INVERSION IN THE PLANE
BERKELEY MATH CIRCLE

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Contents

Note: All objects lie in the plane, unless otherwise specified. The expression "object A touches object B" refers to tangent objects, e.g. lines and circles.

1. Definition of Inversion in the Plane

Definition 1. Let $k(O, r)$ be a circle with center O and radius r. Consider a function on the plane, $I : \mathbb{R}^2 \to \mathbb{R}^2$, sending a point $X \not\equiv O$ to the point on the half line $OX^{\to}$, X_1, defined by
$$OX \cdot OX_1 = r^2.$$
Such a function I is called an *inversion of the plane* with center O and radius r (write $I(O, r)$.)

It is immediate that I is *not* defined at p.O. But if we compactify $\mathbb{R}^2$ to a sphere by adding one extra point O_∞, we could define $I(O) = O_\infty$ and $I(O_\infty) = O$.

An inversion of the plane can be equivalently described as follows. If $X \in k$, then $I(X) = X$. If X lies outside k, draw a tangent from X to k and let X_2 be the point of tangency. Drop a perpendicular X_2X_1 towards the segment OX with $X_1 \in OX$, and set $I(X) = X_1$. The case when X is inside k, $X \not\equiv O$, is treated in a reverse manner: erect a perpendicular XX_2 to OX, with $X_2 \in k$, draw the tangent to k at point X_2 and let X_1 be the intersection of this tangent with the line OX; we set $I(X) = X_1$.

Properties of Inversion

Some of the basic properties of a plane inversion $I(O, r)$ are summarized below:

- I^2 is the identity on the plane.
- If $A \not\equiv B$, and $I(A) = A_1, I(B) = B_1$, then $\triangle OAB \sim \triangle OB_1A_1$. Consequently,

$$A_1B_1 = \frac{AB \cdot r^2}{OA \cdot OB}.$$

- If l is a line with $O \in l$, then $I(l) = l$.
- If l is a line with $O \notin l$, then $I(l)$ is a circle k_1 with diameter OM_1, where $M_1 = I(M)$ for the orthogonal projection M of O onto l.
- If k_1 is a circle through O, then $I(k_1)$ is a line l: reverse the previous construction.
- If $k_1(O_1, r_1)$ is a circle not passing through O, then $I(k_1)$ is a circle k_2 defined as follows: let A and B be the points of intersection of the line OO_1 with k_1, and let $A_1 = I(A)$ and $B_1 = I(B)$; then k_2 is the circle with diameter A_1B_1. Note that the center O_1 of k_1 does *not* map to the center O_2 of k_2.

Note that two circles are perpendicular if their tangents at a point of intersection are perpendicular; following the same rule, a line and a circle will be perpendicular if the line passes through the center of the circle. In general, the angle between a line and a circle is the angle between the line and the tangent to the circle at a point of intersection with the line.

- Inversion preserves angles between figures: let F_1 and F_2 be two figures (lines, circles); then

$$\angle(F_1, F_2) = \angle(I(F_1), I(F_2)).$$

Problems

(1) Given a point A and two circles k_1 and k_2, construct a third circle k_3 so that k_3 passes through A and is tangent to k_1 and k_2.

(2) Given two points A and B and a circle k_1, construct another circle k_2 so that k_2 passes through A and B and is tangent to k_1.

(3) Given circles k_1, k_2 and k_3, construct another circle k which tangent to all three of them.

(4) Let k be a circle, and let A and B be points on k. Let s and q be any two circles tangent to k at A and B, respectively, and tangent to each other at M. Find the set traversed by the point M as s and q move in the plane and still satisfy the above conditions.

(5) Circles k_1, k_2, k_3 and k_4 are positioned in such a way that k_1 is tangent to k_2 at point A, k_2 is tangent to k_3 at point B, k_3 is tangent to k_4 at point C, and k_4 is tangent to k_1 at point D. Show that A, B, C and D are either collinear or concyclic.

