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The Statistics of Statistical Arbitrage

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Hedge funds sometimes use mathematical techniques to "capture" the short-term volatility of stocks and perhaps other types of securities. This sort of strategy resembles market making and is sometimes considered a form of statistical arbitrage. This study shows that for the universe of large-capitalization U.S. stocks, even quite naive techniques can achieve remarkably high information ratios. The methods used are quite general and should be applicable also to other asset classes.

Market makers in financial markets generate profits by buying low and selling high over short time intervals. This process occurs naturally because, as market makers, they offer a stock for sale at a higher price than they are willing to pay for it and because the more urgent buyers and sellers have to accept the market makers' terms. Market making, particularly that of NYSE specialists, has been studied in the normative context of academic finance; this approach is represented by the seminal papers of Hasbrouck and Sofianos (1993) and Madhavan and Smidt (1993).

High-speed trading strategies similar to market making have putatively been used by hedge funds in recent years. This type of strategy has sometimes been referred to as *statistical arbitrage*—or perhaps *stat arb,* in the abbreviated patois of the Street. Statistical arbitrage of this nature can be studied in the context of portfolio behavior and is hence amenable to the methods of stochastic portfolio theory (Fernholz 2002). In this article, we use these methods to examine the potential profitability of such a strategy applied to large-capitalization U.S. stocks, but the methodology is quite general and should be applicable also to other asset classes.

Dynamic stock portfolios can be constructed that behave like market makers. Equal-weighted portfolios are dynamic portfolios in which each of the stocks has the same constant weight. In an equal-weighted portfolio, if a stock rises in price relative to the others, it generates a sell trade in the stock; if the price declines, it generates a buy. Therefore, such a portfolio will sell on upticks and buy on downticks, the way a market maker would. Our goal is to estimate the return and risk parameters of equal-weighted portfolios and use these parameters to determine the efficacy of statistical arbitrage in the U.S. stock market. First, we calculate the logarithmic return for an equal-weighted portfolio; then, we use this calculation to analyze the behavior of portfolios of large-cap U.S. stocks. Finally, we propose a hedging strategy to control the risk of the high-speed trading, and we estimate the information ratio of the hedged strategy.

Logarithmic Return of Portfolios

Roughly speaking, the logarithmic return (log return) of a financial asset is the change in the natural logarithm of the asset's value. Sometimes, the log return yields a clearer picture of asset behavior than is available from the usual arithmetic return—particularly in the case of certain stock portfolios.[1] In this section, we calculate the log return for equal-weighted portfolios and investigate the implications regarding the behavior of these portfolios.

First, let us recall the difference between the *arithmetic* return and the *logarithmic* return of a given financial asset. Consider a financial asset represented by its value X at time t. The value X is assumed to be positive, and it varies over time. If a period of time dt passes and the value of the asset moves from X to $X + dX$, then the *arithmetic* return over the period is dX/X. The corresponding *log* return is defined to be $d\log X$, which in this example would be equal to $\log(X + dX) - \log X$. In all cases, the term "log" is understood to be the natural logarithm.

Now, let us consider an equity market and portfolios in this market. Suppose we have a market of n stocks represented by their prices $X_1, \ldots, X_n$, which we think of as positive-valued, time-dependent random processes. Because we are considering intraday price behavior, it is reasonable to assume that the number of shares of each company

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is constant and that the stocks pay no dividends. In this case, over a period of time dt, the ith stock will have return dX_i/X_i and log return $d\log X_i$.

Suppose we have a portfolio represented by its value $P > 0$ and with weights, or proportions, $p_1, \ldots, p_n$, where p_i represents the proportion of the portfolio value invested in X_i, so that $p_1 + \ldots + p_n = 1$. With this notation, the amount that the portfolio has invested in X_i is equal to p_iP. A positive p_i weight implies that P holds a long position in X_i; a negative p_i weight implies a short position. In general, the weights are all time-dependent random processes and there is no simple relationship among the weights, the stock prices, and the portfolio value. There *is* a simple relationship, however, among the portfolio return, the weights, and the stock returns. This relationship is given by

