5S.1
Graphical Representation of One-Dimensional, Transient Conduction in the Plane Wall, Long Cylinder, and Sphere

In Sections 5.5 and 5.6, one-term approximations have been developed for transient, one-dimensional conduction in a plane wall (with symmetrical convection conditions) and radial systems (long cylinder and sphere). The results apply for $Fo > 0.2$ and can conveniently be represented in graphical forms that illustrate the functional dependence of the transient temperature distribution on the Biot and Fourier numbers.

Results for the plane wall (Figure 5.6*a*) are presented in Figures 5S.1 through 5S.3. Figure 5S.1 may be used to obtain the *midplane* temperature of the wall, $T(0, t) \equiv T_o(t)$, at any time during the transient process. If T_o is known for particular values of Fo and Bi, Figure 5S.2 may be used to determine the corresponding temperature at any location *off the midplane.* Hence Figure 5S.2 must be used in conjunction with Figure 5S.1. For example, if one wishes to determine the surface temperature ($x^* = \pm 1$) at some time t, Figure 5S.1 would first be used to determine T_o at t. Figure 5S.2 would then be used to determine the surface temperature from knowledge of T_o. The

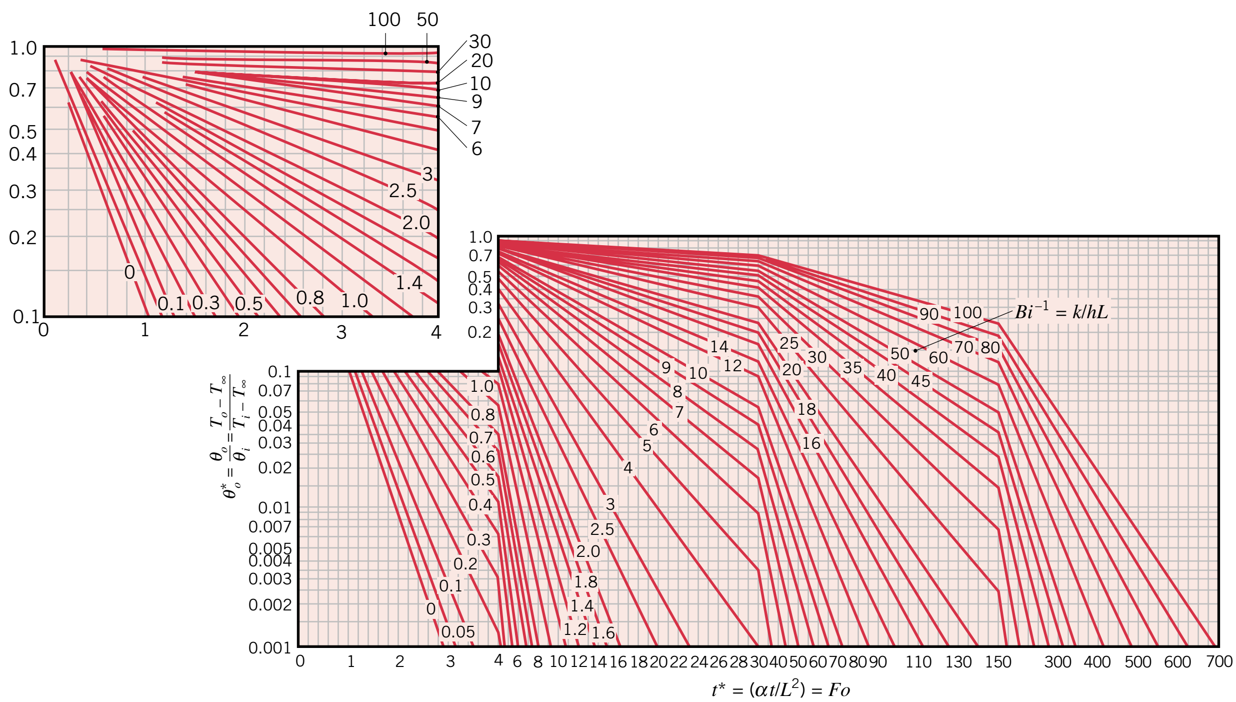


FIGURE 5S.1 Midplane temperature as a function of time for a plane wall of thickness $2L$ [1]. Used with permission.

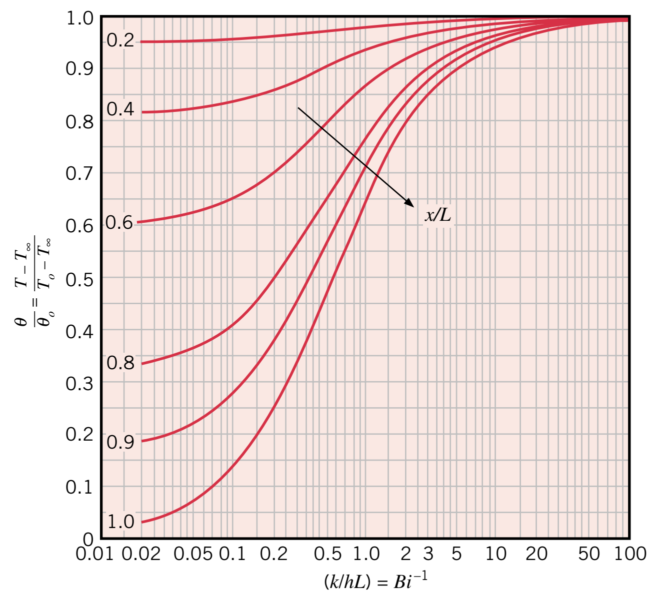


FIGURE 5S.2 Temperature distribution in a plane wall of thickness $2L$ [1]. Used with permission.

procedure would be inverted if the problem were one of determining the time required for the surface to reach a prescribed temperature.