(6) Circles k_1, k_2, k_3 and k_4 intersect cyclicly pairwise in points $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{C_1, C_2\}$, and $\{D_1, D_2\}$. (k_1 and k_2 intersect in A_1 and A_2, k_2 and k_3 intersect in B_1 and B_2, etc.)

   (a) Prove that if A_1, B_1, C_1, D_1 are collinear (concyclic), then A_2, B_2, C_2, D_2 are also collinear (concyclic).
   (b) Prove that if A_1, A_2, C_1, C_2 are concyclic, then B_1, B_2, D_1, D_2 are also concyclic.

(7) (Ptolemy's Theorem) Let $ABCD$ be inscribed in a circle k. Prove that the sum of the products of the opposite sides equals the product of the diagonals of $ABCD$:

$$AB \cdot DC + AD \cdot BC = AC \cdot BD.$$

Further, prove that for any four points A, B, C, D: $AB \cdot DC + AD \cdot BC \geq AC \cdot BD$. When is equality achieved?

(8) Let k_1 and k_2 be two circles, and let P be a point. Construct a circle k_0 through P so that $\angle(k_1, k_0) = \alpha$ and $\angle(k_1, k_0) = \beta$ for some given angles $\alpha, \beta \in [0, \pi)$.

(9) Given three angles $\alpha_1, \alpha_2, \alpha_3 \in [0, \pi)$ and three circles k_1, k_2, k_3, two of which do not intersect, construct a fourth circle k so that $\angle(k, k_i) = \alpha_i$ for $i = 1, 2, 3$.

(10) Construct a circle k^* so that it goes through a given point P, touches a given line l, and intersects a given circle k at a right angle.

(11) Construct a circle k which goes through a point P, and intersects given circles k_1 and k_2 at angles $45°$ and $60°$, respectively.

(12) Let $ABCD$ and $A_1B_1C_1D_1$ be two squares oriented in the same direction. Prove that AA_1, BB_1 and CC_1 are concurrent if $D \equiv D_1$.

(13) Let $ABCD$ be a quadrilateral, and let k_1, k_2, and k_3 be the circles circumscribed around $\triangle DAC$, $\triangle DCB$, and $\triangle DBA$, respectively. Prove that if $AB \cdot CD = AD \cdot BC$, then k_2 and k_3 intersect k_1 at the same angle.

(14) In the quadrilateral $ABCD$, set $\angle A + \angle C = \beta$.

   (a) If $\beta = 90°$, prove that that $(AB \cdot CD)^2 + (BC \cdot AD)^2 = (AC \cdot BD)^2$.

(b) If $\beta = 60^\circ$, prove that $(AB \cdot CD)^2 + (BC \cdot AD)^2 = (AC \cdot BD)^2 + AB \cdot BC \cdot CD \cdot DA$.

(15) Let k_1 and k_2 be two circles intersecting at A and B. Let t_1 and t_2 be the tangents to k_1 and k_2 at point A, and let $t_1 \cap k_2 = \{A, C\}$, $t_2 \cap k_1 = \{A, D\}$. If $E \in AB^{\rightarrow}$ such that $AE = 2AB$, prove that $ACED$ is concyclic.

(16) Let OL be the inner bisector of $\angle POQ$. A circle k passes through O and $k \cap OP^{\rightarrow} = \{A\}$, $k \cap OQ^{\rightarrow} = \{B\}$, $k \cap OL^{\rightarrow} = \{C\}$. Prove that, as k changes, the following ratio remains constant:

$$\frac{OA + OB}{OC}.$$

(17) Let a circle k^* be inside a circle k, $k^* \cap k = \emptyset$. We know that there exists a sequence of circles $k_0, k_1, ..., k_n$ such that k_i touches k, k^* and k_{i-1} for $i = 1, 2, ..., n+1$ (here $k_{n+1} = k_0$.) Show that, instead of k_1, one can start with *any* circle k_1' tangent to both k and k^*, and still be able to fit a "ring" of n circles as above. What is n is terms of the radii of and the distance between the centers of k and k^*?

(18) Circles k_1, k_2, k_3 touch pairwise, and all touch a line l. A fourth circle k touches k_1, k_2, k_3, so that $k \cap l = \emptyset$. Find the distance from the center of k to l given that radius of k is 1.

