Classification of the Finite Subgroups of the Rotation Group

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Preliminaries

The **special orthogonal group** is defined to be
$SO_3 = \{A \in GL_3(\mathbb{R}) | A^t A = I, \det A = 1\}$.

A matrix A represents a rotation in $\mathbb{R}^3$ about the origin if and only if $A \in SO_3$.

Hence SO_3 is also called the **rotation group** in $\mathbb{R}^3$.

Main Theorem

Every finite subgroup G of SO_3 is one of the following:

C_k: the **cyclic group**

D_k: the **dihedral group**

T: the **tetrahedral group**

O: the **octahedral group**

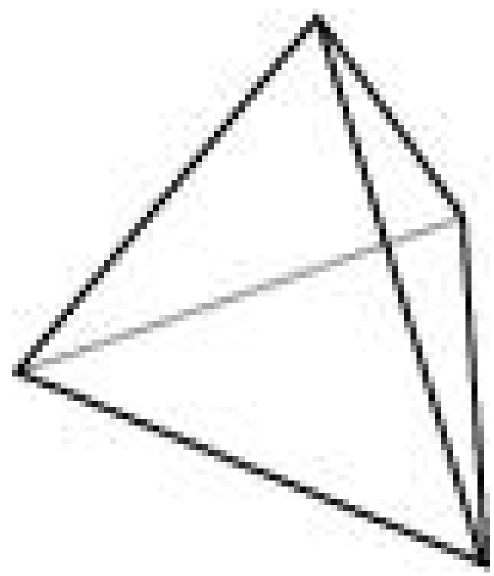
I: the **icosahedral group**

Descriptions

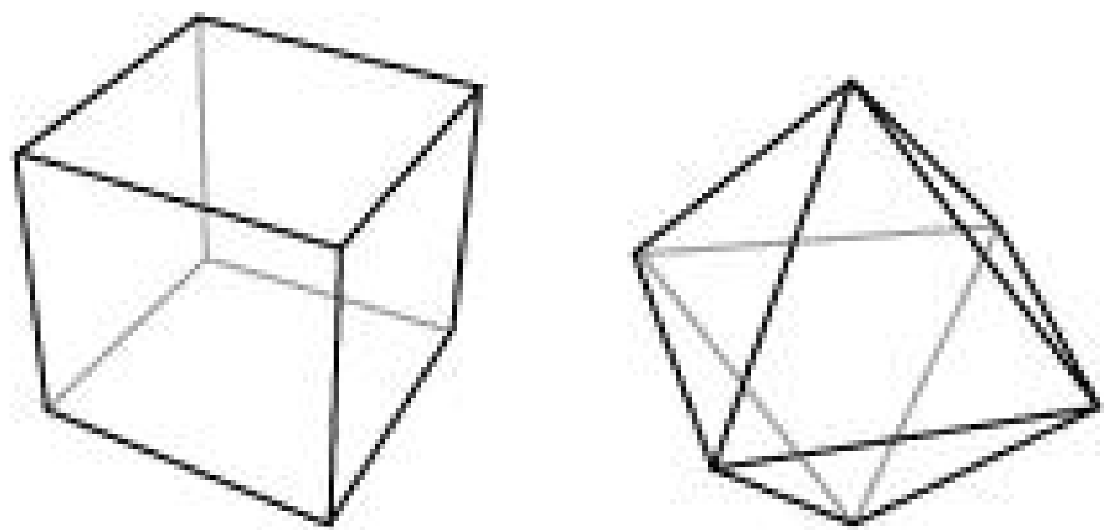
C_k: the **cyclic group** of rotations by multiples of $2\pi/k$ about a line

