# DIRICHLET SERIES

PETE L. CLARK

## 1. Introduction

In considering the arithmetical functions $f : \mathbb{N} \to \mathbb{C}$ as a ring under pointwise addition and "convolution":

$$f * g(n) = \sum_{d_1 d_2 = n} f(d_1)g(d_2),$$

we employed that old dirty trick of abstract algebra. Namely, we introduced an algebraic structure without any motivation and patiently explored its consequences until we got to a result that we found useful (Möbius Inversion), which gave a sort of retroactive motivation for the definition of convolution.

This definition could have been given to an 18th or early 19th century mathematical audience, but it would not have been very popular: probably they would not have been comfortable with the Humpty Dumpty-esque redefinition of multiplication.[1] Mathematics at that time did have commutative rings: rings of numbers, of matrices, of functions, but not rings with a "funny" multiplication operation defined for no better reason than mathematical pragmatism.

So despite the fact that we have shown that the convolution product is a useful operation on arithmetical functions, one can still ask what $f * g$ "really is." There are (at least) two possible kinds of answers to this question: one would be to create a general theory of convolution products of which this product is an example and there are other familiar examples. Another would be to show how $f * g$ is somehow a more familiar multiplication operation, albeit in disguise.

To try to take the first approach, consider a more general setup: let $(M, \bullet)$ be a commutative monoid. Recall from the first homework assignment that this means that $M$ is a set endowed with a binary operation $\bullet$ which is associative, commutative, and has an identity element, say $e$: $e \bullet m = m \bullet e = m$ for all $m \in M$. Now consider the set of all functions $f : M \to \mathbb{C}$. We can add functions in the obvious "pointwise" way:

$$(f + g)(m) := f(m) + g(m).$$

We could also multiply them pointwise, but we choose to do something else, defining

$$(f * g)(m) := \sum_{d_1 \bullet d_2 = m} f(d_1)g(d_2).$$

With the assistance of Richard Francisco and Diana May.

[1] Recall that Lewis Carroll – or rather Charles L. Dodgson (1832-1898) – was a mathematician.

But not so fast! For this definition to make sense, we either need some assurance that for all $m \in M$ the set of all pairs $d_1, d_2$ such that $d_1 \cdot d_2 = m$ is finite (so the sum is a finite sum), or else some analytical means of making sense of the sum when it is infinite. But let us just give three examples:

Example 1: $(M, \bullet) = (\mathbb{Z}^+, \cdot)$. This is the example we started with – and of course the set of pairs of positive integers whose product is a given positive integer is finite.

Example 2: $(M, \bullet) = (\mathbb{N}, +)$. This is the "additive" version of the previous example:

$$(f * g)(n) = \sum_{i+j=n} f(i)g(j).$$

Of course this sum is finite: indeed, for $n \in \mathbb{N}$ it has exactly $n+1$ terms. As we shall see shortly, this "additive convolution" is closely related to the Cauchy product of infinite series.

Example 3: $(M, \bullet) = (\mathbb{R}, +)$. Here we have seem to have a problem, because for functions $f, g : \mathbb{R} \to \mathbb{C}$, we are defining

$$(f * g)(x) = \sum_{d_1+d_2=x} f(d_1)g(d_2) = \sum_{y \in \mathbb{R}} f(x-y)g(y),$$

and although it is possible to define a sum over all real numbers, it turns out never to converge unless $f$ and $g$ are zero for the vast majority of their values.[2] However, there is a well-known replacement for a "sum over all real numbers": the integral. So one should probably define

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Here still one needs some conditions on $f$ and $g$ to ensure convergence of this "improper" integral. It is a basic result of analysis that if

$$\int_{-\infty}^{\infty} |f| < \infty, \ \int_{-\infty}^{\infty} |g| < \infty,$$

then the convolution product is well-defined. The convolution is an all-important operation in harmonic analysis: roughly speaking, it provides a way of "mixing together" two functions. Like any averaging process, it often happens that $f * g$ has nicer properties than its component functions: for instance, when $f$ and $g$ are absolutely integrable in the above sense, then $f * g$ is not only absolutely integrable but also continuous.

The most important property of this convolution is its behavior with respect to the **Fourier transform**: for a function $f : \mathbb{R} \to \mathbb{C}$ and $y \in \mathbb{R}$, one defines

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)e^{-2\pi ix}dy.$$

[2]More precisely, if $S$ is an arbitrary set of real numbers, it makes sense to define $\sum_{x_i \in S} = x$ if for all $\epsilon > 0$, there exists a finite subset $T \subset S$ such that for all finite subsets $T' \supset T$ we have $|\sum_{x_i \in T} x_i - x| < \epsilon$. (This is a special case of a **Moore-Smith limit**.) It can be shown that such a sum can only converge if the set of indices $i$ such that $x_i \neq 0$ is finite or countably infinite.

Then one has the following identity:

$$\widehat{f * g} = \hat{f} \cdot \hat{g}.$$

In other words, there is a natural type of "transform" $f \mapsto \hat{f}$ under which the convolution becomes the more usual pointwise product.

Now the question becomes: is there some similar type of "transform" $f \mapsto \hat{f}$ which carries functions $f : M \to \mathbb{C}$ to some other space of functions and under which the convolution product becomes the pointwise product?

The answer is well-known to be "yes" if $M$ is a locally compact abelian group (e.g. $\mathbb{Z}, \mathbb{Z}/N\mathbb{Z}, \mathbb{R}^n, \ldots$), and the construction is in fact rather similar to the above: this is the setting of **abstract Fourier analysis**. But our Examples 1 and 2 involve monoids that are not groups, so what we are looking for is not *exactly* a Fourier transform. So let us come back to earth by looking again at Examples 1 and 2.