$$\frac{dP}{P} = \sum_{i=1}^{n} p_i \frac{dX_i}{X_i} \tag{1}$$

(see, e.g., Markowitz 1952). A more complicated equation for the portfolio log return exists, however, and it is precisely the "complication" that provides the analytical tool we use in our analysis. For log return, Equation 1 becomes

$$d\log P = \sum_{i=1}^{n} p_i d\log X_i + \gamma_p^* dt \tag{2}$$

with γ_p^* given by

$$\gamma_p^* = \frac{1}{2}\sum_{i=1}^{n} p_i \sigma_{ip}^2, \tag{3}$$

where σ_{ip}^2 is the variance (rate) of X_i relative to the portfolio (which means that σ_{ip} is the tracking error of X_i versus P). A derivation of Equation 2 and Equation 3 can be found in Appendix A, and complete details are available in Fernholz 2002. The quantity γ_p^* is the *excess growth rate* of P, and because relative variances are nonnegative, for a long-only portfolio, γ_p^* will be nonnegative. Equation 2 shows that for a long-only portfolio, the log return of the portfolio exceeds the weighted average of the log returns of its component stocks, and the difference is exactly the excess growth rate.

Consider an equal-weighted portfolio P in which $p_i = 1/n$ for $i = 1, \ldots, n$. In this case, Equation 2 becomes

$$\begin{aligned} d\log P &= \sum_{i=1}^{n} \frac{1}{n} d\log X_i + \gamma_p^* dt \\ &= d\log \sqrt[n]{X_1 \cdots X_n} + \gamma_p^* dt. \end{aligned} \tag{4}$$

Hence, the log return of P depends only on the change in the geometric mean of the stock prices and the excess growth of the portfolio. It follows from Equation 3 that for the equal-weighted portfolio, γ_p^* is simply one-half the average of the relative variances, σ_{ip}^2, of the stocks.

By its nature, an equal-weighted portfolio will sell a stock when its price rises relative to the rest of the portfolio and buy when the price declines. Accordingly, our plan is to use an equal-weighted portfolio to act as a market maker and thus exploit the short-term volatility generated by the difference in price between buy and sell trades. To determine the efficacy of this strategy, we must estimate the value of γ_p^* in Equation 4, which we do in the next section.

Remember that real stock portfolios cannot have precisely constant weights because a trader cannot trade the stocks back to the weights as fast as the stock prices move. Instead, real portfolios are rebalanced at discrete time intervals back to weights that are nearly constant. Nevertheless, it is widely known and accepted in mathematical finance that stochastic differential equations such as Equation 1 or Equation 2 provide a close approximation to discrete trading strategies for real portfolios as long as the trading intervals are short enough (see Fouque, Papanicolaou, and Sircar 2000). In practice, portfolio strategies with trading intervals of a month or less can be closely approximated by continuous models of the form we are considering (Fernholz 2002). Keep in mind, however, that in any application, the parameters in a continuous model must be compatible with the trading intervals used in the strategy. We develop this idea in the next section.

Estimation of Excess Growth Rates

In this section, we estimate the excess growth rate for an equal-weighted portfolio of large-cap U.S. stocks—specifically, those stocks included in either the S&P 500 Index or the Russell 1000 Index. These stocks are traded on the NYSE or NASDAQ, and the data we use are the individual transaction prices, or "prints," of the trades that occurred on these exchanges each day.

The estimation of the excess growth rate depends on the estimation of relative variances because for an equal-weighted portfolio, the excess growth rate is simply one-half the average of the relative variances of the stocks in the portfolio. To estimate these relative variances, the time series of stock prices must be sampled. To sample a time series, a particular length of sampling interval must be chosen, and the value of the variance estimate may depend on this choice.[2] This dependence is particularly likely if there is trading noise being caused by the difference between the bid and the asked prices. In these circumstances, shorter sampling intervals usually result in higher estimates of daily variance than longer sampling intervals. A mathematical model for trading noise is included in Appendix B, but we make no assumption that real stocks follow such a model.