Graphical results for the energy transferred from a plane wall over the time interval t are presented in Figure 5S.3. These results were generated from Equation 5.46. The dimensionless energy transfer Q/Q_o is expressed exclusively in terms of Fo and Bi.

Results for the infinite cylinder are presented in Figures 5S.4 through 5S.6, and those for the sphere are presented in Figures 5S.7 through 5S.9, where the Biot number is defined in terms of the radius r_o.

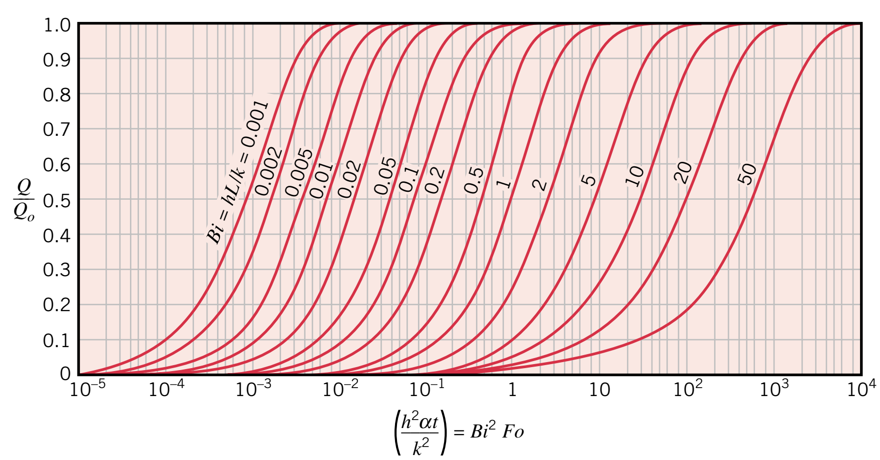


FIGURE 5S.3 Internal energy change as a function of time for a plane wall of thickness $2L$ [2]. Adapted with permission.

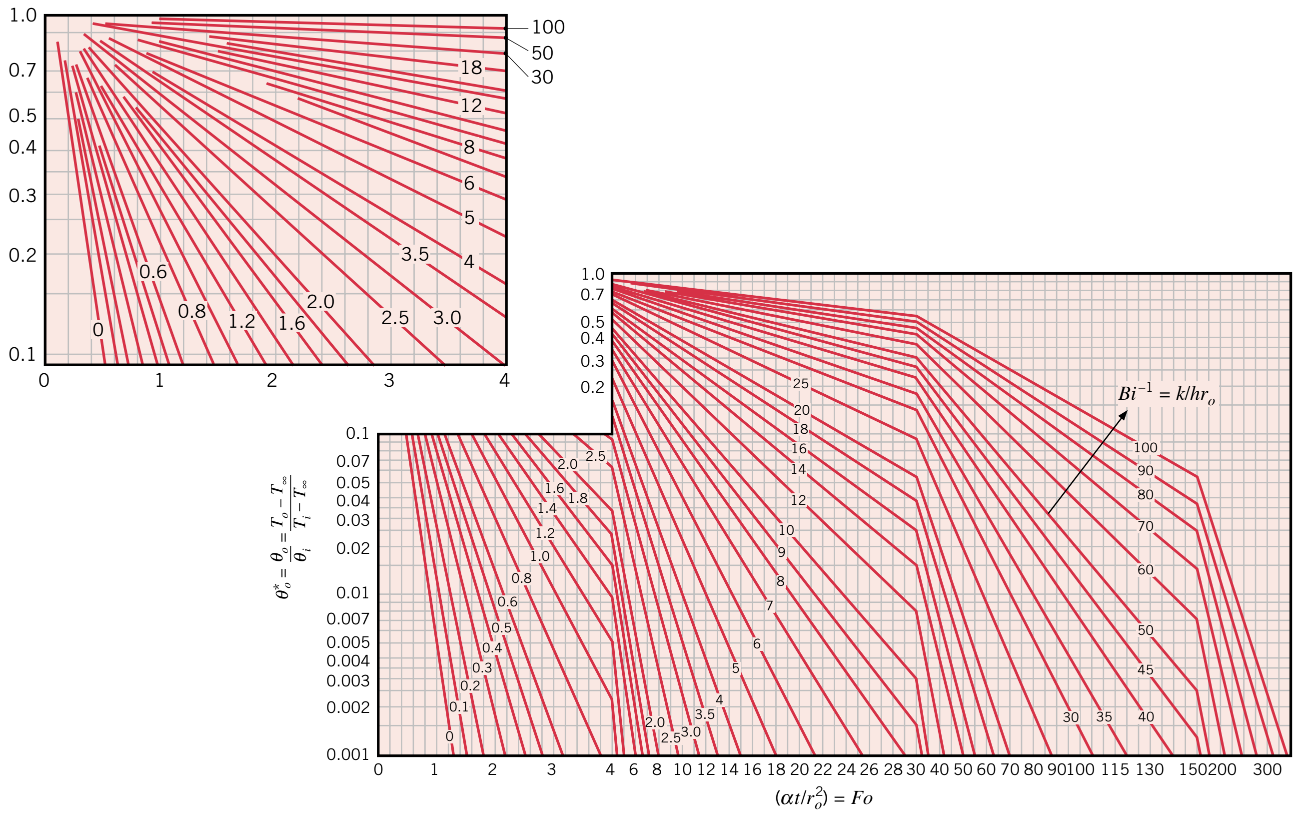


FIGURE 5S.4 Centerline temperature as a function of time for an infinite cylinder of radius r_o [1]. Used with permission.