In the case of Example 2, the construction we are looking for is just:

$$f \iff \{f(n)\}_{n=0}^{\infty} \mapsto F(x) = \sum_{n=0}^{\infty} f(n)x^n.$$

That is, to the sequence $\{f(n)\}$ we associate the corresponding **power series** $F(x) = \sum_n f(n)x^n$. One can look at this construction both *formally* and *analytically.*

The formal construction is purely algebraic: the ring of formal power series $\mathbb{C}[[t]]$ consists of all expressions of the form $\sum_{n=0}^{\infty} a_n x^n$ where the $a_n$'s are complex numbers. We define addition and multiplication in the "obvious ways":

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n := \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$

$$(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n) := \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0) x^n.$$

The latter definition seems obvious because it is consistent with the way we multiply polynomials, and indeed the polynomials $\mathbb{C}[t]$ sit inside $\mathbb{C}[[t]]$ as the subring of all formal expressions $\sum_n a_n x^n$ with $a_n = 0$ for all sufficiently large $n$. Now note that this definition of multiplication is just the convolution product in the additive monoid $(\mathbb{N}, +)$:

$$a_0 b_n + \ldots + a_n b_0 = (a * b)(n).$$

It is not immediately clear that anything has been gained. For instance, it is, technically, not for free that this multiplication law of formal power series is associative (although of course this is easy to check). Nevertheless, one should not underestimate the value of this purely formal approach. Famously, there are many nontrivial results about sequences $f_n$ which can be proved just by simple algebraic manipulations of the "generating function" $F(x) = \sum_n f_n x^n$. For example:

**Theorem 1.** *Let $a_1, \ldots, a_k$ be a coprime set of positive integers, and define $r(N)$ to be the number of solutions to the equation*

$$x_1 a_1 + \ldots + x_k a_k = N$$

*in non-negative integers $x_1, \ldots, x_k$. Then as $N \to \infty$,*

$$r(N) \sim \frac{N^{k-1}}{(k-1)!(a_1 \cdots a_k)}.$$

Nevertheless we also have and need an *analytic* theory of power series, i.e., of the study of properties of $F(x) = \sum_n a_n x^n$ viewed as a function of the complex variable $x$. This theory famously works out very nicely, and can be summarized as follows:

**Theorem 2.** *(Theory of power series) Let $\sum_n a_n x^n$ be a power series with complex coefficients. Let $R = (\limsup_n |a_n|^{\frac{1}{n}})^{-1}$. Then:*
*a) The series converges absolutely for all $x \in \mathbb{C}$ with $|x| < R$, and diverges – indeed, the general term tends to infinity in modulus – for all $x$ with $|x| > R$.*
*b) The convergence is uniform on compact subsets of the open disk of radius $R$ (about $0$), from which it follows that $F(x)$ is a complex analytic function on this disk.*
*c) If two power series $F(x) = \sum_n a_n x^n$, $G(x) = \sum_n b_n x^n$ are defined and equal for all $x$ in some open disk of radius $R > 0$, then $a_n = b_n$ for all $n$.*

In particular, it follows from Cauchy's theory of products of absolutely convergent series that if $F(x) = \sum_n a_n x^n$ and $G(x) = \sum_n b_n x^n$ are two power series convergent on some disk of radius $R > 0$, then on this disk the function $FG$ – the product of $F$ and $G$ in the usual sense – is given by the power series $\sum_n (a*b)(n)x^n$. In other words, with suitable growth conditions on the sequences, we get that the product of the transforms is the transform of the convolutions, as advertised.

Now we return to the case of interest: $(M, \bullet) = (\mathbb{Z}^+, \cdot)$. The transform that does the trick is $f \mapsto D(f,s)$, where $D(f,s)$ is the **formal Dirichlet series**

$$D(f,s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

To justify this, suppose we try to formally multiply out

$$D(f,s)D(g,s) = \left(\sum_{m=1}^{\infty} \frac{f(m)}{m^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right).$$

We will get one term for each pair $(m,n)$ of non-negative integers, so the product is (at least formally) equal to

$$\sum_{(m,n)} \frac{f(m)g(n)}{m^s n^s} = \sum_{(m,n)} \frac{f(m)g(n)}{(mn)^s},$$

where in both sums $m$ and $n$ range over over all positive integers. To make a Dirichlet series out of this, we need to collect all the terms with a given denominator, say $N^s$. The only way to get 1 in the denominator is to have $m = n = 1$, so the first term is $\frac{f(1)g(1)}{1^s}$. Now to get a 2 in the denominator we could have $m = 1$, $n = 2$ – giving the term $\frac{f(1)g(2)}{2^s}$ – or also $m = 2, n = 1$ – giving the term $\frac{f(2)g(1)}{2^s}$, so all in all the numerator of the "$2^s$-term" is $f(1)g(2) + f(2)g(1)$.

Aha. In general, to collect all the terms with a given denominator $N^s$ in the product involves summing over all expressions $f(m)g(n)$ with $mn = N$. In other words, we have the following formal identity:

$$D(f,s)\cdot D(g,s) = (\sum_{n=1}^{\infty}\frac{f(n)}{n^s})(\sum_{n=1}^{\infty}\frac{g(n)}{n^s}) = \sum_{n=1}^{\infty}\frac{\sum_{d|n} f(d)g(n/d)}{n^s} = D(f*g,s).$$

Thus we have attained our goal: under the "transformation" which associates to an arithmetical function its Dirichlet series $D(f,s)$, Dirichlet convolution of arithmetical functions becomes the usual multiplication of functions!

There are now several stages in the theory of Dirichlet series:

Step 1: Explore the purely formal consequences: that is, that identities involving convolution and inversion of arithmetical functions come out much more cleanly on the Dirichlet series side.

Step 2: Develop the theory of $D(f,s)$ as a function of a complex variable $s$. It is rather easy to tell when the series $D(f,s)$ is *absolutely* convergent. In particular, with suitable growth conditions on $f(n)$ and $g(n)$, we can see that

$$D(f,s)D(g,s) = D(f*g,s)$$

holds not just formally but also as an equality of functions of a complex variable. In particular, this leads to an "analytic proof" of the Möbius Inversion Formula.

On the other hand, unlike power series there can be a region of the complex plane with nonempty interior in which the Dirichlet series $D(f,s)$ is only conditionally convergent (that is, convergent but not absolutely convergent). We will present, without proofs, the basic results on this more delicate convergence theory.

In basic analysis we learn to abjure conditionally convergent series, but they lie at the heart of analytic number theory. In particular, in order to prove Dirichlet's theorem on arithmetic progressions one studies the Dirichlet series $L(\chi,s)$ attached to a **Dirichlet character** $\chi$ (a special kind of arithmetical function we will define later on), and it is extremely important that for all $\chi \neq \mathbf{1}$, there is a "critical strip" in the complex plane for which $L(\chi,s)$ is only conditionally convergent. We will derive this using the assumed results about conditional convergence of Dirichlet series and a convergence test, **Dirichlet's test**, from advanced calculus.[3] Finally, as an example of how much more content and subtlety lies in conditionally convergent series, we will use Dirichlet series to give an analytic continuation of the zeta function to the right half-plane (complex numbers with positive real part), which allows for a rigorous and concrete statement of the Riemann hypothesis.