2. Radical Axes

Definition 2. The *degree* of point A with respect to a circle $k(O, R)$ is defined as

$$d_k(A) = OA^2 - R^2.$$

This is simply the square of the tangent segment from A to k. Let M be the midpoint of AB in $\triangle ABC$, and CH – the altitude from C, with $H \in AB$. Mark the sides BC, CA and AB by a, b and c, respectively. Then

$$|a^2 - b^2| = |BH^2 - AH^2| = c|BH - AH| = 2c \cdot MH, \tag{1}$$

where M is the midpoint of AB.

Definition 3. The *radical axis* of two circles k_1 and k_2 is the geometric place of all points which have the same degree with respect to k_1 and k_2: $\{A \,|\, d_{k_1}(A) = d_{k_2}(A)\}$.

Let P be one of the points on the radical axis of $k_1(O_1, R_1)$ and $k_2(O_2, R_2)$. We have by (1):

$$PO_1^2 - R_1^2 = PO_2^2 - R_2^2 \;\Rightarrow\; |R_1^2 - R_2^2| = |PO_1^2 - PO_2^2| = 2O_1O_2 \cdot MH,$$

where M is the midpoint of O_1O_2, and H is the orthogonal projection of P onto O_1O_2. Then

$$MH = \frac{|R_1^2 - R_2^2|}{2O_1O_2} = \text{constant} \;\Rightarrow\; \text{point } H \text{ is constant.}$$

(Show that the direction of $MH^{\rightarrow}$ is the same regardless of which point P on the radical axis we have chosen.) Thus, the radical axis is a subset of a line $\perp O_1O_2$. The converse is easy.

Lemma 1. *Let $k_1(O_1, R_1)$ and $k_2(O_2, R_2)$ be two nonconcentric circles circles, with $R_1 \geq R_2$, and let M be the midpoint of O_1O_2. Let H lie on the segment MO_2, so that*

$$HM = (R_1^2 - R_2^2)/2O_1O_2.$$

Then the radical axis of $k_1(O_1, R_1)$ and $k_2(O_2, R_2)$ is the line l, perpendicular to O_1O_2 and passing through H.

What happens with the radical axis when the circles are concentric? In some situations it is convenient to have the circles concentric. In the following fundamental lemma, we achieve this by applying both ideas of inversion and radical axis.

Lemma 2. *Let k_1 and k_2 be two nonintersecting circles. Prove that there exists an inversion sending the two circles into concentric ones.*

Proof: If the radical axis intersects O_1O_2 in point H, let $k(H, \mathrm{d}_{k_i}(H))$ intersect O_1O_2 in A and B. Apply inversion wrt $k'(A, AB)$. Then $I(k)$ is a line l through B, $l \perp O_1O_2$. But $k_1 \perp k$, hence $I(k_1) \perp l$, i.e. the center of $I(k_1)$ lies on l. It also lies on O_1O_2, hence $I(k_1)$ is centered at B. Similarly, $I(k_2)$ is centered at B. $\square$

Problems

(19) The radical axis of two intersecting circles passes through their points of intersection.

(20) The radical axes of three circles intersect in one point, provided their centers do not lie on a line.

(21) Given two circles k_1 and k_2, find the geometric place the centers of the circles k perpendicular to both k_1 and k_2.

(22) A circle k is tangent to a line l at a point P. Let O be diametrically opposite to P on k. For some points $T, S \in k$ set $OT \cap l = T_1$ and $OS \cap l = S_1$. Finally, let SQ and TQ be two tangents to k meeting in point Q. Set $OQ \cap l = \{Q_1\}$. Prove that Q_1 is the midpoint of T_1S_1.

(23) Consider $\triangle ABC$ and its circumscribed and inscribed circles K and k, respectively. Take an arbitrary point A_1 on K, draw through A_1 a tangent line to k and let it intersect K in point B_1. Now draw through B_1 another tangent line to k and let it intersect K in point C_1. Finally, draw through C_1 a third tangent line to k and let it intersect K in point D_1. Prove that D_1 coincides with A_1. In other words, prove that any triangle $A_1B_1C_1$ inscribed in K, two of whose sides are tangent to k, must have its third side also tangent to k so that k is the inscribed circle for $\triangle A_1B_1C_1$ too.