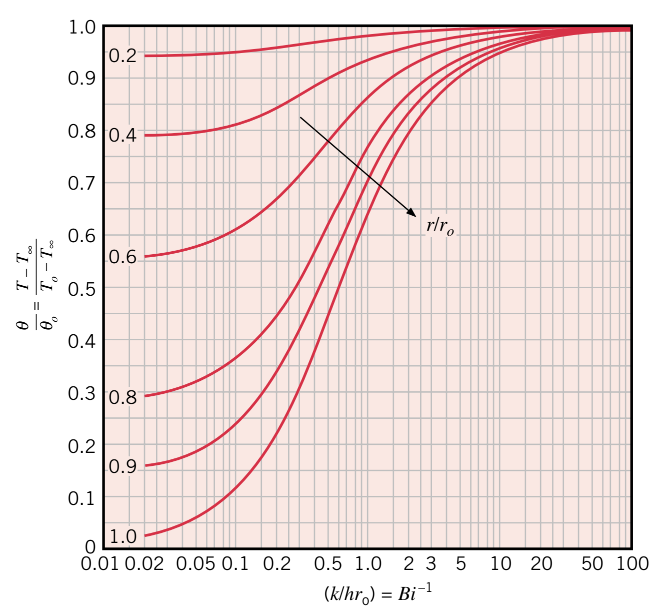


FIGURE 5S.5 Temperature distribution in an infinite cylinder of radius r_o [1]. Used with permission.

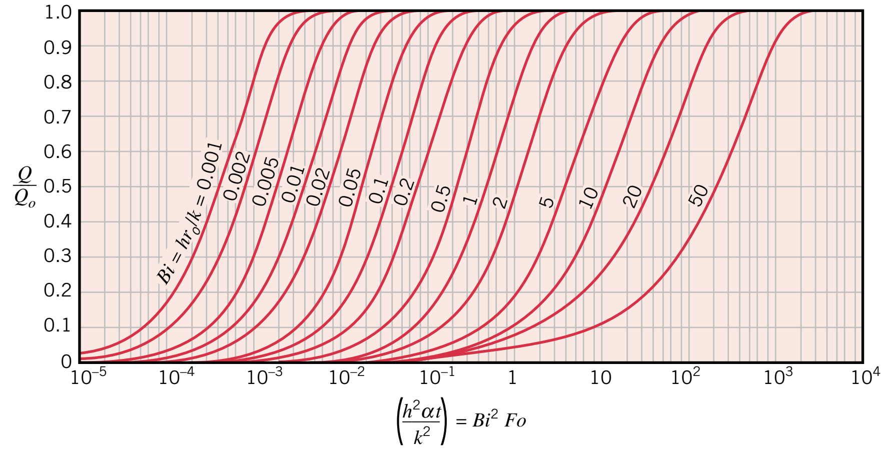


FIGURE 5S.6 Internal energy change as a function of time for an infinite cylinder of radius r_o [2]. Adapted with permission.

The foregoing charts may also be used to determine the transient response of a plane wall, an infinite cylinder, or sphere subjected to a *sudden change in surface temperature*. For such a condition it is only necessary to replace T_∞ by the prescribed surface temperature T_s and to set Bi^{-1} equal to zero. In so doing, the convection coefficient is tacitly assumed to be infinite, in which case $T_\infty = T_s$.