## 2. Some Dirichlet series identities

Example 1: If $f = \mathbf{1}$ is the constant function 1, then by definition $D(\mathbf{1},s)$ is what is probably the most single important function in all of mathematics, the **Riemann zeta function**:

$$\zeta(s) = D(\mathbf{1},s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

[3]P.G.L. Dirichlet propounded his convergence test with this application in mind.

Example 2: Let $f(n) = d(n)$, the divisor function, so $D(d, s) = \sum_n \frac{d(n)}{n^s}$. But we also know that $d = \mathbf{1} * \mathbf{1}$. On Dirichlet series this means that we multiply: so that $D(d, s) = D(\mathbf{1}, s)D(\mathbf{1}, s)$, and we get that

$$D(d, s) = \zeta(s) \cdot \zeta(s) = \zeta^2(s).$$

Example 3: Since $\delta(1) = 1$ and $\delta(n) = 0$ for all $n > 1$, we have $D(\delta, s) = \frac{1}{1^s} + \sum_{n=2}^{\infty} \frac{0}{n^s} = 1$. Thus the Dirichlet series of the $\delta$ – the multiplicative identity for convolution – is just the constant function 1, the multiplicative identity in the "usual" sense of multiplication functions.

Example 4: What is $D(\mu, s)$? Since $\mu * \mathbf{1} = \delta$, we must have

$$1 = D(\iota, s) = D(\mu, s)D(\mathbf{1}, s) = D(\mu, s)\zeta(s),$$

so

$$D(\mu, s) = \frac{1}{\zeta(s)}.$$

Probably this is the most important such identity: it relates *combinatorial* methods (the Möbius function is closely related to the inclusion-exclusion principle) to *analytical* methods. More on this later.

We record without proof the following further identities, whose derivations are similarly straightforward. Some notational reminders: we write $\iota$ for the function $n \mapsto n$; $\iota_k$ for the function $n \mapsto n^k$; and $\lambda$ for the function $n \mapsto (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime divisors of $n$ counted with multiplicity.

$$D(\iota, s) = \zeta(s-1).$$

$$D(\iota_k, s) = \zeta(s-k).$$

$$D(\sigma, s) = \zeta(s)\zeta(s-1).$$

$$D(\sigma_k, s) = \zeta(s)\zeta(s-k).$$

$$D(\varphi, s) = \frac{\zeta(s-1)}{\zeta(s)}.$$

$$D(\lambda, s) = \frac{\zeta(2s)}{\zeta(s)}.$$

## 3. Euler products

Our first task is to make formal sense of an infinite product of infinite series, which is unfortunately somewhat technical. Suppose that we have an infinite indexing set $P$ and for each element of $p$ of $P$ an infinite series whose first term is 1:

$$\sum_{n=0}^{\infty} a_{p,n} = 1 + a_{p,1} + a_{p,2} + \ldots.$$

Then by the infinite product $\prod_{p \in P} \sum_n a_{p,n}$ we mean an infinite series whose terms are indexed by the infinite direct sum $\mathcal{T} = \bigoplus_{p \in P} \mathbb{N}$. Otherwise put, an element

$t \in \mathcal{T}$ is just a function $t : P \to \mathbb{N}$ such that $t(p) = 0$ for all but finitely many $p$ in $P$.[4] Then by $\prod_{p \in P} \sum_n a_{p,n}$ we mean the formal infinite series

$$\sum_{t \in \mathcal{T}} \prod_{p \in P} a_{p,t(p)}.$$

Note well that for each $t$, since $t(p) = 0$ except for finitely many $p$ and since $a_{p,0} = 1$ for all $p$, the product $\prod_{p \in P} a_{p,t(p)}$ is really a finite product. Thus the series is well-defined "formally" – that is, merely in order to write it down, no notion of limit of an infinite process is involved.

Let us informally summarize the preceding: to make sense of a formal infinite product of the form

$$\prod_p (1 + a_{p,1} + a_{p,2} + \ldots + a_{p,n} + \ldots),$$

we give ourselves one term for each possible product of one term from the first series, one term from the second series, and so forth, but we are only allowed to choose a term which is different from the $a_{p,0} = 1$ term finitely many times.

With that out of the way, recall that when developing the theory of arithmetical functions, we found ourselves in much better shape under the hypothesis of **multiplicativity**. It is natural to ask what purchase we gain on $D(f,s)$ by assuming the multiplicativity of $f$. The answer is that multiplicativity of $f$ is equivalent to the following formal identity:

$$D(f,s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots\right). \tag{1}$$

Here the product extends over all primes. The fact that this identity holds (as an identity of formal series) follows from the uniqueness of the prime power factorization of positive integers.

An expression as in (1) is called an **Euler product expansion**. If $f$ is moreover completely multiplicative, then $\frac{f(p^k)}{p^{ks}} = (\frac{f(p)}{p^s})^k$, and each factor in the product is a geometric series with ratio $\frac{f(p)}{p^s}$, so we get

$$D(f,s) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

In particular $f = \mathbf{1}$ is certainly completely multiplicative, so we get the identity

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

which we used in our study of the primes. We also get

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right), \tag{2}$$

[4] The property that $t(p) = 0$ except on a finite set is, by definition, what distinguishes the infinite direct sum from the infinite direct product.

and, plugging in $s = 2$,

$$\frac{6}{\pi^2} = \frac{1}{\zeta(2)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \prod_p \left(1 - \frac{1}{p^2}\right).$$

But not so fast! We changed the game here: so far (2) expresses a *formal* identity of Dirichlet series. In order to be able to plug in a value of $s$, we need to discuss the *convergence* properties of Dirichlet series and Euler products. In particular, since we did not put any particular ordering on our formal infinite product, in order for the sum to be meaningful we need the series involved to be *absolutely* convergent. It is therefore to this topic that we now turn.

## 4. Absolute convergence of Dirichlet series

Let us first study the *absolute convergence* of Dirichlet series $\sum_n \frac{a_n}{n^s}$. That is, we will look instead at the series $\sum_n \frac{|a_n|}{n^\sigma}$, where $s = \sigma + it$.[5]

**Theorem 3.** *Suppose a Dirichlet series $D(s) = \sum_n \frac{a_n}{n^s}$ is absolutely convergent at some complex number $s_0 = \sigma_0 + it_0$. Then it is also absolutely convergent at all complex numbers $s$ with $\sigma = \Re(s) > s_0$.*

Proof: If $\sigma = \Re(s) > \sigma_0 = \Re(s_0)$, then $n^{-\sigma} > n^{\sigma_0}$ for all $n \in \mathbb{Z}^+$, so

$$\sum_{n=1}^{\infty} |\frac{a_n}{n^s}| = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}} = \sum_{n=1}^{\infty} |\frac{a_n}{n^{s_0}}| < \infty.$$

It follows that the **domain of absolute convergence** of a Dirichlet series $D(f, s)$ is one of the following:

(i) The empty set. (I.e., for no $s$ does the series absolutely converge.)
(ii) $(-\infty, \infty)$.
(iii) A half-infinite interval of the form $(S, \infty)$.
(iv) A half-infinite interval of the form $[S, \infty)$.

