Chapter 7

The Rotation Group and Its Representations

7.1 The Groups $SO(3)$ and $SU(2)$

The rotation group $SO(3)$ is of fundamental importance in modern physical theories. Many physical systems admit $SO(3)$ as a symmetry group, a fact which is related to the conservation of angular momentum for such systems. Moreover, the theory of spin and isotopic spin of particles is intimately related to the rep theory of $SO(3)$ and its locally isomorphic companion $SU(2)$. The theory of hypergeometric functions is associated with the study of the Lie algebra of $SO(3)$. Finally, a knowledge of the rep theory of the rotation group and its Lie algebra is indispensible for an understanding of the more complicated rep theory of the classical groups.

Recall that $SO(3) = SO(3, R)$ is the group of all 3×3 real matrices such that $A^t A = E_3$ and $\det A = +1$ (see Section 2.1). This is the natural realization of $SO(3)$ as a transformation group on R_3. We have shown that $SO(3)$ is a three-parameter Lie group whose Lie algebra $so(3)$ consists of all 3×3 real matrices $\mathcal{a}$ such that $\mathcal{a}^t = -\mathcal{a}$. As a convenient basis for $so(3)$ we choose three tangent matrices to the one-parameter groups of rotations about the x, y, and z axes, respectively. The rotations about the z axis are

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.1}$$

This is a one-parameter subgroup of $SO(3)$ with tangent matrix

$$\mathfrak{L}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2}$$

at the identity. Similarly

$$\mathfrak{L}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathfrak{L}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{1.3}$$

are tangent matrices to one-parameter subgroups of rotations about the x and y axes, respectively. We have

$$\exp \varphi\mathfrak{L}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}, \quad \exp\varphi\mathfrak{L}_2 = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}. \tag{1.4}$$

Since these three tangent matrices are linearly independent, they form a basis for $so(3)$. As the reader can easily verify, the commutation relations of the basis vectors are

$$[\mathfrak{L}_1, \mathfrak{L}_2] = \mathfrak{L}_3, \quad [\mathfrak{L}_3, \mathfrak{L}_1] = \mathfrak{L}_2, \quad [\mathfrak{L}_2, \mathfrak{L}_3] = \mathfrak{L}_1. \tag{1.5}$$

In Section 5.4 we showed that $SU(2)$ was also a three-parameter real Lie group. As the reader can easily verify, every $A \in SU(2)$ can be written in the form

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{1.6}$$

where $|\alpha|^2 + |\beta|^2 = 1$. If $A, A_1, A_2 \in SU(2)$ then

$$A^{-1} = \bar{A}^t = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}, \quad A_1A_2 = \begin{pmatrix} \alpha_1\alpha_2 - \beta_1\bar{\beta}_2, & \alpha_1\beta_2 + \beta_1\bar{\alpha}_2 \\ -\bar{\beta}_1\alpha_2 - \bar{\alpha}_1\bar{\beta}_2, & -\bar{\beta}_1\beta_2 + \bar{\alpha}_1\bar{\alpha}_2 \end{pmatrix}.$$

The Lie algebra $su(2) = L(SU(2))$ consists of all 2×2 complex skew-Hermitian matrices $\mathfrak{a}$ of trace zero:

$$\mathfrak{a} = \begin{pmatrix} ix_3, & -x_2 + ix_1 \\ x_2 + ix_1, & -ix_3 \end{pmatrix}, \quad x_j \in R. \tag{1.7}$$

As a basis for $su(2)$ we choose the elements

$$\mathfrak{J}_1 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}, \quad \mathfrak{J}_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \mathfrak{J}_3 = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}. \tag{1.8}$$

A direct computation shows that these matrices satisfy the commutation relations (1.5). Thus $so(3)$ and $su(2)$ are isomorphic Lie algebras, so $SO(3)$ and $SU(2)$ are locally isomorphic Lie groups. However, this isomorphism is not global.

To exhibit explicitly the relation between $SO(3)$ and $SU(2)$, consider the adjoint rep of $SU(2)$ on its Lie algebra:

$$\mathfrak{A} \longrightarrow \mathfrak{B} = A\mathfrak{A}A^{-1} \in su(2), \qquad \mathfrak{A} \in su(2), \quad A \in SU(2). \tag{1.9}$$

(See Section 5.6.)

Now $\det \mathfrak{B} = \det (A\mathfrak{A}A^{-1}) = \det \mathfrak{A}$. Therefore, writing

$$\mathfrak{B} = \begin{pmatrix} iy_3, & -y_2 + iy_1 \\ y_2 + iy_1, & -iy_3 \end{pmatrix} \tag{1.10}$$

we find

$$y_1^2 + y_2^2 + y_3^2 = \det \mathfrak{B} = \det \mathfrak{A} = x_1^2 + x_2^2 + x_3^2. \tag{1.11}$$

According to (1.9) the y_j are linear combinations of the x_k:

$$y_j = \sum_{k=1}^{3} R(A)_{jk} x_k, \qquad j = 1, 2, 3. \tag{1.12}$$

Since (1.9) defines a rep of $SU(2)$ the 3×3 matrices $R(A)$ satisfy $R(AB) = R(A)R(B)$ for all $A, B \in SU(2)$. Moreover, from (1.11) and (1.12), $R(A)^t R(A) = E_3$, i.e., $R(A) \in O(3)$. The rep $A \to R(A)$ is continuous and $SU(2)$ is connected. Thus $\det R(A)$ is a continuous function of A, and since $R(E_2) = E_3$, we conclude that $\det R(A) = +1$ for all $A \in SU(2)$. We have shown that $R(A) \in SO(3)$ and $A \to R(A)$ is a homomorphism of $SU(2)$ into $SO(3)$.

We now verify that this homomorphism covers $SO(3)$. Let $R \in SO(3)$ and set $y_j = \sum R_{jk} x_k$. Defining $\mathfrak{A}, \mathfrak{B} \in su(2)$ by (1.7) and (1.10) we find $\operatorname{tr} \mathfrak{A} = \operatorname{tr} \mathfrak{B} = 0$, $\det \mathfrak{A} = \det \mathfrak{B} = x_1^2 + x_2^2 + x_3^2 = q^2$, so the Hermitian matrices $i\mathfrak{A}$ and $i\mathfrak{B}$ have the same eigenvalues, $\pm iq$. Therefore, $i\mathfrak{A}$ and $i\mathfrak{B}$ are similar and there exists a unitary matrix B such that $\mathfrak{B} = B\mathfrak{A}B^{-1}$. Now $|\det B| = 1$ for B unitary, so $B = e^{i\theta}A$, where $e^{2i\theta} = \det B$ and $A \in SU(2)$. Thus $\mathfrak{B} = A\mathfrak{A}A^{-1}$, so $R = R(A)$ and the homomorphism $A \to R(A)$ maps $SU(2)$ onto $SO(3)$. Finally, the relation

$$(-A)\mathfrak{A}(-A)^{-1} = A\mathfrak{A}A^{-1}$$

shows that $R(A) = R(-A)$, so two elements of $SU(2)$ map onto a single element of $SO(3)$. Note: The matrix $-A \in SU(2)$ if $A \in SU(2)$.

The reader can check that $R(A) = E_3$ if and only if $A = \pm E_2$. Thus, $SO(3)$ is isomorphic to the factor group $SU(2)/\{\pm E_2\}$. Exactly two elements of $SU(2)$ map onto one element of $SO(3)$. (Since $-A$ is far from E_2 when A is close to E_2 it is clear that this map is locally an isomorphism.)

Writing $\alpha = a + ib$, $\beta = c + id$, $a, b, c, d \in R$, in (1.6) we see that the only restriction on these four real parameters is $a^2 + b^2 + c^2 + d^2 = 1$.

Topologically, $SU(2)$ is homeomorphic to the unit sphere S_4 in four-dimensional space. If $A \in SU(2)$ is a point on this sphere then $-A$ is the point on the other end of the diameter of S_4 passing through A. Topologically, $SO(3)$ is homeomorphic to the projective space obtained by identifying opposite ends of each diameter in S_4. We say that $SU(2)$ is a **covering group** of $SO(3)$ and that it covers $SO(3)$ twice. (For a more geometrical derivation of the relationship between $SU(2)$ and $SO(3)$ see Gel'fand et al, [1].)

The **Euler angles** (φ, θ, ψ) form a convenient coordinate system for $SU(2)$. Consider the product

$$(1.13)\quad \begin{aligned} A(\varphi, \theta, \psi) &= (\exp \varphi \mathcal{J}_3)(\exp \theta \mathcal{J}_1)(\exp \psi \mathcal{J}_3) \\ &= \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos \frac{1}{2}\theta & i \sin \frac{1}{2}\theta \\ i \sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} e^{i(\varphi+\psi)/2} \cos \frac{1}{2}\theta & ie^{i(\varphi-\psi)/2} \sin \frac{1}{2}\theta \\ ie^{i(\psi-\varphi)/2} \sin \frac{1}{2}\theta & e^{-i(\varphi+\psi)/2} \cos \frac{1}{2}\theta \end{pmatrix}. \end{aligned}$$

It follows that any $A \in SU(2)$ is determined by Euler angles (φ, θ, ψ), where

$$(1.14)\quad |\alpha| = \cos \tfrac{1}{2}\theta, \qquad \arg \alpha = \tfrac{1}{2}(\varphi + \psi), \qquad \arg \beta = \tfrac{1}{2}(\varphi - \psi + \pi),$$

$$(1.15)\quad \begin{aligned} &\cos \tfrac{1}{2}\theta = |\alpha|, \qquad \sin \tfrac{1}{2}\theta = |\beta|, \qquad \varphi = \arg \alpha + \arg \beta - \tfrac{1}{2}\pi, \\ &\psi = \arg \alpha - \arg \beta + \tfrac{1}{2}\pi, \qquad |\alpha\beta| \neq 0. \end{aligned}$$

If we restrict the Euler angles to the domain

$$(1.16)\quad 0 \leq \varphi < 2\pi, \qquad 0 \leq \theta \leq \pi, \qquad -2\pi \leq \psi < 2\pi,$$

then for $|\alpha\beta| \neq 0$, (φ, θ, ψ) are uniquely determined. (Recall that the argument of a complex number is determined only up to an integer multiple of 2π.) However, if $|\alpha\beta| = 0$, an infinite number of Euler angles describe the same group element. The Euler angles are coordinates on the sphere S_4 somewhat analogous to the coordinates latitude and longitude on the sphere S_3 in three-space. All points on S_3 have unique values of latitude and longitude except the poles, where the longitude becomes indeterminant. The Euler angles are still very useful despite this drawback because the set on which they are indeterminant has lower dimension than three. Thus, if we integrate a function over $SU(2)$ using the invariant measure, the behavior of the function on this set will have no effect on the integral.

Clearly, the Euler angles of the product of two group elements can be expressed as analytic functions of the Euler angles of the factors. The results are given by expressions (2.16).

The invariant measure on $SU(2)$ can be computed directly from the formulas of Section 6.1. Let $A(\varphi, \theta, \psi) \in SU(2)$. Then

$$A^{-1}\frac{\partial A}{\partial \varphi} = \begin{pmatrix} \frac{1}{2}i\cos\theta & -\frac{1}{2}e^{-i\psi}\sin\theta \\ \frac{1}{2}e^{i\psi}\sin\theta & -\frac{1}{2}i\cos\theta \end{pmatrix} = (\sin\psi\sin\theta)\mathcal{J}_1 + (\cos\psi\sin\theta)\mathcal{J}_2 + (\cos\theta)\mathcal{J}_3$$

$$A^{-1}\frac{\partial A}{\partial \theta} = \begin{pmatrix} 0 & \frac{1}{2}ie^{-i\psi} \\ \frac{1}{2}ie^{i\psi} & 0 \end{pmatrix} = (\cos\psi)\mathcal{J}_1 - (\sin\psi)\mathcal{J}_2$$

$$A^{-1}\frac{\partial A}{\partial \psi} = \begin{pmatrix} \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i \end{pmatrix} = \mathcal{J}_3.$$

Thus

$$V_A(\varphi, \theta, \psi) = \left| \det\begin{pmatrix} \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \sin\theta$$

and

$$(1.17) \quad dA = \sin\theta\, d\varphi\, d\theta\, d\psi, \quad 0 \le \varphi < 2\pi, \ 0 \le \theta \le \pi, \ -2\pi \le \psi < 2\pi.$$

Since $SU(2)$ is compact, dA is both left- and right-invariant. The volume of the group is

$$(1.18) \quad V = \int_{SU(2)} dA = \int_{-2\pi}^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta = 16\pi^2.$$

Note that the Euler angles φ, ψ are indeterminant only for $\theta = 0, \pi$ and these points make no contribution to the integral.

Now that we have successfully parameterized $SU(2)$ we use the homomorphism $A \to R(A)$ to parametrize $SO(3)$. The one-parameter group $\exp t\mathcal{J}_1$ in $SU(2)$ maps onto the one-parameter group $R(\exp t\mathcal{J}_1)$ in $SO(3)$. Thus R induces a Lie algebra isomorphism which maps $\mathcal{J}_1$ to $\mathcal{L}_1' = (d/dt)R(\exp t\mathcal{J}_1)|_{t=0}$. By direct computation from (1.9) and (1.12) we see that $\mathcal{L}_1' = \mathcal{L}_1$. Similarly, $\mathcal{J}_2$ maps to $\mathcal{L}_2$ and $\mathcal{J}_3$ maps to $\mathcal{L}_3$.

Thus

$$(1.19) \quad R(A) = R(\exp\varphi\mathcal{J}_3)R(\exp\theta\mathcal{J}_1)R(\exp\psi\mathcal{J}_3) = (\exp\varphi\mathcal{L}_3)(\exp\theta\mathcal{L}_1)(\exp\psi\mathcal{L}_3),$$

or from (1.1) and (1.4),

(1.20)

$$R(A) = \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta, & -\cos\varphi\sin\psi - \sin\varphi\cos\psi\cos\theta, & \sin\varphi\sin\theta \\ \sin\varphi\cos\psi + \cos\varphi\sin\psi\cos\theta, & -\sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\theta, & -\cos\varphi\sin\theta \\ \sin\psi\sin\theta, & \cos\psi\sin\theta, & \cos\theta \end{pmatrix}.$$

Since $R(A) = R(-A)$, two different sets of Euler angles determine the same rotation matrix. Indeed it is easy to check from (1.20) that $R(A(\varphi, \theta, \psi)) = R(A(\varphi, \theta, \psi \pm 2\pi))$. Thus, to uniquely associate a rotation matrix $R(\varphi, \theta, \psi)$ with each set of Euler angles it is enough to restrict the angles to the domain

$$0 \leq \varphi < 2\pi, \qquad 0 \leq \theta \leq \pi, \qquad 0 \leq \psi < 2\pi, \tag{1.21}$$

i.e., ψ now runs over a domain of 2π rather than 4π radians. In the cases $\theta = 0, \pi$ only the sum $\varphi + \psi$ is determined by R, but this exceptional set is of lower dimension than three.

Since $SO(3)$ and $SU(2)$ are locally isomorphic, the invariant measure on $SO(3)$ must be given by (1.17), again except that the domain of the variables φ, θ, ψ is given by (1.21) rather than by (1.16). Thus the volume of $SO(3)$ is $8\pi^2$, half that of $SU(2)$.

Let $\mathbf{T}$ be a rep of $SO(3)$ by operators $\mathbf{T}(R)$. Then the operators $\mathbf{T}'(A) = \mathbf{T}(R(A))$, $A \in SU(2)$, define a rep of $SU(2)$ such that $\mathbf{T}'(-A) = \mathbf{T}'(A)$. Conversely, if $\mathbf{S}$ is a rep of $SU(2)$ such that $\mathbf{S}(-A) = \mathbf{S}(A)$ for all $A \in SU(2)$ then the operators $\mathbf{S}'(R(A)) = \mathbf{S}(A)$ define a rep of $SO(3)$. Thus, there is a 1–1 relationship between reps of $SO(3)$ and those reps $\mathbf{S}$ of $SU(2)$ such that $\mathbf{S}(-A) = \mathbf{S}(A)$, i.e., such that $\mathbf{S}(-E_2)$ is the identity operator.

Since $SU(2)$ and $SO(3)$ are compact groups, the problem of constructing all reps of these groups reduces to the problem of constructing all finite-dimensional unitary irred reps. Suppose $\mathbf{S}$ is a unitary irred rep of $SU(2)$ on an m-dimensional vector space. Now $-E_2 \in SU(2)$ commutes with all $A \in SU(2)$, so $\mathbf{S}(-E_2)$ commutes with all operators $\mathbf{S}(A)$. But $\mathbf{S}$ is irred, so by the Schur lemmas, $\mathbf{S}(-E_2) = \alpha\mathbf{E}$, where $\mathbf{E}$ is the identity operator. Since $(-E_2)^2 = E_2$ we have $\alpha^2 = 1$, or $\alpha = \pm 1$. Thus, $\mathbf{S}(-E_2) = \pm\mathbf{E}$. If the plus sign occurs then $\mathbf{S}$ is called **integral** and it defines an irred rep of $SO(3)$. However, if the minus sign occurs then $\mathbf{S}$ does not define a single-valued rep of $SO(3)$. [It is frequently stated that $\mathbf{S}$ defines a **double-valued** rep of $SO(3)$, i.e., two operators are associated with a single group element.] We shall call these reps **half-integral.**

In quantum mechanics the half-integral reps of $SU(2)$ appear even though one is initially concerned only with the rotation group $SO(3)$. The reason for this is that the states of a quantum mechanical system are given by rays in Hilbert space rather than by vectors. Thus the vectors $e^{i\gamma}\mathbf{v}$, $0 \leq \gamma < 2\pi$, all correspond to the same state for fixed $\mathbf{v}$ in the Hilbert space $\mathcal{H}$. A rotation $\mathbf{R}$ of 2π radians about the z axis will transform this state into itself. However, $\mathbf{Rv}$ need not be $\mathbf{v}$. In fact if $\mathbf{Rv} = e^{i\gamma}\mathbf{v}$ then the state will be mapped into itself. It is possible to show that for any action of $SO(3)$ as a continuous transformation group on the states of $\mathcal{H}$ we can always choose the state vectors $\mathbf{v}$ so γ is either 0 or π (Wigner [1]). The case $\gamma = \pi$ actually occurs, e.g., the electron wave functions, so we are led to consider double-valued reps of $SO(3)$.

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Let $\mathcal{G}$ be a real n-dimensional matrix Lie algebra. The **complexification** $\mathcal{G}_c$ of $\mathcal{G}$ is the complex n-dimensional Lie algebra consisting of all complex linear combinations of elements in the real algebra $\mathcal{G}$. It is easy to check that isomorphic real Lie algebras have isomorphic complexifications. Let $\mathcal{K}$ be a complex n-dimensional matrix Lie algebra. A subset $\mathcal{K}_r$ is a **real form** of $\mathcal{K}$ if $\mathcal{K}_r$ is a real n-dimensional Lie algebra. A given complex Lie algebra may have several nonisomorphic real forms. If $\mathcal{K}_r$ is a real form of $\mathcal{K}$, then $(\mathcal{K}_r)_c$ is an n-dimensional complex Lie algebra and $(\mathcal{K}_r)_c \subseteq \mathcal{K}$. Since $\mathcal{K}$ is n-dimensional, $(\mathcal{K}_r)_c = \mathcal{K}$. Conversely, if $\mathcal{G}_c$ is the complexification of $\mathcal{G}$ it is obvious that $\mathcal{G}$ is a real form of $\mathcal{G}_c$.

Now $sl(2) = sl(2, \mathfrak{C})$ is the complexification of $su(2)$. Indeed, if we set

$$\mathcal{J}^{\pm} = \pm\mathcal{J}_2 + i\mathcal{J}_1, \qquad \mathcal{J}^3 = -i\mathcal{J}_3, \tag{1.22}$$

where $i = \sqrt{-1}$ and the $\mathcal{J}_k$ are given by (1.8), we find that $\mathcal{J}^{\pm}, \mathcal{J}^3$ form a basis for a three-dimensional complex Lie algebra with commutation relations

$$[\mathcal{J}^3, \mathcal{J}^{\pm}] = \pm\mathcal{J}^{\pm}, \qquad [\mathcal{J}^+, \mathcal{J}^-] = 2\mathcal{J}^3. \tag{1.23}$$

Comparing these relations with (10.9), Section 5.10, we see $sl(2) \cong (su(2))_c$. Furthermore, $su(2)$ is a real form of $sl(2)$.

It is clear from these remarks that any rep $\mathbf{T}$ of $su(2)$ on a complex vector space V induces a rep of $sl(2)$. Indeed, $\mathbf{T}(\mathcal{J}^{\pm}) = \pm\mathbf{T}(\mathcal{J}_2) + i\mathbf{T}(\mathcal{J}_1)$, $\mathbf{T}(\mathcal{J}^3) = -i\mathbf{T}(\mathcal{J}_3)$. Conversely, any rep of $sl(2)$ on V induces a rep of $su(2)$ by restriction. One of these reps is irred if and only if the other is irred.

Thus, to find the finite-dimensional irred reps of $su(2)$ it is enough to compute the finite-dimensional irred reps of $sl(2)$ and restrict these reps to $su(2)$. Then the results can be exponentiated to obtain irred reps of $SU(2)$.

7.2 Irreducible Representations of $SU(2)$

In Section 5.10 we constructed a family of finite-dimensional irred reps of $SL(2)$. The rep $\mathbf{D}^{(u)}$, $2u = 0, 1, 2, \ldots$, is defined by operators

$$[\mathbf{T}(A)f](z) = (bz + d)^{2u} f\left(\frac{az + c}{bz + d}\right), \tag{2.1}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad f \in \mathcal{V}^{(u)},$$

acting on the $(2u + 1)$-dimensional space of polynomials of order $2u$. The corresponding rep of $sl(2)$ is given by

$$J^3 h_j = (j - u)h_j, \qquad J^+ h_j = (j - 2u)h_{j+1}, \qquad J^- h_j = -jh_{j-1}, \tag{2.2}$$

where $h_j(z) = z^j$, $0 \le j \le 2u$, is a basis for $\mathcal{V}^{(u)}$. By the remarks at the end of the preceding section, (2.2) also defines an irred rep of $su(2)$. We need only express the Lie derivatives J_k, $k = 1, 2, 3$, corresponding to $\mathcal{J}_k$ in terms of $J^{\pm}, J^3$ and use (2.2) to compute the action of J_k on a basis for $\mathcal{V}^{(u)}$. In particular,

$$J^{\pm} = \pm J_2 + iJ_1, \qquad J^3 = -iJ_3. \tag{2.3}$$

We now exponentiate each rep of $su(2)$ to see if it defines a **global** irred rep of $SU(2)$.

If we consider $SL(2)$ as a real Lie group of dimension six then $SU(2)$ is a connected Lie subgroup. Thus, to obtain the group reps of $SU(2)$ induced by the reps of $su(2)$ we restrict the operators $\mathbf{T}(A)$, (2.1), to $A \in SU(2)$, (1.6):

$$[\mathbf{T}(A)f](z) = (\beta z + \bar{\alpha})^{2u} f\left(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}\right), \qquad f \in \mathcal{V}^{(u)}. \tag{2.4}$$

We shall again denote these $(2u + 1)$-dimensional reps of $SU(2)$ by the symbol $\mathbf{D}^{(u)}$. The $\mathbf{D}^{(u)}$ are irred because their associated Lie algebra reps are irred.

Note that

$$[\mathbf{T}(-E_2)f](z) = (-1)^{2u} f(z), \tag{2.5}$$

or $\mathbf{T}(-E_2) = (-1)^{2u}\mathbf{E}$. Thus, for $u = 0, 1, 2, \ldots$ the reps are **integral** and define irred reps of $SO(3)$. On the other hand, for $u = \frac{1}{2}, \frac{3}{2}, \ldots$ the $\mathbf{D}^{(u)}$ are **half-integral** and yield double-valued reps of $SO(3)$. We shall show later that the $\mathbf{D}^{(u)}$ constitute all the irred reps of $SU(2)$ and the $\mathbf{D}^{(u)}$ for u an integer constitute all the irred reps of $SO(3)$.

Since $SU(2)$ is compact there must exist an inner product $(-, -)$ on $\mathcal{V}^{(u)}$ with respect to which $\mathbf{D}^{(u)}$ is unitary. Thus,

$$(\mathbf{T}(A)f, \mathbf{T}(A)h) = (f, h), \qquad A \in SU(2) \tag{2.6}$$

for all $f, h \in \mathcal{V}^{(u)}$. Let $\exp tJ_k = \mathbf{T}(\exp t\mathcal{J}_k)$, where the $\mathcal{J}_k$ form a basis for $su(2)$. Substituting into (2.6), differentiating with respect to t, and setting $t = 0$, we find

$$(J_k f, h) = -(f, J_k h), \qquad k = 1, 2, 3, \tag{2.7}$$

i.e., $J_k^* = -J_k$. Thus the operators J_k are skew-Hermitian. Stated another way, the operators iJ_k are Hermitian, $i = \sqrt{-1}$. It follows from (2.3) that $(J^+)^* = J^-$, $(J^-)^* = J^+$, and $(J^3)^* = J^3$.

