The Shape and Dynamics of the Rikitake Attractor

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Abstract

The Rikitake dynamical system is a model which attempts to explain the irregular polarity switching of the geomagnetic field. The system exhibits Lorenz-type chaos and orbiting around two unstable fixed points. We employ techniques in the vein of the Lyapunov function method in an attempt to characterize the dynamics on the attractor of the system.

1 Introduction

Geophysicists have long been puzzled by one striking aspect of the Earth's geomagnetic field. The dipole has reversed its polarity many times over geological history. The average interval between geomagnetic polarity reversals is about 7×10^5 years. However, the time series of reversals is highly irregular: there have been intervals as long as 3×10^7 years when the polarity apparently remained unchanged. One model which attempts to explain the reversal of the Earth's magnetic field is the Rikitake system [Rikitake]. This system describes the currents of two coupled dynamo disks. The governing equations are:

$$\left.\begin{aligned} \dot{x} &= -\mu x + zy \\ \dot{y} &= -\mu y + (z-a)x \\ \dot{z} &= 1 - xy \end{aligned}\right\} = \vec{f}(x,y,z) \tag{1}$$

Here a and μ are parameters which we will assume to be nonnegative. The system (1) has two fixed points, at

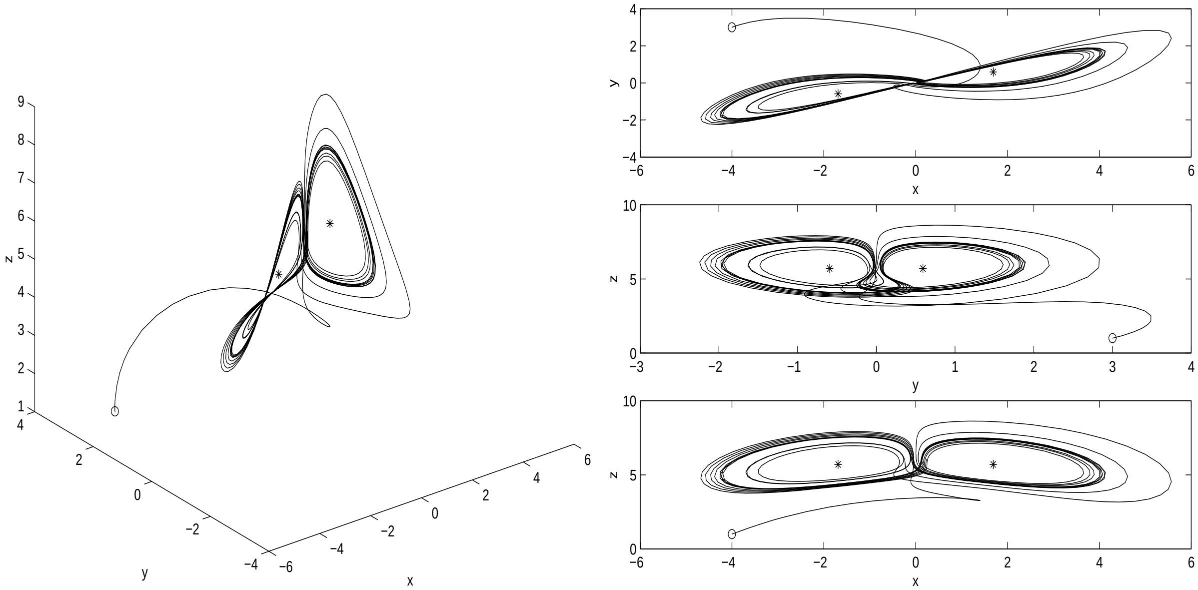
$$\vec{x}_1 = (K,\ (-1+\sqrt{a^2+4\mu^2})K/2\mu,\ \mu K^2)$$
$$\vec{x}_2 = (-K, (1-\sqrt{a^2+4\mu^2})K/2\mu, \mu K^2)$$

where

$$K = \frac{\sqrt{\frac{a}{\mu} + \frac{\sqrt{a^2+4\mu^2}}{\mu}}}{\sqrt{2}}$$

The linearized system about either of the fixed points $\vec{x}_1$ and $\vec{x}_2$ has eigenvalues

$$-2\mu,\ -\frac{i(a^2+4\mu^2)^{\frac{1}{4}}}{\sqrt{\mu}},\ \frac{i(a^2+4\mu^2)^{\frac{1}{4}}}{\sqrt{\mu}}$$