Notice that in all cases, there is a unique $\sigma_{\text{ac}} \in [-\infty, \infty]$ such that:

(AAC1) For all $s$ with $\Re(s) > \sigma_{\text{ac}}$, $D(s)$ is absolutely convergent.
(AAC2) For all $s$ with $\Re(s) < \sigma_{\text{ac}}$, $D(s)$ is **not** absolutely convergent.

This unique $\sigma_{\text{ac}}$ is called the **abscissa of absolute convergence** of $D(s)$.

Example 1 (Type i): $D(s) = \sum_n \frac{2^n}{n^s}$.
This series does not converge (absolutely or otherwise) for any $s \in \mathbb{C}$: no matter what $s$ is, $|2^n \cdot n^{-s}| \to \infty$: exponentials grow faster than power functions. So $\sigma_{\text{ac}} = \infty$.

Example 2 (Type ii): A trivial example is the zero series – $a_n = 0$ for all $n$, or

[5] In other words, for a complex number $s$ we write $\sigma$ for its real part and $t$ for its imaginary part. This seemingly unlikely notation was introduced in a fundamental paper of Riemann, and remains standard to this day.

for that matter, any series with $a_n = 0$ for all sufficiently large $n$: these give finite sums. Or we could take $a_n = 2^{-n}$ and now the series converges absolutely independent of $s$. So $\sigma_{\mathrm{ac}} = -\infty$.

Example 3 (Type iii): $\zeta(s) = \frac{1}{n^s}$ is absolutely convergent for $s \in (1, \infty)$. So $\sigma_{\mathrm{ac}} = 1$.

Example 4 (Type iv): For $a_n = \frac{1}{(\log n)^2}$, the domain of absolute convergence is $[1, \infty)$.

The following result gives a sufficient condition for $\sigma_{\mathrm{ac}} = 1$:

**Proposition 4.** *Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series.*
*a) Suppose that there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n$. Then $\sigma_{\mathrm{ac}} \leq 1$.*
*b) suppose that the sequence $a_n$ does* **not** *tend to 0. then $\sigma_{\mathrm{ac}} \geq 1$.*
*c) In particular if the sequence $a_n$ is bounded but not convergent to 0, then $\sigma_{\mathrm{ac}} = 1$.*

Proof: a) Suppose $|a_n| \leq M$ for all $n$ and also that $\sigma = \Re(s) > 1$. Then

$$\sum_n |\frac{a_n}{n^s}| \leq M \sum_n \frac{1}{n^\sigma} = M\zeta(\sigma) < \infty.$$

b) The Dirichlet series at 0 is $\sum_n \frac{a_n}{n^0} = \sum_n a_n$. Of course this series can only be convergent (absolutely or otherwise) if $a_n \to 0$. Part c) follows immediately from a) and b).

Definition: We say that an arithmetic function $a_n : \mathbb{Z}^+ \to \mathbb{C}$ has **polynomial growth of order N** if there exist positive real numbers $C$ and $N$ such that $|a_n| \leq Cn^N$ for all $n \in \mathbb{Z}^+$. We say that a function has **polynomial growth** if it has polynomial growth of order $N$ for some $N \in \mathbb{R}^+$.

**Proposition 5.** *Suppose $\{a_n\}$ has polynomial growth of order $N$. Then the associated Dirichlet series $D(s) = \frac{a_n}{n^s}$ has $\sigma_{\mathrm{ac}} \leq N + 1$.*

Proof: By hypothesis, there exists $C$ such that $|a_n| \leq Cn^N$ for all $n \in \mathbb{Z}^+$. If $\sigma = \Re(s) > N + 1$, then there exists $\epsilon > 0$ such that $\sigma > N + 1 + \epsilon$. Then

$$\sum_n |\frac{a_n}{n^\sigma}| \leq \sum_n \frac{|a_n|}{n^{N+1+\epsilon}} \leq C \sum_n \frac{n^N}{n^{N+1+\epsilon}} = C \sum_n \frac{1}{n^{1+\epsilon}} < \infty.$$

**Corollary 6.** *Let $f(n)$, $g(n)$ be arithmetical functions with polynomial growth of order $N$. Then*

$$D(f, s)D(g, s) = D(f * g, s)$$

*is an equality of functions defined on $(N + 1, \infty)$.*

This follows easily from the theory of absolute convergence and the Cauchy product.

**Theorem 7.** *(Uniqueness Theorem) Let $f(n)$, $g(n)$ be arithmetical functions whose Dirichlet series are both absolutely convergent in the halfplane $\sigma = \Re(s) > \sigma_0$. Suppose there exists an infinite sequence $s_k$ of complex numbers, with $\sigma_k = \Re(s_k) > \sigma_0$ for all $k$ and $\sigma_k \to \infty$ such that $D(f, s_k) = D(g, s_k)$ for all $k$. Then $f(n) = g(n)$ for all $n$.*

Proof: First we put $h(n) := f(n) - g(n)$, so that $D(h,s) = D(f,s) - D(g,s)$. Then our assumption is that $D(h,s_k) = 0$ for all $k$, and we wish to show that $h(n) = 0$ for all $n$.

So suppose not, and let $N$ be the least $n$ for which $h(n) \neq 0$. Then

$$D(h,s) = \sum_{n=N}^{\infty} \frac{h(n)}{n^s} = \frac{h(N)}{N^s} + \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s},$$

so

$$h(N) = N^s D(h,s) - N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Taking now $s = s_k$ we have that for all $k \in \mathbb{Z}^+$,

$$h(N) = -N^{s_k} \sum_{n=N+1}^{\infty} \frac{h(n)}{n^{-s_k}}.$$

Fix a $\sigma > \sigma_0$, and choose a $k$ such that $\sigma_k > \sigma$. Then

$$|h(N)| \leq N^{\sigma_k} \sum_{n=N+1}^{\infty} |h(n)| n^{-\sigma_k} \leq \frac{N^{\sigma_k}}{(N+1)^{\sigma_k - c}} \sum_{n=N+1}^{\infty} |h(n)| n^{-c} \leq C \left( \frac{N}{N+1} \right)^{\sigma_k},$$

for some constant $C$ independent of $n$ and $k$. Since $N$ is a constant, letting $\sigma_k \to \infty$ the right hand side approaches 0, thus $h(N) = 0$, a contradiction.