The relations

$$(J^3 h_j, h_k) = (h_j, J^3 h_k), \qquad (J^+ h_j, h_k) = (h_j, J^- h_k)$$

together with (2.2) imply

$$(h_j, h_k) = 0, \qquad j \neq k, \tag{2.8}$$

$$(2u - j)\,\|h_{j+1}\|^2 = (j + 1)\,\|h_j\|^2, \qquad j = 0, 1, \ldots, 2u - 1. \tag{2.9}$$

Thus the basis vectors $h_j(z) = z^j$ are mutually orthogonal. Expression (2.9) shows the relationship between the norms of the basis vectors. We can normalize the inner product by choosing $\|h_0\|$ arbitrarily. Then (2.9) will fix the remaining norms. We now choose an ON basis $\{f_m\}$ for $\mathcal{V}^{(u)}$. The basis vectors will be labeled by the eigenvalue $m = j - u$ of f_m, with respect to J^3, rather then the parameter j. Normalizing $h_0(z) = 1$ by $\|h_0\|^2 = (2u)!$ we obtain the relation $\|h_j\|^2 = (2u - j)!\, j!$. Therefore, the vectors

$$(2.10)\qquad f_m(z) = \frac{(-1)^j h_j(z)}{[(2u-j)!\, j!]^{1/2}} = \frac{(-z)^{u+m}}{[(u-m)!\,(u+m)!]^{1/2}},$$
$$m = -u, -u+1, \ldots, u-1, u,$$

form an ON basis for $\mathcal{V}^{(u)}$. It follows from (2.2) that

$$(2.11)\qquad J^3 f_m = m f_m, \qquad J^{\pm} f_m = [(u \pm m + 1)(u \mp m)]^{1/2} f_{m\pm 1}.$$

The matrix elements of the rep $\mathbf{D}^{(u)}$ with respect to the ON basis $\{f_m\}$ are

$$T^u_{nm}(A) = (\mathbf{T}(A) f_m, f_n)$$

or

$$[\mathbf{T}(A) f_m](z) = \sum_{n=-u}^{u} T^u_{nm}(A) f_n(z), \qquad -u \le m \le u.$$

Thus,

(2.12)
$$g(A, z) = \frac{(\beta z + \bar{\alpha})^{u-m}(\alpha z - \bar{\beta})^{u+m}}{[(u-m)!\,(u+m)!]^{1/2}} = \sum_{n=-u}^{u} T^u_{nm}(A) \frac{(-1)^{n-m} z^{u+n}}{[(u-n)!\,(u+n)!]^{1/2}}.$$

Equating powers of z on both sides of this expression, we obtain

$$(2.13)\qquad T^u_{nm}(A) = \left[\frac{(u+m)!\,(u-n)!}{(u+n)!\,(u-m)!}\right]^{1/2} \frac{\alpha^{u+n}\bar{\alpha}^{u-n}\bar{\beta}^{m-n}}{\Gamma(m-n+1)}$$
$$\times\, {}_2F_1\left(-u-n, m-u; m-n+1; -\left|\frac{\beta}{\alpha}\right|^2\right).$$

In terms of the Euler angles (1.13) this reads

(2.14)
$$T^u_{nm}(\varphi, \theta, \psi) = i^{n-m}\left[\frac{(u+m)!\,(u-n)!}{(u+n)!\,(u-m)!}\right]^{1/2} \frac{e^{i(n\varphi+m\psi)}(\sin\theta)^{m-n}(1+\cos\theta)^{u+n-m}}{2^u\Gamma(m-n+1)}$$
$$\times\, {}_2F_1\left(-u-n, m-u; m-n+1; \frac{\cos\theta - 1}{\cos\theta + 1}\right)$$
$$= i^{n-m}\left[\frac{(u+m)!\,(u-n)!}{(u+n)!\,(u-m)!}\right]^{1/2} e^{i(n\varphi+m\psi)} P_u^{-n,m}(\cos\theta)$$

(see the Symbol Index). By suitably manipulating these formulas we could obtain many other expressions for the matrix elements. Note the simple dependence of T^u_{nm} on φ and ψ. The group property

$$(2.15)\qquad T^u_{nm}(A_1 A_2) = \sum_{j=-u}^{u} T^u_{nj}(A_1) T^u_{jm}(A_2)$$

defines an addition theorem obeyed by the matrix elements. To apply the addition theorem when the $T^u_{nm}(A)$ are parametrized by the Euler angles it is necessary to compute the Euler angles (φ, θ, ψ) of a product $A(\varphi, \theta, \psi) = A_1(\varphi_1, \theta_1, \psi_1)A_2(\varphi_2, \theta_2, \psi_2)$. A straightforward though tedious computation yields

$$\begin{aligned} \cos\theta &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\varphi_2+\psi_1), \\ e^{i\varphi} &= (e^{i\varphi_1}/\sin\theta)(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\cos(\varphi_2+\psi_1) \\ &\qquad + i\sin\theta_2\sin(\varphi_2+\psi_1)), \\ e^{i(\varphi+\psi)/2} &= (e^{i(\varphi_1+\psi_1)/2}/\cos\tfrac{1}{2}\theta)(\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\, e^{i(\varphi_2+\psi_1)/2} \\ &\qquad - \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\, e^{-i(\varphi_2+\psi_1)/2}), \end{aligned} \tag{2.16}$$

and the addition theorems are obtained by substituting (2.14) and (2.16) into (2.15). The unitary property of the operators $\mathbf{T}(A)$ implies

$$T^u_{nm}(A^{-1}) = \overline{T^u_{mn}(A)}, \tag{2.17}$$

or in Euler angles,

$$(-1)^{m-n}P_u^{-n,m}(\cos\theta) = \frac{(u+n)!\,(u-m)!}{(u-n)!\,(u+m)!}P_u^{-m,n}(\cos\theta).$$

Also, $|T^u_{nm}(A)| \leq 1$ or

$$|P_u^{-n,m}(\cos\theta)| \leq \left[\frac{(u+n)!\,(u-m)!}{(u+m)!\,(u-n)!}\right]^{1/2}, \qquad 0 \leq \theta \leq \pi.$$

We can obtain an integral expression for the matrix elements by setting $z = e^{i\gamma}$ in (2.12), multiplying by $e^{-i(u+n)\gamma}$, and integrating both sides of the resulting expression from 0 to 2π:

$$\begin{aligned} T^u_{nm}(\varphi, \theta, \psi) &= \frac{(-1)^{n-m}}{2\pi}\left[\frac{(u-n)!\,(u+n)!}{(u-m)!\,(u+m)!}\right]^{1/2} \\ &\quad\times \int_0^{2\pi}\left(i\sin\frac{\theta}{2}e^{i\gamma} + \cos\frac{\theta}{2}\right)^{u-m} \\ &\quad\times\left(\cos\frac{\theta}{2}e^{i\gamma} + i\sin\frac{\theta}{2}\right)^{u+m} e^{-i\gamma(u+n)}\,d\gamma. \end{aligned} \tag{2.18}$$

The matrix elements $T^l_{0m}(\varphi, \theta, \psi)$, l, m, integers, are proportional to the spherical harmonics $Y_l^m(\theta, \psi)$. Indeed

(2.19)

$$T^l_{0m}(\varphi, \theta, \psi) = i^m\left(\frac{4\pi}{2l+1}\right)^{1/2} Y_l^m(\theta, \psi) = i^m\left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\psi},$$

where the $P_l^m(\cos\theta)$ are the associated Legendre functions. Moreover,

$$T^l_{00}(\varphi, \theta, \psi) = P_l(\cos\theta), \tag{2.20}$$

where $P_l(\cos\theta)$ is the lth Legendre polynomial.

According to the general theory of Section 6.2, the matrix elements $T^u_{nm}(A)$ satisfy the orthogonality relations

$$\text{(2.21)}\qquad \int_{SU(2)} T^{u_1}_{n_1m_1}(A)\overline{T^{u_2}_{n_2m_2}(A)}\,dA = \frac{16\pi^2}{2u_1+1}\,\delta_{n_1n_2}\,\delta_{m_1m_2}\,\delta_{u_1u_2}.$$

Thus,

$$\int_{-2\pi}^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\; T^{u_1}_{n_1m_1}(\varphi,\theta,\psi)\overline{T^{u_2}_{n_2m_2}(\varphi,\theta,\psi)}\sin\theta = \frac{16\pi^2}{2u_1+1}\,\delta_{n_1n_2}\,\delta_{m_1m_2}\,\delta_{u_1u_2}.$$

The ψ and φ integrations are trivial, while the θ integration gives

$$\int_0^{\pi} P^{n,m}_u(\cos\theta)P^{n,m}_v(\cos\theta)\sin\theta\,d\theta = \frac{2}{2u+1}\,\frac{(u-n)!\,(u-m)!}{(u+n)!\,(u+m)!}\,\delta_{uv}.$$

For $n = m = 0$ these are the orthogonality relations for the Legendre polynomials. Note: By definition,

$$\text{(2.22)}\qquad P^{0,-m}_u(\cos\theta) = P_u{}^m(\cos\theta), \qquad P^{0,0}_u(\cos\theta) = P_u(\cos\theta),$$

where $P_u{}^m$, P_u are Legendre functions. At this point we know only that the functions $\{(2u+1)^{1/2}T^u_{nm}(A)\}$ form an ON set in $L_2(SU(2))$, but later we will show that they form a basis, i.e., the $\mathbf{D}^{(u)}$ constitute a complete set of irred reps of $SU(2)$.

We now compute the character $\chi^{(u)}(A)$ of $\mathbf{D}^{(u)}$. By definition,

$$\text{(2.23)}\qquad \chi^{(u)}(A) = \sum_{m=-u}^{u} T^u_{mm}(A).$$

This expression is too complicated to compute easily. On the other hand we know $\mathbf{T}^{(u)}(BAB^{-1}) = \mathbf{T}^{(u)}(A)$ for all $A, B \in SU(2)$. From elementary matrix theory, every $A \in SU(2)$ can be diagonalized by a unitary similarity transformation. Indeed, there exists a number τ, $-2\pi \le \tau < 2\pi$, and a $B \in SU(2)$ such that

$$BAB^{-1} = \begin{pmatrix} e^{i\tau/2} & 0 \\ 0 & e^{-i\tau/2} \end{pmatrix}.$$

Therefore, the conjugacy classes in $SU(2)$ are labeled by the parameter τ. Passing from $SU(2)$ to $SO(3)$ by the usual homomorphism we see that A represents a rotation through angle τ about a fixed axis. [In $SO(3)$, two rotations about distinct axes are conjugate if and only if they have the same rotation angle.]

We have shown that A is conjugate to the group element C with Euler parameters $(0, 0, \tau)$, or $\alpha = e^{i\tau/2}$, $\beta = 0$. By (2.12), $T^u_{mm}(C) = e^{im\tau}$. Thus,

$$\text{(2.24)}\qquad \chi^{(u)}(A) = \sum_{m=-u}^{u} e^{im\tau} = \frac{e^{i(u+1)\tau} - e^{-iu\tau}}{e^{i\tau}-1} = \frac{\sin[(u+\frac{1}{2})\tau]}{\sin(\tau/2)},$$

where we have used the formula for the sum of a geometric series. It is not difficult to express $\chi^{(u)}(A)$ directly in terms of the parameters of A, but the expression is not very enlightening. For $u = l = 0, 1, 2, \ldots$ the formula $\chi^{(l)}(R(A)) = \sin[(l + \frac{1}{2})\tau]/\sin(\tau/2)$ gives the character of the rep $\mathbf{D}^{(l)}$ of $SO(3)$ where $R(A)$ is a rotation through the angle τ about a fixed axis. In this case $\tau \pm 2\pi$ yield the same value as τ.

Let $\mathbf{D}^{(u)}, \mathbf{D}^{(v)}$ be irred reps of $SU(2)$ and consider the tensor product $\mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)}$. This rep is $(2u + 1)(2v + 1)$-dimensional and its character is $\chi^{(u)} \otimes \chi^{(v)}(A) = \chi^{(u)}(A)\chi^{(v)}(A)$. We can determine the decomposition of $\mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)}$ into a direct sum of irred reps of $SU(2)$ by expressing $\chi^{(u)} \otimes \chi^{(v)}$ as a sum of simple characters. Now

$$\chi^{(u)} \otimes \chi^{(v)}(A) = \sum_{m=-u}^{u} \sum_{n=-v}^{v} e^{i(m+n)\tau} = \sum_{w=u-v}^{u+v} \sum_{k=-w}^{w} e^{ik\tau} = \sum_{w=|u-v|}^{u+v} \chi^{(w)}(A),$$

where we have assumed $u \geq v$. [Note: The term $e^{ik\tau}$ occurs $\min(u + v + 1 - |k|, 2v + 1)$ times in the above expansion.] In general

$$\chi^{(u)} \otimes \chi^{(v)}(A) = \sum_{w=|u-v|}^{u+v} \chi^{(w)}(A). \tag{2.25}$$

Therefore,

$$\mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)} \cong \mathbf{D}^{(u+v)} \oplus \mathbf{D}^{(u+v-1)} \oplus \cdots \oplus \mathbf{D}^{(|u-v|)}. \tag{2.26}$$

This expression is known as the **Clebsch–Gordan series.** Note that each irred rep which occurs on the right-hand side of (2.26) has multiplicity one. Thus, the decomposition of the rep space into irred subspaces is unique and independent of basis. In Section 7.7 we discuss this decomposition in detail.

7.3 Irreducible Representations of *sl*(2)

In Section 7.1, we showed that $sl(2)$ is the complexification of the real Lie algebra $su(2) \cong so(3)$. Therefore, there is a 1–1 relationship between irred reps of $sl(2)$ and irred reps of $su(2)$. To determine all finite-dimensional irred reps ρ of these Lie algebras it is enough to classify (up to isomorphism) all finite-dimensional complex vector spaces V and operators $J^{\pm}, J^3$ on V satisfying the commutation relations

$$[J^3, J^{\pm}] = \pm J^{\pm}, \qquad [J^+, J^-] = 2J^3, \tag{3.1}$$

such that V is irred under the J-operators. Here $J^3 = -iJ_3$, $J^{\pm} = \pm J_2 + iJ_1$, and $J_k = \rho(\mathcal{J}_k)$. The operators (3.1) will prove to be much more convenient for computations than the J_k.

The following computation should be familiar to those readers who have studied quantum mechanics. A good understanding of this procedure is essential since similar methods will be used to construct the irred reps of all the classical groups.

Let ρ be a finite-dimensional irred rep of $sl(2)$ on V. As the reader can verify, the **Casimir operator**

$$C = -(J_1)^2 - (J_2)^2 - (J_3)^2 = J^+J^- + J^3J^3 - J^3 \tag{3.2}$$

commutes with $J^\pm, J^3$. By the Schur lemmas, C must be a multiple of the identity operator on V, $C = \lambda \mathbf{E}$.

Let $\mathbf{h}_q \in V$ be an eigenvector of J^3 with eigenvalue q: $J^3\mathbf{h}_q = q\mathbf{h}_q$. Now $[J^3, J^+]\mathbf{h}_q = J^+\mathbf{h}_q$, or $J^3(J^+\mathbf{h}_q) = (q+1)J^+\mathbf{h}_q$. Thus, either $J^+\mathbf{h}_q = \boldsymbol{\theta}$ or $J^+\mathbf{h}_q$ is an eigenvector of J^3 with eigenvalue $q + 1$. Similarly the commutation relation $[J^3, J^-] = -J^-$ implies that $J^-\mathbf{h}_q = \boldsymbol{\theta}$ or $J^-\mathbf{h}_q$ is an eigenvector of J^3 with eigenvalue $q - 1$. By a simple induction argument

$$J^3(J^+)^k\mathbf{h}_q = (q+k)(J^+)^k\mathbf{h}_q, \quad J^3(J^-)^k\mathbf{h}_q = (q-k)(J^-)^k\mathbf{h}_q, \quad k = 0, 1, \ldots.$$

Since V is finite-dimensional there exists an integer $r \geq 0$ such that $(J^+)^r\mathbf{h}_q \neq \boldsymbol{\theta}$ and $(J^+)^{r+1}\mathbf{h}_q = \boldsymbol{\theta}$. Set $(J^+)^r\mathbf{h}_q = \mathbf{f}_u$, where $u = q + r$. Then $J^3\mathbf{f}_u = u\mathbf{f}_u$. Similarly there is an integer $s \geq 0$ such that $(J^-)^s\mathbf{f}_u \neq \boldsymbol{\theta}$, $(J^-)^{s+1}\mathbf{f}_u = \boldsymbol{\theta}$. We will show that the eigenvectors $\mathbf{f}_m$, $m = u, u-1, \ldots, u-s$, where $\mathbf{f}_m = (J^-)^{u-m}\mathbf{f}_u$ form a basis for V.

Now $C\mathbf{f}_u = \lambda\mathbf{f}_u$. On the other hand, by (3.1) and (3.2),

$$C\mathbf{f}_u = (J^-J^+ + J^3J^3 + J^3)\mathbf{f}_u = J^-J^+\mathbf{f}_u + u(u+1)\mathbf{f}_u.$$

Since $J^+\mathbf{f}_u = \boldsymbol{\theta}$ we obtain $\lambda = u(u+1)$. Applying C to $\mathbf{f}_{u-s}$ we find

$$C\mathbf{f}_{u-s} = u(u+1)\mathbf{f}_{u-s} = (J^+J^- + J^3J^3 - J^3)\mathbf{f}_{u-s} = (u-s)(u-s-1)\mathbf{f}_{u-s},$$

since $J^-\mathbf{f}_{u-s} = \boldsymbol{\theta}$. Thus, $u(u+1) = (u-s)(u-s-1)$ or $s = 2u$. It follows that $2u$ is a nonnegative integer. Since $J^3\mathbf{f}_m = m\mathbf{f}_m$, $-u \leq m \leq u$, we obtain

$$C\mathbf{f}_m = u(u+1)\mathbf{f}_m = (J^+J^- + J^3J^3 - J^3)\mathbf{f}_m = J^+\mathbf{f}_{m-1} + m(m-1)\mathbf{f}_m,$$

or $J^+\mathbf{f}_m = (u-m)(u+m+1)\mathbf{f}_{m+1}$, $u - 1 \geq m \geq -u$. We have shown that the $(2u+1)$-dimensional subspace of V spanned by the $\{\mathbf{f}_m\}$ is invariant and irred under ρ. Since ρ is irred, this subspace must be V itself. The rep ρ is now completely determined:

$$J^3\mathbf{f}_m = m\mathbf{f}_m, \quad J^-\mathbf{f}_m = \mathbf{f}_{m-1}, \quad J^+\mathbf{f}_m = (u-m)(u+m+1)\mathbf{f}_{m+1}, \quad -u \leq m \leq u. \tag{3.3}$$

(On the right-hand sides of these expressions we adopt the convention: $\mathbf{f}_m = \boldsymbol{\theta}$ if m is not an eigenvalue of J^3.) Conversely, if $2u$ is a nonnegative integer then the operators $J^\pm, J^3$ defined by (3.3) determine an irred rep $\mathbf{D}^{(u)}$ of $sl(2)$. If $u \neq v$ then $\mathbf{D}^{(u)}$ is not equivalent to $\mathbf{D}^{(v)}$ since the two reps have different dimensions.

The rep $\mathbf{D}^{(u)}$ uniquely determines and is determined by the eigenvalues $-u, \ldots, +u$ of J^3. However, the basis vectors are not uniquely determined.

If $\{\gamma_m: -u \le m \le u\}$ is a set of nonzero complex constants then the eigenvectors $\{\mathbf{f}_m' = \gamma_m \mathbf{f}_m\}$ also form a basis for V. If the constants are chosen such that $\gamma_{m+1}/\gamma_m = [(u+m+1)(u-m)]^{1/2}$, $-u \le m \le u-1$, then relations (3.3) become

$$J^3\mathbf{f}_m = m\mathbf{f}_m, \quad J^{\pm}\mathbf{f}_m = [(u \mp m)(u \pm m + 1)]^{1/2}\mathbf{f}_{m\pm 1}, \quad C\mathbf{f}_m = u(u+1)\mathbf{f}_m, \tag{3.4}$$

where we have omitted the prime on $\mathbf{f}_m'$. Note that expressions (3.4) and (2.11) are identical. Thus the reps $\mathbf{D}^{(u)}$, $2u = 0, 1, 2, \ldots$, of $SU(2)$ constructed in the preceding section constitute all the bounded irred reps of $SU(2)$, up to equivalence. [Furthermore, the reps of $SL(2)$ constructed in Section 5.10 constitute all finite-dimensional irred reps of $SL(2)$ as a complex Lie group.]

Another useful basis for V is obtained by setting $\gamma_{m+1}/\gamma_m = -(u+m+1)$. Relations (3.3) become

$$J^3\mathbf{f}_m = m\mathbf{f}_m, \qquad J^{\pm}\mathbf{f}_m = (-u \pm m)\mathbf{f}_{m\pm 1}, \qquad C\mathbf{f}_m = u(u+1)\mathbf{f}_m. \tag{3.5}$$

Although we have confined ourselves to a search for finite-dimensional reps, expressions (3.5) can also be used to construct infinite-dimensional irred reps of $sl(2)$. (Here we mean V is infinite-dimensional in the algebraic sense. We do not consider V as a Hilbert space.) Indeed if $2u$ is a complex number, not a nonnegative integer, and V is a vector space generated by the vectors $\{\mathbf{f}_m\}$, $m = -u, -u+1, -u+2, \ldots$, then expressions (3.5) define an irred rep $\uparrow_u$ of $sl(2)$ on V, as the reader can verify. Since $J^-\mathbf{f}_{-u} = \boldsymbol{\theta}$ the operator J^3 has a lowest eigenvalue $-u$, i.e., an eigenvalue whose real part is least. However, J^3 has no highest eigenvalue. The rep $\uparrow_u$ is said to be **bounded below**. The reps $\mathbf{D}^{(u)}$ are bounded both below and above. Using similar techniques one can use expressions (3.5) to construct infinite-dimensional reps which are bounded above but not below or which are bounded neither above nor below. A systematic study of such reps is undertaken by Miller [1].

We have already seen the infinite-dimensional reps $\uparrow_u$. In Section 5.10 we constructed the local multiplier rep

$$[\mathbf{T}(A)f](z) = (bz+d)^{2u} f\left(\frac{az+c}{bz+d}\right), \qquad A \in SL(2) \tag{3.6}$$

of $SL(2)$ on the space $\mathfrak{A}$ of all functions analytic in a neighborhood of $z = 0$. Here $2u$ is not a nonnegative integer. As a basis for $\mathfrak{A}$ we choose the functions $h_j(z) = z^j$, $j = 0, 1, \ldots$. The Lie derivatives associated with (3.6) are easily computed to be

$$J^+ = -2uz + z^2(d/dz), \qquad J^- = -d/dz, \qquad J^3 = -u + z(d/dz). \tag{3.7}$$

Setting $f_m(z) = h_j(z) = z^j$, where $m + u = j$, we find

$$
\begin{aligned}
&J^+ f_m = (-2uz + z^2\, d/dz) z^{m+u} = (m-u) f_{m+1}, \\
(3.8)\qquad &J^- f_m = -dz^{m+u}/dz = -(m+u) f_{m-1}, \qquad J^- f_{-u} = 0, \\
&J^3 f_m = (-u + z\, d/dz) z^{m+u} = m f_m, \\
&\qquad\qquad m = -u, -u+1, -u+2, \ldots .
\end{aligned}
$$

Thus the local multiplier rep (3.6) induces the irred rep $\uparrow_u$ of $sl(2)$. Conversely, the infinite-dimensional rep $\uparrow_u$ induces the local multiplier rep (3.6), which we will also call $\uparrow_u$. Note that the group rep $\uparrow_u$ is purely local.

7.4 Expansion Theorems for Functions on $SU(2)$

We have shown that the $(2u+1)$-dimensional reps $\mathbf{D}^{(u)}$, $2u = 0, 1, 2, \ldots$, constitute a complete set of nonequivalent irred unitary reps of $SU(2)$. Thus, by the Peter–Weyl theorem, the functions $\varphi^u_{nm}(\varphi, \theta, \psi) = (2u+1)^{1/2} T^u_{nm}(\varphi, \theta, \psi)$, $-u \le m, n \le u$, $2u = 0, 1, \ldots$, constitute an ON basis for $L_2(SU(2))$. [Here we use Euler coordinates on $SU(2)$ for the matrix elements (2.14).] The matrix elements satisfy orthogonality relations (2.21). Furthermore, if $f \in L_2(SU(2))$ then

$$
(4.1)\qquad f(\varphi, \theta, \psi) \sim \sum_{2u=0}^{\infty} \sum_{n,m=-u}^{u} a^u_{nm} \varphi^u_{nm}(\varphi, \theta, \psi),
$$

where

(4.2)
$$
a^u_{nm} = (f, \varphi^u_{nm}) = \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, f(\varphi, \theta, \psi) \overline{\varphi^u_{nm}(\varphi, \theta, \psi)} \sin\theta .
$$

The Parseval equality reads

$$
(4.3)\qquad (f, f) = \sum_{2u=0}^{\infty} \sum_{m,n=-u}^{u} |a^u_{nm}|^2 .
$$

With simple modifications these results apply to functions in $L_2(SO(3))$. The modifications are (1) u takes only integral values, (2) the volume of $SO(3)$ is $8\pi^2$ rather than $16\pi^2$, and (3) the variable ψ runs over the range $0 \le \psi < 2\pi$ rather than $-2\pi \le \psi < 2\pi$.

Some particular cases of (4.1) are of special interest. Suppose $f(\theta, \psi) \in L_2(SO(3))$ is independent of the variable φ. If we think of (θ, ψ) as latitude and longitude, we can consider f as a function on the unit sphere S_3, square-integrable with respect to the area measure on S_3. Since the φ-dependence of $\varphi^u_{nm}(\varphi, \theta, \psi)$ is $e^{in\varphi}$, it follows from (4.2) that $a^u_{nm} = 0$ unless $n = 0$. The only possible nonzero coefficients are a^u_{0m}, where $u = l = 0, 1, 2, \ldots$.