(a) (b)

Figure 1: Numerical Solution of the Rikitake System (1). The circle (o) is the initial condition. Stars (*) are the fixed points $\vec{x}_1$ and $\vec{x}_2$. In this simulation $a = 5$ and $\mu = 2$. The simulation was run to time $t = 100$. Figure (a) shows the 3-dimensional flow, while (b) shows the projection of the flow onto the planes

So there is a stable and a center manifold through each of the fixed points. Thus the dynamics take place on the center manifold, on which the fixed points are both unstable (see [Ito]).

2 General Dynamics

We can get an idea of the dynamics by solving the system numerically. To this end, we employ a fourth order adaptive Runge-Kutta-Fehlberg algorithm to solve (1). In figure (1) we see the results of the simulation. The system is attracted toward the fixed points, where it oscillates around one or the other fixed point. Occasionally the oscillations switch from one fixed point to the other. Physically this corresponds to a switch in the polarity of the Earth's dipole.

3 The Attractor

The Rikitake system is a dissipative system for all positive values of μ because the divergence of the vector field, $\nabla \cdot \vec{f}(x, y, z) = -2\mu$, is negative. Hence 3-dimensional volumes in the phase space contract at a uniform exponential rate. This suggests that the dynamics may tend to an "attractor" as $t \to \infty$. The word "attractor" has many meanings, so we will define what we mean by it in this context.

Consider all initial conditions in a closed ball B_ρ of radius ρ in the phase space. Let $B_\rho(t)$ be the image of B_ρ under the system's evolution for a time t. Consider the set

$$A_\rho = \cap_{t>0} B_\rho(t)$$

This is the set of all points in the phase space which can be reached by starting at some point in B_ρ at an arbitrarily long time in the past. We define the global attractor as

$$A = \cup_{\rho>0} A_\rho$$

The global attractor is the set of all points in phase space which can be reached by starting from some point at arbitrarily long times in the past. Some of the properties of A are obvious:

- A is invariant under the evolution.
- The distance of any solution from A vanishes as $t \to \infty$

So if we start from any initial condition we will get arbitrarily close to the attractor as $t \to \infty$

Understanding the dynamics, then, entails having a picture of what the global attractor looks like.

Numerical simulations of the Rikitake system suggest that the system has an attractor which is bounded. That is, solutions seem to enter a ball around the origin from which they never escape. This is not true however, as we can see from the following example.

Take as an initial condition $(x(0), y(0), z(0)) = (0, 0, k)$. Then

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = 1$$

The solution of this system is

$$x = 0, \quad y = 0, \quad z = t + k \tag{2}$$

The flow approaches ∞ in the l^2 norm as $t \to \infty$!

We suspect, however, that this is a special case, in that if x and y are not both zero, then the flow will tend toward a bounded set. Whether or not this is true depends partly on whether or not the solution (2) is stable. We take up this question next.

4 (In)Stability of the z-Axis Solution

First of all, notice that the z-axis is an invariant manifold. If the flow is on the z-axis ($x = y = 0$) it never escapes. On the other hand, if the flow is not on the z-axis, it can never enter it. (The solution (2) is symmetric in time.) Therefore, if we can show that for points arbirtrarily close to the z-axis, the flow takes them back to a bounded attractor, then we can regard the z-axis as unimportant to the dynamics of the system.

