Asset & Derivatives Pricing
Lecture 4: Exotic Derivatives & Structured Products

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Disclaimer

This document is for research or educational purposes only and is not intended to promote any financial investment or security.

1. Exotic Derivatives

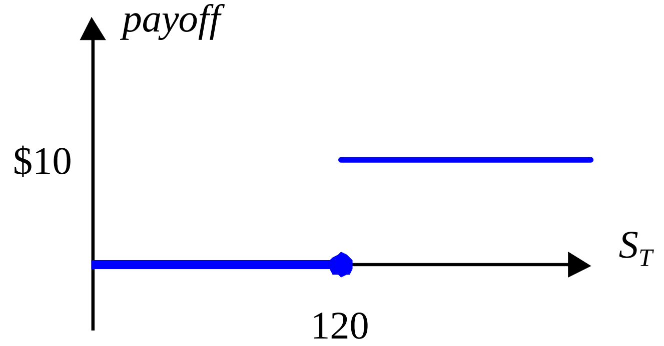
- Definition: ANY derivative security which is NOT a European or American vanilla call or put on a single underlying S
  - Examples: barrier options, digital options, asian options, lookback options, ladder options, variance swaps, any derivative security on multiple underlyings (quanto options, basket options, worst-of/best-of/rainbow options…)
- With the development of Structured Products, particularly in Europe, some exotic options (barriers, asians…) have become standardised and are often traded by vanilla option traders

Digital Options

- A digital option (also known as binary option) is the simplest kind of exotic derivative: it pays off a fixed amount A if $S_T > K$ (digital call) or $S_T < K$ (put), and otherwise pays nothing.

- Example: digital call, S_0 = \$100, K = \$120, A = \$10

$$Payoff = \begin{cases} \$10 \text{ if } S_1 \geq 120 \\ 0 \text{ if } S_1 < 120 \end{cases}$$



Digital Options: fair value

- A closed-form formula is easily derived in the lognormal / Black-Scholes model:

$$Digital\ Call_0 = Ae^{-rT}\Pr\left(\left[S_T \geq K\right]\right) = Ae^{-rT}N(d_2)$$

$$\text{where } d_2 = \frac{\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

- Q: Can you guess the formula for digital puts?

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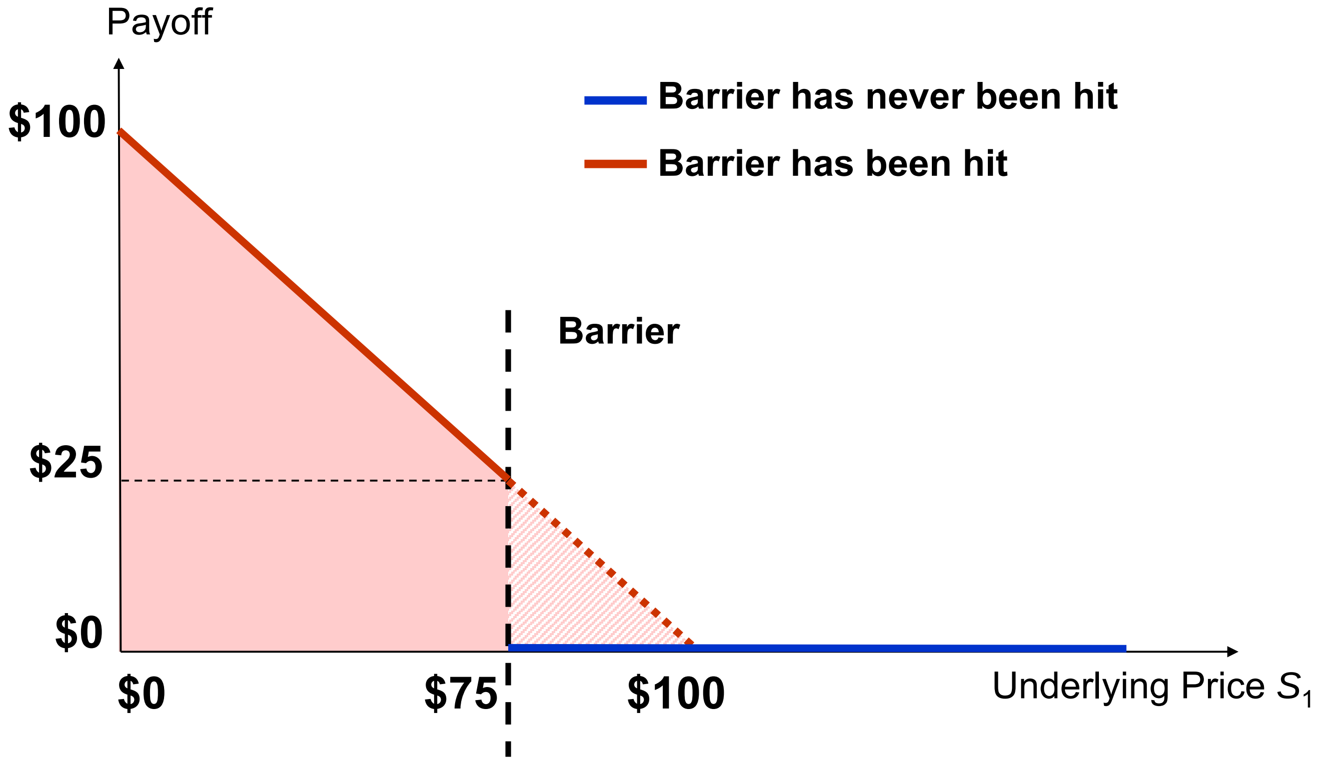
Barrier Options: Knock-in