By (2.19)

$$\varphi_{0m}^{l}(\varphi, \theta, \psi) = (4\pi)^{1/2} Y_l^m(\theta, \psi), \tag{4.4}$$

where Y_l^m is a spherical harmonic. Thus,

$$f(\theta, \psi) \sim \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_m{}^l Y_l^m(\theta, \psi), \tag{4.5}$$

where

$$c_m{}^l = \int_0^{2\pi} d\psi \int_0^{\pi} d\theta\, f(\theta, \psi)\overline{Y_l^m(\theta, \psi)} \sin\theta, \qquad (Y_l^m, Y_{l'}^{m'}) = \delta_{ll'}\,\delta_{mm'}. \tag{4.6}$$

This is the expansion of a function on the sphere as a linear combination of spherical harmonics. As usual, (4.5) converges in the norm of $L_2(SO(3))$, not necessarily pointwise.

If $f(\theta) \in L_2(SO(3))$ is a function of θ alone then the coefficients a_{nm}^{u} are zero unless $n = m = 0$. From (2.20),

$$\varphi_{00}^{l}(\varphi, \theta, \psi) = (2l + 1)^{1/2} P_l(\cos\theta), \qquad l = 0, 1, 2, \ldots, \tag{4.7}$$

where

$$P_l(x) = {}_2F_1\left(l + 1, -l; 1, \frac{1 - x}{2}\right) = 2^{-l}(1 + x)^l\, {}_2F_1\left(-l, -l; 1; \frac{x - 1}{x + 1}\right) \tag{4.8}$$

is a Legendre polynomial of order l. The coefficient of x^l in the expansion of $P_l(x)$ is nonzero and $P_l(1) = 1$. The expansion of $f(\theta)$ becomes

$$\begin{gathered} f(\theta) \sim \sum_{l=0}^{\infty} c_l P_l(\cos\theta), \qquad c_l = \tfrac{1}{2}(2l + 1) \int_0^{\pi} f(\theta) P_l(\cos\theta) \sin\theta\, d\theta, \\ \int_0^{\pi} P_l(\cos\theta) P_k(\cos\theta) \sin\theta\, d\theta = 2\delta_{kl}/(2l + 1). \end{gathered} \tag{4.9}$$

Expressions (4.9) can be simplified by introduction of the new variable $x = \cos\theta$, $0 \le \theta \le \pi$.

The reader can construct some examples of the above expansions by considering the generating function (2.12) and the addition theorem (2.15). Other examples can be obtained by manipulation of the integral expression (2.18) for the matrix elements. If $n = m = 0$, $u = l$, (2.18) becomes

$$P_l(\cos\theta) = (1/2\pi) \int_0^{2\pi} (\cos\theta + i \sin\theta \cos\gamma)^l\, d\gamma. \tag{4.10}$$

Setting $z = e^{i\gamma}$, we can write this last equation as a contour integral

$$P_l(\cos\theta) = \frac{1}{2\pi i} \oint \left[\cos\theta + \frac{i}{2} \sin\theta\,(z + z^{-1})\right]^l \frac{dz}{z}, \tag{4.11}$$

where the contour is a simple closed curve surrounding the origin. The change of variable $z = [t - \cos\theta + (t^2 - 2t\cos\theta + 1)^{1/2}]/(i\sin\theta)$ transforms (4.11) into

$$P_l(\cos\theta) = \frac{1}{2\pi i}\oint \frac{t^l\,dt}{(t^2 - 2t\cos\theta + 1)^{1/2}},$$

where the contour can be chosen as the circle $|t| = r > 1$. Setting $s = t^{-1}$, we find

$$(4.12) \qquad P_l(\cos\theta) = \frac{1}{2\pi i}\oint \frac{s^{-l-1}\,ds}{(s^2 - 2s\cos\theta + 1)^{1/2}}, \qquad |s| = r^{-1}.$$

The analytic function $(s^2 - 2s\cos\theta + 1)^{-1/2} = \sum c_n s^n$ possesses a power series expansion convergent for $|s| < 1$. It follows from (4.12) and the Cauchy integral theorem that $c_n = P_n(\cos\theta)$:

$$(4.13) \qquad h(s, x) = (s^2 - 2sx + 1)^{-1/2} = \sum_{n=0}^{\infty} s^n P_n(x), \qquad -1 \le x \le 1.$$

One can check that $h(s, \cos\theta) \in L_2(SO(3))$ for $|s| < 1$, so this is an example of the expansion (4.9). This generating function is often used to define the Legendre polynomials. Let $P_n'(x) = (d/dx)P_n(x)$.

Theorem 7.1.

(a) $P_n(1) = 1$;

(b) $P_n(-1) = (-1)^n$;

(c) $(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) - nP_{n-1}(x)$;

(d) $(1 - x^2)P_n'(x) + nxP_n(x) = nP_{n-1}(x)$;

(e) $(1 - x^2)P_n'(x) - (n + 1)xP_n(x) = -(n + 1)P_{n+1}(x)$;

(f) $[(1 - x^2)P_n'(x)]' + n(n + 1)P_n(x) = 0$, $n = 0, 1, 2, \ldots$.

Proof. These results follow from (4.13). (a) $h(s, 1) = (1 - s)^{-1} = \sum s^n$. (b) $h(s, -1) = (1 + s)^{-1} = \sum (-s)^n$. (c) $(s^2 - 2sx + 1)\,\partial h/\partial s = (s - x)h$. Now compare coefficients of s^n on both sides of this equality. (d) Follows from the identity $(1 - x^2)(\partial h/\partial x) + xs(\partial h/\partial s) = s^2(\partial h/\partial s) + sh$. (e) Follows from the identity $(1 - x^2)(\partial h/\partial x) - xs(\partial h/\partial s) - xh = -\partial h/\partial s$. (f) An easy consequence of (d) and (e). Q.E.D.

Any identity we can obtain for the generating function implies an identity for the Legendre polynomials. Thus, the identity $s\,\partial h/\partial s = (x - s)\,\partial h/\partial x$ implies

$$(4.14) \qquad nP_n(x) = xP_n'(x) - P_{n-1}'(x).$$

Identities such as (c)–(e) which relate different Legendre polynomials are called **recurrence formulas**. The differential equation (f) is the **Legendre equation**. Here we have derived these results by manipulation of the generating function $h(s, x)$, but we shall see that all these identities, including the generating function, have a simple group-theoretic interpretation.

7.5 New Realizations of the Irreducible Representations

From an abstract point of view we have completely classified the irred reps of $SU(2)$ and $SO(3)$. We have obtained simple realizations or models of these reps in which the underlying vector spaces consist of polynomials in one complex variable. In actual physical or geometrical systems, however, the group action may appear far different from that in our models. In other words, even though two group reps are abstractly equivalent they may appear physically or geometrically quite different. For this reason it is useful to survey some of the distinct realizations of the reps $\mathbf{D}^{(u)}$ which appear in mathematical physics.

For our first model we consider the natural action of $SO(3)$ as a transformation group on R_3:

$$(5.1) \qquad \mathbf{x} \longrightarrow A^{-1}\mathbf{x}, \qquad A \in SO(3), \qquad \mathbf{x} = (x, y, z) \in R_3.$$

(The inverse is necessary to conform to the definition of a Lie transformation group as given in Section 5.9.) Using the basis $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ for $so(3)$ as defined by (1.1)–(1.3) and computing the corresponding Lie derivatives we find

$$(5.2) \quad L_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad L_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad L_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

As guaranteed by the general theory, these Lie derivatives satisfy the commutation relations

$$(5.3) \qquad [L_1, L_2] = L_3, \qquad [L_2, L_3] = L_1, \qquad [L_3, L_1] = L_2$$

and generate a Lie algebra isomorphic to $so(3)$. The Lie derivatives (5.2) are essentially the angular momentum operators of quantum mechanics. We shall construct models of the reps $\mathbf{D}^{(u)}$ where the action of the group and Lie algebra is given by (5.1) and (5.2), and the underlying vector space consists of functions on R_3.

First of all we define operators

$$(5.4) \qquad L^{\pm} = \mp L_2 + iL_1, \qquad L^3 = iL_3,$$

which satisfy the commutation relations (1.23) and form a basis for the complex Lie algebra $sl(2)$. [Note: These operators are not identical with (2.3). Nevertheless they satisfy the same commutation relations:

$$[L^3, L^{\pm}] = \pm L^{\pm}, \qquad [L^+, L^-] = 2L^3.$$

The choice (5.4) is more convenient for the computation to follow.] The action (5.1) of $SO(3)$ on R_3 is not transitive. In particular $x^2 + y^2 + z^2$ is invariant under the group. Any sphere of radius r and center at $\mathbf{0}$ is mapped into itself. To exploit this property we introduce spherical coordinates r, θ, φ:

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta, \tag{5.5}$$
$$r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi.$$

Then the L-operators become

$$L^{\pm} = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right), \qquad L^3 = -i\frac{\partial}{\partial\varphi}, \tag{5.6}$$

independent of r. We now look for realizations of $\mathbf{D}^{(u)}$ such that the basis space $\mathcal{V}^{(u)}$ is a space of analytic functions of θ, φ and the operators $L^{\pm}, L^3$ are given by (5.6). According to expressions (3.4) we must find basis functions $f_m(\theta, \varphi) = Y_u{}^m(\theta, \varphi)$ for $\mathcal{V}^{(u)}$ such that

$$\begin{aligned} L^3 Y_u{}^m &= mY_u{}^m, \qquad L^{\pm}Y_u{}^m = [(u \mp m)(u \pm m + 1)]^{1/2}Y_u^{m\pm 1}, \\ CY_u{}^m &= (L^+L^- + L^3L^3 - L^3)Y_u{}^m = u(u+1)Y_u{}^m. \end{aligned} \tag{5.7}$$

Since $L^3 = -i\partial/\partial\varphi$ we have

$$-i\,\partial Y_u{}^m/\partial\varphi = mY_u{}^m, \qquad Y_u{}^m(\theta, \varphi) = Q_u{}^m(\theta)e^{im\varphi},$$

where $Q_u{}^m(\theta)$ is yet to be determined. The equation $L^+Y_u{}^u = 0$ becomes

$$(d/d\theta)Q_u{}^u - u\cot\theta\, Q_u{}^u = 0,$$

whose solution is

$$Q_u{}^u = c_u\sin^u\theta = c_u(1 - \cos^2\theta)^{u/2},$$

where c_u is an arbitrary nonzero constant. We can now use the "lowering operator" L^- to obtain the functions $Q_u{}^m$ recursively from $Q_u{}^u$:

$$-(d/d\theta)Q_u^{m+1} - (m+1)(\cot\theta)Q_u^{m+1} = [(u+m+1)(u-m)]^{1/2}Q_u{}^m. \tag{5.8}$$

A straightforward induction argument and (5.8) yield the explicit expressions

$$Q_u{}^m(\theta) = c_u\left[\frac{(u+m)!}{(2u)!\,(u-m)!}\right]^{1/2}(1-\cos^2\theta)^{-m/2}\frac{d^{u-m}(1-\cos^2\theta)^u}{d(\cos\theta)^{u-m}}, \tag{5.9}$$
$$-u \leq m \leq u.$$

The equation $L^-Y_u^{-u} = 0$ applied to (5.9) yields the condition

$$-\frac{dQ_u^{-u}}{d\theta} + u(\cot\theta)Q_u^{-u} = \frac{c_u}{(2u)!}(1-\cos^2\theta)^{(u+1)/2}\frac{d^{2u+1}(1-\cos^2\theta)^u}{d(\cos\theta)^{2u+1}} \equiv 0.$$

This condition can be satisfied only if $u = l$ is an integer. For $u = \frac{1}{2}, \frac{3}{2}, \ldots$, our construction fails. This is not surprising since the angular momentum operators (5.2) were obtained from an action of $SO(3)$ as a transformation group. For $u = l$, however, we have found a highest weight vector $Y_l{}^l$ and a lowest weight vector Y_l^{-l}. By copying the construction of the reps $\mathbf{D}^{(l)}$ in

Section 7.3, the reader can check that the functions Y_l^m satisfy all the relations (5.7):

$$\pm\frac{d}{d\theta}Q_l^m - m\cot\theta\, Q_l^m = [(l\mp m)(l\pm m+1)]^{1/2}Q_l^{m\pm 1},$$

(5.10)
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}Q_l^m\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]Q_l^m = 0,\quad -l\le m\le l,$$

where the last expression is obtained by writing $CY_l^m = l(l+1)Y_l^m$ in terms of differential operators.

The constant c_l is usually fixed by the requirement

$$\int_0^{2\pi}\int_0^{\pi}|Y_l^l(\theta,\varphi)|^2\sin\theta\,d\theta\,d\varphi = 1$$

or

(5.11)
$$c_l = \frac{(-1)^l}{2^l l!}\left[\frac{(2l+1)!}{4\pi}\right]^{1/2},$$

where the phase factor $(-1)^l$ is introduced to conform to convention.

The basis functions $Y_l^m(\theta,\varphi)$ are just the spherical harmonics. To show this explicitly we obtain some new expressions for the matrix elements $T^u_{nm}(A)$ derived in Section 7.2. From (2.12). $T^u_{nm}(A)$ is, to within a constant factor, the coefficient of z^{u+n} in the Taylor series expansion of $g(A,z)$. Thus,

$$T^u_{nm}(A) = (-1)^{n-m}\left[\frac{(u-n)!}{(u+n)!}\right]^{1/2}\frac{d^{u+n}g(A,z)}{dz^{u+n}}\bigg|_{z=0}.$$

In terms of the functions $P_u^{n,m}(\cos\theta)$, (2.14), this reads

(5.12)
$$P_u^{-n,m}(\cos\theta) = \frac{i^{n-m}}{(u+m)!}\frac{d^{u+n}}{dz^{u+n}}\times\left[\left(iz\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{u-m}\left(z\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{u+m}\right]_{z=0}.$$

Setting $y = (iz\sin\theta+\cos\theta-1)/2$, we find $dy = \frac{1}{2}i\sin\theta\,dz$ and

(5.13)
$$P_u^{-n,m}(\cos\theta) = \frac{(-1)^{u+n}}{2^u(u+m)!}(1-\cos\theta)^{(n-m)/2}(1+\cos\theta)^{(n+m)/2}\times\frac{d^{u+n}[(1-\cos\theta)^{u+m}(1+\cos\theta)^{u-m}]}{d(\cos\theta)^{u+n}}.$$

In particular, from (2.14), (2.16), (2.19), and (2.22) we obtain the expressions

(5.14)
$$P_l^n(\cos\theta) = \frac{(l+n)!\,(-1)^l}{(l-n)!\,2^l l!}(1-\cos^2\theta)^{-n/2}\frac{d^{l-n}(1-\cos^2\theta)^l}{d(\cos\theta)^{l-n}}$$

for the associated Legendre functions and

(5.15)
$$Y_l^m(\theta,\varphi) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\varphi},\qquad -l\le m\le l,$$

for the spherical harmonics. This last expression agrees with (5.9) and (5.11), so the basis functions for the realization (5.7) are just spherical harmonics.

We have already seen that special functions appear in Lie theory as matrix elements of group reps. The above example shows that they also appear as basis functions in the underlying vector space of a group rep.

Now that we have found realizations for the reps $\mathbf{D}^{(l)}$ of $so(3)$ we can determine the action of $SO(3)$ on these realizations. Indeed $SO(3)$ acts on R_3 according to (5.1). It is not difficult to show that the resulting identity is

$$T^l_{0n}(AB) = \sum_{m=-l}^{l} T^l_{mn}(B)T^l_{0m}(A), \tag{5.16}$$

a special case of (2.15). Recall that

$$T^l_{0m}(A(\varphi, \theta, \psi)) = i^m\left(\frac{4\pi}{2l+1}\right)^{1/2} Y_l^m(\theta, \varphi).$$

Since $r^2 = x^2 + y^2 + z^2$ is invariant under the action of $SO(3)$ the set $\{f(r)Y_l^m(\theta, \varphi)\colon -l \leq m \leq l\}$ forms a basis for a realization of the irred rep $\mathbf{D}^{(l)}$. Here $f(r)$ is an arbitrary nonzero function. It follows that $L_2(R_3)$, the Hilbert space of all Lebesgue square-integrable functions on R_3, decomposes into a direct sum of irred reps $\mathbf{D}^{(l)}$, each $\mathbf{D}^{(l)}$ with infinite multiplicity.

An important special case of these considerations is the space $\mathcal{W}^l$ of all homogeneous polynomials $u(x, y, z)$ with degree l in x, y, z which satisfy Laplace's equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{5.17}$$

It is easy to show that under the action (5.1) of $SO(3)$ any solution of Laplace's equation is mapped into another solution. Furthermore, any homogeneous polynomial of degree l is mapped into another homogeneous polynomial of degree l. Thus, $\mathcal{W}^l$ is a finite-dimensional space invariant under the action of $SO(3)$. We shall decompose $\mathcal{W}^l$ into a direct sum of irred subspaces. Introducing the change of variable $\xi = x + iy$, $\eta = x - iy$, we see that every $u \in \mathcal{W}^l$ can be written uniquely in the form

$$u = \sum a_{nm}\xi^n\eta^m z^{l-n-m},$$

where

$$4(\partial^2 u/\partial\xi\,\partial\eta) + (\partial^2 u/\partial z^2) = 0$$

and n, m run over all nonnegative integers such that $0 \leq n + m \leq l$. Thus

$$\sum_{n,m} a_{nm}[4nm\xi^{n-1}\eta^{m-1}z^{l-n-m} + (l-n-m)(l-n-m-1)\xi^n\eta^m z^{l-n-m-2}] = 0,$$

or

$$4(n+1)(m+1)a_{n+1,m+1} + (l-m-n)(l-m-n-1)a_{n,m} = 0. \tag{5.18}$$

It follows from this expression that once values are prescribed for the $2l+1$ independent constants $a_{0,m}$, $0 \le m \le l$, and $a_{n,0}$, $1 \le n \le l$, the remaining constants are uniquely determined. Thus $\mathcal{W}^l$ is $(2l+1)$-dimensional. It is clear that the polynomial

$$r^l Y_l^l(\theta, \varphi) = \left[\frac{(2l+1)(2l)!}{4\pi}\right]^{1/2} \frac{(-1)^l \xi^l}{2^l l!}$$

belongs to $\mathcal{W}^l$. Now $\mathcal{W}^l$ is invariant under the operators $L^\pm$, L^3, expressions (5.6). From (5.7) we see that the $2l+1$ linearly independent functions $r^l Y_l^m(\theta, \varphi)$, $-l \le m \le l$, all lie in $\mathcal{W}^l$. Since $\mathcal{W}^l$ is $(2l+1)$-dimensional it transforms irreducibly under the rep $\mathbf{D}^{(l)}$.

A well-known model of the rep $\mathbf{D}^{(u)}$ of $SL(2)$ is defined on the $(2u+1)$-dimensional space $\mathcal{P}^u$ of homogeneous polynomials of degree $2u$ in the complex variables z_1, z_2. The group action is

$$(5.19) \qquad (z_1, z_2) \longrightarrow (z_1, z_2)A = (z_1, z_2)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad A \in SL(2).$$

Thus,

$$(5.20) \qquad [\mathbf{T}(A)p](z_1, z_2) = p(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2), \qquad p \in \mathcal{P}^u.$$

To see the connection between this expression and our previous models, set $w = z_1/z_2$. Then any $p \in \mathcal{P}^u$ can be written uniquely as $p(z_1, z_2) = z_2^{2u} p(w, 1)$, where $p(w, 1) = h(w)$ is a polynomial in w of order at most $2u$. We can factor z_2^{2u} from both sides of (5.20) to obtain the result

$$(5.21) \qquad [\mathbf{T}(A)h](w) = (\beta w + \delta)^{2u} h\left(\frac{\alpha w + \gamma}{\beta w + \delta}\right).$$

This expression is identical with the model (2.1) of $\mathbf{D}^{(u)}$. Restricting (5.20) to the subgroup $SU(2)$, we get a model of the rep $\mathbf{D}^{(u)}$ for this subgroup.

We have seen (5.20) before. Indeed, if we let V be the two-dimensional space $V = \{az_1 + bz_2 : a, b \in \mathfrak{C}\}$ then $\mathcal{P}^u$ can be identified with the $(2u+1)$-dimensional subspace of completely symmetric tensors in $V^{\otimes 2u}$. This subspace is determined by the Young frame $[2u]$, i.e., the frame with one row and $2u$ columns. The action (5.20) of $SL(2)$ on this subspace is induced by the action (5.19) of $SL(2)$ on V. In Section 4.3 we showed that $[2u]$ determined an irred rep of $GL(2)$. Now we see that the restriction of this rep to $SL(2)$ and then to $SU(2)$ remains irred. The other irred reps $[f_1, f_2]$, $f_1 \ge f_2$, of $GL(2)$ also restrict to irred reps of $SL(2)$. However, as we shall show later, on restriction to $SL(2)$ we have the equivalences $[f_1, f_2] \cong [f_1 - f_2, 0]$, so the frames $[f_1] = [f_1, 0]$, $f_1 = 0, 1, 2, \ldots$, exhaust the irred reps of $SL(2)$. In Chapter 9 we will study the irred reps of $SL(n)$ and $SU(n)$, and demonstrate the relationship between these reps and Young diagrams.

For our next example we construct a model of the infinite-dimensional local rep $\uparrow_{-1/2}$ of $SL(2)$. Consider the operators

(5.22) $$J^{\pm} = t^{\pm 1}\left((x^2 - 1)\frac{\partial}{\partial x} \pm xt\frac{\partial}{\partial t} + \frac{x}{2}\right), \qquad J^3 = t\frac{\partial}{\partial t},$$

acting on a space of analytic functions of x and t. These operators satisfy the commutation relations

$$[J^3, J^{\pm}] = \pm J^{\pm}, \qquad [J^+, J^-] = 2J^3$$

of $sl(2)$. In order to construct a model of $\uparrow_{-1/2}$ we must find functions $f_k(x, t) = g_k(x)t^{k+(1/2)}$, $k = 0, 1, 2, \ldots$, such that

(5.23) $$J^3 f_k = (k + \tfrac{1}{2})f_k, \qquad J^+ f_k = (k+1)f_{k+1}, \qquad J^- f_k = -kf_{k-1},$$

[see (3.6) and (3.8)]. It follows that the functions $g_k(x)$ satisfy the recurrence relations

(5.24) $$\begin{aligned} (x^2 - 1)g_k' + (k+1)xg_k &= (k+1)g_{k+1}, \\ (x^2 - 1)g_k' - kxg_k &= -kg_{k-1}. \end{aligned}$$

Furthermore the relation $(J^+J^- + J^3J^3 - J^3)f_k = -\frac{1}{4}f_k$ implies that the $g_k(x)$ satisfy the second-order differential equation

(5.25) $$[(x^2 - 1)(d^2/dx^2) + 2x(d/dx) - k(k+1)]g_k(x) = 0,$$
$$k = 0, 1, 2, \ldots.$$

Expressions (5.24) determine the $g_k(x)$ up to a multiplicative constant. Indeed the relation $J^- f_0 = 0$ implies $g_0'(x) = 0$, or $g_0(x) = c$. If we set $c = 1$ we can uniquely determine the remaining $g_k(x)$ from the first of the recurrence formulas (5.24). The second recurrence formula and the differential equation (5.25) are consequences of the commutation relations and do not have to be verified explicitly for the $g_k(x)$. Rather then determine the $g_k(x)$ recursively we compare our recurrence formulas with Theorem 7.1 to obtain

(5.26) $$g_k(x) = P_k(x), \qquad f_k(x, t) = P_k(x)t^{k+(1/2)}.$$

Thus the Legendre polynomials define a model of $\uparrow_{-1/2}$. The operators (5.22) determine a local Lie multiplier rep $\mathbf{T}$ of $SL(2)$. In particular,

$$\mathbf{T}(\exp \alpha\mathcal{J}^3)f(x, t) = f(x, te^{\alpha})$$

$$\mathbf{T}(\exp \beta\mathcal{J}^{\pm})f(x, t) = Q_{\pm}^{-1/4} f\left(\frac{x - \beta t^{\pm 1}}{Q_{\pm}^{1/2}}, tQ_{\pm}^{\mp 1/2}\right),$$
$$Q_{\pm} = \beta^2 t^{\pm 2} - 2\beta x t^{\pm 1} + 1.$$

Just as in (10.22), Section 5.10, we could use these results to compute $\mathbf{T}(A)$ for any $A \in SL(2)$. However, we shall not do this here. The matrix elements $B_{lk}(A)$, (10.26), of the operators $\mathbf{T}(A)$ with respect to the basis f_k are model-independent. That is, they are completely determined by the relations (5.23) and are independent of our particular realization of this rep. We have

(5.27)
$$\mathbf{T}(A)f_k = \sum_{l=0}^{\infty} B_{lk}(A)f_l, \qquad k = 0, 1, 2, \ldots, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2).$$

For certain group elements A the functions $B_{lk}(A)$ are very simple. For example,

$$(5.28) \qquad B_{lk}(\exp(-b\mathcal{J}^+)) = \begin{cases} (-b)^{l-k} l!/k!\,(l-k)!, & l \geq k, \\ 0, & l < k, \end{cases}$$

$$(5.29) \qquad B_{lk}(\exp(-c\mathcal{J}^-)) = \begin{cases} c^{k-l} k!/l!\,(k-l)!, & k \geq l, \\ 0, & k < l. \end{cases}$$

[Note: The reader can obtain these results directly from relations (5.23).] Substituting (5.26) and (5.28) into (5.27) and simplifying, we obtain

(5.30)
$$(1 + b^2 - 2bx)^{-(k+1)/2} P_k\left(\frac{x - b}{(1 + b^2 - 2bx)^{1/2}}\right) = \sum_{l=0}^{\infty} b^l \binom{l+k}{l} P_{k+l}(x),$$

where

$$\binom{n}{m}$$

is the binomial coefficient. This expression makes sense for $|b| < |x \pm (x^2 - 1)^{1/2}|$. For $k = 0$, (5.30) reduces to the standard generating function

$$(1 + b^2 - 2bx)^{-1/2} = \sum_{l=0}^{\infty} b^l P_l(x).$$

Similarly, by substituting (5.26) and (5.29) into (5.27) we obtain

$$(5.31) \qquad (1 + c^2 + 2cx)^{k/2} P_k\left(\frac{x + c}{(1 + c^2 + 2cx)^{1/2}}\right) = \sum_{l=0}^{k} \binom{k}{l} c^l P_l(x).$$

The point of this example is that identities such as (5.30) and (5.31) have a group-theoretic interpretation. Using the same operators (5.22) we could construct models of each of the irred reps $\uparrow_u$. The basis functions are essentially the Gegenbauer polynomials $C_k^{-u}(x)$ and our method yields generating functions and relations for the $C_k^{-u}(x)$.