To this end, we examine the stability of the solution (2). Let $(\delta x, \delta y, \delta z)$ be a small perturbation of (2). That is,

$$x = \delta x, \quad y = \delta y, \quad z = t + k + \delta z$$

Then $(\delta x, \delta y, \delta z)$ satisfies

$$\left\{ \begin{array}{ll} \dot{\delta x} = & -\mu\delta x + \delta y(t + k + \delta z) \\ \dot{\delta y} = & -\mu\delta y + (t + k + \delta z - a)\delta x \\ \dot{\delta z} = & -\delta x\delta y \end{array} \right. \tag{3}$$

The solution (2) is stable if and only if the $\vec{0}$ solution to (3) is stable. We perform the standard stability analysis of (3). We first linearize about $\vec{0}$, giving

$$\left\{ \begin{array}{ll} \dot{\delta x} = & -\mu\delta x + (t + k)\delta y \\ \dot{\delta y} = & (t + k - a)\delta x - \mu\delta y \end{array} \right. \tag{4}$$

(In the linearization $\dot{\delta z} = 0$.) This is equivalent to the ODE

$$(t+k)\ddot{X} + (2\mu(t+k) - 1)\dot{X} + (\mu^2(t+k) - (t+k)^2(t+k-a) - \mu)X = 0 \tag{5}$$

where we have let $X = \delta x$. The solution (2) is stable, then, if and only if $X = 0$ is a stable solution of (5), or, if and only if $(\delta x, \delta y) = (0, 0)$ is a stable solution of (4).

Notice that (5) can be written as

$$\dot{X} = \frac{(t+k)^2(t+k-a) + \mu}{2\mu(t+k) - 1}X - \frac{t+k}{2\mu(t+k) - 1}\ddot{X} \tag{6}$$

We can make a (non-rigorous) argument that the 0 solution to (5) is unstable as follows. The coefficient of X in (6) increases without bound as t increases while the coefficient of $\ddot{X}$ converges to $\frac{1}{2\mu}$. This suggests that for large t the ratio of $\dot{X}$ to X will be large, meaning that X will tend to increase. Thus the 0 solution is unstable.

We can also make the following argument: If $x(0)$ and $y(0)$ are not both zero then

$$\begin{aligned}
\tfrac{d}{dt}(xy) = \quad \dot{x}y + x\dot{y} &= (-\mu x + zy)y + x(-\mu y + (z-a)x) \\
&= z(x^2 + y^2) - ax^2 - 2\mu xy \\
&\geq z(x^2 + y^2) - \mu(x^2 + y^2) - ax^2 \\
&= (z - \mu)(x^2 + y^2) - ax^2 \\
&> 0 \text{ if } z > \mu + a,
\end{aligned}$$

since x and y are never both zero. Thus as z increases the quantity xy is increasing, pushing (x, y) out from the z-axis, making the z-axis solution unstable.

It seems reasonable to conjecture that the z-axis is very unstable. However, it also seems likely that a trajectory can go arbirtrarily far from the origin before being attracted back toward the attractor, by starting sufficiently close to the z-axis.

5 Bounds on the Attractor

We now attempt to find a bound on the attractor by employing a technique discussed in [Doering] which uses Lyapunov-type functions to characterize the dynamics.

The problem is that the attractor is not bounded! The existence of the $z-$axis solution (2) makes it impossible to ever find a bound on the attractor of the Rikitake system (1). However, in view of our previous remarks on the likelihood of the $z-$axis solution being unstable, it may be possible to bound "the attractor minus the point $z = \infty$." That is we may be able to find a bounding surface S such that if $x(0)$ and $y(0)$ are not both zero, then the flow eventually goes inside the surface S and never leaves. Another glance at figure (1) should convince the reader that this is possible.

The method employed in [Doering] is as follows. Look for a Lyapunov-type quantity $K(x, y, z)$ such that K decreases for large values of (x, y, z), i.e. $\frac{d}{dt}K < 0$ for large x, y, z. As $t \to \infty$, then, K decreases to at least its maximal value over the set $\frac{d}{dt}K \geq 0$. Thus, as $t \to \infty$ (x, y, z) must lie inside the surface S defined by

$$K(x, y, z) = K_{\max},$$

where

$$K_{\max} = \max\{K(x, y, z) | \frac{d}{dt}K(x, y, z) \geq 0\}. \tag{7}$$

The value $K_{\max}$ in (7) can be found by solving the constrained maximization problem. Since the attractor of the system is exactly the set of all limit points as $t \to \infty$, the attractor must lie inside the surface $K(x, y, z) = K_{\max}$.