- A knock-in barrier option is a call or put which is only activated if the underlying asset reaches a certain barrier level *H* before the maturity date *T*, and otherwise has zero payoff.

- Example: Knock-in Barrier Put, $S_0 = K = \$100$, $H = \$75$, $T = 1$ year

$$Payoff = \begin{cases} \max(0, 100 - S_1) \text{ if } S_t < 75 \text{ at any time } t \leq 1 \\ 0 \text{ if } S_t > 75 \text{ at all times } t \leq 1 \end{cases}$$

Example: Knock-in put



Barrier Options: Knock-out

- ►A knock-out barrier option is a call or put which is de-activated if the underlying asset reaches a certain barrier level *H* before the maturity date *T.*

- ►Example: Knock-out Barrier Call, $S_0 = K = \$100$, $H = \$120$, $T = 1$ year

$$Payoff = \begin{cases} \max(0, S_1 - 100) \text{ if } S_t < 120 \text{ at all times } t \le 1 \\ 0 \text{ if } S_t \ge 120 \text{ at any time } t \le 1 \end{cases}$$

Barrier options

- Q: Do you expect barrier options to be cheaper or more expensive than their vanilla equivalents?
- A: …
- Q: What can you say about the price of a portfolio made of a Knock-in barrier call and a Knock-out barrier call (same strike *K*, same barrier level *H*)?
- A: …

Asian options

- An Asian option is a call or put whose payoff formula involves an average of the underlying price
- Example: 1-year call, monthly Asian-out, S_0 = \$100, K = \$110

$$Payoff = \max\left(0, \frac{1}{12}\sum_{m=1}^{12} S_{m/12} - 110\right)$$

	Terminal Underlying Price S_T	**Floating Underlying Level S_{avg}**
Fixed Strike K	**Vanilla**: ▶ Max(0, S_T – K) ▶ Max(0, K – S_T)	**Asian-in**: ▶ Max(0, S_{avg} – K) ▶ Max(0, K – S_{avg})
Floating Strike K_{avg}	**Asian-out**: ▶ Max(0, S_T – K_{avg}) ▶ Max(0, K_{avg} – S_T)	**Asian-in+out**: ▶ Max(0, S_{avg} – K_{avg}) ▶ Max(0, K_{avg} – S_{avg})

Asian options Q&A

- Q: Are Asian-out options cheaper/more expensive than their vanilla equivalent?