D_k: the **dihedral group** of symmetries of a regular k-gon

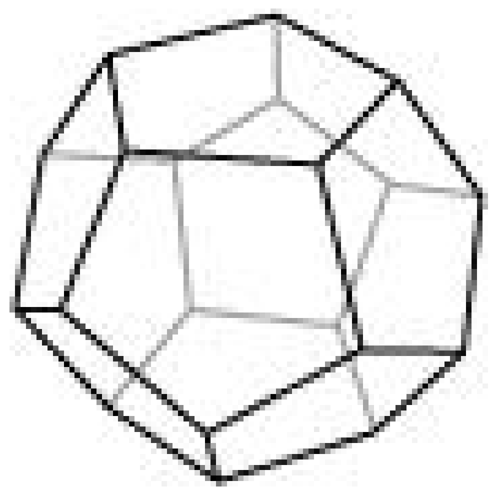
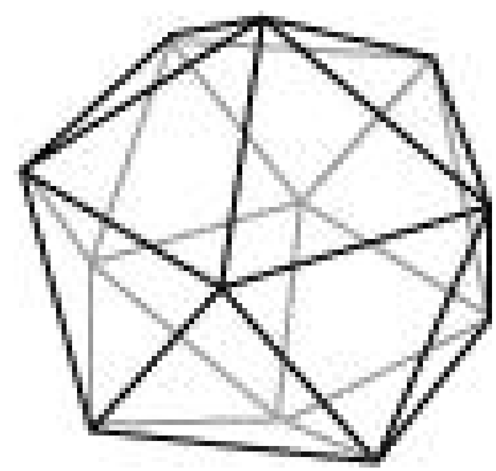
T: the **tetrahedral group** of twelve rotations carrying a regular tetrahedron to itself



O: the **octahedral group** of order 24 of rotations of a cube or regular octahedron



I: the **icosahedral group** of order 60 of rotations of a regular dodecahedron or regular icosahedron

Proof of the Main Theorem

Useful Definitions:

Let G be a finite subgroup of SO_3, and $g \in G \setminus \{1\}$.

Then g is a rotation about a unique line, call it ℓ.

g fixes the two points in $l \cap S^2$.

Call these points the **poles** of g.

Define $P := \{p \in S^2 : gp = p \text{ for some } g \in G \setminus \{1\}\}$.

Proof of the Main Theorem, con't

Lemma:

P is carried to itself by the action of G on S^2.

Proof:

Let $p \in P$ be a pole of some $g \in G$.

Let $x \in G$ be arbitrary.

It suffices to show that $xp \in P$.

Consider $xgx^{-1} \in G$: $xgx^{-1}(xp) = x(gp) = xp$, so xgx^{-1} fixes xp.

Thus $xp \in P$.

Proof of the Main Theorem, con't

Notation:

Let G be a finite subgroup of SO_3, and $\text{N} = |G|$.

Fix $p \in P$, and let ℓ be the line through $(0, p)$.

Let G_p be the stabilizer of p in G, which is the group of rotations about ℓ. Let $r_p = |G_p|$.

Let O_p be the orbit of p in G, and $n_p = |O_p|$.

Then $|G_p||O_p| = r_p n_p = \text{N} = |G| \; \forall p \in P$.

Proof of the Main Theorem, con't

Calculating the Number of Poles:

$\exists\ r_p - 1$ elements of G which have p as a pole.

Each $g \in G \setminus \{1\}$ has two poles.

Thus $\sum_{p \in P}(r_p - 1) = 2(\mathrm{N} - 1)$.

If p and p' are in the same orbit, then $r_p = r_{p'}$.

There are n_p terms in the sum corresponding to a given orbit O_p, which implies that there are $n_p(r_p - 1)$ poles.

Proof of the Main Theorem, con't

Calculating the Number of Orbits:

Number the orbits O_1, O_2,.., with $n_i = |O_i|$ and $r_i = |G_p|\ \forall p \in O_i$.

Then $\sum_i n_i(r_i - 1) = 2\mathrm{N} - 2$.

Since $n_i r_i = \mathrm{N}$, we have

$$2 - \frac{2}{\mathrm{N}} = \sum_i (1 - \frac{1}{r_i}).$$

Then $2 - \dfrac{2}{\mathrm{N}} < 2$ and $1 - \dfrac{1}{r_i} \geq \dfrac{1}{2}$

Hence $\exists$ at most 3 orbits.