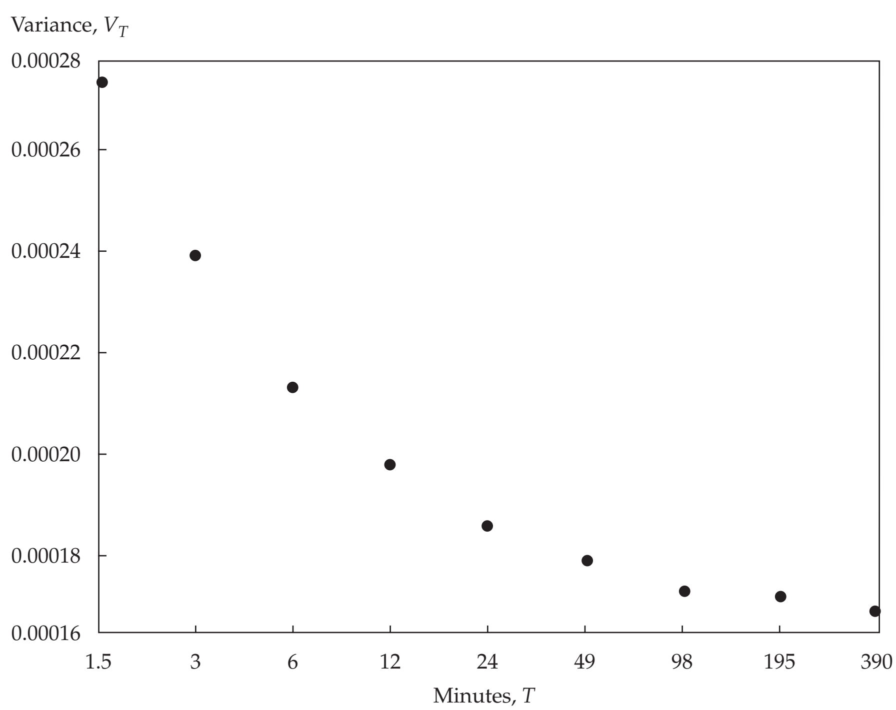
We let V_T represent the average daily relative variance of the stocks in the equal-weighted portfolio, estimated by sampling at intervals of length T the time series for the log return of each stock relative to the equal-weighted portfolio. The graph of V_T versus T is a form of *variogram*.[3] **Figure 1** shows a variogram for the year 2005 for large-cap U.S. stocks. As can be seen, the estimate of daily variance decreases from 0.000273 for a 1.5-minute sampling interval to 0.000169 for a 390-minute interval, which is the total number of minutes the exchanges in the United States are open each business day. By Equation 3, these variances correspond to daily excess growth rates of about 0.0137 percent for a sampling interval of 1.5 minutes and about 0.0085 percent for a sampling interval of 390 minutes. To annualize these numbers, we multiply by 250, the number of trading days in 2005. With this multiple, we have annual excess growth rates of about 3.41 percent for the 1.5-minute sampling interval and about 2.11 percent for the 390-minute sampling interval. Figure 1 indicates that sampling intervals between 1.5 and 390 minutes will produce estimated excess growth rates between these two values.

With estimates of annual excess growth that range from 2.11 percent to 3.41 percent, which value is appropriate for use in Equation 4? The appropriate estimate corresponds to the sampling interval closest to the interval at which the portfolio will be rebalanced back to equal weights. Hence, for a high-speed trader who trades continuously, the 1.5 minute interval is appropriate, whereas for a trader who trades only at the open and close of the market, the 390-minute interval should be used. So, the high-speed trader will generate 3.41 percent of excess growth a year, and the open/close trader will generate only 2.11 percent. More frequent trading allows the portfolio to capture more volatility.

A Hedged Strategy

An investment in an equal-weighted portfolio can be risky because, although the γ_p^* term in Equation 4 is always positive, the $\log \sqrt[n]{X_1 \quad X_n}$ term can vary quite a bit. If, however, we consider two equal-weighted portfolios, each rebalanced at different intervals, the $\log \sqrt[n]{X_1 \quad X_n}$ terms should be about equal even though, as discussed, there can be a systematic difference in the γ_p^* terms.