Another interesting model of $\uparrow_u$ is obtained from a consideration of the operators

(5.32)
$$J^+ = t\left(z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - z - u\right), \qquad J^- = t^{-1}\left(z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} - u\right), \qquad J^3 = t\frac{\partial}{\partial t},$$

acting on a space of analytic functions of the complex variables z, t. As the reader can easily verify, these operators satisfy the commutation relations of $sl(2)$. To construct a realization of $\uparrow_u$ we must find functions $f_{k-u}(z, t) = g_k(z)t^{-u+k}$ such that

$$(5.33)\qquad \begin{aligned} &J^3 f_m = m f_m, \qquad J^{\pm} f_m = (-u \pm m) f_{m\pm 1},\\ &C f_m = (J^+J^- + J^3J^3 - J^3) f_m = u(u+1) f_m. \end{aligned}$$

Thus the special functions $g_k(z)$ satisfy

$$(5.34)\qquad z g_k{}' + (k - 2u - z) g_k = (k - 2u) g_{k+1}, \qquad z g_k{}' - k g_k = -k g_{k-1},$$

$$(5.35)\qquad z g_k'' - (2u + z) g_k{}' + k g_k = 0, \qquad k = 0, 1, 2, \ldots.$$

The functions $g_k(z)$ are determined to within a multiplicative constant by these relations. Indeed the relation $J^- f_u = 0$ implies $g_0{}' = 0$ or $g_0(z) = c$. Setting $c = 1$ we can then uniquely determine all of the $g_k(z)$ recursively from the first formula (5.34). The solutions are

$$(5.36)\qquad g_k(z) = \frac{\Gamma(-2u)k!}{\Gamma(k - 2u)} L_k^{(-2u-1)}(z), \qquad k = 0, 1, 2, \ldots,$$

where $L_k^{(\alpha)}(z)$ is a generalized Laguerre polynomial of order k and $\Gamma(z)$ is the gamma function (see the Symbol Index). Recall that $2u \neq 0, 1, \ldots$. The function $L_k(z) = L_k^{(0)}(z)$ is an (ordinary) Laguerre polynomial. The $L_k^{(-2u-1)}(z)$ satisfy the Laguerre differential equation (5.35).

A direct computation shows that the operators (5.32) determine a local multiplier rep $\mathbf{T}$ of $SL(2)$ given by

(5.37)

$$\mathbf{T}(A)f(z, t) = (d + bt)^u (a + c/t)^u \exp\left(\frac{bzt}{d + bt}\right) \times f\left(\frac{zt}{(at + c)(bt + d)}, \frac{at + c}{bt + d}\right), \qquad \left|\frac{c}{at}\right| < 1, \quad \left|\frac{bt}{d}\right| < 1.$$

The matrix elements $B_{lk}(A)$ of the $\mathbf{T}(A)$ with respect to the basis $f_{k-u}(z, t)$ are given by (10.26), Section 5.10. Substituting these expressions into

$$\mathbf{T}(A) f_{k-u} = \sum_{l=0}^{\infty} B_{lk}(A) f_{l-u}$$

and simplifying, we obtain identities for the Laguerre polynomials. For example, from (5.28) there follows

$$(5.38)\qquad (1 - b)^{2u-k} \exp\left(\frac{-bz}{1 - b}\right) L_k^{(-2u-1)}\left(\frac{z}{1 - b}\right) = \sum_{l=0}^{\infty} \binom{l + k}{l} b^l L_{k+l}^{(-2u-1)}(z), \qquad |b| < 1.$$

For $k = 0$, $L_0^{(\alpha)}(z) = 1$ and this expression simplifies to a well-known generating function for the Laguerre polynomials:

$$(5.39)\qquad (1-b)^{2u}\exp\left(\frac{-bz}{1-b}\right)=\sum_{l=0}^{\infty} b^l L_l^{(-2u-1)}(z), \qquad |b|<1.$$

Similarly, the matrix elements (5.29) yield the identity

$$(1+c)^k L_k^{(-2u-1)}\left(\frac{z}{1+c}\right)=\sum_{l=0}^{k}\binom{k-2u-1}{l} c^l L_{k-l}^{(-2u-1)}(z).$$

It is shown by Vilenkin [1] and Miller [1] that all hypergeometric and confluent hypergeometric functions can be obtained as basis functions in models of irred reps of $sl(2)$. Furthermore, in the work of Miller [1] it is shown how to derive such models in a systematic fashion.

7.6 Applications to Physics

Here we present a few of the many applications of the rep theory of $SO(3)$ and $SU(2)$ to problems in mathematical physics. In Section 3.8 we studied the relationship between symmetry and perturbation theory in quantum mechanics. Though our discussion was limited to finite symmetry groups it carries over without change to compact Lie symmetry groups.

Recall that the Hamiltonian $\mathbf{H}$ of a nonrelativistic quantum mechanical system containing k particles with masses $m_1, \ldots, m_k$ is

$$(6.1)\qquad \mathbf{H}=\sum_{j=1}^{k}(-1/2m_j)\,\Delta_j+V(\mathbf{x}_1,\ldots,\mathbf{x}_k),$$

where $V(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is the potential function and $\mathbf{x}_j \in R_3$ designates the coordinates of the jth particle. (We are using units in which $\hbar = 1$.) The Hilbert space $\mathcal{H}$ consists of all Lebesgue square-integrable functions $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)$,

$$\|\Psi\|^2=\int_{R_3{}^k}|\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_k)|^2\,d\mathbf{x}<\infty, \qquad d\mathbf{x}=d^3_{\mathbf{x}_1}\cdots d^3_{\mathbf{x}_k}.$$

The inner product on $\mathcal{H}$ is

$$(\Psi,\Phi)=\int_{R_3{}^k}\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_k)\bar{\Phi}(\mathbf{x}_1,\ldots,\mathbf{x}_k)\,d\mathbf{x}.$$

We can define a unitary rep $\mathbf{T}$ of $SO(3)$ on $\mathcal{H}$ by

$$(6.2)\qquad [\mathbf{T}(A)\Psi](\mathbf{x}_1,\ldots,\mathbf{x}_k)=\Psi(A^{-1}\mathbf{x}_1,\ldots,A^{-1}\mathbf{x}_k), \qquad A\in SO(3),$$

It is an elementary computation to verify $\mathbf{T}(AB) = \mathbf{T}(A)\mathbf{T}(B)$ and $(\mathbf{T}(A)\Psi, \mathbf{T}(A)\Phi) = (\Psi, \Phi)$ for all $A, B \in SO(3)$ and $\Psi, \Phi \in \mathcal{H}$.

Now $SO(3)$ is a symmetry group of $\mathbf{H}$ provided $\mathbf{T}(A)\mathbf{H} = \mathbf{H}\mathbf{T}(A)$ for all $A \in SO(3)$, i.e., provided $V(A\mathbf{x}_1, \ldots, A\mathbf{x}_k) = V(\mathbf{x}_1, \ldots, \mathbf{x}_k)$. If $SO(3)$

is a symmetry group and λ is an eigenvalue of $\mathbf{H}$ then the eigenspace

$$W_\lambda = \{\Psi \in \mathcal{H} : \mathbf{H}\Psi = \lambda\Psi\} \tag{6.3}$$

is invariant under $\mathbf{T}$. By the results of Section 3.7, we can decompose W_λ into a direct sum of subspaces irred under $\mathbf{T}$:

$$W_\lambda = \sum_{l=0}^{\infty} \sum_{i=1}^{a_l} \oplus W_i^{(l)}.$$

Here $\mathbf{T} \mid W_i^{(l)}$ is equivalent to the irred rep $\mathbf{D}^{(l)}$ and a_l is the multiplicity of $\mathbf{D}^{(l)}$ in $\mathbf{T}$. For simplicity we assume $\dim W_\lambda < \infty$, though this assumption could be removed with a little care. Then, only a finite number of the a_l are nonzero. Furthermore, if $SO(3)$ is a maximal symmetry group and there is no accidental degeneracy then only one of the a_l is nonzero.

The most important (and common) case in which $SO(3)$ appears as a symmetry group is the one where the potential takes the form

$$V = V(\|\mathbf{x}_i - \mathbf{x}_j\|, \|\mathbf{x}_i\|). \tag{6.4}$$

That is, V depends only on the mutual distances between particles and/or their distances from a common point. A special case is $V(\mathbf{x}) = V(\|\mathbf{x}\|)$, a single-particle, radially symmetric potential. These potentials admit the larger symmetry group $O(3)$. Indeed $V(A\mathbf{x}_1, \ldots, A\mathbf{x}_k) = V(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ for all $A \in O(3)$.

The irred reps of the compact group $O(3)$ can easily be obtained from those of $SO(3)$. Indeed $SO(3)$ is a normal subgroup of index two in $O(3)$. The left coset decomposition of $O(3)$ is

$$O(3) = \{SO(3), I \cdot SO(3)\},$$

where the inversion $I = -E_3$. Let $\mathbf{D}$ be an irred unitary rep of $O(3)$. Since I commutes with all elements of $O(3)$, $\mathbf{D}(I)$ must be a multiple $\alpha\mathbf{E}$ of the identity operator. But $\mathbf{D}(I)^2 = \mathbf{D}(I^2) = \mathbf{D}(E_3) = \mathbf{E}$, so $\alpha = \pm 1$. Since $\mathbf{D}$ is irred and $\mathbf{D}(I) = \pm\mathbf{E}$ it follows that $\mathbf{D} \mid SO(3)$ is still irred. Therefore $\mathbf{D} \mid SO(3) \cong \mathbf{D}^{(l)}$, $l = 0, 1, 2, \ldots$. We conclude that there are two families $\mathbf{D}_+^{(l)}$, $\mathbf{D}_-^{(l)}$ of irred reps of $O(3)$. Their definitions are

$$\mathbf{D}_+^{(l)}(IA) = \mathbf{D}_+^{(l)}(A) = \mathbf{D}^{(l)}(A), \tag{6.5}$$

$$\mathbf{D}_-^{(l)}(IA) = -\mathbf{D}_-^{(l)}(A) = -\mathbf{D}^{(l)}(A), \qquad A \in SO(3). \tag{6.6}$$

The $\mathbf{D}_+^{(l)}$ are called **positive** reps and the $\mathbf{D}_-^{(l)}$ **negative** reps. Here $\dim \mathbf{D}_\pm^{(l)} = 2l + 1$.

Returning to the study of a system with potential (6.4), we see that each irred subspace $W_i^{(l)}$ of W_λ will transform according to $\mathbf{D}_\pm^{(l)}$. In a one-particle system with central potential $V(\|\mathbf{x}\|)$ we can say more. The space $W_i^{(l)}$ consists of functions $\Psi(\mathbf{x}) = \Psi(x, y, z)$ transforming irreducibly under $\mathbf{D}_\pm^{(l)}$, hence under the rep $\mathbf{D}^{(l)}$ of $SO(3)$. Thus, we can find a basis for $W_i^{(l)}$

of the form $f_m(\mathbf{x}) = j_l(r)\, Y_l^m(\theta, \varphi)$, $-l \le m \le l$, where the Y_l^m are spherical harmonics and r, θ, φ are spherical coordinates. The inversion I maps $\mathbf{x}$ to $-\mathbf{x}$, or in terms of spherical coordinates, (r, θ, φ) to $(r, \pi - \theta, \pi + \varphi)$. In our study of Laplace's equation (5.17) we showed that $r^l Y_l^m(\theta, \varphi)$ is a homogeneous polynomial of degree l in x, y, and z. Thus, under inversion $Y_l^m(\theta, \varphi) \longrightarrow Y_l^m(\pi - \theta, \pi + \varphi) = (-1)^l Y_l^m(\theta, \varphi)$. If l is even then $W_l^{(l)}$ transforms under the representation $\mathbf{D}_+^{(l)}$ of $O(3)$; if l is odd then $W_l^{(l)}$ transforms under $\mathbf{D}_-^{(l)}$. The sign of $(-1)^l$ is sometimes called the **parity** of the rep. In this example the symmetry of the Schrödinger equation under rotations has completely determined the angular dependence of the eigenfunctions. Only the radial dependence $j_l(r)$ remains to be determined from the dynamics of the problem. The well-known separation of variables method applied to the Schrödinger equation yields a second-order ordinary differential equation for $j_l(r)$:

$$(6.7)\qquad \frac{1}{r}\frac{d}{dr}\left[r^2 \frac{d}{dr} j_l(r)\right] + \left[\frac{l(l+1)}{r} + V(r)\right] j_l(r) = \lambda j_l(r).$$

The permissible solutions of this equation are those such that $j_l(r) Y_l^m(\theta, \varphi) \in \mathcal{H}$, i.e., $\int_0^\infty |j_l(r)|^2\, r^2\, dr < \infty$. Only for certain values of λ, the eigenvalues, do there exist solutions belonging to $\mathcal{H}$.

The characters of $\mathbf{D}_\pm^{(l)}$ are easily obtained from the characters of the reps $\mathbf{D}^{(l)}$ of $SO(3)$. If R is a rotation through the angle τ about some axis then

$$(6.8)\qquad \chi_\pm^{(l)}(R) = \chi^{(l)}(R) = \{\sin[(l + \tfrac{1}{2})\tau]\}/\sin \tfrac{1}{2}\tau.$$

In the limit as $\tau \longrightarrow 0$ we get $\chi_\pm^{(l)}(E_3) = 2l + 1$. If S is a rotation through the angle τ followed by an inversion, then

$$(6.9)\qquad \chi_+^{(l)}(S) = -\chi_-^{(l)}(S) = \{\sin(l + \tfrac{1}{2})\tau]\}/\sin \tfrac{1}{2}\tau.$$

Suppose the k-particle system with Hamiltonian (6.1) is an atom or molecule. If this system is put into a crystal the new Hamiltonian is

$$(6.10)\qquad \mathbf{H}_1 = \sum_{j=1}^{k} (-1/2m_j)\, \Delta_j + V(\mathbf{x}_1, \ldots, \mathbf{x}_k) + V_1(\mathbf{x}_1, \ldots, \mathbf{x}_k),$$

where V_1 is the potential due to the crystal. Let G be the maximal point symmetry group of this crystal. Note that G is a finite subgroup of $O(3)$. Thus, the symmetry of the system is reduced from $O(3)$ to G under the perturbing potential V_1. If λ is an eigenvalue of $\mathbf{H}$ whose eigenspace transforms according to the $(2l + 1)$-dimensional rep $\mathbf{D}_+^{(l)}$ (or $\mathbf{D}_-^{(l)}$) then under the perturbing potential this degenerate energy level splits into energy levels whose eigenspaces transform according to irred reps of G. We can determine this splitting directly from the simple characters of $O(3)$ and G.

Suppose the eigenspace W_λ of $\mathbf{H}$ transforms according to $\mathbf{D}_+^{(l)}$. Then the restriction of $\mathbf{D}_+^{(l)}$ to the subgroup G splits into a direct sum of irred reps of G:

$$\mathbf{D}_+^{(l)} \mid G \cong \sum_{\mu=1}^{\kappa} \oplus\, a_\mu \mathbf{T}^{(\mu)}.$$

The character $\psi_l(A)$ of $\mathbf{D}_+^{(l)} \mid G$ is obtained from $\chi_+^{(l)}(A)$ by restricting A to G. Since G is a crystallographic point group it contains only rotations or rotation-inversions through the angles $0, \pm\pi/3, \pm\pi/2, \pm 2\pi/3$, and π. Now from (6.8)

$$\text{(6.11)} \qquad \chi^{(l)}\left(\pm\frac{2\pi}{n}\right) = \frac{\sin[(2\pi l/n) + (\pi/n)]}{\sin(\pi/n)}, \qquad n = 2, 3, 4, 6.$$

For fixed n this expression is periodic in l with period n. Thus, to evaluate ψ_l for any G it is enough to compute (6.11) for $0 \le l \le n$.

For example, suppose $G = O$, the octahedral group. The conjugacy classes are $E, \mathfrak{C}_4{}^2, \mathfrak{C}_2, \mathfrak{C}_4, \mathfrak{C}_3$, so O contains only twofold, threefold, and fourfold rotation axes. We compute ψ_l on these conjugacy classes for $0 \le l \le 5$:

l	E	$3\mathfrak{C}_4{}^2$	$6\mathfrak{C}_2$	$6\mathfrak{C}_4$	$8\mathfrak{C}_3$
0	1	1	1	1	1
1	3	−1	−1	1	0
2	5	1	1	−1	−1
3	7	−1	−1	−1	1
4	9	1	1	1	0
5	11	−1	−1	1	−1

Indeed, $\chi^{(l)}(E_3) = 2l + 1$, $\chi^{(l)}(\pi) = (-1)^l$, and so on. Using the character table of O, (6.22) of Section 3.6, we can write $\psi_l = \sum a_\mu^{(l)} \chi^{(\mu)}$ and compute the multiplicity $a_\mu^{(l)}$ of $\mathbf{T}^{(\mu)}$ in $\mathbf{D}_+^{(l)} \mid O$. The results are

(6.12)

$$\psi_0 = \chi^{(1)}, \qquad \psi_1 = \chi^{(4)}, \qquad \psi_2 = \chi^{(3)} + \chi^{(5)}, \qquad \psi_3 = \chi^{(2)} + \chi^{(4)} + \chi^{(5)},$$

$$\psi_4 = \chi^{(1)} + \chi^{(3)} + \chi^{(4)} + \chi^{(5)}, \qquad \psi_5 = \chi^{(3)} + 2\chi^{(4)} + \chi^{(5)}.$$

The interpretation of the expansion for ψ_4, for example, is that a ninefold degenerate energy level of $\mathbf{H}$ splits into four energy levels under the perturbation, one of the split levels is nondegenerate, one is twofold degenerate, and two are threefold degenerate. We can continue in this fashion to compute the splitting of an arbitrary $(2l + 1)$-fold energy level under a perturbation with octahedral symmetry. Since O contains only proper rotations the splitting for $\mathbf{D}_-^{(l)} \mid O$ is exactly the same as the splitting for $\mathbf{D}_+^{(l)} \mid O$.

If G contains rotation-inversions the determination of the splitting of the energy levels is analogous to that given above except that the results for $\mathbf{D}_-^{(l)} \mid G$ differ from those for $\mathbf{D}_+^{(l)} \mid G$.

Some Lie subgroups of $O(3)$ are of importance for perturbation theory calculations. Suppose we break the symmetry of a rotationally symmetric system by introducing a perturbing potential which transforms like the z component of an axial vector in xyz space. Then the symmetry group of the perturbed Hamiltonian will be $C_{\infty h} = C_\infty \times \{E, I\}$, where $C_\infty \cong U(1)$ is the group of all rotations about the z axis and $\{E, I\}$ consists of the identity element and the inversion I. (The choice of the z axis is arbitrary. Any other axis of symmetry would do.) As an example of such a perturbation consider an electron in a spherically symmetric field. If a uniform magnetic field parallel to the z axis is applied to this system, the perturbed system has symmetry $C_{\infty h}$. (We are ignoring the spin of the electron. This complication will be considered in Section 7.8).

Since $C_{\infty h}$ is abelian its irred reps are one-dimensional. Furthermore, since $I^2 = E$ and I commutes with all elements of $C_{\infty h}$ it follows that $\mathbf{T}(I) = \pm 1$ for any irred rep $\mathbf{T}$. We already know the irred reps of C_∞. They are denoted by the integer m: $\chi^{(m)}(C(\theta)) = e^{im\theta}$, $m = 0, \pm 1, \ldots$, where $C(\theta)$ is a rotation through the angle θ about the z axis. It follows that the irred reps of $C_{\infty h}$ are $\psi_\pm^{(m)}$, where

$$(6.13) \qquad \psi_\pm^{(m)}(C(\theta)) = e^{im\theta}, \qquad \psi_\pm^{(m)}(C(\theta)I) = \pm e^{im\theta}, \qquad m = 0, \pm 1, \ldots.$$

Suppose the eigenspace W_λ of the unperturbed Hamiltonian transforms according to the irred rep $\mathbf{D}_+^{(l)}$ with character $\chi_+^{(l)}$, (6.8) and (6.9). Now $\mathbf{D}_+^{(l)} \mid C_{\infty h}$ has character $\chi_+^{(l)}(C(\theta)) = \chi_+^{(l)}(C(\theta)I) = \sum_{m=-l}^{l} e^{im\theta} = \sum_{m=-l}^{l} \psi_+^{(m)}(\theta)$. Therefore, under the perturbing potential the degenerate energy level splits completely into $2l + 1$ simple sublevels, each with parity $+1$. Similarly $\mathbf{D}_-^{(l)} \mid C_{\infty h}$ has character $\chi_-^{(l)} \mid C_{\infty h} = \sum_{m=-l}^{l} \psi_-^{(m)}$, so the degenerate energy level splits into $2l + 1$ simple sublevels with parity -1. In the case where the perturbing potential is a magnetic field this splitting of energy levels is called the **Zeeman effect**.

It was shown in Section 2.9 that a molecule whose atoms all lie on a single line L possesses the symmetry group $C_{\infty v}$ consisting of all rotations about L and reflections in all planes in which L lies. Furthermore, if the molecule is also invariant with respect to the reflection σ in a plane perpendicular to L then the symmetry group is $D_{\infty h} = C_{\infty v} \times \{E, \sigma\}$. This occurs if the molecule is symmetric about its center of mass.

If L is the z axis then $C_{\infty v}$ is generated by the rotations $C(\varphi)$ and the reflection σ_v in the xz plane. It is easy to verify that this group has a 2×2 matrix realization

$$C(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \qquad \sigma_v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $C(\varphi)\sigma_v = \sigma_v C(-\varphi)$ and $\sigma_v{}^2 = E_2$. The rotations $C(\pm\varphi)$ form a conjugacy class and every reflection is conjugate to σ_v. Let $\mathbf{T}$ be a unitary irred rep of $C_{\infty v}$ on V. Then $\mathbf{T} \mid C_\infty$ splits into a direct sum of irred reps $\chi^{(m)}(\varphi) = e^{im\varphi}$ of C_∞. Suppose the nonzero vector $\mathbf{f}_m$ in V transforms according to $\chi^{(m)}$: $\mathbf{T}(C(\varphi))\mathbf{f}_m = e^{im\varphi}\mathbf{f}_m$. Then $\mathbf{T}(C(\varphi))\mathbf{T}(\sigma_v)\mathbf{f}_m = \mathbf{T}(\sigma_v)\mathbf{T}(C(-\varphi))\mathbf{f}_m = e^{-im\varphi}\mathbf{T}(\sigma_v)\mathbf{f}_m$. Thus $\mathbf{T}(\sigma_v)\mathbf{f}_m = \mathbf{f}_{-m}$ is a nonzero vector transforming according to $\chi^{(-m)}$ and $\mathbf{T}(\sigma_v)\mathbf{f}_{-m} = \mathbf{T}^2(\sigma_v)\mathbf{f}_m = \mathbf{f}_m$. Since V is irred under $C_{\infty v}$ it follows that $\mathbf{f}_{\pm m}$ generate V. If $m \neq 0$ then V is two-dimensional and with respect to the basis $\{\mathbf{f}_{\pm m}\}$ we obtain the matrix reps E_m:

$$(6.14) \quad T(C(\varphi)) = \begin{pmatrix} e^{im\varphi} & 0 \\ 0 & e^{-im\varphi} \end{pmatrix}, \quad T(\sigma_v) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad m = 1, 2, 3, \ldots.$$

The characters are $\chi^{(m)}(\varphi) = 2\cos m\varphi$, $\chi^{(m)}(\sigma_v) = 0$. If $m = 0$, then the irreducibility of $\mathbf{T}$ and the property $\mathbf{T}^2(\sigma_v) = \mathbf{E}$ imply V is one-dimensional, $\mathbf{T}(C(\varphi)) = 1$ and $\mathbf{T}(\sigma_v) = \pm 1$. Thus, we get two one-dimensional reps

$$(6.15) \quad A_1\colon\ \mathbf{T}(C(\varphi)) = 1, \quad \mathbf{T}(\sigma_v) = 1, \qquad A_2\colon\ \mathbf{T}(C(\varphi)) = 1, \quad \mathbf{T}(\sigma_v) = -1.$$

The E_m, A_1, A_2 are a complete set of irred reps of $C_{\infty v}$. The character table is

(6.16)

$C_{\infty v}$	$C(\varphi)$	σ_v
A_1	1	1
A_2	1	-1
E_m	$2\cos m\varphi$	0

Now suppose the eigenspace W_λ of a Hamiltonian with spherical symmetry transforms according to $\mathbf{D}_+^{(l)}$, and introduce a perturbing potential with symmetry $C_{\infty v}$. Then the character of $\mathbf{D}_+^{(l)} \mid C_{\infty v}$ becomes $\chi(\varphi) = \sin(l + \frac{1}{2})\varphi/\sin(\varphi/2)$ and $\chi(\sigma_v) = \chi(\pi) = (-1)^l$. Clearly,

$$(6.17) \qquad \mathbf{D}_+^{(l)} \mid C_{\infty v} = E_l \oplus E_{l-1} \oplus \cdots \oplus E_1 \oplus A_k,$$

where $k = 1$ if l is even and $k = 2$ if l is odd. Thus, the $(2l + 1)$-degenerate eigenvalue λ splits into l eigenvalues with multiplicity two and one single eigenvalue. Similarly if W_λ transforms according to $\mathbf{D}_-^{(l)}$ a simple computation yields

$$(6.18) \qquad \mathbf{D}_-^{(l)} \mid C_{\infty v} = E_l \oplus E_{l-1} \oplus \cdots \oplus E_1 \oplus A_j,$$

where $j = 1$ if l is odd and $j = 2$ if l is even. In the case where our system contains only one particle the parity is $(-1)^l$, so we always get the identity rep A_1 in (6.17) and (6.18).