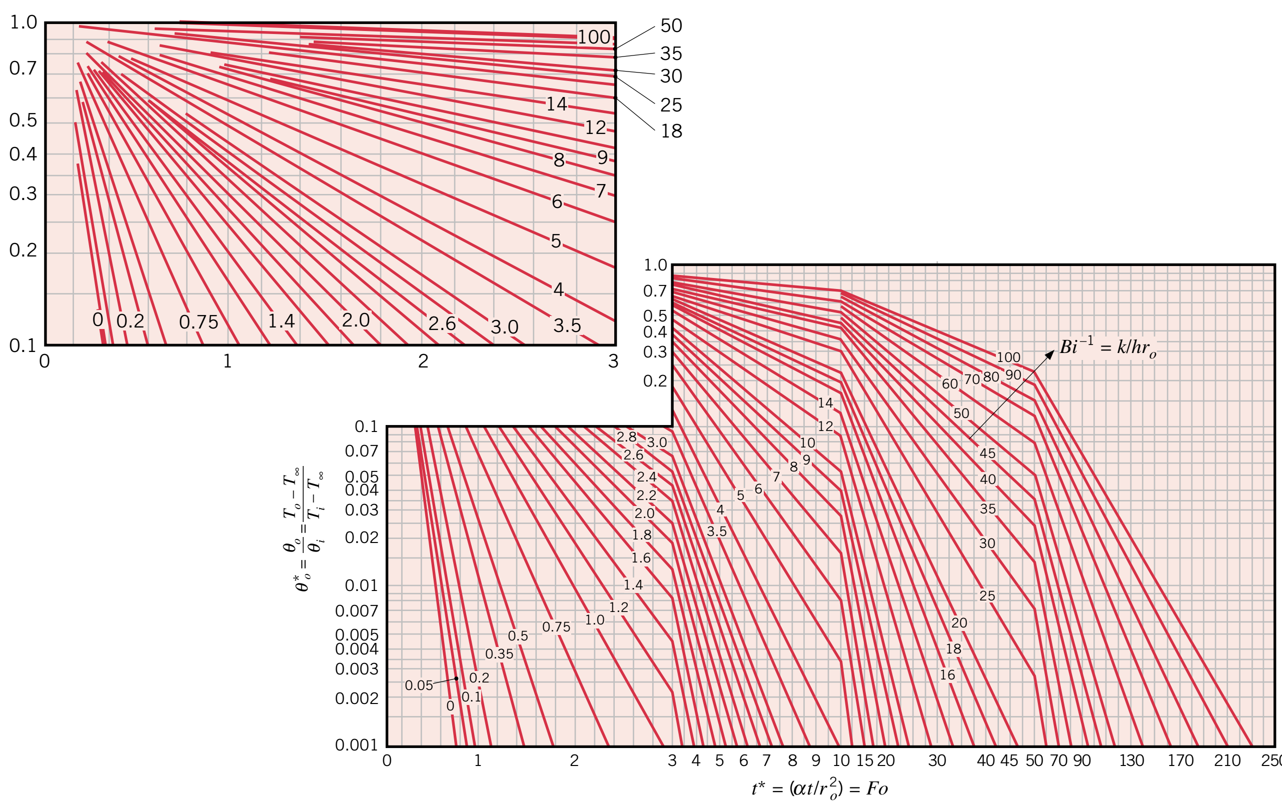


FIGURE 5S.7 Center temperature as a function of time in a sphere of radius r_o [1]. Used with permission.

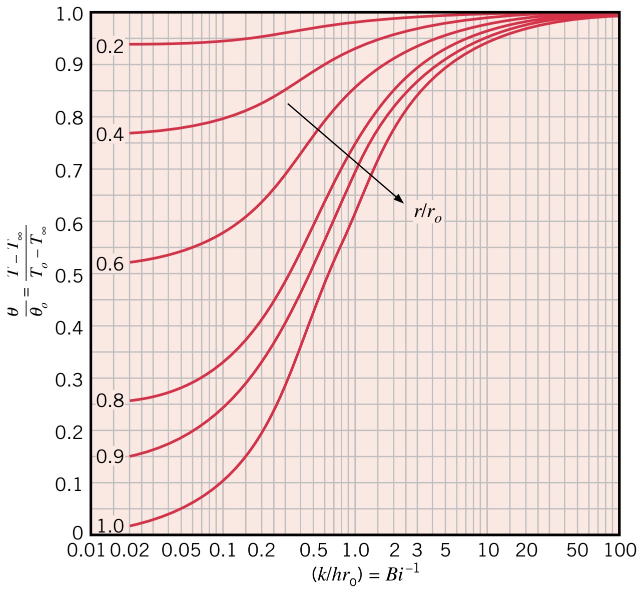


FIGURE 5S.8 Temperature distribution in a sphere of radius r_o [1]. Used with permission.

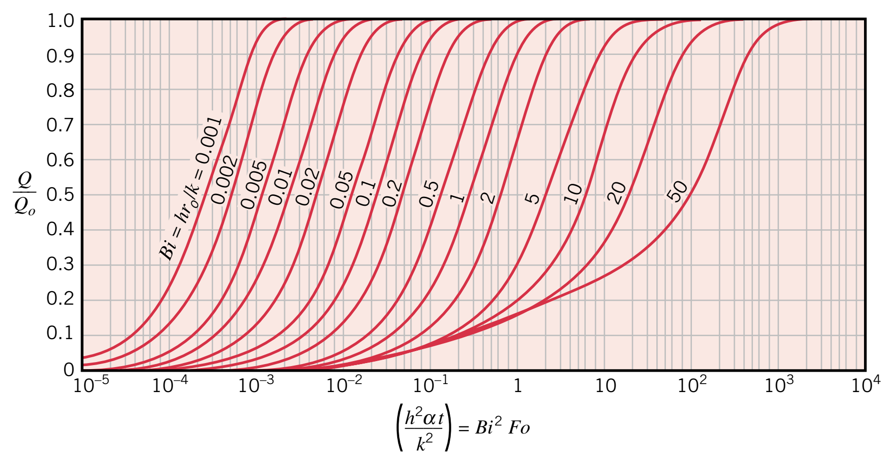


FIGURE 5S.9 Internal energy change as a function of time for a sphere of radius r_o [2]. Adapted with permission.

References

1. Heisler, M. P., *Trans. ASME,* **69,** 227–236, 1947.
2. Gröber, H., S. Erk, and U. Grigull, *Fundamentals of Heat Transfer,* McGraw-Hill, New York, 1961.

5S.2
Analytical Solution of Multidimensional Effects

Transient problems are frequently encountered for which two- and even three-dimensional effects are significant. Solution to a class of such problems can be obtained from the one-dimensional analytical results of Sections 5.5 through 5.7.

Consider immersing the *short* cylinder of Figure 5S.10, which is initially at a uniform temperature T_i, in a fluid of temperature $T_\infty \neq T_i$. Because the length and diameter are comparable, the subsequent transfer of energy by conduction will be significant for both the r and x coordinate directions. The temperature within the cylinder will therefore depend on r, x, and t.

