 = \frac{(2+a_1)\dots(2+a_p)xc_2}{(2+b_1)\dots(2+b_q)3}$$

-Substitute for c_2

$$c_3 = \frac{a_1(1+a_1)(2+a_2)\dots a_p(1+a_p)(2+a_p)x^3c_0}{b_1(1+b_1)(2+b_1)\dots b_q(1+b_q)(2+b_q)3*2*1}$$

We then get the general form

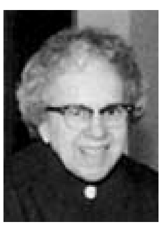
$$c_n = \frac{(a_1)_n(a_2)_n(a_3)_n\dots(a_p)_n}{(b_1)_n(b_2)_n\dots(b_q)_n} * \frac{x^n}{n!}c_0$$

Where $(a_i)_n = a_i * (1 + a_i) * \dots *((n-1) + a_i)$ and similarly for the b sub i's.

When rewritten as the following we get what we were trying to show

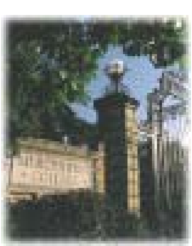
$$\sum_{n=0}^{\infty} c_n = c_0 \sum_{n=0}^{\infty} \frac{(a_1)_n\dots(a_p)_n}{(b_1)_{n\dots}(b_q)_n} * \frac{x^n}{n!}$$

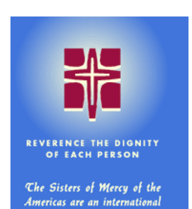
The History of Sister Celine



Why is Sister Celine's research so important? "The subject of computerized proofs of identities begins with the Ph. D. thesis of Sister Mary Celine Fasenmyer at the University of Michigan in 1945 (Petkovsek et. al., 54)." At the University of Michigan, Sister Celine developed a method for finding recurrence relations for hypergeometric polynomials. In the past, the customary method was hit-or-miss when seeking a new recurrence relation. She was one of the first to find a systematic approach! Sister Celine's algorithm also yielded general existence theorems for the recurrence relations satisfied by hypergeometric sums.

Sister Mary Celine was born in Crown, Pennsylvania October 4, 1906, to George and Cecilia Fasenmyer. When Mary was only one year old, her mother died, and three





years later, her father remarried a woman named Josephine, who was 25 years younger than he. After graduating from high school, Mary taught school for ten years. She then earned an AB degree in mathematics and a minor in physics at the Catholic Mercyhurst College in Erie, Pennsylvania, which was run by the Sisters of Mercy. The same year (1933) she took her vows and joined the Sisters of Mercy becoming Sister Celine. The Sisters of Mercy had a strong educational program as well as social and medical programs which were dedicated to the care of the sick, the elderly and orphans, both in their homes and in hospitals. Approximately five years later, Sister Celine resumed her formal studies at the University of Pittsburgh to work for her doctorate in mathematics, where she was supervised by Earl Rainville. Her thesis showed how a person can find recurrence relations that are satisfied by sums of hypergeometric terms by a purely algorithmic method. She developed this method further in two later papers. The first paper was "Some generalized hypergeometric polynomials" which was in the *Bulletin of the American Mathematical Society in 1947.* It examined special sets of generalized hypergeometric polynomials containing cases including Legendre's, Jacobi's, and Bateman's polynomials. In this paper, she also proposed some additional relations but without proof (pure recurrence relations, contiguous polynomial relations and integral relations). The second paper was entitled "On Recurrence Relations" which was published in the *American Mathematical Monthly* in 1949. It illustrated the algorithms which she discovered during her doctoral work. Her pioneering research went largely unnoticed at the time! Sister Celine then returned to Mercyhurst College, where she taught mathematics as a professor for many years. The teaching of mathematics became her primary, professional goal and she no longer engaged in research. It is doubtful that she was aware that her thesis would prove to be so important until after she retired. It wasn't until 1978 that the significance of Sister Celine's methods was realized! Doron Zeilberger had written about the importance of Sister Celines's theorem: "I remember feeling that I was about to connect to a parallel universe that had always existed but which until then had remained very well hidden, and I was about to find out what sort of creatures lived there（O'Connor, 2）."