In the case of the Lorenz system studied in [Doering], it is found that the Lorenz attractor must lie inside the surface $K(x, y, z) = K_{\max}$, where

$$\begin{array}{l} K(x,y,z) = x^2 + y^2 + (z - r - \sigma)^2 \\ K_{\max} = \frac{b^2(r+\sigma)^2}{4(b-1)} \text{ for } b \geq 2 \end{array}$$

So the Lorenz attractor lies inside a sphere. Other such functions yield similar bounds with different shapes, such as a cylinder inside which the attractor must lie, and so on. With the variety of bounds found on the attractor, a fairly complete picture is developed of where the Lorenz attractor lies.

The question is, can we do the same for the Rikitake system? Because of its similarity to the Lorenz system, we are led to believe we can, so we attempt the same technique.

Consider the function

$$K(x, y, z) = x^2 + y^2 \tag{8}$$

Then the derivative of K with respect to t along the flow is

$$\frac{d}{dt}K = -2\mu(x^2 + y^2) - 2xy(a - 2z).$$

Now consider the set B, of all initial conditions such that $z(t)$ is bounded as $t \to \infty$, i.e.

$$B = \{(x(0), y(0), z(0)) : |z(t)| < C \text{ as } t \to \infty\} \tag{9}$$

Then, as $t \to \infty$, $\frac{d}{dt}K < 0$ for x, y large. Thus K will decrease to at least its maximum value on the set $\frac{d}{dt}K \geq 0$, in the limit as $t \to \infty$. Thus, as $t \to \infty$

$$\begin{array}{lll} K(x,y) & = x^2 + y^2 & \leq K_{\max}, \quad \text{where} \\ K_{\max} & = \max\{K(x,y) | \frac{d}{dt}K \geq 0\} & \end{array}$$

This means that as $t \to \infty$ x and y lie inside the cylinder $x^2 + y^2 = K_{\max}$ if $(x(0), y(0), z(0)) \in B$.

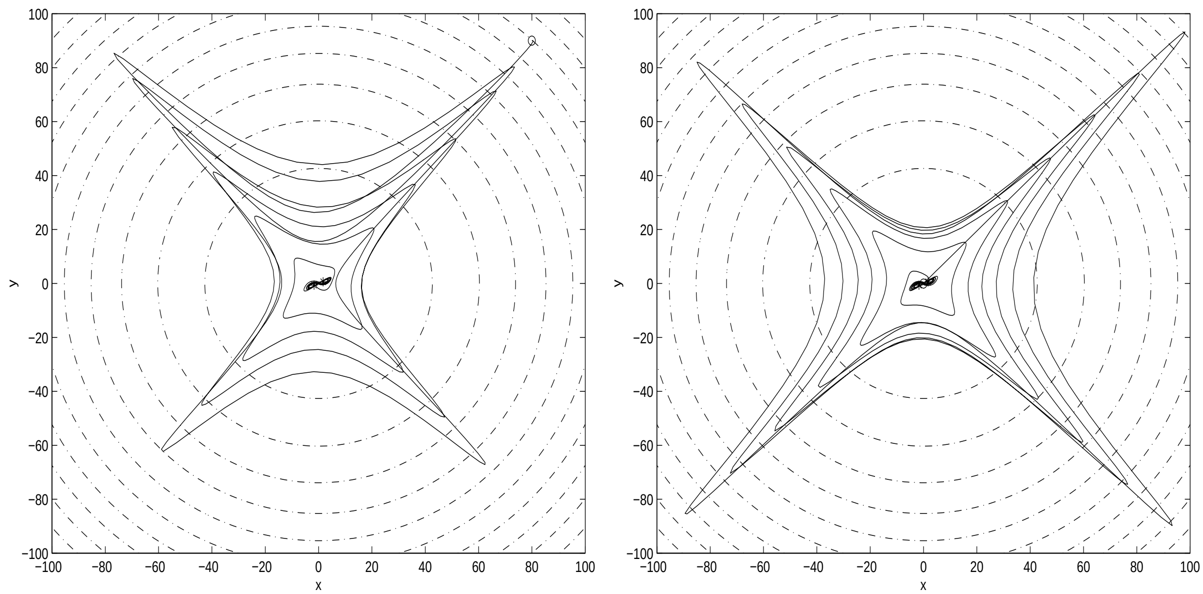
Thus, for all initial conditions inside B, the flow tends toward an attractor which lies inside the cylinder $K \leq K_{\max}$. We are led to believe, furthermore, that B is "most of $\mathbf{R}^3$," so that for "most" initial conditions the flow tends to a set, a "local attractor," bounded by the cylinder.