Assuming constant properties and no generation, the appropriate form of the heat equation is, from Equation 2.24,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where x has been used in place of z to designate the axial coordinate. A closed-form solution to this equation may be obtained by the separation of variables method. Although we will not consider the details of this solution, it is important to note that the end result may be expressed in the following form:

$$\frac{T(r,x,t)-T_\infty}{T_i-T_\infty} = \left.\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

That is, the two-dimensional solution may be expressed as a *product* of one-dimensional solutions that correspond to those for a plane wall of thickness $2L$ and an infinite cylinder of radius r_o. For $Fo > 0.2$, these solutions are provided by the one-term approximations of Equations 5.40 and 5.49, as well as by Figures 5S.1 and 5S.2 for the plane wall and Figures 5S.4 and 5S.5 for the infinite cylinder.

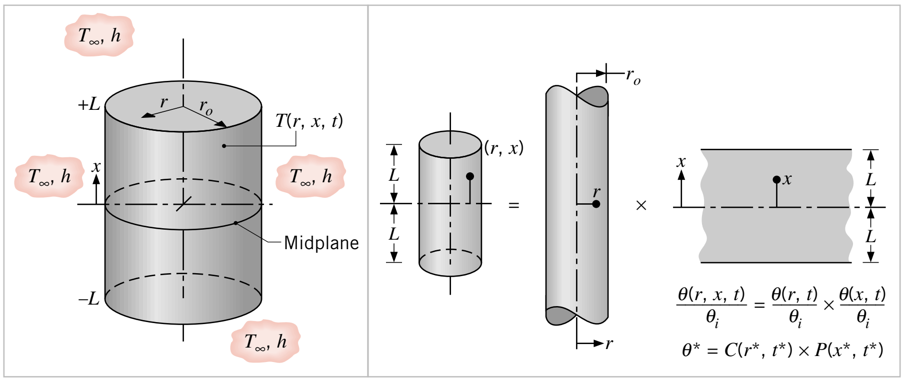


FIGURE 5S.10 Two-dimensional, transient conduction in a short cylinder. (*a*) Geometry. (*b*) Form of the product solution.

Results for other multidimensional geometries are summarized in Figure 5S.11. In each case the multidimensional solution is prescribed in terms of a product involving one or more of the following one-dimensional solutions:

$$S(x, t) \equiv \left.\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Semi-infinite}\\ \text{solid}}} \tag{5S.1}$$

$$P(x, t) \equiv \left.\frac{T(x, t) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\ \text{wall}}} \tag{5S.2}$$

$$C(r, t) \equiv \left.\frac{T(r, t) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\ \text{cylinder}}} \tag{5S.3}$$

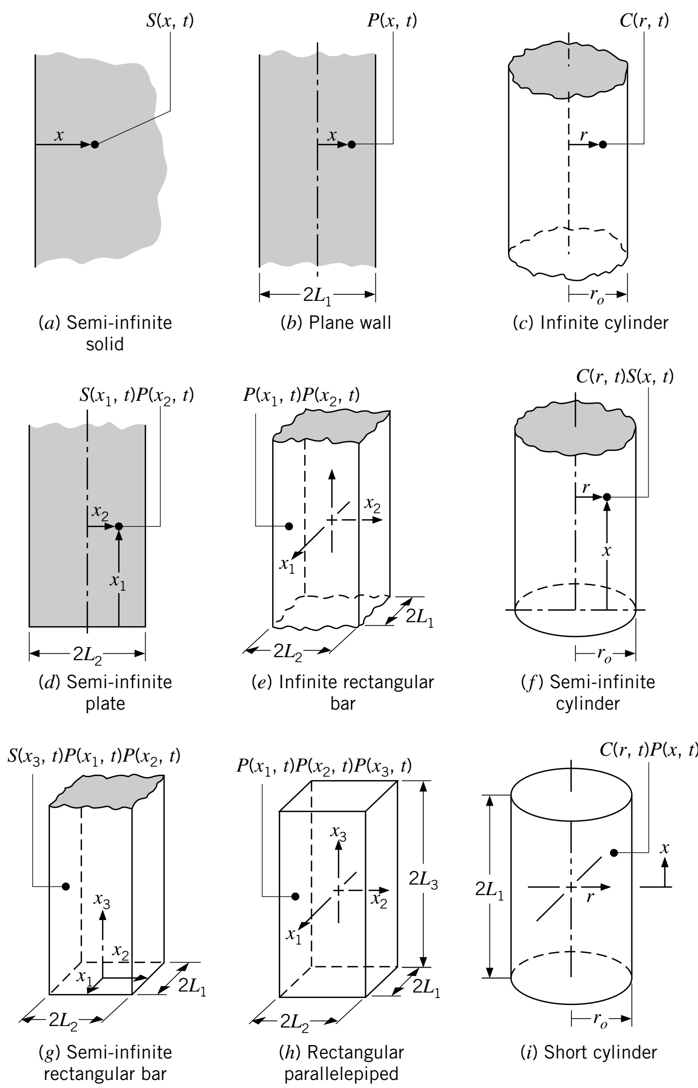


FIGURE 5S.11 Solutions for multidimensional systems expressed as products of one-dimensional results.