- A: …

Proxy: vanilla FV with $\sigma/\sqrt{3}$

Lookback options

- A lookback option is a call or put whose payoff formula involves an extremum of the underlying price
- Example: 1-year lookback call, monthly monitoring, S_0 = \$100, K = \$110

$$Payoff = \max\left(0, \max_{m=1,..,12} S_{m/12} - 110\right)$$

	Terminal Underlying Price S_T	**Floating Underlying Level $S_{max/min}$**
Fixed Strike *K*	**Vanilla**: ▶ $\max(0, S_T - K)$ ▶ $\max(0, K - S_T)$	**Lookback**: ▶ $\max(0, S_{max} - K)$ ▶ $\max(0, K - S_{min})$
Floating Strike	**Floating-strike Lookback**: ▶ $S_T - S_{min}$ ▶ $S_{max} - S_T$	**Lookback straddle**: ▶ $S_{max} - S_{min}$

Lookback options Q&A

- Q: Are fixed-strike lookback options cheaper/more expensive than their vanilla equivalent?

- A: …

Proxy: twice as much for ATM

Ladder options

- A ladder option is a call or put which locks-in the highest/lowest step level reached by the underlying

- Example: 1-year ladder call, S_0 = K = \$100, \$10 steps

$$Payoff = \max\left(0, \max(S_{lock-in}, S_1) - 100\right)$$

Basket Options

▶A basket option is a call or put on a portfolio of several underlying assets

▶Example: 1-year basket put on 3 assets $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, equal weights, strike K = 90%, notional amount $1mn

$$Payoff = 1{,}000{,}000 \times \max\left[0, 90\% - \frac{1}{3}\left(\frac{S_1^{(1)}}{S_0^{(1)}} + \frac{S_1^{(2)}}{S_0^{(2)}} + \frac{S_1^{(3)}}{S_0^{(3)}}\right)\right]$$

▶Q: What will the payoff be if the price of each underlying asset goes down 25% after 1 year?

▶A Best-of Option is a call or put on the best-performing underlying asset among several [*in the sense of the ROR* $(S_T - S_0) / S_0$]

▶Example: 1-year best-of call on 3 assets $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, strike K = 110%, notional amount $1mn

$$Payoff = 1{,}000{,}000 \times \max\left[0, \max\left(\frac{S_1^{(1)}}{S_0^{(1)}}, \frac{S_1^{(2)}}{S_0^{(2)}}, \frac{S_1^{(3)}}{S_0^{(3)}}\right) - 110\%\right]$$

▶Similarly, a Worst-of Option is a call or put on the worst-performing underlying asset among several

2. Structured Products

Structured Products

- Definition: a combination of securities packaged ("structured") into a single security ("product")

  - Example: 3-year capital guaranteed upside note on S&P500, 70% participation, $1mn notional

$$Payoff = 1{,}000{,}000 \times \max\left(100\%, 100\% + 70\% \times \frac{SPX_3 - SPX_0}{SPX_0}\right)$$

  - This payoff can be decomposed as follows:

$$Payoff = 1{,}000{,}000 + \frac{700{,}000}{SPX_0}\max\left(0, SPX_3 - SPX_0\right)$$

Structured Products (cont'd)

$$Payoff = 1{,}000{,}000 + \frac{700{,}000}{SPX_0} \boxed{\max\left(0, SPX_3 - SPX_0\right)}$$

Fixed cash flow **Vanilla call payoff**

- This decomposition means that this particular structured product is replicated with a portfolio of:
  - A $1,000,000 zero-coupon bond
  - A fixed quantity of European vanilla calls on S&P500
- The cost of these two ingredients determine the fair value of the structured product.