**Corollary 8.** *Let $D(s) = \sum_n \frac{a_n}{n^s}$ be a Dirichlet series with abscissca of absolute convergence $\sigma_{\mathrm{ac}}$. Suppose that for some $s$ with $\Re(s) > \sigma_{\mathrm{ac}}$ we have $D(s) = 0$. Then there exists a halfplane in which $D(s)$ is absolutely convergent and never zero.*

Proof: If not, we have an infinite sequence $\{s_k\}$ of complex numbers, with real parts tending to infinity, such that $D(s_k) = 0$ for all $k$. By the Uniqueness Theorem this implies that $a_n = 0$ for all $n$ and thus $D(s)$ is identically zero in its halfplane of absolute convergence, contrary to our assumption.

**Corollary 9.** *(MIF for polynomially growing functions) If $f(n)$ is an arithmetical function with polynomial growth and $F(n) = \sum_{d|n} f(n)$, then $f(n) = \sum_{d|n} F(d)\mu(n/d)$.*

Surely this was the first known version of the Möbius inversion formula. Of course as Hardy and Wright remark in their classic text, the "real" proof of MIF is the purely algebraic one we gave earlier, but viewing things in terms of "honest" functions has a certain appeal.

Moreover, the theory of absolute convergence of infinite products (see e.g. [1, §11.5]) allows us to justify our formal Euler product expansions:

**Theorem 10.** *(Theorem 11.7 of [1]) Suppose that $D(f,s) = \sum_n \frac{f(n)}{n^s}$ converges absolutely for $\sigma > \sigma_{ac}$. If $f$ is multiplicative we have an equality of functions*

$$D(f,s) = \prod_p \left( 1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots \right),$$

*valid for all $s$ with $\Re(s) > \sigma_{ac}$. If $f$ is completely multiplicative, this simplifies to*

$$D(f,s) = \prod_p \left( 1 - \frac{f(p)}{p^s} \right)^{-1}.$$

Euler products are ubiquitous in modern number theory: they play a prominent role in (e.g.!) the proof of Fermat's Last Theorem.

## 5. Conditional convergence of Dirichlet series

Let $D(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series. We assume that the abscissa of absolute convergence $\sigma_{ac}$ is finite.

**Theorem 11.** *There exists a real number $\sigma_c$ with the following properties:*
*(i) If $\Re(s) > \sigma_c$, then $D(f,s)$ converges (not necessarily absolutely).*
*(ii) If $\Re(s) < \sigma_c$, then $D(f,s)$ diverges.*

Because the proof of this result is already somewhat technical, we defer it until $\S X.X$ on general Dirichlet series, where we will state and prove a yet stronger result.

Definition: $\sigma_c$ is called the **abscissa of convergence**.

Contrary to the case of absolute convergence we make no claims about the convergence or divergence of $D(f,s)$ along the line $\Re(s) = \sigma$: this is quite complicated.

**Proposition 12.** *We have*

$$0 \leq \sigma_{ac} - \sigma_c \leq 1.$$

Proof: Since absolutely convergent series are convergent, we evidently must have $\sigma_{ac} \geq \sigma$. On the other hand, let $s = \sigma + it$ be a complex number such that $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges. Of course this implies that $\frac{a_n}{n^s} \to 0$ as $n \to \infty$, and that in turn implies that there exists an $N$ such that $n \geq N$ implies $|\frac{a_n}{n^s}| = \frac{|a_n|}{n^\sigma} \geq 1$. Now let $s'$ be any complex number with real part $\sigma + 1 + \epsilon$ for any $\epsilon > 0$. Then for all $n \geq N$,

$$|\frac{a_n}{n^{s'}}| = \frac{|a_n|}{n^\sigma} \cdot \frac{1}{n^{1+\epsilon}} \leq \frac{1}{n^{1+\epsilon}},$$

so by comparison to a $p$-series with $p = 1 + \epsilon > 1$, $D(f,s')$ is absolutely convergent.

It can be a delicate matter to show that a series is convergent but not absolutely convergent: there are comparatively few results that give criteria for this. The following one – sometimes encountered in an advanced calculus class – will serve us well.

**Proposition 13.** *(Dirichlet's Test) Let $\{a_n\}$ be a sequnece of complex numbers and $\{b_n\}$ a sequence of real numbers. Suppose both of the following hold:*
*(i) There exists a fixed $M$ such that for all $N \in \mathbb{Z}^+$, $|\sum_{n=1}^{N} a_n| \leq M$ (bounded partial sums);*
*(ii) $b_1 \geq b_2 \geq \ldots \geq b_n \geq \ldots$ and $\lim_n b_n = 0$.*
*Then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.*

Proof: Write $S_N$ for $\sum_{n=1}^{N}$, so that by $(i)$ we have $|S_N| \leq M$ for all $N$. Fix $\epsilon > 0$, and choose $N$ such that $b_N < \frac{1}{\epsilon 2M}$. Then, for all $m, n \geq N$:

$$|\sum_{k=m}^{n} a_k b_k| = \sum_{k=m}^{n} (S_k - S_{k-1}) b_k|$$

$$= |\sum_{k=m}^{n} S_k b_k - \sum_{k=m-1}^{n-1} S_k b_{k+1}|$$

$$= |\sum_{k=m}^{n-1} S_k(b_k - b_{k+1}) + S_n b_n - S_{m-1} b_m|$$

$$\leq \sum_{k=m}^{n-1} |S_k||b_k - b_{k+1}| + |S_n||b_n| + |S_{m-1}||b_m|$$

$$\leq M\left(\sum_{k=m}^{n-1} |b_k - b_{k+1}| + |b_n| + |b_m|\right) = 2Mb_m \leq 2Mb_N < \epsilon.$$

Therefore the series satisfies the Cauchy criterion and hence converges.[6]

**Theorem 14.** *Let $\{a_n\}_{n=1}^{\infty}$ be a complex sequence.*
*a) Suppose that the partial sums $\sum_{n=1}^{N} a_n$ are bounded. Then the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has $\sigma_c \leq 0$.*
*b) Assume in addition that $a_n$ does not converge to 0. Then $\sigma_{ac} = 1$, $\sigma_c = 0$.*

Proof: By Proposition 4, $\sigma_{\mathrm{ac}} = 1$. For any real number $\sigma > 0$, by taking $b_n = \frac{1}{n^\sigma}$ the hypotheses of Proposition 13 are satisfied, so that $D(\sigma) = \sum_n \frac{a_n}{n^\sigma}$ converges. The smallest right open half-plane which contains all positive real numbers $\sigma$ is of course $\Re(s) > 0$, so $\sigma \leq 0$. By Proposition 12 we have $1 = \sigma_{ac} \leq 1 + \sigma$, so we conclude that $\sigma = 0$.