Finding the smallest such cylinder entails finding the smallest value of C in (9). In figure (2) we see the flow projected onto the $x - y$ plane, superimposed onto the cylinders $x^2 + y^2 = k$. The flow eventually ends up inside a minimal cylinder from which it never escapes.

It is possible that there is a C such that B is all of $\mathbf{R}^3$ "minus the $z-$axis." If this is true then for any initial condition such that $x(0)$ and $y(0)$ are not both zero the flow eventually ends up inside the cylinder $x^2 + y^2 = K_{\max}$ and never leaves. This would then be a bound on "the attractor minus the point $z = \infty$."

6 Semipermeable Surfaces

Following the method presented in [Giacomini] for bounding attractors, we look for families of surfaces through which the flow is "one-way.". We search for a surface S for which, if $\vec{N}$ is



(a) (b)

Figure 2: Projection of the flow onto the $x - y$ plane. The dash-dot curves are the projections of the cylinders $x^2 + y^2 = k$. The circle (o) is the initial condition, stars (*) are the fixed points. $a = 5$ and $\mu = 2$. In figure (a) we see an initial condition far from the origin which is attracted towards the attractor near the origin. In (b) we have a point near the $z-$axis getting pushed out and then back toward the attractor near the origin. There appears to be a minimal cylinder inside which the attractor is trapped.

the normal to the surface and $\vec{T}$ is the tangent to the flow, then $\vec{N} \cdot \vec{T}$ has the same sign on the surface S. If S is an orientable surface on which $\vec{N} \cdot \vec{T}$ always has the same sign, then the flow can go through the surface in only one direction. Such a surface is called in [Giacomini] a *semipermeable* surface.

Finding such semipermeable surfaces will be a great aid in finding the shape of the attractor. For example, suppose we have a family of spheres of radius R about the origin such that for every $R > K$, the flow through the sphere goes only one-way, toward the origin. In this case the only action of the flow outside the sphere of radius K is to head toward the origin. Then we would know that the attractor of the system was contained inside the sphere of radius K.

Numerous families of such surfaces exist for the Lorenz system. In [Giacomini] these surfaces are used to obtain bounds on the shape of the Lorenz attractor. The intersection of these bounds provides a sharper estimate for the location of the Lorenz attractor than that obtained in [Doering].

There is no hope of finding a set of semipermeable spheres about the origin for the Rikitake system because of the unbounded solution on the z-axis, (2). No matter how large a sphere we consider there is always one trajectory which flows out of the sphere. But, perhaps there are other such semipermeable surfaces which can give us some ideas about the shape of the attractor.

The tangent to the flow at (x, y, z) is just $\vec{f}(x, y, z)$. So if S is a surface defined as the level set of some function $g(x, y, z)$, then $\vec{N} \cdot \vec{T} = \nabla g(x, y, z) \cdot \vec{f}(x, y, z)$.