The x coordinate for the semi-infinite solid is measured from the surface, whereas for the plane wall it is measured from the midplane. In using Figure 5S.11 the coordinate origins should carefully be noted. The transient, three-dimensional temperature distribution in a rectangular parallelepiped, Figure 5S.11h, is then, for example, the product of three one-dimensional solutions for plane walls of thicknesses $2L_1$, $2L_2$, and $2L_3$. That is,

$$\frac{T(x_1, x_2, x_3, t) - T_\infty}{T_i - T_\infty} = P(x_1, t) \cdot P(x_2, t) \cdot P(x_3, t)$$

The distances x_1, x_2, and x_3 are all measured with respect to a rectangular coordinate system whose origin is at the center of the parallelepiped.

The amount of energy Q transferred to or from a solid during a multidimensional transient conduction process may also be determined by combining one-dimensional results, as shown by Langston [1].

EXAMPLE 5S.1

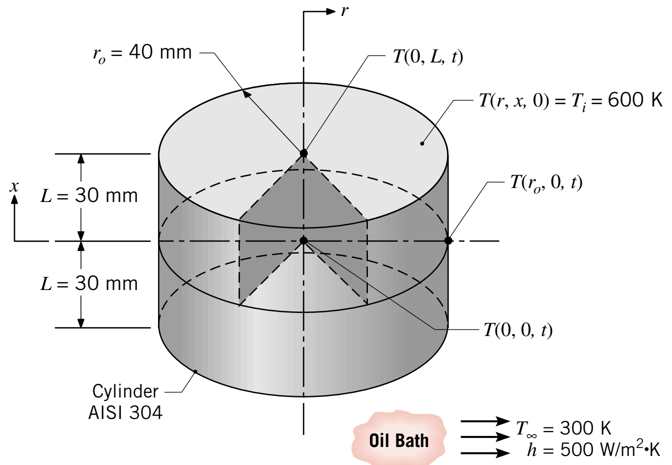
In a manufacturing process stainless steel cylinders (AISI 304) initially at 600 K are quenched by submersion in an oil bath maintained at 300 K with $h = 500 \text{ W/m}^2 \cdot \text{K}$. Each cylinder is of length $2L = 60$ mm and diameter $D = 80$ mm. Consider a time 3 min into the cooling process and determine temperatures at the center of the cylinder, at the center of a circular face, and at the midheight of the side. Note that Problem 5.124 requires a numerical solution of the same problem using *FEHT*.

SOLUTION

Known: Initial temperature and dimensions of cylinder and temperature and convection conditions of an oil bath.

Find: Temperatures $T(r, x, t)$ after 3 min at the cylinder center, $T(0, 0, 3 \text{ min})$, at the center of a circular face, $T(0, L, 3 \text{ min})$, and at the midheight of the side, $T(r_o, 0, 3 \text{ min})$.

Schematic:



Assumptions:

1. Two-dimensional conduction in r and x.
2. Constant properties.

Properties: Table A.1, stainless steel, AISI 304 [$T = (600 + 300)/2 = 450$ K]: $\rho = 7900$ kg/m^3, $c = 526$ J/kg · K, $k = 17.4$ W/m · K, $\alpha = k/\rho c = 4.19 \times 10^{-6}$ m^2/s.

Analysis: The solid steel cylinder corresponds to case (i) of Figure 5S.11, and the temperature at any point in the cylinder may be expressed as the following product of one-dimensional solutions.

$$\frac{T(r, x, t) - T_\infty}{T_i - T_\infty} = P(x, t)C(r, t)$$

where $P(x, t)$ and $C(r, t)$ are defined by Equations 5S.2 and 5S.3, respectively. Accordingly, for the center of the cylinder,

$$\frac{T(0, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

Hence, for the plane wall, with

$$Bi^{-1} = \frac{k}{hL} = \frac{17.4\text{ W/m}\cdot\text{K}}{500\text{ W/m}^2\cdot\text{K} \times 0.03\text{ m}} = 1.16$$

$$Fo = \frac{\alpha t}{L^2} = \frac{4.19 \times 10^{-6}\text{ m}^2\text{/s} \times 180\text{ s}}{(0.03\text{ m})^2} = 0.84$$

it follows from Equation 5.41 that

$$\theta_o^* = \frac{\theta_o}{\theta_i} = C_1 \exp(-\zeta_1^2 Fo)$$

where, with $Bi = 0.862$, $C_1 = 1.109$ and $\zeta_1 = 0.814$ rad from Table 5.1. With $Fo = 0.84$,

$$\frac{\theta_o}{\theta_i} = \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} = 1.109 \exp[-(0.814\text{ rad})^2 \times 0.84] = 0.636$$

Similarly, for the infinite cylinder, with

$$Bi^{-1} = \frac{k}{hr_o} = \frac{17.4\text{ W/m}\cdot\text{K}}{500\text{ W/m}^2\cdot\text{K} \times 0.04\text{ m}} = 0.87$$

$$Fo = \frac{\alpha t}{r_o^2} = \frac{4.19 \times 10^{-6}\text{ m}^2\text{/s} \times 180\text{ s}}{(0.04\text{ m})^2} = 0.47$$

it follows from Equation 5.49c that

$$\theta_o^* = \frac{\theta_o}{\theta_i} = C_1 \exp(-\zeta_1^2 Fo)$$

where, with $Bi = 1.15$, $C_1 = 1.227$ and $\zeta_1 = 1.307$ rad from Table 5.1. With $Fo = 0.47$,

$$\left.\frac{\theta_o}{\theta_i}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} = 1.109 \exp[-(1.307\text{ rad})^2 \times 0.47] = 0.550$$