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Basic terminology

- ►The vast majority of structured products are packaged as notes ("bonds") issued by a bank
- ►Structured notes have very similar characteristics to government bonds: maturity, notional amount (capital), coupons.
  - ►Capital guaranteed notes: the investor is guaranteed(*) to receive at least his/her capital back at maturity
  - ►Non-capital guaranteed notes: the investor may lose all or part of his/her capital (but will never be asked for an additional payment or 'margin call')

(*) By the note issuer (which is a bank subject to default risk)

Termsheet example

Issuer	Lezard Brothers (S&P: AA-, Moody's: Aa2)
Notional Amount	USD 1,000,000
Issue Date	8 September 2008
Maturity Date	8 September 2011
Coupons	Zero
Underlying Index	S&P 500 (SPX Index)
Redemption Amount	On the Maturity Date, the notes will redeem a USD amount calculated in accordance with the following formula:

$$1{,}000{,}000 \times \max\left(100\%, 100\% + 70\% \times \frac{SPX_3 - SPX_0}{SPX_0}\right)$$

Basic Terminology (cont'd)

▶ Q: Which redemption formula is capital guaranteed?

$$1)\ 1,000,000 \times \left[100\% \times \max\left(0, 70\% \times \frac{SPX_3 - SPX_0}{SPX_0}\right)\right]$$

$$2)\ 1,000,000 \times \left[100\% + \max\left(0, 70\% \times \frac{SPX_3 - SPX_0}{SPX_0}\right)\right]$$

$$3)\ 1,000,000 \times \max\left(0, 100\% + 70\% \times \frac{SPX_3 - SPX_0}{SPX_0}\right)$$

Linearity

- Structured Products which can be decomposed into a portfolio of “basic” securities (including “light” exotics) are called ‘**linear**’.

- There are yet more complex structured products which combine several features in a correlated fashion; those are called ‘**non-linear**’. For example:

$$Payoff = \begin{cases} 1{,}000{,}000 \times \max\left(100\%, 100\% + 150\% \times \dfrac{SPX_3 - SPX_0}{SPX_0}\right) & \text{if } NKY_3 \geq NKY_0 \\ 1{,}000{,}000 & \text{otherwise} \end{cases}$$

The frontier between linear and non-linear isn't clear-cut: it very much depends on the definition of “basic” securities

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Structuring / Financial Engineering

- Structurers or Financial Engineers are front-office specialists who:
  - Work with traders to calculate the fair value of a wide range of structured products, based on an analysis of their financial risks [**analytical skills**]
  - Work with sales & clients to propose ad-hoc solutions (payoffs) responding to their investment needs [**commercial skills**]
  - Develop new products based on current or anticipated market trends [**creative skills**]

Fair value

- In the old days, structured products were all linear and finding their fair value was about breaking down a payoff into its basic components (reverse-engineering)
- Today, structured products are increasingly non-linear and finding their fair value is about understanding whether and how their embedded financial risks can be hedged, as well as calculating the corresponding hedging costs:
  1. Program the payoff algorithm, including certain relevant adjustments (e.g. barrier shifting to mitigate Greek letters…)
  2. Select an appropriate model ("Black-Scholes", "Local Volatility", "Heath-Jarrow-Morton"…)
  3. Select an appropriate numerical method (Monte-Carlo, Finite Differences, closed-form formulas)

Fair value example

- Example: 3-year capital guaranteed upside note on S&P500, 70% participation, \$1mn notional

$$1,000,000 \times \max\left(100\%, 100\% + 70\% \times \frac{SPX_3 - SPX_0}{SPX_0}\right)$$

- Data: 3-year interest rate = 2.49% p.a., 3-year ATM vanilla call on S&P500 = 17.32% x SPX_0

$$FV = 1,000,000 \times \left(\frac{100\%}{(1+2.5\%)^3} + 70\% \times 17.32\%\right)$$

$$= \$1,049,839.41$$

Fair value example (cont'd)

- <u>Problem</u>: the fair value is higher than the notional amount!
- <u>Question</u>: what should the participation level '*x*' be such that FV = 1,000,000?

$$Payoff = 1{,}000{,}000 \times \max\left(100\%, 100\% + x \times \frac{SPX_3 - SPX_0}{SPX_0}\right)$$

- <u>Answer</u>: $$1{,}000{,}000 \times \left(\frac{100\%}{(1+2.5\%)^3} + x \times 17.32\%\right) = 1{,}000{,}000$$

$$\rightarrow x = \frac{1 - \dfrac{100\%}{1.025^3}}{17.32\%} = 41.22\%$$