**Theorem 15.** *(Theorem 11.11 of* [1]*) A Dirichlet series $D(f,s)$ converges uniformly on compact subsets of the half-plane of convergence $\Re(s) > \sigma$.*

Suffice it to say that, in the theory of sequences of functions, "uniform convergence on compact subsets" is the magic incantation. As a consequence, we may differentiate and integrate Dirichlet series term-by-term. Also:

**Corollary 16.** *The function $D(f,s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ defined by a Dirichlet series in its half-plane $\Re(s) > \sigma$ of convergence is complex analytic.*

## 6. Dirichlet series with non-negative real coefficients

Suppose we are given a Dirichlet series $D(s) = \sum_n \frac{a_n}{n^s}$ with the property that for all $n$, $a_n$ is real and non-negative. There is more to say about the analytic theory of such series. First, the non-negativity hypothesis ensures that for any real $s$, $D(s)$ is a series with non-negative terms, so its absolute convergence is equivalent to its convergence. Thus:

**Lemma 17.** *For a Dirichlet series with non-negative real coefficients, the abscissae of convergence and absolute convergence coincide.*

Thus one of the major differences from the the theory of power series is eliminated for Dirichlet series with non-negative real coefficients. Another critical property of all complex power series is that the radius of convergence $R$ is as large as conceivably possible, in that the function necessarily has a singularity somewhere on the

[6]This type of argument is known as **summation by parts**.

boundary of the disk $|z - z_0| < R$ of convergence. This property need not be true for an arbitrary Dirichlet series. Indeed the series

$$D(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \ldots,$$

has $\sigma = 0$ but extends to an analytic function on the entire complex plane.[7] However:

**Theorem 18.** *(Landau) Let $D(s) = \sum_n \frac{a_n}{n^s}$ be a Dirichlet series, with $a_n$ real and non-negative for all $n$. Suppose that for a real number $\sigma$, $D(s)$ converges in the half-plane $\Re(s) > \sigma$, and that $D(s)$ extends to an analytic function in a neigborhood of $\sigma$. Then $\sigma$ strictly exceeds the abscissa of convergence $\sigma_c$.*

Proof (Kedlaya): Suppose on the contrary that $D(s)$ extends to an analytic function on the disk $|s - \sigma| < \epsilon$, for some $\epsilon > 0$ but $\sigma = \sigma_c$. Choose $c \in (\sigma, \sigma + \epsilon/2)$, and write

$$D(s) = \sum_n a_n n^{-c} n^{c-s} = \sum_n a_n n^{-c} e^{(c-s)\log n}$$
$$= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{a_n n^{-c} (\log n)^k}{k!} (c-s)^k.$$

Here we have a double series with all coefficients non-negative, so it must converge absolutely on the disk $|s-c| < \frac{\epsilon}{2}$. In particular, viewed as a Taylor series in $(c-s)$, this must be the Taylor series expansion of $D(s)$ at $s = c$. Since $D(s)$ is assumed to be holomorphic in the disk $|s - c| < \frac{\epsilon}{2}$, this Taylor series is convergent there. In particular, choosing any real number $\sigma'$ with $\sigma - \frac{\epsilon}{2} < \sigma' < \sigma$, we have that $D(\sigma')$ is absolutely convergent. But this implies that the original Dirichlet series is convergent at $\sigma'$, contradiction!

For example, it follows from Landau's theorem that the Riemann zeta function $\zeta(s) = \sum_n \frac{1}{n^s}$ must have a singularity at $s = 1$, since otherwise there would exist some $\sigma < 1$ such that the series converges in the entire half-plane $\Re(s) > \sigma$.

Of course this is a horrible illustration of the depth of Landau's theorem, since we used the fact that $\zeta(1) = \infty$ in order to compute the abscissa of convergence of the zeta function! We will see a much deeper application of Landau's theorem during the proof of Dirichlet's theorem on primes in arithmetic progressions.

## 7. Characters and L-series

Let $f : \mathbb{Z}^+ \to \mathbb{C}$ be an arithmetic function.

Recall that $f$ is said to be **completely mutliplicative** if $f(1) \neq 0$ and for all $a, b \in \mathbb{Z}$, $f(ab) = f(a)f(b)$. The conditions imply $f(1) = 1$.

For $N \in \mathbb{Z}^+$, we say a function $f$ is $N$-**periodic** if it satisfies:

($P_N$) For all $n \in \mathbb{Z}^+$, $f(n + N) = f(n)$.

[7] We will see a proof of the former statement shortly, but not the latter. More generally, it is true for the $L$-function associated to any primitive Dirichlet character.

An arithmetic function is **periodic** if it is $N$-periodic for some $N \in \mathbb{Z}^+$.

Remark: A function $f : \mathbb{Z} \to \mathbb{C}$ is said to be $N$-periodic if for all $n \in \mathbb{Z}$, $f(n+N) = f(n)$. It is easy to see that any $N$-periodic arithmetic function admits a unique extension to an $N$-periodic function with domain $\mathbb{Z}$.

Note that if $f$ is $N$-periodic it is also $kN$-periodic for every $k \in \mathbb{Z}^+$. Conversely, we define the **period** $P$ of a periodic function to be the least positive integer $N$ such that $f$ is $N$-periodic, then it is easy to see that $f$ is $N$-periodic iff $P \mid N$.