Figure 1. Estimated Average Daily V_T by Minute Intervals: Large-Cap U.S. Stocks, 2005

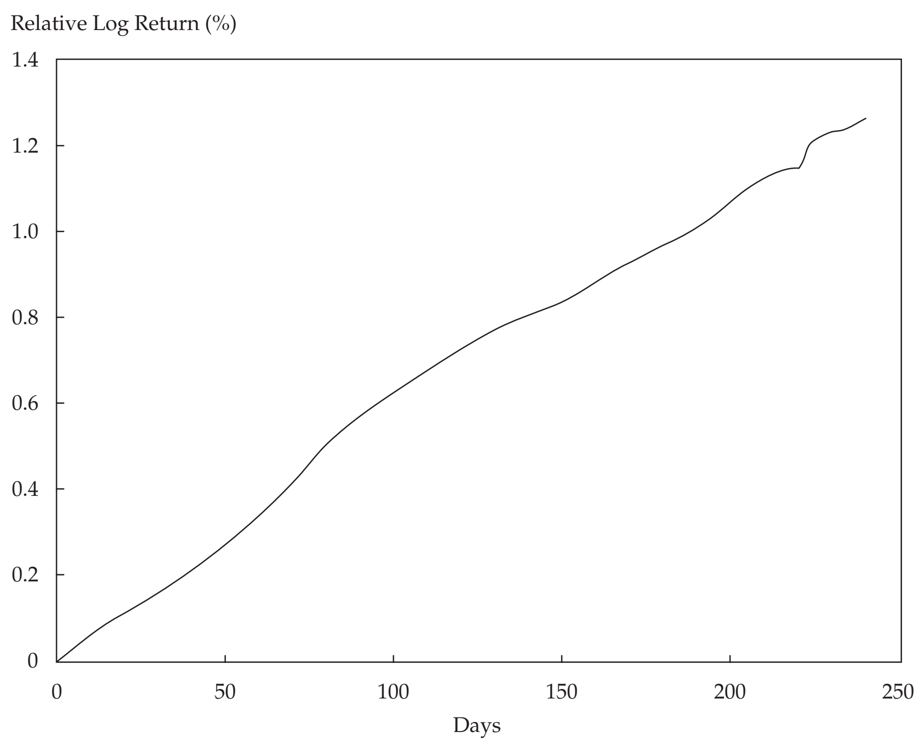


Consider two investments, one in equal-weighted portfolio $P_{1.5}$ that is rebalanced at intervals of about 1.5 minutes and one in equal-weighted portfolio P_{390} that is rebalanced only at the open and close of the market. Suppose a high-speed trader buys \$100 of $P_{1.5}$, sells \$100 of P_{390} short, and combines these two investments in a single hedged long–short portfolio. A great deal of cancellation will occur in the holdings of the long–short portfolio, so the total stock position will be nowhere near \$100 long and \$100 short. Indeed, at the market open, both portfolios will be equal weighted and thus will cancel exactly, so the hedged portfolio will have no stock holdings at all. During the trading day, portfolio $P_{1.5}$ will buy on downticks and sell on upticks of the individual stocks, but its total long–short holdings will remain close to market neutral throughout the day. At the close, both $P_{1.5}$ and P_{390} will be rebalanced back to equal weights, so the long and short holdings will cancel again and the high-speed trader will be left with just the gain, or loss, for the day.

The value of $P_{1.5}$ and P_{390} can be approximated by equations of the form of Equation 4, where the value of $\gamma^*_{1.5}$ will correspond to a variance estimate based on a sampling interval of 1.5 minutes and γ^*_{390} will correspond to a variance estimate based on a sampling interval of 390 minutes. The result is that $\gamma^*_{1.5} \cong 3.41\%$ but $\gamma^*_{390} \cong 2.11\%$. So, over a year, the high-speed trader, who is long \$100 of $P_{1.5}$ and short \$100 of P_{390}, will accumulate about \$1.30. Of course, we must assume that the high-speed trader is a broker/dealer; otherwise, trading commissions would come into play and the strategy would most likely fail.

Figure 2 shows the relative log return of the simulated equal-weighted portfolio that was rebalanced every 1.5 minutes versus the equal-weighted portfolio that was rebalanced only at the open and the close of the market each day. The simulation was run over the year 2005, the same year used for estimating the excess growth rates.[4] Figure 2 indicates that the relative log return was about 1.26 percent, which is consistent with the 1.30 percent annual log return calculated in the previous paragraph applied to the reduced 241-trading-day year. The mean and standard deviation for the daily log returns in Figure 2 are, respectively, about 0.0052 percent and 0.0026 percent, for a daily information ratio of about 2.0. These data imply an annual information ratio of about 32 for the long–short combination.