For example, consider the surfaces $S_\pm$ defined by $g(x, y, z) = x/y = \pm 1$. Then

$$\begin{aligned}
\nabla g(x, y, z) \cdot \vec{f}(x, y, z) &= (1/y, -x/y^2, 0) \cdot (-\mu x + zy, -\mu y + (z-a)x, 1 - xy) \\
&= a\tfrac{x^2}{y^2} + z(1 - \tfrac{x^2}{y^2}) \\
&= a
\end{aligned}$$

on $S_\pm$ since $\frac{x^2}{y^2} = 1$ on $S_\pm$. In this case $S_\pm$ are not orientable surfaces, so we cannot say that the flow passes through the surfaces in only one direction. The normals to the surfaces point in different directions on the surfaces depending on which quadrant of the $x - y$ plane (x, y) is in. $\vec{N}$ points "down and right" on S_+ in Quadrant I. $\vec{N}$ points "up and left" on S_+ in Quadrant III.

Since $\vec{N} \cdot \vec{T} = a > 0$, the flow goes through the line $y = x$ "down and right" in Quadrant I and "up and left" in Quadrant III. Similarly, the flow goes through the line $y = -x$ "down and left" in Quadrant IV and "up and right" in Quadrant II. The dynamics on the lines $y = \pm x$ can be seen in figure (3).

The method employed in [Giacomini] for finding families of semipermeable surfaces is as follows. First, find a first integral, a quantity $I(x, y, z, t)$ such that $\frac{d}{dt}K = 0$, for the system, for a particular choice of the parameters. Then look for a family of semipermeable surfaces S_k of the same form as I, which is valid for all values of the parameters.

Following this method, we notice that, for $a = 0$,

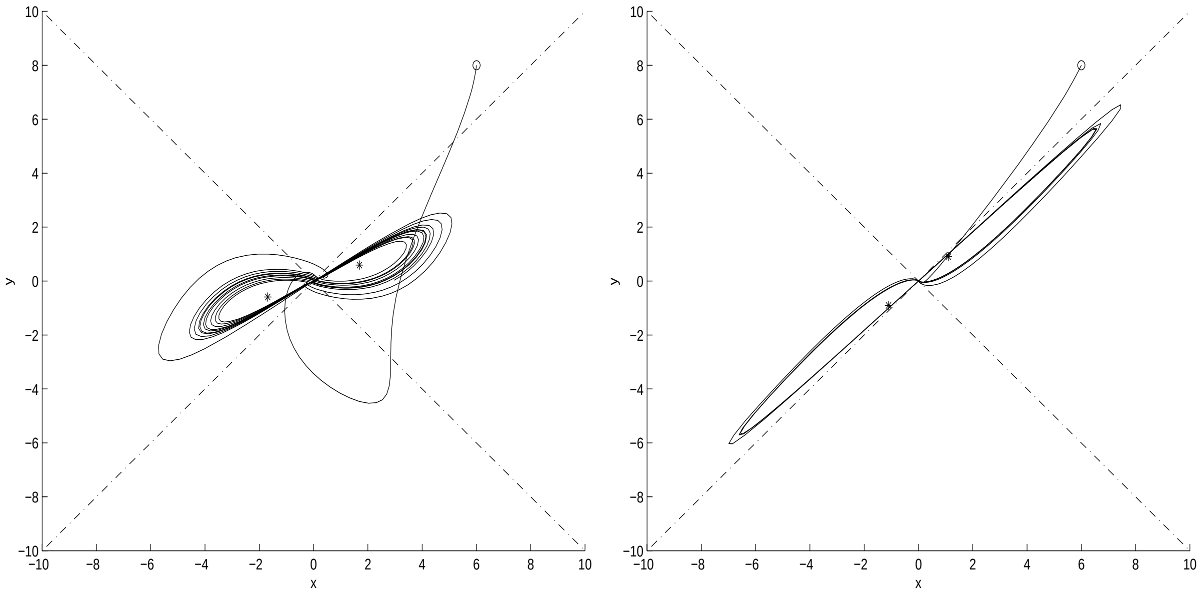
$$I(x, y, z, t) = e^{2\mu t}(x^2 - y^2)$$

is a first integral for the system (1). So we look for a family of semipermeable surfaces S_k of the form $g(x, y, z) = k$ where

$$g(x, y, z) = a_1(x - b_1)^2 - a_2(y - b_2)^2 + a_3 z.$$

On such a surface, the normal to the surface, dotted with the tangent to the flow is

$$\nabla g \cdot \vec{f} = 2a_1(b_1 - x)(-yz + x\mu) - 2a_2(b_2 - y)(ax - xz + y\mu).$$