Hence, for the center of the cylinder,

$$\frac{T(0, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = 0.636 \times 0.550 = 0.350$$

$$T(0, 0, 3\text{ min}) = 300\text{ K} + 0.350(600 - 300)\text{ K} = 405\text{ K} \qquad \triangleleft$$

The temperature at the center of a circular face may be obtained from the requirement that

$$\frac{T(0, L, 3\text{ min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

where, from Equation 5.40b,

$$\frac{\theta^*}{\theta_o^*} = \frac{\theta}{\theta_o} = \cos(\zeta_1 x^*)$$

Hence, with $x^* = 1$, we have

$$\frac{\theta(L)}{\theta_o} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} = \cos(0.814\text{ rad} \times 1) = 0.687$$

Hence

$$\left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}}$$

$$\left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} = 0.687 \times 0.636 = 0.437$$

Hence

$$\frac{T(0, L, 3\text{ min}) - T_\infty}{T_i - T_\infty} = 0.437 \times 0.550 = 0.240$$

$$T(0, L, 3\text{ min}) = 300\text{ K} + 0.24(600 - 300)\text{ K} = 372\text{ K} \qquad \triangleleft$$

The temperature at the midheight of the side may be obtained from the requirement that

$$\frac{T(r_o, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

where, from Equation 5.49b,

$$\frac{\theta^*}{\theta_o^*} = \frac{\theta}{\theta_o} = J_0(\zeta_1 r^*)$$

With $r^* = 1$ and the value of the Bessel function determined from Table B.4,

$$\frac{\theta(r_o)}{\theta_o} = \left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} = J_0(1.307\text{ rad} \times 1) = 0.616$$

Hence

$$\left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} = \left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

$$\left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} = 0.616 \times 0.550 = 0.339$$

Hence

$$\frac{T(r_o, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = 0.636 \times 0.339 = 0.216$$

$$T(r_o, 0, 3\text{ min}) = 300\text{ K} + 0.216(600 - 300)\text{ K} = 365\text{ K} \qquad \triangleleft$$

Comments:

1. Verify that the temperature at the edge of the cylinder is $T(r_o, L, 3\text{ min}) = 344$ K.
2. The Heisler charts of Section 5S.1 could also be used to obtain the desired results. Accessing these charts, one would obtain $\theta_o/\theta_i|_{\text{Plane wall}} \approx 0.64$, $\theta_o/\theta_i|_{\text{Infinite cylinder}} \approx 0.55$, $\theta(L)/\theta_o|_{\text{Plane wall}} \approx 0.68$, and $\theta(r_o)/\theta_o|_{\text{Infinite cylinder}} \approx 0.61$, which are in good agreement with results obtained from the one-term approximations.
3. The *IHT Models*, *Transient Conduction* option for the *Plane Wall* and *Infinite Cylinder* may be used to calculate temperature ratios required for the foregoing product solution.

Reference

1. Langston, L.S., *Int. J. Heat Mass Transfer*, **25**, 149–150, 1982.

Problems

One-Dimensional Conduction: The Plane Wall

5S.1 Consider the thermal energy storage unit of Problem 5.11, but with a masonry material of $\rho = 1900$ kg/m^3, $c = 800$ J/kg · K, and $k = 0.70$ W/m · K used in place of the aluminum. How long will it take to achieve 75% of the maximum possible energy storage? What are the maximum and minimum temperatures of the masonry at this time?

5S.2 An ice layer forms on a 5-mm-thick windshield of a car while parked during a cold night for which the ambient temperature is −20°C. Upon start-up, using a new defrost system, the interior surface is suddenly exposed to an airstream at 30°C. Assuming that the ice behaves as an insulating layer on the exterior surface, what interior convection coefficient would allow the exterior surface to reach 0°C in 60 s? The windshield thermophysical properties are $\rho = 2200$ kg/m^3, $c_p = 830$ J/kg · K, and $k = 1.2$ W/m · K.

One-Dimensional Conduction: The Long Cylinder

5S.3 Cylindrical steel rods (AISI 1010), 50 mm in diameter, are heat treated by drawing them through an oven 5 m long in which air is maintained at 750°C. The rods enter at 50°C and achieve a centerline temperature of 600°C before leaving. For a convection coefficient of 125 W/m$^2 \cdot$ K, estimate the speed at which the rods must be drawn through the oven.

5S.4 Estimate the time required to cook a hot dog in boiling water. Assume that the hot dog is initially at 6°C, that the convection heat transfer coefficient is 100 W/m$^2 \cdot$ K, and that the final temperature is 80°C at the centerline. Treat the hot dog as a long cylinder of 20-mm diameter having the properties: $\rho = 880$ kg/m^3, $c = 3350$ J/kg $\cdot$ K, and $k = 0.52$ W/m $\cdot$ K.

5S.5 A long bar of 70-mm diameter and initially at 90°C is cooled by immersing it in a water bath that is at 40°C and provides a convection coefficient of 20 W/m$^2 \cdot$ K. The thermophysical properties of the bar are $\rho =$ 2600 kg/m^3, $c = 1030$ J/kg $\cdot$ K, and $k = 3.50$ W/m $\cdot$ K.

(a) How long should the bar remain in the bath in order that, when it is removed and allowed to equilibrate while isolated from any surroundings, it achieves a uniform temperature of 55°C?

(b) What is the surface temperature of the bar when it is removed from the bath?