Figure 2. Simulated Cumulative Value of log $P_{1.5}$ – log P_{390}: Large-Cap U.S. Stocks, 2005



An annual information ratio of 32 makes high-speed trading an attractive investment strategy—and one that could be safely combined with considerable leverage. If this appears to be too good to be true, what hidden problems might there be that could interfere with the strategy? Probably the most difficult problem for a real high-speed trader would be gaining access to adequate order flow. In our simulation, we assumed that every minute and a half we could execute a trade, but a real market would present competition for order flow. Hence, the performance in our simulation may be accurate for the combined class of high-speed traders, but each member of the class would have to compete with the others for order flow. However, a trader should be satisfied with an information ratio of less than 32: Indeed, a trading strategy with an annual information ratio of just 3 would produce a negative year only about once every seven centuries, at least if the log return of the strategy followed a normal distribution. (No wonder specialists' seats on the NYSE were hereditary!)

What amount of trading would be needed to generate the \$1.30 profit we calculated? For a stock with a daily relative variance of 0.000273, the corresponding 1.5-minute relative standard deviation will be about 0.00103. With this value of σ, the expected relative log price change over a minute and a half will be about $\pm\sigma\sqrt{2/\pi} = \pm 0.00824$. With about 256 intervals of a minute and a half in a trading day, the expected cumulative absolute change in the log prices corresponding to these trades will be about 21.1 percent. Hence, the daily trading on a \$100 equal-weighted portfolio that is traded about every minute and a half will amount to approximately \$21.10, which gives an annual trading of about \$5,275 for a 250-day year. The annual rate of return on trading will be 1.30/5275 = 2.5 bps.

The profit enjoyed by our high-speed trader as market maker comes at the expense of conventional portfolio managers, who access the market more slowly to adjust their portfolio holdings. Hence, the 2.5 bps gained by the high-speed trader should be comparable to the trading costs paid by these conventional managers. The market impact (i.e., trading cost excluding commissions) for institutional managers of U.S. stocks has been estimated by the Plexus Group (2006) to be about 12 bps (it is referred to as "Broker Impact" in that publication). Why is this cost to conventional portfolio managers greater than the 2.5 bp gain of our high-speed trader? Several factors contribute to this difference. First, because the high-speed trader had to pay 2.11 percent for the hedging portfolio, he collected only 1.30 percent a year of the 3.41 percent a year generated by his long portfolio. The 2.11 percent hedging expense of a high-speed trader does not revert, however, to the conventional portfolio managers, so they will experience the full 3.41 percent generated by the long portfolio as trading cost. Second, institutional equity managers are likely to make larger trades than the average stockholder, and larger trades are likely to have greater market impact and, therefore, cost more to execute. Third, if the stock price fails to revert back to its previous value after a trade, the high-speed trader will earn nothing, but whether or not the price reverts, all the price changes preceding a trade will still be measured as trading cost by the portfolio managers. When these factors are considered, the 2.5 bps generated by the high-speed trader appears to be reasonably consistent with the 12 bps of market-impact cost experienced by institutional managers.

Might there be hidden risks that were not evident in our simulation? Probably not significant risks, at least if our high-speed trader exercises reasonable caution. Even in strong bull or bear markets, the average daily drift of stock prices is a small fraction (probably about 5 percent) of their daily standard deviation, so normal market movement should not have much effect. Of course, a crash like the one on Black Monday, 19 October 1987, could be, and indeed was, a disaster for some market makers. However, our high-speed trader is not a true market maker and has no obligation to maintain an orderly market: He can simply sit out nasty days like Black Monday and let the market fend for itself.