We can achieve $C_{\infty v}$ symmetry by introducing into a spherically symmetric system a perturbing potential which transforms like the z component of a polar vector in xyz space, e.g., an electron in a uniform electric field parallel to the z axis. The splitting of the energy levels due to this perturbation is called the **Stark effect**.

The group $D_{\infty h}$ can be written as the direct product $D_{\infty h} = C_{\infty v} \times \{E, I\}$. Thus, the irred reps can be obtained almost immediately from (6.16). The conjugacy classes are determined by the elements $C(\pm\varphi), \sigma_v, I, C(\pm\varphi)I, \sigma_v I$. If $\mathbf{T}$ is an irred rep of $D_{\infty h}$ then $\mathbf{T}(I) = \pm\mathbf{E}$. The character table is

(6.19)

$D_{\infty h}$	$C(\varphi)$	σ_v	I	$C(\varphi)I$	$\sigma_v I$
A_1^+	1	1	1	1	1
A_1^-	1	1	-1	-1	-1
A_2^+	1	-1	1	1	-1
A_2^-	1	-1	-1	-1	1
E_m^+	$2\cos m\varphi$	0	1	$2\cos m\varphi$	0
E_m^-	$2\cos m\varphi$	0	-1	$-2\cos m\varphi$	0

Here $D_{\infty h}$ has four one-dimensional reps and two infinite families $E_m^{\pm}$ of two-dimensional reps. The determination of the splitting of $\mathbf{D}_{\pm}^{(l)} | D_{\infty h}$ is left to the reader.

The use of irred reps of symmetry groups to label the state vectors is of much more importance than perturbation theory alone would indicate. Suppose a quantum mechanical system is in the state $\Psi \in \mathcal{H}$ at time $t = 0$. Then at any other time t the system is in the state $\Psi(t)$, where $\Psi(t)$ is the unique solution of the (time-dependent) Schrödinger equation

$$i\,\partial\Psi(t)/\partial t = \mathbf{H}\Psi(t), \qquad \Psi(0) = \Psi. \tag{6.20}$$

Formally, the solution is $\Psi(t) = [\exp(-it\mathbf{H})]\Psi$. Since $\exp(-it\mathbf{H})$ is a unitary operator, the norm $\|\Psi(t)\|$ is independent of t. To make precise sense out of these statements we would have to employ some sophisticated techniques from functional analysis. (In particular we would need Stone's theorem; see Riesz–Sz.–Nagy [1].) Expression (6.20) is not always well-defined since there exist vectors $\Psi \in \mathcal{H}$ such that $\mathbf{H}\Psi$ has no meaning. Nevertheless, one can show that for the usual Hamiltonians of quantum mechanics there is a dense subspace of $\mathcal{H}$ on which (6.20) does make sense and on which the formal computations to follow can be rigorously justified.

Suppose $\mathbf{T}$ is a unitary rep of $SO(3)$ on $\mathcal{H}$ such that $\mathbf{T}(A)\mathbf{H} = \mathbf{H}\mathbf{T}(A)$ for all $A \in SO(3)$. Let W be an invariant subspace of $\mathcal{H}$ such that $\mathbf{T}|W$ transforms according to the irred rep $\mathbf{D}^{(l)}$. Then there is an ON basis $\{\Psi_m^{(l)}\}$ for W such that $\mathbf{T}(A)\Psi_m^{(l)} = \sum_n D_{nm}(A)\Psi_n^{(l)}$, where $\{D_{nm}(A)\}$ is a unitary matrix realization of $\mathbf{D}^{(l)}$. Let $\Psi_m^{(l)}(t)$ be solutions of (6.20) such that $\Psi_m^{(l)}(0) = \Psi_m^{(l)}$. Since $\mathbf{H}$ commutes with the operators $\mathbf{T}(A)$ it follows that $\mathbf{T}(A)\Psi_m^{(l)}(t) - \sum D_{nm}(A)\Psi_n^{(l)}(t) = \Phi(t)$ is a solution of (6.20) with initial condition $\Phi(0) \equiv 0$. Thus, $\Phi(t) \equiv 0$ and the vectors $\{\Psi_m^{(l)}(t)\}$ form an ON basis for the rep $\mathbf{D}^{(l)}$ at any time t. We conclude that l and m are good quantum numbers for the system. A state $\Psi_m^{(l)}(t)$ which transforms like the mth basis vector in a realization of $\mathbf{D}^{(l)}$ at one instant of time transforms like the mth basis vector in $\mathbf{D}^{(l)}$ at any time. Physicists refer to this as the conservation of angular momentum. Although this analysis applies only to $SO(3)$, similar results can be easily obtained for any compact symmetry group of $\mathbf{H}$.

It is worthwhile to point out the connection between the time-dependent and time-independent Schrödinger equations. Suppose $\Psi \in \mathcal{H}$ is a nonzero solution of the Schrödinger equation $\mathbf{H}\Psi = \lambda\Psi$. (We assume $\mathbf{H}$ is independent of t.) Then Ψ is an eigenvector of $\mathbf{H}$ with eigenvalue λ. Furthermore, the one-parameter family $\Psi(t) = e^{-it\lambda}\Psi$ is the unique solution of the Schrödinger equation

$$i\,\partial\Psi(t)/\partial t = \mathbf{H}\Psi(t), \qquad \Psi(0) = \Psi.$$

Since the vectors $e^{-it\lambda}\Psi$ belong to the same ray in $\mathcal{H}$ for all t, it follows that any eigenstate of $\mathbf{H}$ remains fixed with passage of time.

As a final application we investigate the quantum mechanical interpretation of the Lie algebra $so(3) \cong su(2)$. As usual we consider the unitary rep $\mathbf{T}$, (6.2), of $SO(3)$ on $\mathcal{H}$. Then $\mathbf{T}$ induces a rep (also called $\mathbf{T}$) of $so(3)$ on $\mathcal{H}$:

(6.21) $$\mathbf{T}(\mathfrak{a}) = (d/dt)\mathbf{T}(\exp t\mathfrak{a})|_{t=0}, \qquad \mathfrak{a} \in so(3).$$

In particular, the operators $\mathbf{T}(\mathcal{L}_j) = \boldsymbol{\mathcal{L}}_j$ are

(6.22) $$\boldsymbol{\mathcal{L}}_1 = \sum_{j=1}^{k}\left(z_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial z_j}\right), \qquad \boldsymbol{\mathcal{L}}_2 = \sum_{j=1}^{k}\left(x_j\frac{\partial}{\partial z_j} - z_j\frac{\partial}{\partial x_j}\right),$$
$$\boldsymbol{\mathcal{L}}_3 = \sum_{j=1}^{k}\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right), \qquad \mathbf{x}_j = (x_j, y_j, z_j),$$

where the $\mathcal{L}_j$ are given by (1.2) and (1.3). As the reader can easily verify, the $\boldsymbol{\mathcal{L}}_j$ satisfy the commutation relations (1.5) of $so(3)$ and they form a basis for the Lie algebra of operators $\mathbf{T}(\mathfrak{a})$. Proceeding formally by differentiating the identity

$$(\mathbf{T}(\exp t\mathfrak{a})\Psi, \mathbf{T}(\exp t\mathfrak{a})\Phi) = (\Psi, \Phi), \qquad \Psi, \Phi \in \mathcal{H},$$

with respect to t we obtain

(6.23) $$(\mathbf{T}(\mathfrak{a})\Psi, \Phi) + (\Psi, \mathbf{T}(\mathfrak{a})\Phi) = 0$$

at $t = 0$. Thus $\mathbf{T}^*(\mathfrak{a}) = -\mathbf{T}(\mathfrak{a})$ and the operators $i\mathbf{T}(\mathfrak{a})$, $i = \sqrt{-1}$, are **symmetric**. In particular the operators $\mathbf{L}_j = i\boldsymbol{\mathcal{L}}_j$ are symmetric and satisfy the commutation relations

(6.24) $$[\mathbf{L}_1, \mathbf{L}_2] = i\mathbf{L}_3, \qquad [\mathbf{L}_3, \mathbf{L}_1] = i\mathbf{L}_2, \qquad [\mathbf{L}_2, \mathbf{L}_3] = i\mathbf{L}_1.$$

The $\mathbf{L}_j$ are called the **angular momentum operators**. If the Hamiltonian $\mathbf{H}$ commutes with the operators $\mathbf{T}(A)$, $A \in SO(3)$, then by differentiating the identity $\mathbf{T}(\exp t\mathfrak{a})\mathbf{H} = \mathbf{H}\mathbf{T}(\exp t\mathfrak{a})$ at $t = 0$ we find $\mathbf{T}(\mathfrak{a})\mathbf{H} = \mathbf{H}\mathbf{T}(\mathfrak{a})$ for all $\mathfrak{a} \in so(3)$. In particular the angular momentum operators commute with

H. Conversely, if the angular momentum operators commute with **H** then the operators $\mathbf{T}(A)$, $A \in SO(3)$, commute with **H**.

Unfortunately the above computations are merely formal. The operators $\mathbf{L}_j$ and **H** are not defined on all of $\mathcal{H}$. For instance if **H** is given formally by (6.1) then $\mathbf{H}\Psi(\mathbf{x})$ makes sense only if $\Psi(\mathbf{x})$ can be differentiated twice in each variable. Furthermore the function $\mathbf{H}\Psi(\mathbf{x})$ must belong to $\mathcal{H}$, i.e., $\|\mathbf{H}\Psi\| < \infty$. Since many functions in $\mathcal{H}$ are not differentiable it is clear that $D_{\mathbf{H}}$ cannot be all of $\mathcal{H}$. The problem of defining explicitly the domain of **H**, or any unbounded operator in quantum mechanics, is outside the scope of this book (see Helwig [1]). It can be shown that each of these operators can be defined on a dense (not necessarily closed) subspace of $\mathcal{H}$. However, the subspace varies with the operator. The angular momentum operators make sense only when applied to differentiable functions $\Psi(\mathbf{x})$ such that $\mathbf{L}_j\Psi(\mathbf{x})$ is square-integrable. Furthermore, the meaning of a commutation relation such as $[\mathbf{L}_1, \mathbf{L}_2] = i\mathbf{L}_3$ is not completely clear since the domains of the left- and right-hand sides may not be the same.

However, it can be shown (Helgason [1, p. 440]) that there exists a dense subspace $\mathfrak{D}$ of $\mathcal{H}$ which is contained in the domains of all the operators **H** and $\mathbf{L}_j$. Furthermore $\mathfrak{D}$ is invariant under the restrictions of $\mathbf{H}, \mathbf{L}_j, \mathbf{T}(A)$ to $\mathfrak{D}$ and has the property that all of the above formal computations are rigorously correct for these restricted operators. Thus, the relation

$$\mathbf{T}(\exp \alpha\mathcal{L}_j)\Psi = \sum_{n=0}^{\infty} \frac{(\alpha\mathfrak{L}_j)^n}{n!}\Psi \tag{6.25}$$

is valid for $\Psi \in \mathfrak{D}$. If we accept the fact that $\mathfrak{D}$ exists we can use Lie algebra computations to derive results about infinite-dimensional Lie group reps. Note that the unitary operators $\mathbf{T}(A)$, $A \in SO(3)$, are uniquely determined by the symmetric operators $\mathbf{L}_j$. Indeed $\mathbf{T}(A)$ is uniquely defined on $\mathfrak{D}$ by (6.25). Since $\mathfrak{D}$ is dense in $\mathcal{H}$ and $\mathbf{T}(A)$ is bounded it follows from a standard Hilbert-space argument that $\mathbf{T}(A)$ is uniquely determined on $\mathcal{H}$ (Naimark [2, p. 100]). With these remarks in mind we shall henceforth ignore problems concerning the domains of unbounded operators.

The angular momentum operators can be used to compute the matrix elements of **H** with respect to an ON basis of $\mathcal{H}$. Consider again the unitary rep **T** of $SO(3)$ on $\mathcal{H}$. From the results of Section 6.3 we know that $\mathbf{T} = \sum \oplus a_l\mathbf{D}^{(l)}$, i.e., $\mathcal{H}$ can be decomposed into a direct sum of subspaces irred under **T**. (In general the multiplicities a_l will be infinite.) Thus, there is an ON basis $\{\Psi_{jm}^{(l)}\}$ for $\mathcal{H}$ such that $\mathbf{T}(A)\Psi_{jm}^{(l)} = \sum D_{nm}^{(l)}(A)\Psi_{jn}^{(l)}$ and $1 \le j \le a_l$. We have shown in Section 6.3 how such a basis can be constructed without any knowledge of the Hamiltonian **H**.

Since **H** commutes with the $\mathbf{T}(A)$ it also commutes with the operators $\mathbf{L}^{\pm} = \pm\mathfrak{L}_2 + i\mathfrak{L}_1$ and $\mathbf{L}^3 = -i\mathfrak{L}_3$. Here $(\mathbf{L}^+)^* = \mathbf{L}^-$ and $(\mathbf{L}^3)^* = \mathbf{L}^3$.

The ON basis vectors $\Psi^{(l)}_{jm}$ can be chosen such that

(6.26)
$$\mathbf{L}^3\Psi^{(l)}_{jm} = m\Psi^{(l)}_{jm}, \qquad \mathbf{L}^{\pm}\Psi^{(l)}_{jm} = [(l \pm m + 1)(l \mp m)]^{1/2}\Psi^{(l)}_{jm\pm 1}$$
$$\mathbf{L}\cdot\mathbf{L}\Psi^{(l)}_{jm} = l(l+1)\Psi^{(l)}_{jm}, \qquad \mathbf{L}\cdot\mathbf{L} = \mathbf{L}^+\mathbf{L}^- + \mathbf{L}^3\mathbf{L}^3 - \mathbf{L}^3 = -\sum_{j=1}^{3} \mathfrak{L}_j\mathfrak{L}_j.$$

Now
$$(\mathbf{H}\mathbf{L}^3\Psi^{(l)}_{jm}, \Psi^{(l')}_{j'm'}) = (\mathbf{L}^3\mathbf{H}\Psi^{(l)}_{jm}, \Psi^{(l')}_{j'm'}) = (\mathbf{H}\Psi^{(l)}_{jm}, \mathbf{L}^3\Psi^{(l')}_{j'm'}),$$
so $(m - m')(\mathbf{H}\Psi^{(l)}_{jm}, \Psi^{(l')}_{j'm'}) = 0$ and the matrix element is zero unless $m = m'$. Similarly the relation
$$(\mathbf{H}\mathbf{L}\cdot\mathbf{L}\Psi^{(l)}_{jm}, \Psi^{(l')}_{j'm'}) = (\mathbf{H}\Psi^{(l)}_{jm}, \mathbf{L}\cdot\mathbf{L}\Psi^{(l')}_{j'm'})$$
shows that the matrix element is zero unless $l = l'$. The identity
$$(\mathbf{H}\mathbf{L}^+\Psi^{(l)}_{jm}, \Psi^{(l)}_{j'm+1}) = (\mathbf{H}\Psi^{(l)}_{jm}, \mathbf{L}^-\Psi^{(l)}_{j'm+1})$$
yields $(\mathbf{H}\Psi^{(l)}_{jm+1}, \Psi^{(l)}_{j'm+1}) = (\mathbf{H}\Psi^{(l)}_{jm}, \Psi^{(l)}_{j'm})$, i.e., the matrix elements are independent of m. Thus

(6.27)
$$(\mathbf{H}\Psi^{(l)}_{jm}, \Psi^{(l')}_{j'm'}) = \delta_{ll'}\,\delta_{mm'}\,\lambda(l, j, j'),$$
where $\lambda(l, j, j')$ is independent of m and m'. In Section 7.8 this result will be generalized to obtain information about the matrix elements of operators which do not necessarily commute with the action of $SO(3)$ on $\mathcal{H}$.

7.7 The Clebsch–Gordan Coefficients

In Section 7.2 we derived the Clebsch–Gordan series

(7.1)
$$\mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)} = \sum_{w=|u-v|}^{u+v} \oplus\, \mathbf{D}^{(w)}$$
for the tensor product of two irred reps of $SU(2)$. Recall that we also used the symbol $\mathbf{D}^{(u)}$ to denote the $(2u+1)$-dimensional irred rep of $SL(2)$. Since there is a 1-1 relationship between complex reps of $sl(2)$ and $su(2)$ it follows that expression (7.1) is also valid for $SL(2)$. Furthermore, this same argument shows that any finite-dimensional analytic rep of $SL(2)$ as a complex Lie group can be decomposed into a direct sum of irred reps.

In the following we shall consider (7.1) as a rep of $SL(2)$, but all our results will remain valid on restriction to $SU(2)$. If $\mathbf{D}^{(u)}$, $\mathbf{D}^{(v)}$ are defined on inner product spaces $V^{(u)}$, $V^{(v)}$ then $\mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)}$ is defined on the $(2u+1)(2v+1)$-dimensional space $V^{(u)} \otimes V^{(v)}$. As a convenient ON basis for the rep space we choose $\{\mathbf{f}^{(u)}_m \otimes \mathbf{f}^{(v)}_n : -u \le m \le u,\ -v \le n \le v\}$, where $\{\mathbf{f}^{(u)}_m\}$ is a basis for $V^{(u)}$ such that

(7.2)
$$J^3\mathbf{f}^{(u)}_m = m\mathbf{f}^{(u)}_m, \qquad J^{\pm}\mathbf{f}^{(u)}_m = [(u \pm m + 1)(u \mp m)]^{1/2}\mathbf{f}^{(u)}_{m\pm 1},$$

and $\{\mathbf{f}_u^{(v)}\}$ is defined similarly. [In the future we will call any ON basis $\{\mathbf{f}_m\}$ satisfying (7.2) a **canonical basis.**] Though $\{\mathbf{f}_m^{(u)} \otimes \mathbf{f}_n^{(v)}\}$ is well adapted to show the tensor product character of our rep, it does not clearly exhibit the decomposition of $\mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)}$ into irred reps. From the right-hand side of (7.1) it follows that $V^{(u)} \otimes V^{(v)}$ contains an ON basis of the form

$$\{\mathbf{h}_k^{(w)} : w = u + v, u + v - 1, \ldots, |u - v|, \quad -w \le k \le w\}$$

such that

$$J^3\mathbf{h}_k^{(w)} = k\mathbf{h}_k^{(w)}, \qquad J^{\pm}\mathbf{h}_k^{(w)} = [(w \pm k + 1)(w \mp k)]^{1/2}\mathbf{h}_{k\pm 1}^{(w)}, \tag{7.3}$$

where the J-operators are now those determined by the action of $SL(2)$ on $V^{(u)} \otimes V^{(v)}$. For fixed w the vectors $\{\mathbf{h}_h^{(w)}\}$ are determined up to a phase factor by (7.3). They form an ON basis for the invariant subspace which transforms according to $\mathbf{D}^{(w)}$. The two sets of basis vectors are related by the Clebsch–Gordan (CG) coefficients:

$$\mathbf{h}_k^{(w)} = \sum_{m,n} C(u, m; v, n \,|\, w, k)\mathbf{f}_m^{(u)} \otimes \mathbf{f}_n^{(v)}, \tag{7.4}$$

$$|u - v| \le w \le u + v, \qquad -w \le k \le w,$$

$$C(u, m; v, n \,|\, w, k) = (\mathbf{h}_k^{(w)}, \mathbf{f}_m^{(u)} \otimes \mathbf{f}_n^{(v)}), \tag{7.5}$$

where $(-, -)$ is the inner product on $V^{(u)} \otimes V^{(v)}$. Since $\{\mathbf{h}_k^{(w)}\}$ and $\{\mathbf{f}_m^{(u)} \otimes \mathbf{f}_n^{(v)}\}$ are ON, the matrix formed by the CG coefficients is unitary. Indeed from (7.5)

$$\mathbf{f}_m^{(u)} \otimes \mathbf{f}_n^{(v)} = \sum_{w,k} C(w, k \,|\, u, m; v, n)\mathbf{h}_k^{(w)}, \tag{7.6}$$

where

$$C(w, k \,|\, u, m; v, n) = \overline{C(u, m; v, n \,|\, w, k)}. \tag{7.7}$$

Later we shall see that it is possible to choose $\{\mathbf{h}_k^{(w)}\}$ such that the CG coefficients are real.

The matrix elements of $\mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)}$ with respect to the $\{\mathbf{f}_m^{(u)} \otimes \mathbf{f}_n^{(v)}\}$ basis are $T^u_{mm'}(A)T^v_{nn'}(A)$, where $A \in SL(2)$ and the $T^u_{mm'}(A)$ are the matrix elements of $\mathbf{D}^{(u)}$ with respect to $\{\mathbf{f}_m^{(u)}\}$. On the other hand, the matrix elements with respect to the $\{\mathbf{h}_k^{(w)}\}$ basis are $T^w_{kk'}(A)$. The matrix of $\mathbf{T}^{(u)} \otimes \mathbf{T}^{(v)}(A)$ in one basis is unitary equivalent to the matrix in the other basis. A straightforward computation yields the identity

$$T^u_{mm'}(A)T^v_{nn'}(A) = \sum_{w,k,k'} C(u, m; v, n \,|\, w, k)C(u, m'; v, n' \,|\, w, k')T^w_{kk'}(A), \tag{7.8}$$

expressing the product of two matrix elements as a sum of matrix elements. Since in appropriate parameters $T^u_{mm'}(A)$ is essentially a Jacobi polynomial, (7.8) can be viewed as an identity expanding the product of two Jacobi polynomials as a sum of Jacobi polynomials. If we restrict A to the subgroup

$SU(2)$ then the matrix elements are given by (2.13) and (2.14). Applying the orthogonality relations (2.21), we find

$$\text{(7.9)}\qquad \frac{2w+1}{16\pi^2}\int_{SU(2)} T^u_{mm'}(A)T^v_{nn'}(A)\overline{T^w_{kk'}(A)}\,dA$$
$$= C(u, m; v, n \mid w, k)C(u, m'; v, n' \mid w, k')$$

This expression can be used to explicitly compute the CG coefficients (Wigner [2]). However, we shall adopt another approach which leads to a generating function for the coefficients and yields an independent proof of (7.1).

Consider the model of $\mathbf{D}^{(u)}$ on the vector space $\mathcal{V}^{(u)}$ with ON basis

$$\text{(7.10)}\qquad f_m(z) = \frac{(-z)^{u+m}}{[(u-m)!\,(u+m)!]^{1/2}}, \qquad -u \le m \le u,$$

[see (2.10)]. The action of $SL(2)$ on $\mathcal{V}^{(u)}$ is

$$[\mathbf{T}(A)f](z) = (bz+d)^{2u} f\left(\frac{az+c}{bz+d}\right), \qquad f \in \mathcal{V}^{(u)}, \quad A \in SL(2).$$

The matrix elements of $\mathbf{T}(A)$ with respect to this basis are

$$\mathbf{T}(A)f_m = \sum_{p=-u}^{u} Q^u_{pm}(A)f_p$$

or

$$Q^u_{pm}(A) = \left[\frac{(u+p)!\,(u-p)!}{(u+m)!\,(u-m)!}\right]^{1/2} D_{u+p,u+m}(A)(-1)^{p-m},$$

where $D_{lk}(A)$ is given by (10.22) of Section 5.10. The matrix elements have the symmetric generating function

$$\text{(7.11)}\qquad (1/[2u]!)[(bz+d)+y(az+c)]^{2u} = \sum_{m,p=-u}^{u} f_m(y)Q^u_{pm}(A)f_p(z).$$

In this model the action of the generalized Lie derivatives $J^{\pm}$, J^3 on the basis is described by (7.2).

We can realize $\mathbf{D}^{(u)} \otimes \mathbf{D}^{(v)}$ on the $(2u+1)(2v+1)$-dimensional space $\mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$ with ON basis

$$\text{(7.12)}\qquad f^{(u)}_m \otimes f^{(v)}_n(z, y) = \frac{(-z)^{u+m}(-y)^{v+n}}{[(u+m)!\,(u-m)!\,(v+n)!\,(v-n)!]^{1/2}},$$
$$-u \le m \le u, \quad -v \le n \le v.$$

The action of $SL(2)$ is defined by operators $\mathbf{S}(A)$ such that

$$\text{(7.13)}\qquad [\mathbf{S}(A)f](z, y) = (bz+d)^{2u}(by+d)^{2v} f\left(\frac{az+c}{bz+d}, \frac{ay+c}{by+d}\right)$$

for $f \in \mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$.

The generalized Lie derivatives $J^\pm, J^3$ corresponding to (7.13) are easily computed to be

(7.14)
$$J^3 = -u - v + z\frac{\partial}{\partial z} + y\frac{\partial}{\partial y},$$
$$J^+ = -2uz - 2vy + z^2\frac{\partial}{\partial z} + y^2\frac{\partial}{\partial y}, \qquad J^- = -\frac{\partial}{\partial z} - \frac{\partial}{\partial y}.$$

We will decompose $\mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$ into irred subspaces by explicitly computing the ON bases $\{h_k^{(w)}\}$, (7.3), of these subspaces. The lowest weight vectors $h_{-w}^{(w)}$ satisfy $J^- h_{-w}^{(w)} = 0$, $J^3 h_{-w}^{(w)} = -wh_{-w}^{(w)}$. We will use this property to compute the $h_{-w}^{(w)}$ explicitly and then use relations (7.3) to obtain a set of vectors $\{h_k^{(w)}\}$, $-w \le k \le w$, $|u - v| \le w \le u + v$. By showing that these vectors form an ON basis in $\mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$ we can verify (7.1) independently. Moreover, our explicit expressions for the $h_k^{(w)}(z, y)$ will enable us to compute the CG coefficients.