Sister Celine's Method

In order to understand Sister Celine's Method, we will start with a linear algebra review of the standard method for solving systems of linear equations. First of all we start out with three equations. For example, $2x_1 - 2x_2 + 2x_3 + 4x_4 = 16$, $-3x_1 + 4x_2 – 1x_3 – 2x_4 = -21$, and $1x_1 + 1x_2 + 6x_3 + 11x_4 = 16$. Since we have our system of equations, the next step is to put it into a matrix. After that we want to get the matrix into a row reduced echelon form (RREF). After the matrix is in RREF the solution to the system is apparent. In this case, we do have one variable that is unknown since there are more variables than there are equations in the system. Solving such systems are pivotal in Sister Celine's algorithm and the mechanical processes once used, yield desired solution that is relatively easy to develop as a computer software program that can perform the procedure for us.

Sister Celine's Problem was that she wanted to design and implement an algorithm which sums $f(n) = \sum F(n,k)$. Her idea looks for a recurrence relation that the summands $F(n,k)$ satisfy of the form $\sum\sum a_{i,j}(n)F(n-j, k-I) = 0$ for some i,j that are greater then or equal to one. This is known as the Celine Recurrence. A K-free recurrence has coefficients $a_{i,j}(n)$ that do not depend on k. To solve this we have to find $\{a_{i,j}(n)\}$. Suppose we could solve the recurrence. Then we would know $\{a_{i,j}(n)\}$ i=1…I and j=1…J and $\sum\sum\sum a_{i,j}(n)F(n+i,k+j) = 0$. Then we bring the summation of k inside so we get $\sum\sum a_{i,j}(n) \sum F(n+i,k+j) = 0$. Suppose we have a k-free recurrence. Then our equation would be $\sum\sum a_{i,j}(n)f(n+i) = 0$ and this equation can be explicitly solved. Since we have the equations, we can then take the simplest Celine Recurrence and utilize it which is I=J=1. With this then we have a(n)F(n,k) + b(n)F(n,k+1)+c(n)F(n+1,k) + d(n)F(n+1,k+1) = 0. Therefore, we have f(n)(a(n) + b(n)) + f(n+1)(c(n) + d(n)) = 0. If we solve for f(n+1) we get f(n+1)= -[(a(n) + b(n))/(c(n) + d(n))]f(n) (Petkovsek, 58).

Here's an example where $F(n,k)=\binom{n}{k}$ and $f(n)=\sum\binom{n}{k}$. If we use the equation from before f(n+1)= -[(a(n) + b(n))/(c(n) + d(n))]f(n) and we say f(0) = 1 then we have $a(n)\binom{n}{k}+b(n)\binom{n}{k+1}+c(n)\binom{n+1}{k}+d(n)\binom{n+1}{k+1}=0$. Then we divide by F(n,k) and simplify.

$$a(n)\binom{n}{k}+b(n)\binom{n}{k+1}+c(n)\binom{n+1}{k}+d(n)\binom{n+1}{k+1}=0$$
$$\frac{\binom{n}{k+1}}{\binom{n}{k}}=\frac{n!}{(k+1)!(n-k-1)!}*\frac{k!(n-k)!}{n!}=\frac{n-k}{k+1}$$
$$\frac{\binom{n+1}{k}}{\binom{n}{k}}=\frac{(n+1)!}{k!(n+1-k)!}*\frac{k!(n-k)!}{n!}=\frac{n+1}{n-k+1}$$
$$\frac{\binom{n+1}{k+1}}{\binom{n}{k}}=\frac{(n+1)!}{(k+1)!(n+1-k-1)!}*\frac{k!(n-k)!}{n!}=\frac{n+1}{k+1}$$
$$a(n)+\left(\frac{n-k}{k+1}\right)b(n)+\left(\frac{n+1}{n-k+1}\right)c(n)+\left(\frac{n+1}{k+1}\right)d(n)=0$$

After these computations we have all we need to find the LCD and in this case it is (k+1)(n-k+1). Therefore our equations is a(n)(k+1)(n-k+1) + b(n)(n-k)(n-k+1) + c(n)(n+1)(k+1) + d(n)(n+1)(n-k+1) = 0. After we have this we want to express each coefficient as a polynomial in k. Therefore we have $(k+1)(n-k+1) = -k^2+kn+(n+1)$, (n-k)(n-k+1) = k^2-2kn-k+n^2+n, (k+1)(n+1) = kn+k+n+1, and (n+1)(-k(n+1)) = -k+ n^2+2n-k+1. Given these equations, we have –a(n) + b(n) = 0, n*a(n) + -(2n+1)b(n) + (n+1)c(n) – (n+1)d(n) = 0, and (n+1)a(n) + (n^2+n)b(n) + (n+1)c(n) + $(n+1)^2$d(n) = 0.