(24) Find the distance between the center P of the inscribed circle and the center O of the circumscribed circle of $\triangle ABC$ in terms of the two radii r and R.

(25) We are given $\triangle ABC$ and points $D \in AC$ and $E \in BC$ such that $DE || AB$. A circle k_1 of diameter DB intersects a circle k_2 of diameter AE in M and N. Prove that M and N lie on the altitude CH to AB.

(26) Prove that the altitude of $\triangle ABC$ through C is the radical axis of the circles with diameters the medians AM and BN of $\triangle ABC$.

(27) Find the geometric place of points O which are centers of circles through the end points of diameters of two fixed circles k_1 and k_2.

(28) Construct all radical axes of the four incircles of $\triangle ABC$.

(29) Let A, B, C be three collinear points with B inside AC. On one side of AC we draw three semicircles k_1, k_2 and k_3 with diameters AC, AB and BC, respectively. Let BE be the interior tangent between k_2 and k_3 ($E \in k_1$), and let UV be the exterior tangent to k_2 and k_3 ($U \in k_2$ and $V \in k_3$). Find the ratio of the areas of $\triangle UVE$ and $\triangle ACE$ in terms of k_2 and k_3's radii.

(30) The chord AB separates a circle γ into two parts. Circle γ_1 of radius r_1 is inscribed in one of the parts and it touches AB at its midpoint C. Circle γ_2 of radius r_2 is also inscribed in the same part of γ so that it touches AB, γ_1 and γ. Let PQ be the interior tangent of γ_1 and γ_2, with $P, Q \in \gamma$. Show that $PQ \cdot SE = SP \cdot SQ$, where $S = \gamma_1 \cap \gamma_2$ and $E = AB \cap PQ$.

(31) Let $k_1(O, R)$ be the circumscribed circle around $\triangle ABC$, and let $k_2(T, r)$ be the inscribed circle in $\triangle ABC$. Let $k_3(T, r_1)$ be a circle such that there exists a quadrilateral $AB_1C_1D_1$ inscribed in k_1 and circumscribed around k_3. Calculate r_1 in terms of R and r.

(32) Let $ABCD$ be a square, and let l be a line such that the reflection A_1 of A across l lie on the segment BC. Let D_1 be the reflection of D across l, and let D_1A_1 intersect DC in point P. Finally, let k_1 be the circle of radius r_1 inscribed in $\triangle A_1CP_1$. Prove that $r_1 = D_1P_1$.

(33) In a circle $k(O, R)$ let AB be a chord, and let k_1 be a circle touching internally k at point K so that $KO \perp AB$. Let a circle k_2 move in the region defined by AB and not containing k_1 so that it touches both AB and k. Prove that the tangent distance between k_1 and k_2 is constant.

(34) Prove that for any two circles there exists an inversion which transforms them into congruent circles (of the same radii). Prove further that for any three circles there exists an inversion which transforms them into circles with collinear centers.

(35) Given two nonintersecting circles k_1 and k_2, show that all circles orthogonal to both of them pass through two fixed points and are tangent pairwise.

(36) Given two circles k_1 and k_2 intersecting at points A and B, show that there exist exactly two points in the plane through which there passes no circle orthogonal to k_1 and k_2.

(37) (Brianchon) If the hexagon $ABCDEF$ is circumscribed around a circle, prove that its three diagonals AD, BE and CF are concurrent.

3. Power of a Point wrt a Circle and a Sphere

In the following, we consider $\triangle ABC$ and its circumcircle k with center O, and calculate degrees of distinguished points of $\triangle ABC$ wrt k.

(38) Find the degrees of the medicenter G, orthocenter H and incenter I of $\triangle ABC$ wrt k. Deduce Euler's formula $OI^2 = R^2 - 2Rr$.