The general solution of $J^- f = -(\partial/\partial z + \partial/\partial y)f(z, y) = 0$ is $f(z, y) = \sum_{s=0}^{2q} a_s(z - y)^s$, where the a_s are arbitrary constants and $q = \min(u, v)$. A basis for the q-dimensional solution space is given by the vectors.

$$h_{-w}^{(w)}(z, y) = N_w(z - y)^{u+v-w}, \qquad |u - v| \le w \le u + v,$$

where the N_w are nonzero constants. Indeed

(7.15)
$$J^3 h_{-w}^{(w)} = -wh_{-w}^{(w)}, \qquad J^- h_{-w}^{(w)} = 0.$$

Let $(-,-)$ be the inner product on $\mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$ with respect to which the basis $\{f_m^{(u)} \otimes f_n^{(v)}\}$ is ON. It is easy to check that $(J^3)^* = J^3$ and $(J^+)^* = J^-$ for this inner product. We will choose the constants N_w such that $\|h_{-w}^{(w)}\| = 1$. Thus,

$$\|h_{-w}^{(w)}\|^2 = |N_w|^2 \sum_{j=0}^{u+v-w} (u + v - w)!^2 \frac{(2u - j)!\,(v - u + w + j)!}{j!\,(u + v - w - j)!} = 1.$$

Making use of the identity

(7.16)
$$\sum_{j=0}^{k} \frac{(m + k - j)!\,(n + j)!}{j!\,(k - j)!} = \frac{(m + k)!\,n!}{k!}\,{}_2F_1(-k, n + 1; -m - k; 1)$$
$$= \frac{m!\,n!\,(m + n + k + 1)!}{k!\,(n + m + 1)!}$$

(see Lebedev [1, p. 243] for a proof), we obtain

$$N_w = (-1)^{2v}\left[\frac{(2w + 1)!}{(u + v - w)!\,(u - v + w)!\,(v - u + w)!\,(u + v + w + 1)!}\right]^{1/2},$$

where the phase factor has been added to conform to convention.

Now we define vectors

$$(7.17)\qquad h_k^{(w)}(z, y) = \left[\frac{(w-k)!}{(w+k)!\,(2w)!}\right]^{1/2} (J^+)^{w+k} h_{-w}^{(w)}(z, y), \qquad -w \le k \le w,$$

where J^+ is given by (7.14). It follows immediately that $J^+ h_k^{(w)} = [(w+k+1)(w-k)]^{1/2} h_{k+1}^{(w)}$ in agreement with (7.3). Also, from the proof of relations (3.3) we see that $(J^+)^{2w+1} h_{-w}^{(w)} = J^+ h_w^{(w)} = 0$. Each $h_k^{(w)}(z, y)$ is a homogeneous polynomial of order $u + v + k$ in z and y, and there are a total of $(2u+1)(2v+1)$ such polynomials. We will show that the $\{h_k^{(w)}\}$ form an ON basis for $\mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$.

Lemma 7.1. (a) $J^+ h_k^{(w)} = [(w+k+1)(w-k)]^{1/2} h_{k+1}^{(w)}$; (b) $J^- h_k^{(w)} = [(w-k+1)(w+k)]^{1/2} h_{k-1}^{(w)}$; (c) $J^3 h_k^{(w)} = k h_k^{(w)}$.

Proof. Identity (a) follows directly from (7.17). Identity (c) follows from $J^3 h_{-w}^{(w)} = -w h_{-w}^{(w)}$ and the fact that $J^3(J^+ f) = (k+1)(J^+ f)$ if $J^3 f = kf$. We prove (b) by induction on k. Since $J^- h_{-w}^{(w)} = 0$ the equation holds for $k = -w$. Assume (b) is valid for $k \le l$ where $-w \le l \le w$. Then

$$J^- h_{l+1}^{(w)} = [(w+l+1)(w-l)]^{-1/2} J^- J^+ h_l^{(w)}$$

from (a). Since $J^- J^+ = J^+ J^- - 2J^3$ we have

$$J^- J^+ h_l^{(w)} = (J^+ J^- - 2J^3) h_l^{(w)} = (w+l+1)(w-l) h_l^{(w)}$$

by the induction hypotheses. Therefore, (b) follows for $k = l + 1$. Q.E.D.

Thus, for fixed w the vectors $\{h_l^{(w)}\}$ form a basis for a subspace of $\mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$ which transforms irreducibly under the rep $\mathbf{D}^{(w)}$. Furthermore, by computations analogous to (2.7)–(2.10) we see that the $\{h_l^{(w)}\}$ are ON. The Casimir operator $C = J^+J^- + J^3J^3 - J^3$ is symmetric since J^3 is symmetric and $(J^+J^- f, g) = (J^- f, J^- g) = (f, J^+J^- g)$. Since $Ch_k^{(w)} = w(w+1)h_k^{(w)}$ we obtain

$$w(w+1)(h_k^{(w)}, h_{k'}^{(w')}) = (Ch_k^{(w)}, h_{k'}^{(w')}) = (h_k^{(w)}, Ch_{k'}^{(w')}) = w'(w'+1)(h_k^{(w)}, h_{k'}^{(w')}),$$

so $(h_k^{(w)}, h_{k'}^{(w')}) = \delta_{ww'}\,\delta_{kk'}$, i.e., the $\{h_k^{(w)}\}$ form an ON set. Since the cardinality of this set is equal to the dimension of $\mathcal{V}^{(u)} \otimes \mathcal{V}^{(v)}$ we conclude that $\{h_k^{(w)}\}$ is an ON basis. This proves the validity of the Clebsch–Gordan series (7.1) from a Lie-algebraic viewpoint.

We can use our model to obtain an explicit expression for the coefficients. Since the $\{h_k^{(w)}\}$ for fixed w form a basis for $\mathbf{D}^{(w)}$ we have the identity

$$(7.18)\qquad \mathbf{T}(A)h_m^{(w)} = \sum_{p=-w}^{w} Q_{pm}^{w}(A) h_p^{(w)}, \qquad A \in SL(2),$$

where the matrix elements are given by (7.11). In the case where $A = \exp(-b\mathcal{J}^+)$ and $m = -w$, (7.18) is especially easy to evaluate. Indeed $h_{-w}^{(w)}(z, y)$

$= N_w(z-y)^{u+v-w}$ and $Q^w_{p,-w}(A) = (-b)^{p+w}\{(2w)!/[(w+p)!(w-p)!]\}^{1/2}$. Thus

(7.19)
$$\begin{aligned}
&N_w(bz+1)^{w+u-v}(by+1)^{w+v-u}(z-y)^{u+v-w}\\
&\quad = \sum_{p=-w}^{w}\sum_{m,n}\left[\frac{(2w)!}{(w+p)!\,(w-p)!\,(u+m)!\,(u-m)!\,(v+n)!\,(v-n)!}\right]^{1/2}\\
&\qquad \times C(u,m;v,n\,|\,w,p)\times(-z)^{u+m}(-y)^{v+n}(-b)^{w+p},
\end{aligned}$$

where we have used (7.4) and (7.10). Since $h_p^{(w)}(z, y)$ is homogeneous of order $u + v + p$ in z and y it follows that $C(u, m; v, n\,|\,w, p)$ is nonzero only if $m + n = p$.

Expression (7.19) is a generating function for the CG coefficients. We can write this expression in a more symmetric form by choosing $b = x_3^{-1}$ and introducing the **3-*j* coefficients**

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix} = \frac{(-1)^{j_3-m_3}}{(2j_3+1)^{1/2}}C(j_1, m_1; j_2, m_2\,|\,j_3, -m_3). \tag{7.20}$$

In terms of these quantities, (7.19) becomes

$$\begin{aligned}
&(x_3-x_1)^{j_1-j_2+j_3}(x_2-x_3)^{-j_1+j_2+j_3}(x_1-x_2)^{j_1+j_2-j_3}\\
&\quad\times[(j_1+j_2-j_3)!\,(j_1-j_2+j_3)!\\
&\quad\times(-j_1+j_2+j_3)!\,(j_1+j_2+j_3+1)]^{-1/2}\\
&\quad=\sum_{m_i=-j_i}^{j_i}\{x_1^{j_1+m_1}x_2^{j_2+m_2}x_3^{j_3+m_3}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\\
&\quad\times[(j_1+m_1)!\,(j_2+m_2)!\,(j+m_3)!\\
&\quad\times(j_1-m_1)!\,(j_2-m_2)!\,(j_3-m_3)!]^{-1/2}\}
\end{aligned}\tag{7.21}$$

(We have set $z = -x_1$, $y = -x_2$ in this expression.) Since the left-hand side is homogeneous of degree $j_1 + j_2 + j_3$ in x_1, x_2, x_3, so is the right-hand side. Thus, the 3-*j* coefficients are zero unless $m_1 + m_2 + m_3 = 0$. Furthermore, it follows from the CG series (7.1) that these coefficients are zero unless $j_1 + j_2 + j_3$ and $j_i + m_i$ are integers, and $-j_i \le m_i \le j_i$, $i = 1, 2, 3$. The 3-*j* coefficients have a high degree of symmetry, as is evident from (7.21). Indeed the left-hand side of (7.21) is fixed under an even permutation of the integers 1, 2, 3. As a consequence

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2\\ m_3 & m_1 & m_2\end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1\\ m_2 & m_3 & m_1\end{pmatrix}. \tag{7.22}$$

The left-hand side changes by a phase factor under an odd permutation:

$$\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix} &= (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_2 & j_1 & j_3\\ m_2 & m_1 & m_3\end{pmatrix}\\
&= (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_3 & j_2\\ m_1 & m_3 & m_2\end{pmatrix}.
\end{aligned}\tag{7.23}$$

If we make the substitution $x_i \to x_i^{-1}$ and multiply by $x_1^{2j_1}x_2^{2j_2}x_3^{2j_3}$ the generating function changes by the factor $(-1)^{j_1+j_2+j_3}$. There follows the identity

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{7.24}$$

If we multiply both sides of (7.21) by

$$\frac{(j_1+j_2+j_3+1)^{1/2}\alpha^{-j_1+j_2+j_3}\beta^{j_1-j_2+j_3}\gamma^{j_1+j_2-j_3}}{[(-j_1+j_2+j_3)!\,(j_1-j_2+j_3)!\,(j_1+j_2-j_3)!]^{1/2}}$$

and sum over all j_i for which (7.21) makes sense, we obtain the new generating function

(7.25)

$$\begin{aligned}
\exp[\alpha(x_2-x_3)&+\beta(x_3-x_1)+\gamma(x_1-x_2)] \\
&= \sum_{j_1+j_2+j_3=0}^{\infty} \sum_{m_i=-j_i}^{j_i} \{(j_1+j_2+j_3+1)^{1/2}\alpha^{-j_1+j_2+j_3}\beta^{j_1-j_2+j_3}\gamma^{j_1+j_2-j_3} \\
&\quad \times x_1^{j_1+m_1}x_2^{j_2+m_2}x_3^{j_3+m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\
&\quad \times [(-j_1+j_2+j_3)!\,(j_1-j_2+j_3)!\,(j_1+j_2-j_3)!\,(j_1+m_1)! \\
&\quad \times (j_2+m_2)!\,(j_3+m_3)!\,(j_1-m_1)!\,(j_2-m_2)!\,(j_3-m_3)!]^{-1/2}\}.
\end{aligned}$$

A still higher degree of symmetry can be obtained by making the replacements $x_i \to x_i/y_i$, $\alpha \to y_2y_3\alpha$, $\beta \to y_3y_1\beta$, $\gamma \to y_1y_2\gamma$ in (7.25). Then this expression takes the form

$$\begin{aligned}
\exp(\det B) = \sum_{j_1+j_2+j_3=0}^{\infty} \sum_{m_i=-j_i}^{j_i} & b(j_i, m_i)\alpha^{-j_1+j_2+j_3}\beta^{j_1-j_2+j_3}\gamma^{j_1+j_2-j_3} \\
& \times x_1^{j_1+m_1}y_1^{j_1-m_1}x_2^{j_2+m_2}y_2^{j_2-m_2}x_3^{j_3+m_3}y_3^{j_3-m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix},
\end{aligned} \tag{7.26}$$

where $b(j_i, m_i)$ is completely symmetric under a permutation of the integers 1, 2, 3. Here B is the matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

It is now evident that the symmetries (7.22) and (7.23) correspond to permutations of the columns of B. Under an even permutation $\det B$ remains invariant, while under an odd permutation it changes sign. The identity (7.24) follows from the fact that $\det B$ changes sign under a transposition of the second and third rows of B. Note that a change in sign of $\det B$ is equivalent to multiplication of the right-hand side of (7.26) by $(-1)^{j_1+j_2+j_3}$.

In addition to these symmetries we see that arbitrary permutations of the rows of B lead to new symmetries. Furthermore, since $\det B^t = \det B$ we can obtain a new symmetry by interchanging rows and columns of B. The six column permutations, six row permutations, and the transpose generate a group of $6 \times 6 \times 2 = 72$ symmetries of the 3-j coefficients. This symmetry group was discovered by Regge [1]. The symmetries (7.22)–(7.24) generate a subgroup of order 12.

Substituting (7.20) in the above formulas we can obtain corresponding formulas for the CG coefficients. The most frequently used CG coefficients $C(j_1, m_1; j_2, m_2 \mid j_3, m_3)$ are those for which $j_2 = \frac{1}{2}$ or 1. We can easily compute these special cases from (7.21). For $j_2 = \frac{1}{2}$ the coefficients are zero unless $j_3 = j_1 \pm \frac{1}{2}$ and $m_3 = m_1 + m_2$. The nonzero coefficients are given by

$$
\text{(7.27)} \qquad
\begin{array}{c|cc}
 & m_2 = -\frac{1}{2} & m_2 = \frac{1}{2} \\
\hline
j_3 = j_1 - \frac{1}{2} & \left[\dfrac{j_1 + m_3 + \frac{1}{2}}{2j_1 + 1}\right]^{1/2} & -\left[\dfrac{j_1 - m_3 + \frac{1}{2}}{2j_1 + 1}\right]^{1/2} \\
j_3 = j_1 + \frac{1}{2} & \left[\dfrac{j_1 - m_3 + \frac{1}{2}}{2j_1 + 1}\right]^{1/2} & \left[\dfrac{j_1 + m_3 + \frac{1}{2}}{2j_1 + 1}\right]^{1/2}.
\end{array}
$$

For $j_2 = 1$ the coefficients are zero unless $j_3 = j_1, j_1 \pm 1$, and $m_3 = m_1 + m_2$. The nonzero coefficients $C(j_1, m_2; 1, m_2 \mid i_3, m_3)$ are

$$
\text{(7.28)} \qquad
\begin{array}{c|ccc}
 & m_2 = -1 & m_2 = 0 & m_2 = 1 \\
\hline
j_3 = j_1 - 1 & \left(\dfrac{(j_1 + m_3 + 1)(j_1 + m_3)}{2j_1(2j_1 + 1)}\right)^{1/2} & -\left(\dfrac{(j_1 - m_3)(j_1 + m_3)}{j_1(2j_1 + 1)}\right)^{1/2} & \left(\dfrac{(j_1 - m_3)(j_1 - m_3 + 1)}{2j_1(j_1 + 1)}\right)^{1/2} \\
j_3 = j_1 & \left(\dfrac{(j_1 - m_3)(j_1 + m_3 + 1)}{2j_1(j_1 + 1)}\right)^{1/2} & \dfrac{m_3}{[j_1(j_1 + 1)]^{1/2}} & -\left(\dfrac{(j_1 + m_3)(j_1 - m_3 + 1)}{2j_1(j_1 + 1)}\right)^{1/2} \\
j_3 = j_1 + 1 & \left(\dfrac{(j_1 - m_3)(j_1 - m_3 + 1)}{(2j_1 + 1)(2j_1 + 2)}\right)^{1/2} & \left(\dfrac{(j_1 - m_3 + 1)(j_1 + m_3 + 1)}{(2j_1 + 1)(j_1 + 1)}\right)^{1/2} & \left(\dfrac{(j_1 + m_3)(j_1 + m_3 + 1)}{(2j_1 + 1)(2j_1 + 2)}\right)^{1/2}.
\end{array}
$$

It is not difficult to obtain an explicit expression for an arbitrary CG coefficient. Indeed one can expand one of the generating functions in powers of the independent variables and equate coefficients of like powers. However, the resulting expressions are very complicated (see Hamermesh [1]). For practical (computer) computations it is usually more convenient to use recurrence relations for the CG coefficients. Such relations can be easily derived by differentiating the generating functions with respect to some of the independent variables (Bargmann [2]).

7.8 Applications of the Clebsch–Gordan Series

We return to the study of a k-particle quantum mechanical system as described in Section 7.6. Suppose the Hamiltonian is given by

$$
\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 + \cdots + \mathbf{H}_k, \tag{8.1}
$$

where

$$\mathbf{H}_j = -(1/2m_j)\,\Delta_j + V_j(\mathbf{x}_j), \qquad 1 \le j \le k, \tag{8.2}$$

i.e., $\mathbf{H}$ is a sum of single-particle Hamiltonians. Furthermore, suppose $V_j(A\mathbf{x}_j) = V_j(\mathbf{x}_j)$ for all $A \in SO(3)$, so that each potential function $V_j(\mathbf{x}_j)$ is invariant under $SO(3)$. This system admits the compact symmetry group $G = SO(3) \times SO(3) \times \cdots \times SO(3)$ (k times). Indeed, we can define a unitary rep $\mathbf{S}$ of G on $\mathcal{H}$ by

$$[\mathbf{S}(A_1, \ldots, A_k)\Psi](\mathbf{x}_1, \ldots, \mathbf{x}_k) = \Psi(A_1^{-1}\mathbf{x}_1, \ldots, A_k^{-1}\mathbf{x}_k), \qquad A_j \in SO(3), \quad \Psi \in \mathcal{H}. \tag{8.3}$$

It is easy to check that these operators commute with $\mathbf{H}$.

From Corollary 6.2 it follows that the irred unitary reps of G are products of k unitary irred reps of $SO(3)$. Indeed, the irred reps of G can be denoted $\mathbf{D}^{(l_1,\ldots,l_k)}$, where

$$\mathbf{D}^{(l_1,\ldots,l_k)}(A_1, \ldots, A_k) = \mathbf{D}^{(l_1)}(A_1) \otimes \cdots \otimes \mathbf{D}^{(l_k)}(A_k) \tag{8.4}$$

and $\mathbf{D}^{(l_j)}$ is an irred rep of $SO(3)$.

Suppose λ is an eigenvalue of $\mathbf{H}$ and W_λ is the corresponding eigenspace. If W_λ transforms irreducibly under G according to $\mathbf{D}^{(l_1,\ldots,l_k)}$, the multiplicity of λ is $q = \dim \mathbf{D}^{(l_1,\ldots,l_k)} = (2l_1 + 1)(2l_2 + 1) \cdots (2l_k + 1)$. The functions $\Psi^{l_1}_{m_1}(\mathbf{x}_1) \cdots \Psi^{l_k}_{m_k}(\mathbf{x}_k)$, $-l_j \le m_j \le l_j$ form an ON basis for W_λ where $\Psi^{l_j}_{m_j}(\mathbf{x}_j)$ for fixed j is a canonical ON basis for the rep $\mathbf{D}^{(l_j)}$ and $\Psi^{l_j}_{m_j}$ is an eigenvector of $\mathbf{H}_j$. As we have seen earlier $\Psi^{l_j}_{m_j}(\mathbf{x}) = h_{l_j}(r) Y^{m_j}_{l_j}(\theta, \varphi)$ in spherical coordinates, so the angular dependence of the wave functions is determined. The radial dependence can be obtained only by solving the Schrödinger equation.

In the above system the k particles do not interact with one another. We now consider an interacting system obtained by adding a perturbing potential V' to $\mathbf{H}$:

$$\mathbf{H}' = \mathbf{H} + V'(\mathbf{x}_1, \ldots, \mathbf{x}_k). \tag{8.5}$$

We further assume that V' is invariant under the action $\mathbf{T}$, (6.2), of $SO(3)$ on $\mathcal{H}$ but not under the action $\mathbf{S}$ of G, i.e., the equality

$$V'(A_1\mathbf{x}_1, \ldots, A_k\mathbf{x}_k) = V'(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

holds in general only if $A_1 = \cdots = A_k = A \in SO(3)$. Thus the symmetry group of the perturbed Hamiltonian $\mathbf{H}'$ will be the subgroup of G consisting of all diagonal elements $A \times A \times \cdots \times A$. This subgroup is obviously isomorphic to $SO(3)$. To determine the splitting of the eigenvalue λ under the perturbation we need only express $\mathbf{D}^{(l_1,\ldots,l_k)} \mid SO(3)$ as a direct sum of irred reps of $SO(3)$.

From (8.4) it is clear that this restricted rep is isomorphic to the k-fold tensor product

$$\mathbf{D}^{(l_1,\ldots,l_k)} \mid SO(3) \cong \mathbf{D}^{(l_1)} \otimes \mathbf{D}^{(l_2)} \otimes \cdots \otimes \mathbf{D}^{(l_k)}. \tag{8.6}$$

We can use the CG series to decompose (8.6) into a direct sum of irred reps and thereby obtain the splitting of the energy levels. For example we could use the CG series to decompose $\mathbf{D}^{(l_1)} \otimes \mathbf{D}^{(l_2)}$, tensor the resulting irred reps with $\mathbf{D}^{(l_3)}$, and apply the CG series again, etc. (In case W_λ is not irred under G we can decompose W_λ into a direct sum of G-irred reps and proceed as above.)

In the simplest case $k = 2$ and

$$\mathbf{D}^{(l_1,l_2)} \mid SO(3) \cong \mathbf{D}^{(l_1)} \otimes \mathbf{D}^{(l_2)} \cong \mathbf{D}^{(l_1+l_2)} \oplus \mathbf{D}^{(l_1+l_2-1)} \oplus \cdots \oplus \mathbf{D}^{(|l_1-l_2|)}. \tag{8.7}$$

Here the $(2l_1 + 1)(2l_2 + 1)$-degenerate energy level λ splits into $2\min(l_1, l_2) + 1$ levels and the energy level corresponding to $\mathbf{D}^{(l)}$ is $(2l + 1)$-degenerate. We can use the CG coefficients to decompose W_λ into a direct sum of subspaces transforming under the irred reps of $SO(3)$ given by the right-hand side of (8.7). Indeed a canonical basis for the subspace transforming according to $\mathbf{D}^{(l)}$ is given by

$$h_m{}^l(\mathbf{x}_1, \mathbf{x}_2) = \sum_{m_1 m_2} C(l_1, m_1; l_2, m_2 \mid l, m)\Psi^{l_1}_{m_1}(\mathbf{x}_1)\Psi^{l_2}_{m_2}(\mathbf{x}_2), \qquad -l \leq m \leq l. \tag{8.8}$$

As we have shown in Section 7.6, the computation of matrix elements of $\mathbf{H}'$ with respect to the canonical basis $\{h_m{}^l\}$ is relatively simple because $SO(3)$ is a symmetry group of $\mathbf{H}'$. This basis is far superior to $\{\Psi^{l_1}_{m_1}\Psi^{l_2}_{m_2}\}$ since it explicitly exhibits the $SO(3)$ symmetry. The matrix elements of $\mathbf{H}'$ are needed in quantum mechanical perturbation theory to compute the perturbed eigenvalues (Schiff [1], Landau and Lifshitz [2]).

The decomposition (8.7)–(8.8) is also of great importance in the study of time-varying systems. We look for solutions Ψ of the Schrödinger equation

$$i\,\partial\Psi(\mathbf{x}_1, \mathbf{x}_2, t)/\partial t = \mathbf{H}'\Psi(\mathbf{x}_1, \mathbf{x}_2, t), \tag{8.9}$$

where $\Psi(\mathbf{x}_1, \mathbf{x}_2, t) \in \mathcal{H}$ for each t. Suppose the functions $\Psi_m{}^l(\mathbf{x}_1, \mathbf{x}_2, t)$ are solutions of (8.9) such that $\Psi_m{}^l(\mathbf{x}_1, \mathbf{x}_2, 0) = h_m{}^l(\mathbf{x}_1, \mathbf{x}_2)$, expression (8.8). Then at $t = 0$ the $\Psi_m{}^l$, $-l \leq m \leq l$, form a canonical basis for the irred rep $\mathbf{D}^{(l)}$. According to the results of Section 7.6, the functions $\Psi_m{}^l(\mathbf{x}_1, \mathbf{x}_2, t)$ form a canonical basis for $\mathbf{D}^{(l)}$ at *every* time t. In particular

$$\mathbf{L}^3\Psi_m{}^l = m\Psi_m{}^l, \qquad \mathbf{L} \cdot \mathbf{L}\Psi_m{}^l = l(l + 1)\Psi_m{}^l \tag{8.10}$$

for all t. Thus the quantum numbers l and m are conserved under the interaction.

To see the physical significance of this analysis we consider an (oversimplified) example. Suppose the perturbing potential is a function of time, V'

$= V'(\mathbf{x}_1, \mathbf{x}_2, t)$, such that for all t, V' is $SO(3)$-invariant but not necessarily G-invariant. Furthermore suppose $V' = 0$ for $t \leq 0$ and $t \geq \tau > 0$, where τ is some fixed time. Thus the perturbing potential acts only in the time interval $(0, \tau)$. At all other times $\mathbf{H}' = \mathbf{H}$.