For simplicity, we used equal-weighted portfolios in our example, and even with this naive strategy, the information ratio indicates that a high degree of leverage would be possible. A similar analysis can be carried out for any constant-weighted portfolio with some type of optimization applied to the weights. For example, weights on more volatile stocks might be increased to generate greater γ_p^* or be decreased to control portfolio risk, or the weights could be varied from day to day to adjust for changing market conditions. Optimization of the long–short portfolio might eliminate much of the 2.11 percent hedging expense, and the mean reversion evident in Figure 1 might also be exploited in some manner (see Appendix B). Perhaps with appropriate optimization, high-speed trading could generate a return closer to the 12 bp trading cost paid by institutional equity managers.

Conclusion

We showed that dynamic portfolios in which the portfolio holdings are systematically rebalanced to maintain almost constant weights will capture in a natural way the volatility that is present in stock markets. These portfolios act as market makers by selling on upticks and buying on downticks. We showed that, in the absence of trading commissions, such portfolios can generate remarkably high information ratios.

We thank Jason Greene for his many helpful suggestions and Joseph Runnels for supplying information regarding trading costs.

This article qualifies for 1 PD credit.

Appendix A. Calculation of Portfolio Log Return

Suppose we have a market of n stocks represented by their prices $X_1, \ldots, X_n$, which are positive-valued, time-dependent random processes. We assume that the number of shares of each company is constant and that the stocks pay no dividends. Suppose that a short period of time, dt, passes and the price of the ith stock moves from X_i to $X_i + dX_i$. In this case, the (arithmetic) return of the stock over that period is defined to be dX_i/X_i. The log return of the stock over the same period is defined as $d\log X_i$, and the relationship between these two types of return is

$$\frac{dX_i}{X_i} = d\log X_i + \frac{\sigma_i^2}{2}dt, \tag{A1}$$

where σ_i^2 is the variance rate of X_i at time t (which means that $\sigma_i^2 dt$ is the variance of $\log X_i$ over the time period dt). Equation A1 can be shown to be exact in the infinitesimal limit as the time interval dt tends to zero, and it continues to hold quite accurately in practice for time intervals as long as a month or more, at least for exchange-traded U.S. stocks. Equation A1 is an application of Ito's rule from stochastic calculus, details of which can be found in Karatzas and Shreve (1991); a thorough development of the methods of stochastic portfolio theory can be found in Fernholz (2002).

Suppose we have a portfolio represented by its value P with weights $p_1, \ldots, p_n$, where p_i represents the proportion of the portfolio value invested in X_i so that $p_1 + \ldots + p_n = 1$. A positive p_i weight implies that P holds a long position in X_i; a negative weight implies a short position. In general, the weights are all time-dependent random processes and there is no simple relationship among the weights, the stock prices, and the portfolio value. The classical equation for portfolio return (see Markowitz 1952) states that

$$\frac{dP}{P} = \sum_{i=1}^{n} p_i \frac{dX_i}{X_i}, \tag{A2}$$

and with Equation A2 (which is Equation 1 in the article body), we are equipped to express the log return of P, because $d\log P$ must satisfy an equation corresponding to Equation A1. Hence,

$$\begin{aligned} d\log P &= \frac{dP}{P} - \frac{\sigma_p^2}{2}dt \\ &= \sum_{i=1}^{n} p_i \frac{dX_i}{X_i} - \frac{\sigma_p^2}{2}dt \\ &= \sum_{i=1}^{n} p_i \left(d\log X_i + \frac{\sigma_i^2}{2}dt \right) - \frac{\sigma_p^2}{2}dt, \end{aligned} \tag{A3}$$

where σ_p^2 is the variance rate of P. By rearranging the terms in Equation A3, we see that the log return of P can be written

$$d\log P = \sum_{i=1}^{n} p_i d\log X_i + \frac{1}{2}\left(\sum_{i=1}^{n} p_i \sigma_i^2 - \sigma_p^2 \right) dt. \tag{A4}$$

The expression

$$\gamma_p^* = \frac{1}{2}\left(\sum_{i=1}^{n} p_i \sigma_i^2 - \sigma_p^2 \right) dt \tag{A5}$$

is the *excess growth rate* of P.

The variances in Equation A5 can be replaced by variances relative to any given portfolio. The mathematical proof is in Fernholz (2002), but the idea here is that the portfolio growth rate will not depend on the choice of the *numeraire* asset relative to which variances are calculated. If the variances are measured relative to portfolio P itself, and if we denote them by σ_{ip}^2, then

$$\gamma_p^* = \frac{1}{2}\sum_{i=1}^{n} p_i \sigma_{ip}^2, \tag{A6}$$

where the relative portfolio variance term in Equation A5 measures the variance of P relative to itself, so it vanishes. Equation A6 is Equation 3 in the article text.