Now we are ready to meet the object of our affections:

A **Dirichlet character** is a periodic completely multiplicative arithmetic function.[8]

Example: For an odd prime $p$, define $L_p : \mathbb{Z}^+ \to \mathbb{C}$ by $L_p(n) = (\frac{n}{p})$ (Legendre symbol). The period of $L_p$ is $p$. Notice that $L_p(n) = \pm 1$ if $n$ is prime to $p$, whereas $L_p(n) = 0$ if $\gcd(n,p) > 1$. This generalizes as follows:

**Theorem 19.** *Let $f$ be a Dirichlet character of period $N$.*
*a) If $\gcd(n,N) = 1$, then $f(n)$ is a $\varphi(N)$th root of unity in $\mathbb{C}$ (hence $f(n) \neq 0$).*
*b) If $\gcd(n,N) > 1$, then $f(n) = 0$.*

Proof: Put $d = \gcd(n,N)$. Assume first that $\gcd(n,N) = 1$, so by Lagrange's Theorem $n^{\varphi(N)} \equiv 1 \pmod N$. Then:

$$f(n)^{\varphi(N)} = f(n^{\varphi(N)}) = f(1) = 1.$$

Next assume $d > 1$, and write $n = dn_1$, $N = dN_1$. By assumption $N_1$ properly divides $N$, so is strictly less than $N$. Then $f$ is not $N_1$-periodic, so there exists $m \in \mathbb{Z}^+$ such that

$$f(m + N_1) - f(m) \neq 0.$$

On the other hand

$$f(d)\left(f(m+N_1) - f(m)\right) = f(dm+N) - f(dm) = f(dm) - f(dm) = 0,$$

so

$$f(n) = f(dn_1) = f(d)f(n_1) = 0 \cdot f(n_1) = 0.$$

### 7.1. Period $N$ Dirichlet characters and characters on $U(N)$.

A fruitful perspective on the Legendre character $L(p)$ is that it is obtained from a certain homomorphism from the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ into the multiplicative group $\mathbb{C}^\times$ of complex numbers by extending to $L(0 \pmod p)) := 0$. In fact all Dirichlet characters of a given period can be constructed in this way.

We introduce some further notation: for $N \in \mathbb{Z}^+$, let $U(N) = (\mathbb{Z}/N\mathbb{Z})^\times$ be the unit group, a finite commutative group of order $\varphi(N)$. Let $X(N)$ be the group of characters of $U(N)$, i.e., the group homomorphisms $U(N) \to \mathbb{C}^\times$. We recall from

[8]Recall that by definition a multiplicative function is not identically zero, whence $f(1) = 1$.

[Algebra Handout 2.5, §4] that $X(N)$ is a finite commutative group whose order is is also $\varphi(N)$.[9]

**Proposition 20.** *Let $N$ be a positive integer. There is a bijective correspondence between Dirichlet characters with period $N$ and elements of $X(N) = \mathrm{Hom}(U(N), \mathbb{C}^\times)$.*

Proof: If $f : U(N) \to \mathbb{C}$ is a homomorphism, we extend it to a function from $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by defining $f(0) = 0$ on all residue classes which are not prime to $N$, and then define

$$\tilde{f}(n) := f(n \mod N).$$

In other words, if $q_N : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ is the quotient map, then $\tilde{f} := f \circ q_N$.

Conversely, if $f : \mathbb{Z}^+ \to \mathbb{C}$ is a Dirichlet character mod $N$, then its extension to $\mathbb{Z}$ is $N$-periodic and therefore factors through $\overline{f} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$.

It is easy to see that these two constructions are mutually inverse.

For example, the function $\mathbf{1} : n \to 1$ for all $n$ is the unique Dirichlet character of period 1. The character $\mathbf{1}$ is said to be **trivial**; all other Dirichlet characters are said to be **nontrivial**. Under the correspondence of Proposition 20 it corresponds to the unique homomorphism from the trivial group $\mathbb{Z}/1\mathbb{Z} \to \mathbb{C}$.

## 7.2. Examples.

Example (Principal character): For any $N \in \mathbb{Z}^+$, define $\xi_N : \mathbb{Z}^+ \to \mathbb{C}$ by

$$\xi_N(n) = 1, \gcd(n, N) = 1,$$

$$\xi_N(n) = 0, \gcd(n, N) > 1.$$

This is evidently a Dirichlet character mod $N$, called the **principal character**. It corresponds to the trivial homomorphism $U(N) \to \mathbb{C}^\times$, i.e., the one which maps every element to $1 \in \mathbb{C}$.

Example: $N = 1$: Since $\varphi(1) = 1$, the principal character $\xi_1$ coincides with the trivial character $\mathbf{1}$: this is the unique Dirichlet character modulo 1.

Example: $N = 2$: Since $\varphi(2) = 1$, the principal character $\xi_2$, which maps odd numbers to 1 and even integers to 0, is the unique Dirichlet character modulo 2.

Example: $N = 3$: Since $\varphi(3) = 2$, there are two Dirichlet characters mod 3, the principal one $\xi_3$ and a nonprincipal character, say $\chi_3$. One checks that $\chi_3(n)$ must be 1 if $n = 3k+1$, $-1$ if $n = 3k+2$, and 0 if $n$ is divisible by 3. Thus $\widehat{U(3)} = \{\xi_3, \chi_3\}$.

Example: $N = 4$: Since $\varphi(4) = 2$, there is exactly one nonprincipal Dirichlet character mod 4, $\chi_4$. We must define $\chi_4(n)$ to be 0 if $n$ is even and $(-1)^{\frac{n-1}{2}}$ if $n$ is odd. Thus $\widehat{U(4)} = \{\xi_4, \chi_4\}$. Note that $\xi_4 = \xi_2$.

## 7.3. Conductors and primitive characters.

[9] In fact, $X(N)$ and $U(N)$ are isomorphic groups: Theorem 15, *ibid.*, but this is actually not needed here.

7.4. **Dirichlet L-series.**

By definition, a **Dirichlet L-series** is the Dirchlet series associated to a Dirichlet character:

$$L(\chi, s) = D(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

In particular, taking $\chi = \chi_1 = \mathbf{1}$, we get $L(\chi_1, s) = \zeta(s)$, which has $\sigma_{ac} = \sigma_c = 1$. But this is the exception:

**Theorem 21.** *Let $\chi$ be a nontrivial Dirichlet character. Then for the Dirichlet L-series $L(\chi, s) = D(\chi, s)$, we have $\sigma_{ac} = 1$, $\sigma_c = 0$.*

Proof: It follows from the orthogonality relations [Handout A2.5, Theorem 17] that since $\chi$ is nonprincipal, the partial sums of $L(\chi, s)$ are bounded. Indeed since $|\chi(n)| \leq 1$ for each $n$ and the sum over any $N$ consecutive values is zero, the partial sums are bounded by $N$. Also we clearly have $\chi(n) = 1$ for infinitely many $n$, e.g. for all $n \equiv 1 \pmod N$. So the result follows directly from Theorem 14.

We remark that most of the proof of the Dirichlet's theorem – specifically, that every congruence class $n \in (\mathbb{Z}/N\mathbb{Z})^{\times}$ contains infinitely many primes – involves showing that for every nontrivial character $\chi$, $L(\chi, 1 + it)$ is nonzero for all $t \in \mathbb{R}$. This turns out to be much harder if $\chi$ takes on only real values.