Proof of the Main Theorem, con't

One Orbit:

$$2 - \frac{2}{\mathrm{N}} = 1 - \frac{1}{r} \text{ cannot be, since}$$

$$2 - \frac{2}{\mathrm{N}} \geq 1 \text{ while } 1 - \frac{1}{r} < 1$$

Proof of the Main Theorem, con't

Two Orbits:

$$2 - \frac{2}{\text{N}} = (1 - \frac{1}{r_1}) + (1 - \frac{1}{r_2}) \Rightarrow \frac{2}{\text{N}} = \frac{1}{r_1} + \frac{1}{r_2}$$

So $r_1 = r_2 = \text{N}$.

Then $\exists$ only two poles, p and p', both on a line ℓ and both of which are fixed by all $g \in G$.

So $G = C_{\text{N}}$ is the cyclic group of rotations about ℓ.

Proof of the Main Theorem, con't

Three Orbits:

$$\frac{2}{\mathrm{N}} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - 1$$

Suppose $r_1 \leq r_2 \leq r_3$. Then $r_1 = 2$.

(1) $r_1 = r_2 = 2$: Then $2r_3 = \mathrm{N}$, and $n_3 = 2$.

So $\exists$ two poles, p and p', in O_3.

$\forall g \in G$, g either fixes both or interchanges them.

So G is rotations about a line $\ell = (p, p')$ or rotations by π about a line $\ell' \perp \ell$.

G will be the group of symmetries of a regular r_3-gon, that is, the dihedral group D_{r_3}.

Proof of the Main Theorem, con't

Three Orbits con't:

(2) $r_1 = 2$ but $2 < r_2 \leq r_3$:

Then there are three options for (r_1, r_2, r_3) :

(a) $(r_1, r_2, r_3) = (2, 3, 3)$, $\mathrm{N} = 12$, and $(n_1, n_2, n_3) = (6, 4, 4)$

(b) $(r_1, r_2, r_3) = (2, 3, 4)$, $\mathrm{N} = 24$, and $(n_1, n_2, n_3) = (12, 8, 6)$

(c) $(r_1, r_2, r_3) = (2, 3, 5)$, $\mathrm{N} = 60$, and $(n_1, n_2, n_3) = (30, 20, 12)$

Proof of the Main Theorem, con't

Three Orbits con't:

(2) con't

(a) $\mathrm{N} = 12$:

For $p \in O_3$, let $q \in O_2$ be a pole nearest to p.

Then $G_p = G_3$ operates on O_2 and $r_3 = 3$, so $G_p \cdot q$ is a set of three closest neighbors of p.

i.e. the set obtained by the rotations about p.

$\exists$ 4 equilateral triangles which form a regular tetrahedron.

Thus $G = T$.

Proof of the Main Theorem, con't

Three Orbits con't:

(2) con't

(b) $\mathrm{N} = 24$:

For $p \in O_3$, let $q \in O_2$ be a pole nearest to p.

Then $G_p = G_3$ operates on O_2 and $r_3 = 4$,
so $G_p \cdot q$ is a set of four closest neighbors of p.

i.e. the set obtained by the rotations about p.
$\exists$ 6 squares which form a cube.

Thus $G = O$.

Proof of the Main Theorem, con't

Three Orbits con't:

(2) con't

(c) $\text{N} = 60$:

For $p \in O_3$, let $q \in O_2$ be a pole nearest to p.

Then $G_p = G_3$ operates on O_2 and $r_3 = 5$,
so $G_p \cdot q$ is a set of five closest neighbors to p.

i.e. the set obtained by the rotations about p.
These poles are equally spaced, and so form a regular pentagon in $\mathbb{R}^3$.
$\exists$ 12 pentagons, forming a reg. dodecahedron.

Thus $G = I$.

Final Note:

In each case, $\mathrm{N} = 2n_1$, so n_1 is the number of edges on the polyhedron in $\mathbb{R}^3$.

In each of the cases, one of n_2 and n_3 is the number of vertices and the other is the number of faces; it will not matter which each is.