(a) (b)

Figure 3: Projection of the flow onto the $x-y$ plane. The flow passes through the line y=x "down and right" in Quadrant I and "up and left" in Quadrant III. The flow passes through the line y=-x "down and left" in Quadrant IV and "up and right" in Quadrant II. The dash-dot lines are the lines $x = \pm x$. The circle (o) is the initial condition, stars (*) are the fixed points. In figure (a) $a = 5$ and $\mu = 2$, and in figure (b) $a = 2$ and $\mu = 5$.

Setting $a_1 = a_2 = 1,\ b_1 = b_2 = a_3 = 0$, this is

$$\nabla g \cdot \vec{f} = 2axy - 2\mu(x^2 - y^2).$$

On the surface $S_k = \{(x, y, z)|x^2 - y^2 = k\}$, then

$$\vec{N} \cdot \vec{T} = 2axy - 2k\mu \tag{10}$$

At first sight (10) does not appear to be of much help to us since the value of $\vec{N} \cdot T$ changes sign on S_k for all values of k. In fact, solving $\vec{N} \cdot T = 0$ we obtain

$$\begin{aligned} y = & \ x\tfrac{1}{2\mu}\left(-a \pm \sqrt{a^2 + 4\mu^2}\right) \\ = & \ x\beta_\pm \end{aligned} \tag{11}$$

So, on the lines $y = \beta_\pm x$, $\vec{N} \cdot \vec{T} = 0$ on S_k. $\beta_+\beta_- = -1$, so the lines $y = \beta_\pm x$ are perpendicular.

We have the following situation. On a given surface S_k, the flow passes through the surface in one way on one side of the line $y = \beta_\pm x$ and passes through the surface the opposite way on the other side of the line $y = \beta_\pm x$. The flow is tangent to the surface S_k at the intersection of the lines $y = \beta_\pm x$ and S_k. In figure (4) we see how the flow goes through the surfaces S_k.

The information we have about the surfaces $x^2 - y^2 = k$ and the lines $y = \beta_\pm x$ tell us how the attractor is oriented. Notice that the fixed points $\vec{x}_1$ and $\vec{x}_2$ lie on the line $y = \beta_+ x$ and that this line is where the flow "turns around" and points toward the origin, since on either side of the line the flow is going in opposite ways through the surface $x^2 - y^2 = k$. Thus the attractor is aligned along the line $y = \beta_+ x$, as can be seen in figure (4).