We can then put the coefficients into matrix form and put it into row reduced echelon form:

Calc: $R_2 + nR_1 = R_2$
Calc: $R_3 + (n+1)R_1 = R_3$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ n & -2n+1 & n+1 & -n+1 & 0 \\ n+1 & n^2+n & n+1 & (n+1)^2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -(n+1) & (n+1) & -(n+1) & 0 \\ 0 & (n+1)^2 & (n+1) & (n+1)^2 & 0 \end{bmatrix}$$

Calc: $\frac{1}{(n+1)}R_2 = R_2$, $\frac{1}{(n+1)}R_3 = R_3$

Calc: $R_3 + (n+1)R_2 = R_3$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & (n+1) & 1 & (n+1) & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & (n+2) & 0 & 0 \end{bmatrix}$$

Calc: $\frac{1}{(n+2)}R_3 = R_3$

Calc: $R_2 - R_3 = R_2$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Calc: $R_1 + R_2 = R_1$

Calc: $-1R_1 = R_1$, $-1R_2 = R_2$

$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

a + d = 0	a = -d	d = -1	a(n) = 1	c(n) = 0
b + d = 0	b = -d	a, b = 1	b(n) = 1	d(n) = -1
c = 0	c = 0	c = 0		

f(0)=1

$$f(n+1) = -\left[\frac{1+1}{0-1}\right] f(n) = \left[\frac{2}{1}\right] f(n)$$

$$f(n+1) = 2f(n)$$

$$f(0) = 1$$

So to solve this system:

f(1)=2(1)=2=2^1

f(2)=2*2=4=2^2

f(3)=2*4=8=2^3

f(4)=2*8=16=2^4

f(n)=2^n

Therefore: $\sum_k \binom{n}{k} = 2^n$ by using Sister Celine's Method.

We define a proper hypergeometric function as:

$$F(n,k) = P(n,k) \frac{\prod_{i=1}^{U} (a_i n + b_i k + c_i)!}{\prod_{i=1}^{V} (u_i n + v_i k + w_i)!} * x^k$$

where it satisfies the following conditions:

1. P(n) is a polynomial,

2. ai's, bi's, ui's, vi's, ci's, wi's are in z

3. $0 \leq u, v <$ infinity

4. x is indeterminate (i.e. a variable)

In our example F(n,k) = 1/ (n+ 3k+1), we see that it does not satisfy the conditions of a proper hypergeometric. However, if we multiply the top and bottom of the function by (n+3k)! we can see that F(n,k) = (n+3k)! / (n+3k+1)! which satisfies our definition of a proper hypergeometric function.

Celine's Theorem, also known as the fundamental theorem, is that we suppose F(n,k) is proper hypergeometric. Then, F(n,k) satisfies a k-free recurrence relation of the form

$$\sum_{i=0}^{I} \sum_{j=0}^{J} a_{i,j}(n) f(n-j, k-i) = 0$$

for every I and J that are positive integers. Then, there will be a pair (I*,J*) that will work with

$$J^* = \sum_s |b_s| + \sum_s |v_s|$$

$$I^* = 1 + \deg(P) + J^* \left[\sum_s |a_s| + \sum_s |u_s| - 1 \right]$$

It holds for every point (n,k) where F(n,k) is not equal to 0; and all the values of F that occur in it are well defined (Petkovsek, 64).

Using Celine's Method in Maple

First we need to be able to identify a hypergeometric series with maple to get the i and j used in her algorithm. To determine whether a series can be put in the form pFq[…] (p and q are the j and i respectfully) in Maple, we can use the convert/hypergeom function. We will first show with $\sum_k \binom{n}{k}^2$ you type into maple the following:

```
> convert(Sum(binomial(n,k)^2, k=0..infinity)hypergeom);
```

$$\frac{\Gamma(2\,n+1)}{\Gamma(n+1)^2}$$

As you can see Maple did not give back an answer in the form of pFq[…] this is because it found an identity in the hypergeometric database that is the answer to our sum. Recall that $\Gamma(z+1) = z!$. Next try the same thing with a more complicated sum $\sum_k \binom{n}{k}\binom{2k}{k}(-2)^{n-k}$ again type in the following:

```
> convert(Sum(binomial(n,k)*binomial(2*k,k)*(-2)^(n-
k),k=0..infinity), hypergeom);
```

$$(-2)^n \operatorname{hypergeom}\left(\left[\frac{1}{2},-n\right],[1],2\right)$$

This time we got the pFq[…] form we are looking for (converted into formal notation it reads $(-2)^n\ {}_2F_1\left[\begin{matrix}1/2, -n\\ 1\end{matrix}; 2\right]$). We now know that we have a hypergeometric function and can now evaluate the sum with Sister Celine's method.