(39) Let A and B lie on the circle k. Find the points on line AB whose degree wrt k equals t^2, where t is the length of a given segment.

(40) Let A and B lie on the circle k. Find the points M for which $MA \cdot MB = MT^2$, where MT is the tangent from M to k.

(41) From a given point A outside circle k with center O draw a line l and denote its intersection points with k by B and C. Draw the tangents at B and C to k and let them intersect at M. Find the locus of points M as line l moves.

(42) Let the medians AM, BN and CP intersect each other in G and intersect the circumcircle k of $\triangle ABC$ in points A_1, B_1 and C_1. Prove that
$$\frac{AG}{GA_1}+\frac{BG}{GB_1}+\frac{CG}{GC_1}=3.$$

(43) $\triangle ABC$ is inscribed in circle $k(O,R)$. Find the locus of points Q inside k for which
$$\frac{AQ}{QQ_1}+\frac{BQ}{QQ_2}+\frac{CQ}{QQ_3}=3$$
where $Q_1 = k \cap AQ$, $Q_2 = k \cap BQ$, $Q_3 = k \cap CQ$.

(44) Let A be a point inside circle k. Consider all chords MN in k such that $\angle MAN = 90°$. For each such chord construct point P symmetric to A wrt MN. Find the locus of all such points P.

(45) Given non–colinear points A, B, C, find point P on line AB for which $PC^2 = PA \cdot PB$.

(46) Given points A, B, M and segment m, construct a circle through A and B such that the tangents from M to k are equal to m.

(47) Given points A, B, C and segments m and n, construct a circle k through A such that the tangents to k through B and C are equal to m and n, respectively.

(48) Given poins A and B and line l which intersects AB, construct a cicrle through A and B cutting a chord from line l of given length d.

(49) Through two given points A and B construct a circle which is tangent to a line p.

(50) On the side AC of $\triangle ABC$ fix point M. Find point X on the side BC whose distance to M equals the sum of the distances from M and X to AB.

(51) Construct a circle k passing through two given points A and B and tangent to a given circle k.

(52) Construct a circle k through two given points A and B and cutting a chord from k of given length d.

(53) Construct a circle through a given point and tangent to a given line and a given circle.

(54) Construct a circle through a given point and tangent to two given circles.

(55) Let a circle k and a point A be given. Through two arbitrary points B and C on k and through A draw a circle k'. Let M be the intersection point of line BC with the tangent to k' at A. Find the locus of points M.

(56) Find the locus of points with equal degrees wrt two given circles.

(57) Given circles $k_1(O_1, R_1)$ and $k_2(O_2, R_2)$, construct a point the tangents through which to k_1 and k_2 are equal and it lies on a given line (or on a given circle).

(58) Given circles $k_1(O_1, R_1)$ and $k_2(O_2, R_2)$, and angle α, construct a point P the tangents through which to k_1 and k_2 are equal and $\angle O_1PA_2 = \alpha$.

(59) Given three circles whose centers are non–collinear, find the locus of points which have equal degrees wrt to the three circles.

(60) Points A, B, C and D lie on a given line l. Find the locus of points P for which the circles through A, B, P, and through C, D, P are tangent at P.

4. Problems From Around the World

(61) (IMO Proposal) The incircle of $\triangle ABC$ touches BC, CA, AB at D, E, F, respectively. X is a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches BC at D also, touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

(62) (Israel, 1995) Let PQ be the diameter of semicircle H. Circle k is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that AB is perpendicular to PQ and is also tangent to k. Prove that AC bisects $\angle PAB$.

(63) (Romania, 1997) Let ABC be a triangle, D a point on side BC, and ω the circumcicle of ABC. Show that the circles tangent to ω, AD, BD and to ω, AD, DC are also tangent to each other if and only if $\angle BAD = \angle CAD$.

(64) (Russia, 1995) We are given a semicircle with diameter AB and center O, and a line which intersects the semicircle at C and D and line AB at M ($MB < MA$, $MD < MC$.) Let K be the second point of intersection of the circumcircles of $\triangle AOC$ and $\triangle DOB$. Prove that $\angle MKO = 90°$.