Appendix B. Variance Estimation with Trading Noise

In this appendix, we develop a model for a stock price process with trading noise. The idea is that a "bid–ask bounce" takes place over the short term but the price series over the long term behaves like a random walk. The model we present here is consistent with Figure 1, but we make no assumption that real stock prices follow such a model. Because of the time-dependent nature of the variance estimates, we explicitly include the time variable for the random processes that we consider here.

The model for stock price X with trading noise that we propose here is

$$\log X(t) = Y(t) + Z(t), \quad \text{(B1)}$$

where Y is a random walk that satisfies

$$dY(t) = \sigma dW_1(t), \quad \text{(B2)}$$

with $\sigma > 0$ constant and W_1 a Brownian motion, and where Z is an Ornstein–Uhlenbeck noise process (see Karatzas and Shreve 1991) that satisfies

$$dZ(t) = -\alpha Z(t)\,dt + \tau dW_2(t), \quad \text{(B3)}$$

where α and τ are positive constants and W_2 is a Brownian motion independent of W_1. In this case,

$$d\log X(t) = -\alpha Z(t)\,dt + \sigma dW_1(t) + \tau dW_2(t). \quad \text{(B4)}$$

The noise process, Z, incorporates a "restoring force," $-\alpha Z(t)$, that returns the process to the origin. The steady-state distribution for Z is normal with mean zero, and the covariance of $Z(s)$ and $Z(t)$ is $(\tau^2/2\alpha)e^{-\alpha|t-s|}$ for $s,t \geq 0$ (see Karatzas and Shreve 1991).

In Equation B4, the variance rate for X is seen to be $\sigma^2 + \tau^2$, but to estimate this rate, the time series for $\log X$ must be sampled at regular intervals. Assume that at time 0, we start process Y at the origin and process Z at its steady-state distribution. If for $0 \leq s < t$ we sample the time series for $\log X$ at s and t, then we find

$$\begin{aligned} \operatorname{var}\left[\log X(t) - \log X(s)\right] &= E\left[\log X(t) - \log X(s)\right]^2 \\ &= E\left[Y(t) - Y(s)\right]^2 \\ &\quad + E\left[Z(t) - Z(s)\right]^2 \\ &= \sigma^2(t-s) + \frac{\tau^2}{\alpha}\left[1 - e^{-\alpha(t-s)}\right]. \end{aligned} \quad \text{(B5)}$$

Hence, the expected value of the estimated variance parameter for $\log X$ with sampling intervals of length $T > 0$ will be

$$\begin{aligned} V_T &= \frac{1}{T}\left[\sigma^2 T + \frac{\tau^2}{\alpha}\left(1 - e^{-\alpha T}\right)\right] \\ &= \sigma^2 + \tau^2\left(\frac{1 - e^{-\alpha T}}{\alpha T}\right), \end{aligned} \quad \text{(B6)}$$

which has a limiting value of $V_0 = \lim_{T\to 0} V_T = \sigma^2 + \tau^2$ and decreases asymptotically toward σ^2 as T increases toward infinity.

Note that the shape of the variogram in Figure 1 is consistent with Equation B6, although Figure 1 is based purely on empirical data with no assumption that stock prices follow a model of the form given in Equations B3 and B4.

Trading noise that can be modeled by a system of the form of Equations B3 and B4 can be predicted to some extent—for example, by the use of Kalman filters (see Kalman and Bucy 1961). With prediction, one might be able to improve upon the hedging strategy described in the article.

Notes

1. The log return is sometimes referred to as the "geometric return" or "compound return" and perhaps by other names.
2. See Fouque et al. (2000), for example, for a discussion of the dependence of the variance estimate on the sampling interval.
3. A discussion of this type of analytical methodology can be found in Fouque et al. (2000).
4. Figure 2 has only 241 days of performance because no tick data were collected for the nine days from 24 October through 3 November as a result of office closures resulting from Hurricane Wilma.

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