Let W_λ be the eigenspace of $\mathbf{H}$ corresponding to eigenvalue λ. The space W_λ transforms irreducibly under G:

$$\mathbf{S} | W_\lambda \simeq \mathbf{D}^{(l_1, l_2)} \tag{8.11}$$

and has the ON basis

$$\{\Psi^{l_1}_{m_1}(\mathbf{x}_1)\Psi^{l_2}_{m_2}(\mathbf{x}_2) \colon -l_j \leq m_j \leq l_j\}.$$

Now suppose Ψ is a solution of (8.9) such that $\Psi(\mathbf{x}_1, \mathbf{x}_2, 0) = \Psi^{l_1}_{m_1}(\mathbf{x}_1)\Psi^{l_2}_{m_2}(\mathbf{x}_2) \in W_\lambda$, i.e., the first particle has quantum numbers l_1, m_1 and the second has quantum numbers l_2, m_2. As t increases, the particles begin to interact. We assume the interaction is **elastic**, i.e., we end up with the same two particles and energy is conserved. No particles are created or destroyed by the interaction.

After time $t = \tau$ the particles are again noninteracting. By conservation of energy, $\Psi(\tau)$ must have energy λ. Thus $\Psi(\tau) \in W_\lambda$, or

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \tau) = \sum_{n_1 n_2} a_{n_1 n_2} \Psi^{l_1}_{n_1}(\mathbf{x}_1)\Psi^{l_2}_{n_2}(\mathbf{x}_2) \tag{8.12}$$

and we can describe the interaction by computing $a_{n_1 n_2}$: $|a_{n_1 n_2}|^2$ is the probability that a system in the state $\Psi^{l_1}_{m_1}\Psi^{l_2}_{m_2}$ at $t = 0$ ends up in the state $\Psi^{l_1}_{n_1}\Psi^{l_2}_{n_2}$ at $t = \tau$. Since $SO(3) \times SO(3)$ is *not* a symmetry group of $\mathbf{H}'$, m_1 and m_2 are not conserved by the interaction. Thus, if particle one starts out in the state $\Psi^{l_1}_{m_1}$ there is no reason to assume that it will end up in this state.

On the other hand, $SO(3)$ *is* a symmetry group of $\mathbf{H}'$. If the system is in the state $h_m{}^l$ at $t = 0$ then it must be in the state $h_m{}^l$ at $t = \tau$. Note that the vectors

$$h_m{}^l(\mathbf{x}_1, \mathbf{x}_2), \qquad |l_1 - l_2| \leq l \leq l_1 + l_2, \qquad -l \leq m \leq l, \tag{8.13}$$

form an ON basis for W_λ. Thus, if $\Psi(0) = h_m{}^l$ then by conservation of angular momentum

$$\Psi(\tau) = b_l h_m{}^l. \tag{8.14}$$

Since $\Psi(\tau)$ is a unit vector we must have $|b_l| = 1$, or $b_l = e^{i\theta_l}$, $0 \leq \theta_l < 2\pi$. Just as in Section 7.6 we can easily show that θ_l is independent of m. The basis $\{h_m{}^l\}$ is clearly more convenient for W_λ than the basis $\{\Psi^{l_1}_{m_1}\Psi^{l_2}_{m_2}\}$. On the strength of conservation of angular momentum alone we have proved that $h_m{}^l$ is merely multiplied by a phase factor $e^{i\theta_l}$. The results of the scattering experiment are determined by the scattering angles θ_l, $|l_1 - l_2| \leq l \leq l_1 + l_2$, which must be computed from the dynamical equations.

Now that we know how the $\{h_m{}^l\}$ transform we can use the CG coefficients to determine how the basis $\{\Psi^{l_1}_{m_1}\Psi^{l_2}_{m_2}\}$ transforms. A straightforward computation yields

$$\Psi^{l_1}_{m_1}\Psi^{l_2}_{m_2} \longrightarrow \sum_{ln_1n_2} C(l_1, m_1; l_2, m_2 \,|\, l, m)e^{i\theta_l}C(l_1, n_1; l_2, n_2 \,|\, l, m)\Psi^{l_1}_{n_1}\Psi^{l_2}_{n_2}. \tag{8.15}$$

Thus the probability that a system in the state $\Psi^{l_1}_{m_1}\Psi^{l_2}_{m_2}$ at $t = 0$ will be found in the state $\Psi^{l_1}_{n_1}\Psi^{l_2}_{n_2}$ at $t = \tau$ is

$$\left|\sum_{l=|l_1-l_2|}^{l_1+l_2} C(l_1, m_1; l_2, m_2 \,|\, l, m)e^{i\theta_l}C(l_1, n_1; l_2, n_2 \,|\, l, m)\right|^2. \tag{8.16}$$

If the system has $k > 2$ particles a similar but more complicated analysis can be used to decompose W_λ into a direct sum of irred subspaces under $SO(3)$. The principal complications arise from the fact that a given irred rep may occur with multiplicity greater than one. Then there is no unique way to decompose W_λ and it may be necessary to relate the various possible decompositions by Racah coefficients (Liubarskii [1]).

In the preceding discussion we have ignored the possibility of spin. However, for many particles such as the electron, the proton and the neutron, physical observations do not agree with the predictions of our theory. To obtain predictions in agreement with experiment it is necessary to postulate more complicated transformation properties of the particle state functions. Intuitively, one may think of a particle with spin, say an orbital electron in an atom, as a billiard ball spinning about its own axis. In addition to its orbital angular momentum the billiard ball possesses an intrinsic spin angular momentum.

To make the discussion concrete we construct the state space of a single nonrelativistic particle with spin s, $2s = 0, 1, 2, \ldots$. The Hilbert space $\mathcal{H}_s$ consists of vector valued functions

$$\Psi(\mathbf{x}) = \begin{pmatrix} \Psi_s(\mathbf{x}) \\ \Psi_{s-1}(\mathbf{x}) \\ \cdot \\ \cdot \\ \cdot \\ \Psi_{-s}(\mathbf{x}) \end{pmatrix} = \sum_{\mu=-s}^{s} \Psi_\mu(\mathbf{x})e_\mu, \tag{8.17}$$

where e_μ is the column vector with a one in row μ and zeros everywhere else. The vector $\Psi(\mathbf{x}) \in \mathcal{H}_s$ if

$$\int_{R_3} \Psi^t(\mathbf{x})\bar{\Psi}(\mathbf{x})\, d\mathbf{x} = \int \sum_{\mu=-s}^{s} |\Psi_\mu(\mathbf{x})|^2\, d\mathbf{x} < \infty$$

and the inner product is

$$(\Theta, \Psi) = \int_{R_3} \Theta^t(\mathbf{x})\overline{\Psi(\mathbf{x})}\, d\mathbf{x} = \int \sum_{\mu=-s}^{s} \Theta_\mu(\mathbf{x})\overline{\Psi_\mu(\mathbf{x})}\, d\mathbf{x}. \tag{8.18}$$

We define a unitary rep $\mathbf{T}$ of $SU(2)$ on $\mathcal{H}_s$ by

$$[\mathbf{T}(A)\mathbf{\Psi}](\mathbf{x}) = T^s(A)\mathbf{\Psi}(R(A^{-1})\mathbf{x}),$$

or in components

$$(8.19) \qquad [\mathbf{T}(A)\mathbf{\Psi}]_\mu(\mathbf{x}) = \sum_{\nu=-s}^{s} T^s_{\mu\nu}(A)\Psi_\nu(R(A^{-1})\mathbf{x}), \qquad -s \le \mu \le s.$$

Here $R(A) \in SO(3)$ is defined by (1.12) and (1.20) and the matrix elements $T^s_{\mu\nu}(A)$ by (2.14). Since the matrices $T^s(A)$ are unitary and satisfy the homomorphism property $T^s(AB) = T^s(A)T^s(B)$, the operators $\mathbf{T}(A)$ are unitary in $\mathcal{H}_s$ and satisfy $\mathbf{T}(AB) = \mathbf{T}(A)\mathbf{T}(B)$. Any vector-valued function $\mathbf{\Psi}(\mathbf{x})$ which transforms under the action of $SU(2)$ according to (8.19) is called a **spinor field of weight s.** (A spinor field need not belong to $\mathcal{H}_s$.) It follows from Section 7.2 that if s is an integer, $\mathbf{T}$ defines a single-valued rep of $SO(3)$, while if s is half-integral, $\mathbf{T}$ is double-valued on $SO(3)$.

In nonrelativistic quantum mechanics it is postulated that the state vectors of the electron, proton, and neutron transform under rotations as spinor fields of weight $\frac{1}{2}$. There are mesons and baryons with spins 0, 1, and $\frac{3}{2}$. The photon in relativistic quantum mechanics has spin one, while the nuclei of various atoms can have spins greater than 1.

We have postulated that the state vectors of a particle with spin belong to $\mathcal{H}_s$ and transform under rotations of space by (8.19). If s is half-integral this postulate seems ambiguous because $\mathbf{T}$ is a double-valued rep of $SO(3)$. Indeed if $R \in SO(3)$ there exists $A \in SU(2)$ such that $R = R(\pm A)$ and $\mathbf{T}(-A)\mathbf{\Psi} = -\mathbf{T}(A)\mathbf{\Psi}$ for $\mathbf{\Psi} \in \mathcal{H}_s$. However $\pm\mathbf{T}(A)\mathbf{\Psi}$ both define the same state (ray) in $\mathcal{H}_s$, so there is no physical contradiction.

In a manner similar to the above construction we can define state spaces for systems containing several particles. As an example we construct the state space for a system containing two electrons. The Hilbert space $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$ consists of all tensor-valued functions $\mathbf{\Psi}(\mathbf{x}_1, \mathbf{x}_2)$ with components $\Psi_{\mu_1\mu_2}(\mathbf{x}_1, \mathbf{x}_2)$, $\mu_1, \mu_2 = \pm\frac{1}{2}$, such that

$$\int_{R_3^2} \sum_{\mu_1\mu_2=-1/2}^{1/2} |\Psi_{\mu_1\mu_2}(\mathbf{x}_1, \mathbf{x}_2)|^2 \, d\mathbf{x}_1 \, d\mathbf{x}_2 < \infty.$$

The inner product is

$$(8.20) \qquad (\mathbf{\Theta}, \mathbf{\Psi}) = \int_{R_3^2} \sum_{\mu_1\mu_2} \Theta_{\mu_1\mu_2}(\mathbf{x}_1, \mathbf{x}_2)\overline{\Psi_{\mu_1\mu_2}(\mathbf{x}_1, \mathbf{x}_2)} \, d\mathbf{x}_1 \, d\mathbf{x}_2.$$

Here the spinor indices and spatial coordinates corresponding to particles one and two are $\mu_1, \mathbf{x}_1$ and $\mu_2, \mathbf{x}_2$, respectively. [Actually, by the **Pauli exclusion principle** the state space is the proper closed subspace of $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$ consisting of vectors $\mathbf{\Psi}$ such that $\Psi_{\mu_1\mu_2}(\mathbf{x}_1, \mathbf{x}_2) + \Psi_{\mu_2\mu_1}(\mathbf{x}_2, \mathbf{x}_1) \equiv 0$. Thus, not all elements of $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$ have physical significance (see Section 9.8).] In a similar manner one can construct state spaces for systems containing an arbitrary (finite) number of particles with arbitrary spin. A state vector $\boldsymbol{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ in a k-particle system has components $\Psi_{\mu_1 \cdots \mu_k}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$. If the jth particle has spin s_j then the index μ_j takes values $-s_j, -s_j + 1, \ldots, s_j$. Under a rotation $R(A)$ the state vector $\boldsymbol{\Psi}$ is transformed to

(8.21)
$$[\mathbf{T}(A)\boldsymbol{\Psi}]_{\mu_1 \cdots \mu_k}(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \sum_{\nu_j = -s_j}^{s_j} T^{s_1}_{\mu_1 \nu_1}(A) \cdots T^{s_k}_{\mu_k \nu_k}(A) \Psi_{\nu_1 \cdots \nu_k}(R(A^{-1})\mathbf{x}_1, \ldots, R(A^{-1})\mathbf{x}_k).$$

The rep $\mathbf{T}$ is single-valued on $SO(3)$ if an even number of spins s_j are half-integral. Otherwise, $\mathbf{T}$ is double-valued.

The rep $\mathbf{T}$ of $SU(2)$ induces a corresponding rep of $su(2)$ defined by operators

$$\boldsymbol{\mathcal{J}} = (d/dt)\mathbf{T}(\exp t\mathcal{J})|_{t=0}, \qquad \mathcal{J} \in su(2).$$

We choose the basis $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, (1.8), and compute the operators $\boldsymbol{\mathcal{J}}_1, \boldsymbol{\mathcal{J}}_2, \boldsymbol{\mathcal{J}}_3$ in the case where $\mathbf{T}$ acts on $\mathcal{H}_s$ according to (8.19):

(8.22)
$$\boldsymbol{\mathcal{J}}_j = \mathcal{S}_j + \mathcal{L}_j, \qquad j = 1, 2, 3,$$

where $\mathcal{S}_j$ is a $(2s + 1) \times (2s + 1)$ matrix

$$\mathcal{S}_j = (d/dt)T^s(\exp t\mathcal{J}_j)|_{t=0}$$

acting on spinor components and $\mathcal{L}_j$ is the differential operator (6.22), $(k = 1)$. [If $s = \frac{1}{2}$ then $T^{1/2}(A) = A$ and $\mathcal{S}_j = \mathcal{J}_j$, see (1.8). These three matrices are called the **Pauli spin matrices.**] The action of the spin matrices on the spinors e_μ is given by

(8.23)
$$\begin{aligned} &\mathbf{S}^3 e_\mu = \mu e_\mu, \qquad \mathbf{S}^\pm e_\mu = [(s \pm \mu + 1)(s \mp \mu)]^{1/2} e_{\mu \pm 1}, \\ &\mathbf{S} \cdot \mathbf{S} e_\mu = (\mathbf{S}_1\mathbf{S}_1 + \mathbf{S}_2\mathbf{S}_2 + \mathbf{S}_3\mathbf{S}_3)e_\mu = s(s + 1)e_\mu, \qquad -s \le \mu \le s, \end{aligned}$$

where $\mathbf{S}^\pm = \pm i\mathbf{S}_2 + \mathbf{S}_1 = \mp \mathcal{S}_2 + i\mathcal{S}_1$ and $\mathbf{S}^3 = -\mathbf{S}_3 = -i\mathcal{S}_3$. Since $\mathbf{T}$ is unitary the operators $\mathbf{J}_j = i\boldsymbol{\mathcal{J}}_j$ are symmetric on $\mathcal{H}_s$ and satisfy the usual commutation relations

(8.24)
$$[\mathbf{J}_1, \mathbf{J}_2] = i\mathbf{J}_3, \qquad [\mathbf{J}_3, \mathbf{J}_1] = i\mathbf{J}_2, \qquad [\mathbf{J}_2, \mathbf{J}_3] = i\mathbf{J}_1.$$

[Compare with (6.24).] In quantum theory the $\mathbf{J}_j$ are called **total** angular momentum operators. Here $\mathbf{J}_j = \mathbf{S}_j + \mathbf{L}_j$, where the self-adjoint matrices $\mathbf{S}_j = i\mathcal{S}_j$ are **spin** angular momentum operators and the symmetric operators $\mathbf{L}_j = i\mathcal{L}_j$ are **orbital** angular momentum operators. Note that the $\mathbf{S}_j$ and $\mathbf{L}_h$ operators commute with one another since the first acts on the spinor induces alone, while the second acts on the coordinates $\mathbf{x}$ alone.

In case $\mathbf{T}$ acts on a k-particle state space according to (8.21), an analogous computation yields

(8.25)
$$\mathbf{J}_j = \sum_{c=1}^{k} (\mathbf{S}_j^{(c)} + \mathbf{L}_j^{(c)}), \qquad j = 1, 2, 3,$$

where $\mathbf{S}_j^{(c)}$ is a $(2s_c + 1) \times (2s_c + 1)$ matrix acting on the spinor indices μ_c and $\mathbf{L}_j^{(c)} = i\mathfrak{L}_j^{(c)}$ is the differential operator (6.22) acting on the coordinates $\mathbf{x}_c$. The commutation relations are again (8.24).

To investigate some of the physical consequences of this formalism we consider a system containing a single electron ($s = \frac{1}{2}$). Suppose the Hamiltonian $\mathbf{K}$ on $\mathcal{H}_{1/2}$ takes the form

(8.26) $$\mathbf{K}\boldsymbol{\Psi} = \begin{pmatrix} \mathbf{H} & 0 \\ 0 & \mathbf{H} \end{pmatrix} \begin{pmatrix} \Psi_{1/2} \\ \Psi_{-1/2} \end{pmatrix}, \qquad \boldsymbol{\Psi} \in \mathcal{H}_{1/2},$$

where $\mathbf{H} = (-1/2m)\Delta + V(\mathbf{x})$, m is the mass of the electron, and $V(\mathbf{x})$ is rotationally invariant. We are assuming that $\mathbf{K}$ is *spin-independent*, i.e., it does not depend on the spinor index μ. Let λ be an eigenvalue of $\mathbf{H}$ acting on the Hilbert space $\mathcal{H}$ (no spin), and assume that the eigenspace W_λ in $\mathcal{H}$ transforms according to the $(2l + 1)$-dimensional irred rep $\mathbf{D}^{(l)}$ of $SO(3)$. Here the action of $SO(3)$ on $\mathcal{H}$ is given by $\Psi(\mathbf{x}) \to \Psi(R^{-1}\mathbf{x})$, $R \in SO(3)$. An ON basis for W_λ is $\{j(r)Y_l^m(\theta, \varphi): -l \leq m \leq l\}$, where $j(r)$ is determined from the solution of $\mathbf{H}\Psi = \lambda\Psi$. It is obvious from (8.26) that the eigenspace W_λ' of $\mathcal{H}_{1/2}$ corresponding to eigenvalue λ is $2(2l + 1)$-dimensional and has an ON basis

(8.27) $$j(r)\begin{pmatrix} Y_l^m(\theta, \varphi) \\ 0 \end{pmatrix}, \qquad j(r)\begin{pmatrix} 0 \\ Y_l^m(\theta, \varphi) \end{pmatrix}, \qquad -l \leq m \leq l.$$

Thus the degeneracy of λ is twice that in a spinless theory. It is easy to check that both the spin operators $\mathbf{S}_j$ and the angular momentum operators $\mathbf{L}_j$ commute with $\mathbf{K}$. Thus $\mathbf{K}$ admits the six-dimensional symmetry group $SU(2) \times SU(2)$ obtained by letting $SU(2)$ act on the spin indices and spatial coordinates independently in (8.19). Clearly, W_λ' transforms according to the irred rep $\mathbf{D}^{(1/2, l)}$ of $SU(2) \times SU(2)$.

Now we introduce a spin-dependent perturbing (matrix) potential $\mathbf{V}'$ such that the perturbed Hamiltonian $\mathbf{K}' = \mathbf{K} + \mathbf{V}'$ is still rotationally invariant, i.e., such that $\mathbf{K}'$ commutes with the operators (8.19). Then $\mathbf{K}'$ will no longer commute with all the spin operators $\mathbf{S}_j$ and orbital angular momentum operators $\mathbf{L}_j$, but will still commute with the operators $\mathbf{J}_j = \mathbf{S}_j + \mathbf{L}_j$. The symmetry group of $\mathbf{K}'$ is the diagonal subgroup of $SU(2) \times SU(2)$ consisting of those elements (A, B) such that $A = B$. Clearly, this subgroup is isomorphic to $SU(2)$. Since

(8.28)
$$\mathbf{D}^{(1/2, l)} \mid SU(2) \cong \mathbf{D}^{(1/2)} \otimes \mathbf{D}^{(l)} \cong \begin{cases} \mathbf{D}^{(l+[1/2])} \oplus \mathbf{D}^{(l-[1/2])}, & l = 1, 2, \ldots, \\ \mathbf{D}^{(1/2)}, & l = 0, \end{cases}$$

as follows from (8.19) and the CG series, we see that for $l \geq 1$ the perturbation splits the $2(2l + 1)$-degenerate eigenvalue λ into two eigenvalues of degeneracy $2l + 2$ and $2l$, respectively. For $l = 0$ the twofold eigenvalue does not split. These predictions are dramatically different than the corresponding predictions for spinless particles, and their experimental verification provides a justification for the introduction of spinor fields into quantum theory.

We can use the CG coefficients to construct a canonical basis for $\mathcal{H}_{1/2}$ corresponding to the decomposition (8.28). Indeed the vectors

$$h_n^{(l+(1/2))} = j(r) \sum_{\mu m} C(l, m; \tfrac{1}{2}, \mu \,|\, l + \tfrac{1}{2}, n) Y_l^m(\theta, \varphi) e_\mu,$$

$$h_n^{(l-(1/2))} = j(r) \sum_{\mu m} C(l, m; \tfrac{1}{2}, \mu \,|\, l - \tfrac{1}{2}, n) Y_l^m(\theta, \varphi) e_\mu,$$

form canonical bases for $\mathbf{D}^{(l+(1/2))}$ and $\mathbf{D}^{(l-(1/2))}$, respectively. This basis is very important in scattering problems involving spin-dependent forces. In such problems spin and orbital angular momentum are not separately conserved but only total angular momentum. Thus s, μ, l, m are not good quantum numbers and only the eigenvalues of $\mathbf{J}^3$ and $\mathbf{J} \cdot \mathbf{J}$ are conserved.

The decomposition of energy eigenstates of a system containing k particles with spins $s_1, \ldots, s_k$ into eigenstates of total angular momentum is analogous to that above.

7.9 Double-Valued Representations of the Crystallographic Groups

We have seen that in a physical system containing particles with spin it is possible that an energy eigenspace W_λ of the rotationally invariant, spin-dependent Hamiltonian $\mathbf{H}$ transforms under a half-integral irred rep $\mathbf{D}^{(u)}$ of $SU(2)$. For example, from (8.21) and the CG series, the eigenspaces of systems containing an *odd* number of electrons transform under half-integral reps. [Those with an even number of electrons transform under integral (single-valued) reps of $SO(3)$.]

Suppose W_λ is such an eigenspace of $\mathbf{H}$ in the Hilbert space $\mathcal{H}$ corresponding to a k-particle system. Now suppose we embed our system in an infinite crystal with crystallographic point symmetry group G (of the first kind). That is, we add to $\mathbf{H}$ the perturbing potential $V'(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ with symmetry group G:

$$\text{(9.1)} \qquad \mathbf{H}' = \mathbf{H} + V', \quad V'(R\mathbf{x}_1, \ldots, R\mathbf{x}_k) = V'(\mathbf{x}_1, \ldots, \mathbf{x}_k), \quad R \in G.$$

We assume V' is spin-independent, i.e., V' is a function and does not affect the spinor indices.

Let G' be the set of all $A \in SU(2)$ such that $R(A) \in G$, where $R(A)$ is defined by (1.20). Since $R(-A) = R(A)$, then $A \in G'$ implies $-A \in G'$. In particular $I = -E_2 \in G'$. Clearly, G' is a group. Since the mapping $A \longrightarrow R(A)$ is 2–1, the order of G' is twice that of G. Furthermore, $\{E_2, I\}$ is a normal subgroup of G such that $G'/\{E_2, I\} \cong G$. According to (9.1), $\mathbf{T}(A)V' = V'\mathbf{T}(A)$ for $A \in G'$ and $\mathbf{T}(A)$ given by (8.21). Thus $\mathbf{T}(A)\mathbf{H}' = \mathbf{H}'\mathbf{T}(A)$

for $A \in G'$ and G' is a symmetry group of $\mathbf{H}'$. If G is the largest point group fixing V', then G' is the largest subgroup of $SU(2)$ which is a symmetry group of $\mathbf{H}'$.

To analyze the splitting of the $(2u + 1)$-degenerate energy level λ under the perturbing potential V' we must decompose the restricted rep $\mathbf{D}^{(u)} | G'$ into a direct sum of irred reps of G'. If $R(\varphi, \theta, \psi) \in G$ has Euler coordinates φ, θ, ψ then the corresponding elements of G' are $A(\varphi, \theta, \psi)$, (1.13), and $-A = IA$. Since I commutes with the elements of G' and $I^2 = E_2$ it follows that $\mathbf{Q}(I) = \pm\mathbf{E}$ for any unitary irred rep $\mathbf{Q}$ of G'. If $\mathbf{Q}(I)$ is the identity operator then $\mathbf{Q}(A) = \mathbf{Q}(-A)$ and the $\mathbf{Q}$ induces a single-valued irred rep of the factor group $G'/\{E_2, I\} \cong G$. We say $\mathbf{Q}$ is **integral**. On the other hand, if $\mathbf{Q}(I) = -E$ then $\mathbf{Q}(-A) = -\mathbf{Q}(A)$ and $\mathbf{Q}$ induces a double-valued rep of G. We say $\mathbf{Q}$ is **half-integral**. The relationship between G and G' is analogous to that between $SO(3)$ and $SU(2)$.

If u is an integer then the operator $\mathbf{T}(I)$ corresponding to the rep $\mathbf{D}^{(u)}$ of $SU(2)$ is the identity. Thus, $\mathbf{D}^{(u)} | G'$ splits into a direct sum of integral irred reps of G'. We get the same splitting as by restricting the single-valued rep $\mathbf{D}^{(u)}$ of $SO(3)$ to G.

However, if u is half-integral (which is the case which concerns us here) then $\mathbf{T}(I) = -\mathbf{E}$ and $\mathbf{D}^{(u)} | G'$ splits into a direct sum of half-integral irred reps of G' (double-valued reps of G).

To determine this splitting we must find the character table for G'. This is a straightforward computation. Given G of order n we express its elements in terms of Euler angles and determine the group G' of order $2n$. Then we use the techniques of Section 3.6 to compute the character table. The integral characters are easy to find since there is a 1–1 relationship between reps of G and integral reps of G'. If χ is a simple character of G then the corresponding integral simple character of G' is $\chi'(A) = \chi'(-A) = \chi(R(A))$, $A \in G'$. Thus it only remains to compute the half-integral characters of G'. Complete tables of these characters are presented by Hamermesh [1] and Liubarskii [1]. Here, we present without proof the table of simple half-integral characters for O' where O is the octahedral group.