(65) (Ganchev, 265) We are given nonintersecting circle k and line g, and two circles k_1 and k_2 which are tangent externally at T, and each is tangent to g and (externally) to k. Find the locus of points T.

(66) (Ganchev, 266) We are given two nonintersecting circles k and K, and two circles k_1 and k_2 which are tangent externally at T, and each is tangent externally to k and K. Find the locus of points T.

(67) (95,4,p.31) Let A be a point outside circle k with center O, and let AP be a tangent from A to k ($P \in k$). Let B denote the foot of the perpendicular from P to line OA. Choose an arbitrary chord CD in k passing through B, and let E be the reflection of D across AO. Prove that A, C and E are collinear.

(68) (IMO'95) Let $A,\ B,\ C$ and D be four distinct points on a line, positioned in this order. The circles k_1 and k_2 with diameters AC and BD intersect in X and Y. Lines XY and BC intersect in Z. Let P be a point on line XY, $P \neq Z$. Line CP intersects k_1 in C and M, and line BP intersects k_2 in B and N. Prove that lines AM, DN and XY are concurrent.

(69) (BO'95 IV) Let $\triangle ABC$ have half–perimeter p. On the line AB take points E and F such that $CE = CF = p$. Prove that the externally inscribed for $\triangle ABC$ circle tangent to side AB is tangent to the circumcircle of $\triangle EFC$.

(70) (BQ'95) Three circles k_1, k_2 and k_3 intersect as follows: $k_1 \cap k_2 = \{A,\ D\}$, $k_1 \cap k_3 = \{B,\ E\}$, $k_2 \cap k_3 = \{C,\ F\}$, so that $ABCDEF$ is a non-selfintersecting hexagon. Prove that $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

(71) (IMO'94, shortlisted) Circles ω, ω_1 and ω_2 are externally tangent to each other in points $C = \omega \cap \omega_1$, $E = \omega_1 \cap \omega_2$ and $D = \omega_2 \cap \omega$. Two parallel lines l_1 and l_2 are tangent to ω, ω_1, and ω, ω_2 at points R, A, and S, B, respectively. Prove that the intersection point of AD and BC is the circumcenter of $\triangle CDE$.

(72) (Kazanluk'97 X) Point F on the base AB of trapezoid $ABCD$ is such that $DF = CF$. Let E be the intersection point of the diagonals AC and BD, and O_1 and O_2 be the circumcenters of $\triangle ADF$ and $\triangle BCF$, respectively. Prove that the lines FE and O_1O_2 are perpendicular.

5. Variations on Sylvester's Theorem

(73) (a) (Sylvester, 1893) Let R be a finite set of points in the plane satisfying the following condition: on every line determined by two points in R there lies at least one other point in R. Prove that all points in R lie on a single line.
(b) Let R be a finite set of points in space satisfying the following condition: on every plane determined by three noncollinear points in R there lies at least one other point in R. Prove that all points in R lie on a single plane.

(74) (a) Let S be a finite set of points in the plane, no three collinear. It is known that on the circle determined by any three points in S there lies a fourth point in S. Prove that all points in S lie on a single circle.
(b) Let S be a finite set of points in the plane, no four coplanar. It is known that on the sphere determined by any four points in S there lies a fifth point in S. Prove that all points in S lie on a single sphere.

(75) (a) Let T be a finite set of lines in the plane, no two parallel, satisfying the following condition: through the intersection point of any two lines in T there passes a third line in T. Prove that all lines in T pass through a single point.

(b) Let T be a finite set of planes in space, no two parallel, satisfying the following condition: through the intersection line of any two planes in T there passes a third plane in T. Prove that all planes in T pass through a some fixed line.

(76) (a) Let Q be a set of n points in the plane. If the total number of lines determined by the points in Q is less than n, prove that all points in Q lie on a single line.

(b) Conversely, let Q be a set of n points in the plane, not all collinear and not all concyclic. Prove that through every point in Q there pass at least $n-1$ circles of Q. (A circle of Q is a line or a circle through 3 points in Q.)