One-Dimensional Conduction: The Sphere

5S.6 A sphere of 80-mm diameter ($k = 50$ W/m $\cdot$ K and $\alpha = 1.5 \times 10^{-6}$ m^2/s) is initially at a uniform, elevated temperature and is quenched in an oil bath maintained at 50°C. The convection coefficient for the cooling process is 1000 W/m$^2 \cdot$ K. At a certain time, the surface temperature of the sphere is measured to be 150°C. What is the corresponding center temperature of the sphere?

5S.7 A spherical hailstone that is 5 mm in diameter is formed in a high-altitude cloud at -30°C. If the stone begins to fall through warmer air at 5°C, how long will it take before the outer surface begins to melt? What is the temperature of the stone's center at this point in time, and how much energy (J) has been transferred to the stone? A convection heat transfer coefficient of 250 W/m$^2 \cdot$ K may be assumed, and the properties of the hailstone may be taken to be those of ice.

5S.8 In a process to manufacture glass beads ($k = 1.4$ W/m $\cdot$ K, $\rho = 2200$ kg/m^3, $c_p = 800$ J/kg $\cdot$ K) of 3-mm diameter, the beads are suspended in an upwardly directed airstream that is at $T_\infty = 15$°C and maintains a convection coefficient of $h = 400$ W/m$^2 \cdot$ K.

(a) If the beads are at an initial temperature of $T_i =$ 477°C, how long must they be suspended to achieve a center temperature of 80°C? What is the corresponding surface temperature?

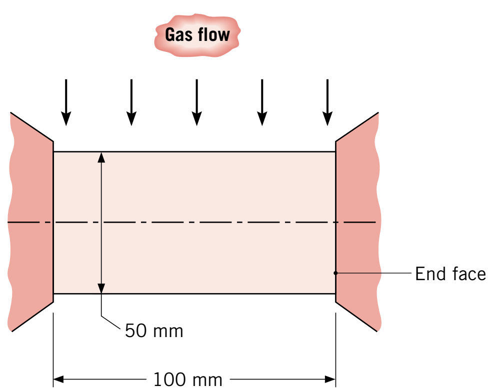
$\boxed{\text{(b)}}$ Compute and plot the center and surface temperatures as a function of time for $0 \leq t \leq 20$ s and $h = 100$, 400, and 1000 W/m$^2 \cdot$ K.

Multidimensional Conduction

5S.9 A long steel (plain carbon) billet of square cross section 0.3 m by 0.3 m, initially at a uniform temperature of 30°C, is placed in a soaking oven having a temperature of 750°C. If the convection heat transfer coefficient for the heating process is 100 W/m$^2 \cdot$ K, how long must the billet remain in the oven before its center temperature reaches 600°C?

5S.10 Fireclay brick of dimensions 0.06 m $\times$ 0.09 m $\times$ 0.20 m is removed from a kiln at 1600 K and cooled in air at 40°C with $h = 50$ W/m$^2 \cdot$ K. What is the temperature at the center and at the corners of the brick after 50 min of cooling?

5S.11 A cylindrical copper pin 100 mm long and 50 mm in diameter is initially at a uniform temperature of 20°C. The end faces are suddenly subjected to an intense heating rate that raises them to a temperature of 500°C. At the same time, the cylindrical surface is subjected to heating by gas flow with a temperature of 500°C and a heat transfer coefficient of 100 W/m$^2 \cdot$ K.



(a) Determine the temperature at the center point of the cylinder 8 s after sudden application of the heat.

(b) Considering the parameters governing the temperature distribution in transient heat diffusion problems, can any simplifying assumptions be justified in analyzing this particular problem? Explain briefly.

5S.12 Recalling that your mother once said that meat should be cooked until every portion has attained a temperature of 80°C, how long will it take to cook a 2.25-kg roast? Assume that the meat is initially at 6°C and that the oven temperature is 175°C with a convection heat transfer coefficient of 15 W/m$^2 \cdot$ K. Treat the roast as a cylinder with properties of liquid water, having a diameter equal to its length.

5S.13 A long rod 20 mm in diameter is fabricated from alumina (polycrystalline aluminum oxide) and is initially at a uniform temperature of 850 K. The rod is suddenly exposed to fluid at 350 K with $h = 500\ W/m^2 \cdot K$. Estimate the centerline temperature of the rod after 30 s at an exposed end and at an axial distance of 6 mm from the end.

5S.14 Consider the stainless steel cylinder of Example 5S.1, which is initially at 600 K and suddenly quenched in an oil bath at 300 K with $h = 500\ W/m^2 \cdot K$. Use the *Transient Conduction*, *Plane Wall and Cylinder* models of *IHT* to obtain the following solutions.

(a) Calculate the temperatures, $T(r, x, t)$, after 3 min at the cylinder center, $T(0, 0, 3\ min)$, at the center of a circular face, $T(0, L, 3\ min)$, and at the midheight of the side, $T(r_o, 0, 3\ min)$. Compare your results with those in the example.

(b) Use the *Explore* and *Graph* options of *IHT* to calculate and plot temperature histories at the cylinder center, $T(0, 0, t)$, and the midheight of the side, $T(r_o, 0, t)$, for $0 \leq t \leq 10$ min. Comment on the gradients occurring at these locations and what effect they might have on phase transformations and thermal stresses. *Hint*: In your sweep over the time variable, start at 1 s rather than zero.

(c) For $0 \leq t \leq 10$ min, calculate and plot temperature histories at the cylinder center, $T(0, 0, t)$, for convection coefficients of $500\ W/m^2 \cdot K$ and $1000\ W/m^2 \cdot K$.