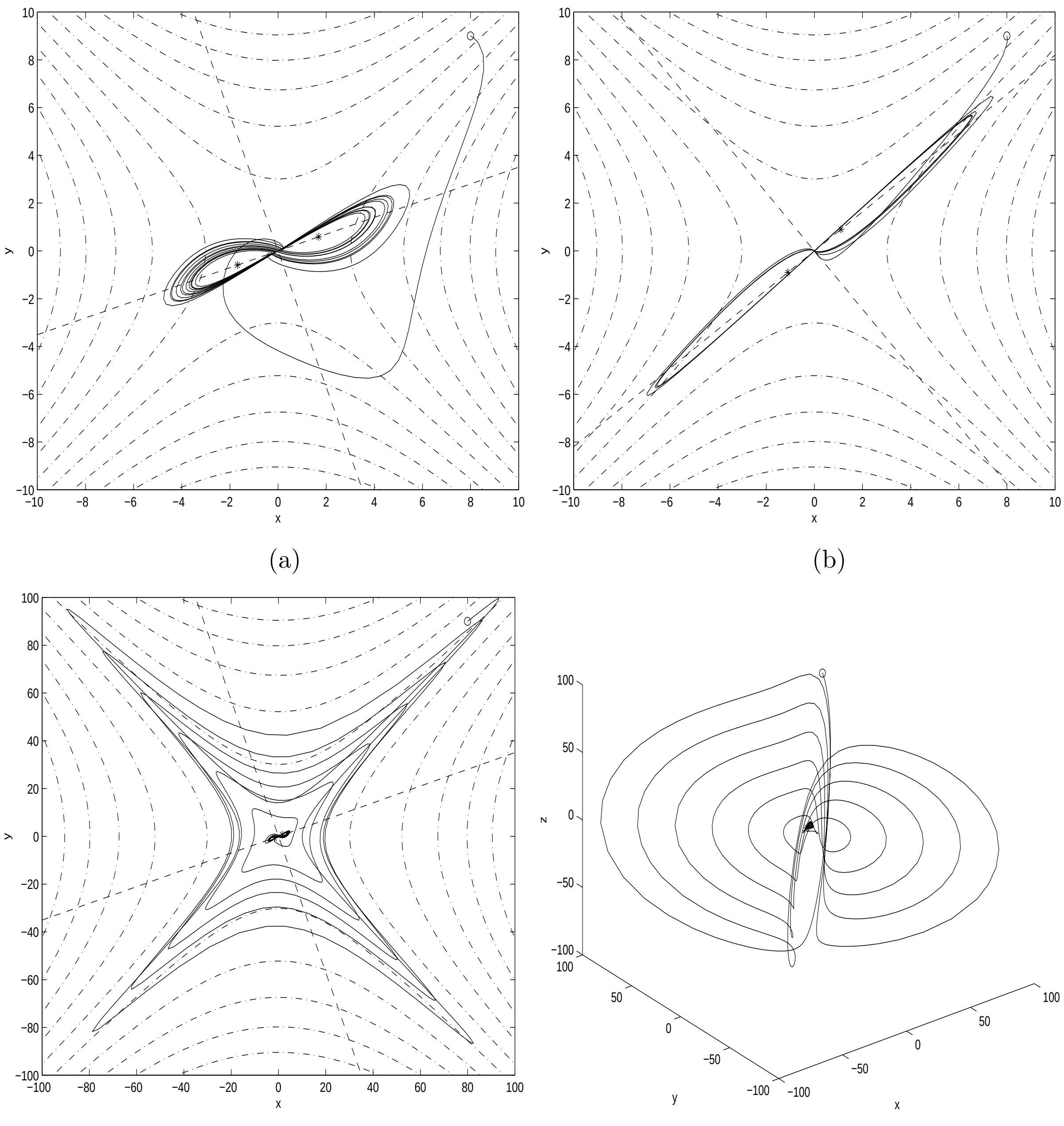


Figure 4: The flow passing through level sets of $g(x,y) = x^2 - y^2$. The dashed lines are the lines $y = \beta_{\pm}$, the dash-dot curves are the curves $x^2 - y^2 = k$. The circle (o) is the initial condition, stars (*) are the fixed points. In figures (a), (c) and (d) $a = 5$ and $\mu = 2$, and in figure (b) $a = 2$ and $\mu = 5$. Figures (c) and (d) show the flow starting from an initial condition far away from the origin. Far from the origin the flow is nearly tangent to the surfaces $x^2 - y^2 = k$. The attractor is aligned along $y = \beta_{+}x$.

7 Conclusion: Is the Attractor Bounded or Not?

While neither of the techniques presented in [Doering] or [Giacomini] gave us the bound on the attractor that we were looking for, they did provide us with some insight into the dynamics of the system.

It is interesting to note that, despite the similarities between the Lorenz and Rikitake systems, the methods failed to give any sort of bound in the Rikitake case. This suggests to us that the methods are not general, and perhaps only produce results in a small number of systems.

We were able to glean information about the behavior for large values of x and y, and surmise that there is a "local" attractor for the system in some sense.

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