Sister Celine's Method With Maple

First to use Sister Celine's Method with maple we need the EKHAD package that is available at the following website: http://www.math.temple.edu/zeilberg. After the package has been loaded, which is done by copying the text in the file you get from the web site, pasting it into maple and pressing return. We can try to find the previous sums using Sister Celine's Method and is entered into Maple in a general form of

celine((vars)->sum, i, j);

Example 1: $\sum_k \binom{n}{k}^2$. Although we already were able to get the answer to this Sum through the hypergeometric database we will still try it with Celine's method. Typing the following into Maple:

```
➢celine((n,k)->(n!/(k!*(n-k)!))^2,2,2);
```

You get back:

```
➢The full recurrence is
➢-(n-1)*b[4]*F(n-2,k-2)-b[4]*(2-2*n)*F(n-2,k-1)-b[4]*(1-
2*n)*F(n-1,k-1)-(n-1)*b[4]*F(n-2,k)-b[4]*(1-2*n)*F(n-1,k)-
b[4]*n*F(n,k) ==0
```

along with some other information we do not need at this time. Now all we have to do is reduce the recurrence. As b[4] is a constant in every term it can be dropped and we get this equation that is in normal notation.

$$n\binom{n}{k}^2-(2n-1)\left\{\binom{n-1}{k}^2+\binom{n-1}{k-1}^2\right\}+(n-1)\left\{\binom{n-2}{k}^2-2\binom{n-2}{k-1}^2+\binom{n-2}{k-2}^2\right\}=0$$

So to find the recurrence of f(n) we sum over all k and get

$$f(n)=\frac{2(2n-1)}{n}f(n-1)=\frac{2^2(2n-1)(2n-3)}{n(n-1)}f(n-2)=\cdots=\frac{(2n)!}{n!^2}$$

Which is exactly what we got before when the convert\hypergeom function returned the identity. Example 2: $\sum_k \binom{n}{k}\binom{2k}{k}(-2)^{(n-k)}$. Enter the sum into the Celine function using the i and j that was found with the hypergeometric test.

```
➢ celine((n,k)->n!*(2*k)!*(-2)^(n-k)/(k!^3*(n-k)!),1,2);
```

We get back from Maple:

```
➢The full recurrence is
➢-b[0]*(8*n-8)*F(n-2,k-1)-(4*n-2)*b[0]* F(n-1,k-1)-b[0]*(4-
4*n)*F(n-2,k)-b[0]*(-4*n+2)*F(n-1,k)+b[0]*n*F(n,k)    ==0
```

Now again all we have to do is reduce the recurrence. As b[0] is a constant in every term it can be dropped so in normal math notation we are left with:

$$n\left\{\binom{n}{k}\binom{2k}{k}(-2)^{(n-k)}\right\}+$$
$$0\left\{\binom{n-1}{k}\binom{2k}{k}(-2)^{((n-1)-k)}+\binom{n-1}{k-1}\binom{2(k-1)}{k-1}(-2)^{((n-1)-(k-1))}\right\}-$$
$$4(n-1)\left\{\binom{n-2}{k}\binom{2k}{k}(-2)^{((n-2)-k)}+\binom{n-2}{k-1}\binom{2(k-1)}{k-1}(-2)^{((n-2)-(k-1))}\right\}=0$$

Since f(0)=1 and f(1)=0 it follows immediately that

$$f(n)=\begin{cases}0 & \text{if } n \text{ is odd}\\ \dbinom{n}{n/2} & \text{if } n \text{ is even}\end{cases}$$

Conclusion

In conclusion, Sister Celine's work revolutionized the application of combinational identities by proving there were algorithmic approaches that have now been implemented in software programs like Maple. This method made finding hypergeometric functions and also finding a recursion using Sister Celine's Method easier.

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