## 8. An explicit statement of the Riemann hypothesis

Let $g$ be the arithmetical function $g(n) = (-1)^{n+1}$. Then:

$$D(g, s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2\sum_{n=1}^{\infty} \frac{1}{(2k)^s} = \zeta(s)(1 - 2^{1-s}).$$

This formal manipulation holds analytically on the region on which all series are absolutely convergent, namely on $\Re(s) > 1$. On the other hand, by Example XX above we know that $D(g, s)$ is convergent for $\Re(s) > 0$. So consider the function

$$Z(s) = \frac{D(g, s)}{1 - 2^{1-s}}.$$

By Corollary 16 the numerator is complex analytic for $\Re(s) > 0$. The denominator is defined and analytic on the entire complex plane, and is zero when $2^{1-s} = e^{(1-s)\log 2} = 1$, or when $1 - s = \frac{2\pi n i}{\log 2}$ for $n \in \mathbb{Z}$, so when $s = s_n = 1 - \frac{2\pi n}{\log 2} i$. But by construction $Z(s) = \zeta(s)$ for $\Re(s) > 1$, so $Z(s)$ is what is called an **meromorphic continuation** of the zeta function.

Remark: All of the zeroes of $2^{1-s}$ are simple (i.e., are not also zeroes for the derivative). It follows that for $n \neq 0$, $Z(s)$ is holomorphic at $s_n$ iff $D(g, s_n) = 0$. We will see in the course of the proof of Dirichlet's theorem that this indeed the case, and thus $Z(s) = \zeta(s)$ is analytic in $\Re(s) > 0$ with the single exception of a simple pole at $s = 1$.

However, our above analysis already shows that $2^{1-s}$ is defined and nonzero in the *critical strip* $0 < \Re(s) < 1$, so that for such an $s$, $Z(s) = 0 \iff D(g, s) = 0$.

We can therefore give a precise statement of the Riemann hypothesis in the following (misleadingly, of course) innocuous form:

**Conjecture 22.** *(Riemann Hypothesis) Suppose $s$ is a zero of the function*

$$D(g,s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

*with* $0 < \Re(s) < 1$. *Then* $\Re(s) = \frac{1}{2}$.

This serves to show once again how the deepest facts (and conjectures!) in analytic number theory turn on cancellation in infinite series.

## 9. General Dirichlet series

Let $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of real numbers which is strictly increasing and with $\lim_{n\to\infty} \lambda_n = \infty$. Given a complex sequence (or "arithmetical function") $a = \{a_n\}_{n=1}^{\infty}$, we may consider the series

$$D_\lambda(a,s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n},$$

called the **general Dirichlet series** associated to the *sequence of exponents* $\lambda$.

The theory we have developed for Dirichlet series can equally well be expressed in this more general context. Why one might want to do this is probably not yet clear, but bear with us for a moment.

In particular, if we define as before $\sigma_{ac}$ (resp. $\sigma_c$) to be the infimum of all real numbers $\sigma$ such that $\sum_{n=1}^{\infty} |a_n| e^{-\sigma\lambda_n}$ converges (resp. such that $D_\lambda(a,\sigma)$ converges), one can prove that $\Re(s) > \sigma_{ac}$ (resp. $\Re(s) > \sigma$) is the largest open half-plane in which $D_\lambda(a,s)$ is absolutely convergent (resp. convergent). Moreover, there are explicit formulas for these abscissae, at least when $\sigma_c \geq 0$ (which holds in all applications we know of). For instance if $\sum_n a_n$ diverges then $\sigma_c \geq 0$.

**Theorem 23.** *([2, §8.2]) Let $D_\lambda(a,s)$ be a general Dirichlet series, and assume that $\sigma_c \geq 0$. Then:*

$$\sigma_{ac} = \limsup_n \frac{\log \sum_{k=1}^{n} |a_k|}{\lambda_n}. \tag{3}$$

$$\sigma_c = \limsup_n \frac{\log |\sum_{k=1}^{n} a_k|}{\lambda_n}. \tag{4}$$

Remark: If $\limsup_n \frac{\log|\sum_{k=1}^{n} a_k|}{\lambda_n} = 0$ and $\sum_n a_n$ diverges, then $\sigma_c = 0$; if $\limsup_n \frac{\log|\sum_{k=1}^{n} a_k|}{\lambda_n} = 0$ and $\sum_n a_n$ converges, then

$$\sigma_c = \limsup_n \frac{1}{\lambda_n} \ln |\sum_{i=1}^{\infty} a_i|.$$

These formulae are highly reminiscent of Hadamard's formula $(\limsup_n |a_n|^{\frac{1}{n}})^{-1}$ for the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$.

But in fact it is no coincidence: just as general Dirichlet series generalize "ordinary" Dirichlet series – which we recover by taking $\lambda_n = \log n$, they also generalize power series – which we essentially recover by taking $\lambda_n = n$. Indeed,

$$\sum_{n=1}^{\infty} a_n e^{-ns} = \sum_{n=1}^{\infty} a_n x^n,$$

with $x = e^{-s}$. This change of variables takes right half-planes to disks around the origin: indeed the open disk $|x| < R$ corresponds to

$$|x| = |e^{-s}| = |e^{-\sigma - it}| = e^{-\sigma} < R,$$

or $\sigma > -\log R$, a right half-plane. Under the change of variables $x = e^{-s}$ the origin $x = 0$ corresponds to some ideal complex number with infinitely large real part.

At first the fact that we have a theory which simultaneously encompasses Dirichlet series and power series seems hard to believe, since the open disks of convergence and of absolute convergence for a power series are identical. However, the analogue of Proposition 12 for general Dirichlet series is

**Proposition 24.** *Let $D_\lambda(a, s)$ be a general Dirichlet series. Then the abscissae of absolute convergence and of convergence are related by:*

$$0 \leq \sigma_{ac} - \sigma_c \leq \limsup_{n \to \infty} \frac{\log n}{\lambda_n}.$$

In the case $\lambda_n = n$ we have $\frac{\log n}{n} \to 0$, and Proposition 24 confirms that $\sigma_{ac} = \sigma_c$.

We leave it as an exercise for the interested reader to compare the formulae (3) and (4) with Hadamard's formula $R^{-1} = \limsup_n |a_n|^{\frac{1}{n}}$ for the radius of convergence of power series. (After making the change of variables $x = e^{-s}$ they are not identical formulae, but it is not too hard to show that they are equivalent in the sense that any of them can be derived from the others without too much trouble.)

## References

[1] Apostol, Tom M. Introduction to analytic number theory. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.

[2] Apostol, Tom M. Modular functions and Dirichlet series in number theory. Second edition. Graduate Texts in Mathematics, 41. Springer-Verlag, New York, 1990.