If $R \in O$ with Euler angles φ, θ, ψ we denote by R^+ the corresponding element in O' with the same Euler angles and set $R^- = -R^+ \in O'$. Now O contains 24 elements in five conjugacy classes: E, $\mathcal{C}_4{}^2(3)$, $\mathcal{C}_2(6)$, $\mathcal{C}_4(6)$, $\mathcal{C}_3(8)$. On the other hand, O' contains 48 elements in eight conjugacy classes: E, I, $\{\mathcal{C}_3{}^+(4), \mathcal{C}_3^{2-}(4)\}$, $\{\mathcal{C}_3^{2+}(4), \mathcal{C}_3{}^-(4)\}$, $\{\mathcal{C}_4{}^+(3), \mathcal{C}_4^{3-}(3)\}$, $\{\mathcal{C}_2{}^+(6), \mathcal{C}_2{}^-(6)\}$, $\{\mathcal{C}_4^{3+}(3), \mathcal{C}_4{}^-(3)\}$, $\{\mathcal{C}_4^{2+}(3), \mathcal{C}_4^{2-}(3)\}$. Thus, O' has eight irred reps of dimensions $n_1, \ldots, n_8$ such that $n_1{}^2 + \cdots + n_8{}^2 = 48$. However, in Section 3.6 we already found five irred reps of O (the integral reps of O') with dimensions 1, 1, 2, 3, 3. Thus there are three half-integral reps of O' with dimensions n_6, n_7, n_8, where $n_6{}^2 + n_7{}^2 + n_8{}^2 = 24$. The only solution with $n_6 \leq n_7 \leq$

n_8 is $n_6 = n_7 = 2$, $n_8 = 4$. The character table can be shown to be

(9.2)

O'	E	I	$\mathcal{C}_3^+(4)$ $\mathcal{C}_3^{2-}(4)$	$\mathcal{C}_3^{2+}(4)$ $\mathcal{C}_3^-(4)$	$\mathcal{C}_4^+(3)$ $\mathcal{C}_4^{3-}(3)$	$\mathcal{C}_4^{3+}(3)$ $\mathcal{C}_4^-(3)$	$\mathcal{C}_4^{2+}(3)$ $\mathcal{C}_4^{2-}(3)$	$\mathcal{C}_2^+(6)$ $\mathcal{C}_2^-(6)$
$(\chi')^{(6)}$	2	-2	1	-1	$\sqrt{2}$	$-\sqrt{2}$	0	0
$(\chi')^{(7)}$	2	-2	1	-1	$-\sqrt{2}$	$\sqrt{2}$	0	0
$(\chi')^{(8)}$	4	-4	-1	1	0	0	0	0

We can use this table to compute the splitting of a $(2u + 1)$-degenerate eigenvalue λ corresponding to the half-integral rep $\mathbf{D}^{(u)}$ of $SU(2)$ under a perturbation with O' symmetry. If $A \in SU(2)$ is similar to $A(0, 0, \tau)$ then the character $\chi^{(u)}(A) = [\sin(u + \frac{1}{2})\tau]/\sin(\tau/2)$. Moreover, $\chi^{(u)}(AI) = -\chi^{(u)}(A)$. With this information we can easily compute the character of $\mathbf{D}^{(u)} | O'$:

(9.3)

$\chi^{(u)}$	E	I	$\mathcal{C}_3^+(4)$ $\mathcal{C}_3^{2-}(4)$	$\mathcal{C}_3^{2+}(4)$ $\mathcal{C}_3^-(4)$	$\mathcal{C}_4^+(3)$ $\mathcal{C}_4^{3-}(3)$	$\mathcal{C}_4^{3+}(3)$ $\mathcal{C}_4^-(3)$	$\mathcal{C}_4^{2+}(3)$ $\mathcal{C}_4^{2-}(3)$	$\mathcal{C}_2^+(6)$ $\mathcal{C}_2^-(6)$
$\chi^{(1/2)}$	2	-2	1	-1	$\sqrt{2}$	$-\sqrt{2}$	0	0
$\chi^{(3/2)}$	4	-4	-1	1	0	0	0	0
$\chi^{(5/2)}$	6	-6	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	0
$\chi^{(7/2)}$	8	-8	1	-1	0	0	0	0
$\chi^{(9/2)}$	10	-10	-1	1	$\sqrt{2}$	$-\sqrt{2}$	0	0

Writing $\chi^{(u)} | O'$ as a linear combination of simple characters, we obtain the results:

(9.4)
$$\chi^{(1/2)} | O' = (\chi')^{(6)}, \qquad \chi^{(3/2)} | O' = (\chi')^{(8)}, \qquad \chi^{(5/2)} | O' = (\chi')^{(7)} + (\chi')^{(8)},$$
$$\chi^{(7/2)} | O' = (\chi')^{(6)} + (\chi')^{(7)} + (\chi')^{(8)}, \qquad \chi^{(9/2)} | O' = (\chi')^{(6)} + 2(\chi')^{(8)}.$$

for $u = \frac{1}{2}, \ldots, \frac{9}{2}$. For example, under the perturbation a sixfold eigenvalue $(u = \frac{5}{2})$ splits into one twofold and one fourfold eigenvalue. Notice that $\chi^{(1/2)} | O'$ and $\chi^{(3/2)} | O'$ are simple, so twofold and fourfold eigenvalues do not split.

7.10 The Wigner–Eckart Theorem and Its Applications

Let $\mathbf{T}$ be a unitary rep of $SU(2)$ on the Hilbert space $\mathcal{H}$. The mapping $\mathbf{Q} \to \mathbf{T}(A)\mathbf{Q}\mathbf{T}^{-1}(A)$ defines a rep of $SU(2)$ on the space $\mathcal{B}(\mathcal{H})$ of all bounded linear operators $\mathbf{Q}$ on $\mathcal{H}$. We could introduce an inner product on $\mathcal{B}(\mathcal{H})$

with respect to which this rep is unitary and then decompose the rep into a direct sum of irred reps $\mathbf{D}^{(u)}$. Rather than carry out such a decomposition we shall merely investigate the irred subspaces of operators.

Let $\mathcal{W}^{(u)}$ be an irred subspace of $\mathcal{B}(\mathcal{H})$ transforming according to $\mathbf{D}^{(u)}$. Then there exists a canonical basis $\{\mathbf{Q}_m: -u \le m \le u\}$ for $\mathcal{W}^{(u)}$ such that

$$\mathbf{T}(A)\mathbf{Q}_m\mathbf{T}^{-1}(A) = \sum_{n=-u}^{u} T^u_{nm}(A)\mathbf{Q}_n. \tag{10.1}$$

Operators with transformation properties (10.1) are called **spherical tensors of rank u**. We shall compute the matrix elements $(\mathbf{Q}_m\mathbf{f}^{u_1}_j, \mathbf{g}^{u_2}_h)$, where $\mathbf{f}^{u_1}_j$ and $\mathbf{g}^{u_2}_h$ belong to canonical ON sets in $\mathcal{H}$ transforming irreducibly under $\mathbf{T}$:

$$\mathbf{T}(A)\mathbf{f}^{u_1}_j = \sum_{k=-u_1}^{u_1} T^{u_1}_{kj}(A)\mathbf{f}^{u_1}_k, \qquad \mathbf{T}(A)\mathbf{g}^{u_2}_h = \sum_{s=-u_2}^{u_2} T^{u_2}_{sh}(A)\mathbf{g}^{u_2}_s. \tag{10.2}$$

Our considerations will also apply to unbounded operators $\mathbf{Q}_m$ on $\mathcal{H}$ provided there is a dense subspace $\mathcal{Z}$ of $\mathcal{H}$ such that (a) the domain of each of the $\mathbf{Q}_m$ contains $\mathcal{Z}$, (b) $\mathcal{Z}$ is invariant under the $\mathbf{T}(A)$, and (c) (10.1) holds on $\mathcal{Z}$.

The group rep (10.1) induces a Lie algebra rep of $su(2)$. Indeed if $J = (d/dt)\mathbf{T}(\exp t\mathcal{J})|_{t=0}$, $\mathcal{J} \in su(2)$, then by setting $A = \exp t\mathcal{J}$ in (10.1) and differentiating with respect to t at $t = 0$ we obtain the Lie algebra rep

$$\mathbf{Q}_m \longrightarrow [J, \mathbf{Q}_m] = J\mathbf{Q}_m - \mathbf{Q}_m J. \tag{10.3}$$

Since the $\mathbf{Q}_m$ form a canonical basis we find

$$[J^3, \mathbf{Q}_m] = m\mathbf{Q}_m, \qquad [J^\pm, \mathbf{Q}_m] = [(u \pm m + 1)(u \mp m)]^{1/2}\mathbf{Q}_{m\pm 1}, \tag{10.4}$$

where

$$J^\pm = \pm J_2 + iJ_1, \qquad J^3 = -iJ_3. \tag{10.5}$$

Spherical tensors appear frequently in quantum mechanics. For example a Hamiltonian $\mathbf{H}$ which commutes with the $\mathbf{T}(A)$ is a spherical tensor of rank zero. As another example we set $\mathcal{H} = L_2(R_3)$ and let $[\mathbf{T}(A)\Psi](\mathbf{x}) = \Psi(A^{-1}\mathbf{x})$ for $A \in SO(3)$, $\Psi \in \mathcal{H}$. Then for fixed integer l the multiplicative operators

$$\mathbf{Q}_m\Psi(r, \theta, \varphi) = r^l Y_l^{-m}(\theta, \varphi)\Psi(r, \theta, \varphi) \tag{10.6}$$

are spherical tensors of rank l. Here, the $Y_l^m(\theta, \varphi)$ are spherical harmonics expressed in spherical coordinates. To verify this we will check the relations (10.4). From (5.4), (5.6), and (10.5) we find

$$J^\pm = e^{\mp i\varphi}\left(\mp\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right), \qquad J^3 = i\frac{\partial}{\partial\varphi}. \tag{10.7}$$

Furthermore, from (5.7)

$$J^3 Y_l^{-m} = mY_l^{-m}, \qquad J^\pm Y_l^{-m} = [(l \pm m + 1)(l \mp m)]^{1/2} Y_l^{-(m\pm 1)}. \tag{10.8}$$

Since the J operators are differential and $\mathbf{Q}_m = r^l Y_l^{-m}$ is multiplicative we find

$$[J, \mathbf{Q}_m]\Psi = J(r^l Y_l^{-m}\Psi) - r^l Y_l^{-m}(J\Psi) = \{J(r^l Y_l^{-m})\}\Psi. \tag{10.9}$$

Together (10.8) and (10.9) yield (10.4) for $u = l$. [Actually the above result is valid for $\mathbf{Q}_m = f(r)Y_l^{-m}(\theta, \varphi)$, where $f(r)$ is arbitrary.]

Let us consider the special case $l = 1$. The canonical basis vectors are

(10.10)
$$\mathbf{f}_1{}^1 = rY_1^{-1} = \left(\frac{3}{8\pi}\right)^{1/2}(x - iy), \qquad \mathbf{f}_1{}^0 = rY_1{}^0 = \left(\frac{3}{4\pi}\right)^{1/2} z,$$
$$\mathbf{f}_1^{-1} = -\left(\frac{3}{8\pi}\right)^{1/2}(x + iy).$$

Multiplying all vectors by $(4\pi/3)^{1/2}$, we see that the vectors

(10.11)
$$(1/\sqrt{2})(x - iy), \quad z, \quad -(1/\sqrt{2})(x + iy)$$

form a canonical basis for $\mathbf{D}^{(1)}$. Note that x, y, z does *not* transform as a canonical basis. Here, the multiplicative operators $\mathbf{Q}_x = x$, $\mathbf{Q}_y = y$, $\mathbf{Q}_z = z$ are the **position operators** of quantum theory.

A similar computation using the same J-operators shows that the differential operators

(10.12)
$$\partial_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \partial_0 = \frac{\partial}{\partial z}, \qquad \partial_{-1} = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

also transform as spherical tensors of rank one. Note that the ∂_j are closely related to the **linear momentum** operators in quantum theory:

$$\mathbf{P}_x = -i\,\partial/\partial x, \qquad \mathbf{P}_y = -i\,\partial/\partial y, \qquad \mathbf{P}_z = -i\,\partial/\partial z.$$

We will compute the matrix elements $(\mathbf{Q}_m \mathbf{f}_j^{u_1}, \mathbf{g}_h^{u_2})$ for a set of spherical tensors of rank u. From (10.1) and (10.2) we obtain

(10.13)
$$(\mathbf{Q}_m \mathbf{f}_j^{u_1}, \mathbf{g}_h^{u_2}) = (\mathbf{T}(A)\mathbf{Q}_m \mathbf{f}_j^{u_1}, \mathbf{T}(A)\mathbf{g}_h^{u_2}) = (\mathbf{T}(A)\mathbf{Q}_m \mathbf{T}^{-1}(A)\mathbf{T}(A)\mathbf{f}_j^{u_1}, \mathbf{T}(A)\mathbf{g}_h^{u_2})$$
$$= \sum_{nks} T^u_{nm}(A) T^{u_1}_{kj}(A) \overline{T^{u_2}_{sh}(A)} (\mathbf{Q}_n \mathbf{f}_k^{u_1}, \mathbf{g}_s^{u_2}).$$

Multiplying the left- and right-hand sides of this equality by dA, integrating over $SU(2)$, and making use of the identity (7.9), we find

(10.14)
$$(\mathbf{Q}_m \mathbf{f}_j^{u_1}, \mathbf{g}_h^{u_2}) = C(u, m; u_1, j \mid u_2, h)N,$$
$$N = \frac{1}{2u_2 + 1} \sum_{uks} C(u, n; u_1\, k \mid u_2, s)(\mathbf{Q}_n \mathbf{f}_k^{u_1}, \mathbf{g}_s^{u_2}).$$

Theorem 7.1 (Wigner–Eckart). If $\{\mathbf{Q}_m\}$ is a set of spherical tensors of rank u then (10.14) holds where N depends on u, u_1, u_2 but not on m, j, and h.

The point of this theorem is that the dependence of the matrix element on m, j, and h is completely determined by the CG coefficient. If for fixed u, u_1, and u_2 we are able to compute one of the nonzero matrix elements (10.14) then we can solve for N and (10.14) will tell us the values of all the matrix elements. The constant N is sometimes called a **reduced** matrix element. From the known properties of the CG coefficients we see that the left-hand side of (10.14) will be zero unless $m + j = h$ and $u_2 = |u - u_1|$, $|u - u_1| + 1, \ldots, u + u_1$.

We have stated the Wigner–Eckart theorem for reps of $SU(2)$, but actually it holds for reps of any finite group or compact Lie group G. Indeed if we denote by $\mathbf{T}^{(u)}$ a complete set of nonequivalent irred unitary reps of G then we can define by (10.1) the operators of rank u where now $A \in G$. Expression (10.13) is unaltered by our generalization. We can integrate (10.13) over G with respect to the invariant measure dA if G is a Lie group or sum over the group if G is finite. Similarly, expression (7.9) is valid for G provided the factor $(2w + 1)/16\pi^2$ is replaced by n_w/V_G, where n_w is the dimension of $\mathbf{T}^{(w)}$. In particular we can define CG coefficients for G in analogy with those for $SU(2)$. (There is one possible complication here. It may be that $\mathbf{T}^{(w)}$ occurs more than once in the decomposition of $\mathbf{T}^{(u)} \otimes \mathbf{T}^{(v)}$. In this case the CG coefficients will need an extra parameter to denote which of the $\mathbf{T}^{(w)}$-subspaces is under consideration.)

We can get a better understanding of the Wigner–Eckart theorem by recalling the discussion of invariant tensors in Section 3.8. Expression (10.13) shows that the tensor $\mathbf{a}$ with components $a_{nks} = (\mathbf{Q}_n \mathbf{f}_k^{u_1}, \mathbf{g}_s^{u_2})$ is an invariant in a tensor space transforming under the rep

$$\mathbf{T}^{(u)} \otimes \mathbf{T}^{(u_1)} \otimes \overline{\mathbf{T}^{(u_2)}} \tag{10.15}$$

of G, where $\overline{\mathbf{T}^{(u_2)}}$ is the rep whose matrix elements are $\overline{T^{u_2}_{sh}}(A)$. Since $\mathbf{a}$ is invariant it must transform according to the identity rep $\mathbf{T}^{(0)}$. Let q be the multiplicity of $\mathbf{T}^{(0)}$ in (10.15) and let $V^{(0)}$ be the subspace of invariant tensors in the tensor space V. Then $q = \dim V^{(0)}$ and $\mathbf{a} \in V^{(0)}$ is nonzero only if $q > 0$. Furthermore, exactly q parameters are needed to uniquely determine $\mathbf{a}$. Let $\chi^{(u)}, \chi^{(u_1)}, \chi^{(u_2)}$ be the characters of $\mathbf{T}^{(u)}, \mathbf{T}^{(u_1)}, \mathbf{T}^{(u_2)}$, respectively. Then $\overline{\chi^{(u_2)}}(A)$ is the character of $\bar{\mathbf{T}}^{(u_2)}$. Since the character of $\mathbf{T}^{(0)}$ is $\chi^{(0)}(A) \equiv 1$ and the character of (10.15) is $\chi^{(u)}\chi^{(u_1)}\overline{\chi^{(u_2)}}$ we find from the orthogonality relations that q is given by

$$q = \int_G \chi^{(u)}(A)\chi^{(u_1)}(A)\overline{\chi^{(u_2)}}(A)\, \delta A = \langle \chi^{(u)}\chi^{(u_1)}\overline{\chi^{(u_2)}}, 1\rangle = \langle \chi^{(u)}\chi^{(u_1)}, \chi^{(u_2)}\rangle. \tag{10.16}$$

On the other hand, the right-hand side of (10.16) is just the multiplicity of $\mathbf{T}^{(u_2)}$ in the tensor product $\mathbf{T}^{(u)} \otimes \mathbf{T}^{(u_1)}$. Thus, we can obtain q from a knowledge of the CG series for irred reps of G. In particular, if $\mathbf{T}^{(u_2)}$ does not appear in the CG series for $\mathbf{T}^{(u)} \otimes \mathbf{T}^{(u_1)}$ then $q = 0$.

In the special case where $G = SU(2)$ the series is

$$\mathbf{D}^{(u)} \otimes \mathbf{D}^{(u_1)} \simeq \mathbf{D}^{(u+u_1)} \oplus \mathbf{D}^{(u+u_1-1)} \oplus \cdots \oplus \mathbf{D}^{(|u-u_1|)}, \tag{10.17}$$

so $q = 1$ if $u_2 = u + u_1, \ldots, |u - u_1|$; otherwise $q = 0$. In the cases where $q = 1$ the space of invariant tensors is one-dimensional and can be determined by specifying a single constant N.

We now give some applications of these results to quantum mechanics. Let $\mathcal{H}$ be the usual Hilbert space corresponding to a k-particle system (without spin) and let the action of $SO(3)$ on $\mathcal{H}$ be given by (6.2). Consider the position operators $\mathbf{Q}_{sj} = x_{sj}$, $s = 1, 2, 3$ $[\mathbf{x}_j = (x_{1j}, x_{2j}, x_{3j}) = (x_j, y_j, z_j)]$, of the jth particle. We will compute the matrix elements

$$(10.18) \quad (\mathbf{Q}_{sj}\Psi^{l_1}_{m_1}, \Psi^{l_2}_{m_2}) = \int_{R_{3}^{k}} x_{sj}\Psi^{l_1}_{m_1}(\mathbf{x}_1, \ldots, \mathbf{x}_k)\overline{\Psi^{l_2}_{m_2}}(\mathbf{x}_1, \ldots, \mathbf{x}_k)\, d\mathbf{x},$$

where the $\Psi^{l_h}_{m_h}$ transform as canonical basis vectors under the representations $\mathbf{D}^{(l_h)}$ of $SO(3)$. According to (10.11) the operators $\mathbf{Q}^{(1)} = 2^{-1/2}(\mathbf{Q}_{1j} - i\mathbf{Q}_{2j})$, $\mathbf{Q}^{(0)} = \mathbf{Q}_{3j}$, $\mathbf{Q}^{(-1)} = -2^{-1/2}(\mathbf{Q}_{1j} + i\mathbf{Q}_{2j})$ determine a spherical tensor of rank one. We first compute the matrix elements

$$(10.19) \qquad (\mathbf{Q}^{(s)}\Psi^{l_1}_{m_1}, \Psi^{l_2}_{m_2}), \qquad s = 1, 0, -1, \quad -l_h \le m_h \le l_h.$$

It is obvious that the matrix elements (10.18) can be determined immediately from (10.19). Since $\mathbf{D}^{(1)} \otimes \mathbf{D}^{(l_1)} \cong \mathbf{D}^{(l_1+1)} \oplus \mathbf{D}^{(l_1)} \oplus \mathbf{D}^{(l_1-1)}$ if $l_1 \ge 1$ and $\mathbf{D}^{(1)} \otimes \mathbf{D}^{(0)} \cong \mathbf{D}^{(1)}$, it follows from our above analysis that for $l_1 \ge 1$ the matrix elements are nonzero only if $l_2 = l_1 + 1, l_1$, or $l_1 - 1$, while for $l_1 = 0$ the matrix elements are zero unless $l_2 = 1$. An explicit expression for the matrix elements is given by (10.14).

If the system contains particles with half-integral spin we can form expressions (10.19) where the l_i take half-integral values. The above analysis is unchanged except for the special case $\mathbf{D}^{(1)} \otimes \mathbf{D}^{(1/2)} \cong \mathbf{D}^{(3/2)} \oplus \mathbf{D}^{(1/2)}$, which implies that for $l_1 = \frac{1}{2}$ the matrix elements are zero unless $l_2 = \frac{3}{2}$ or $\frac{1}{2}$.

Now suppose the group acting on $\mathcal{H}$ (no spins) is $O(3)$. Recall that the irred reps of $O(3)$ are $\mathbf{D}^{(l)}_{\pm}$, where the sign denotes **parity**, (6.5), (6.6). The $\mathbf{Q}^{(s)}$ transform like polar vectors under $O(3)$, hence like $\mathbf{D}^{(1)}_{-}$. It is easy to verify the CG series

$$(10.20) \qquad \mathbf{D}^{(1)}_{-} \otimes \mathbf{D}^{(l)}_{\pm} \cong \begin{cases} \mathbf{D}^{(l+1)}_{\mp} \oplus \mathbf{D}^{(l)}_{\mp} \oplus \mathbf{D}^{(l-1)}_{\mp}, & l \ge 1, \\ \mathbf{D}^{(1)}_{\mp}, & l = 0, \end{cases}$$

The selection rules for the matrix elements follow immediately from (10.20). Again the nonzero matrix elements are given explicitly by (10.14).

We see from these results that (10.19) is always zero if $\Psi^{l_1}_{m_1}$ and $\Psi^{l_2}_{m_2}$ have the same parity. An interesting special case of our analysis occurs for one-particle systems ($k = 1$). In this case $\Psi^{l_1}_{m_1}(\mathbf{x}) = j_{l_1}(r)Y^{m_1}_{l_1}(\theta, \varphi)$, where the $Y^{m_1}_{l_1}(\theta, \varphi)$ are spherical harmonics. Recall that $\{Y_l^m\}$ transforms according to $\mathbf{D}^{(l)}_{+}$ if l is even and $\mathbf{D}^{(l)}_{-}$ if l is odd. Thus $(\mathbf{Q}^{(s)}\Psi^{l_1}_{m_1}, \Psi^{l_2}_{m_2})$ is nonzero only if $l_2 = l_1 \pm 1$. Parity considerations have eliminated the possibility $l_2 = l_1$.

In case the system contains particles of half-integral spin we have to perform our analysis using the group $SU(2) \times \{E, I\}$ rather than $O(3)$, but this changes the above results in no essential manner.

In quantum theory the matrix elements (10.14) may have interpretations other than those given here. For example, expressions of the form (10.19) occur in the study of emission and absorption of light by atoms (Liubarskii [1]). In this case these expressions are related to the lowest-order (dipole) approximation of the transition probability from one state to another. Our results stating that only certain special matrix elements are nonzero are called **selection rules** in this theory. Similarly, the quadrapole approximation of quantum perturbation theory corresponds to the approximation of a set of operators by spherical tensors of rank two and use of the Wigner–Eckart theorem to simplify the matrix element computation.

7.11 Spinor Fields and Invariant Equations

The Euclidean group $E^+(3)$ frequently appears as a symmetry group in classical and quantum physics. Suppose for example that $\mathcal{H}$ is the Hilbert space of a k-particle system (Section 7.6). Then the operators $\mathbf{T}(\mathbf{a}, \mathbf{O})$ given by

$$(11.1)\qquad [\mathbf{T}(\mathbf{a}, \mathbf{O})\Psi](\mathbf{x}_1, \ldots, \mathbf{x}_k) = \Psi(\mathbf{O}^{-1}(\mathbf{x}_1 - \mathbf{a}), \ldots, \mathbf{O}^{-1}(\mathbf{x}_k - \mathbf{a})),$$
$$\mathbf{a} \in R_3, \qquad \mathbf{O} \in SO(3), \qquad \Psi \in \mathcal{H},$$

define a unitary rep of $E^+(3)$ on $\mathcal{H}$. Note that the restriction of $\mathbf{T}$ to $SO(3)$ yields the usual action of $SO(3)$ on $\mathcal{H}$, while the restriction of $\mathbf{T}$ to the translation subgroup R_3 yields

$$(11.2)\qquad [\mathbf{T}(\mathbf{a}, \mathbf{E})\Psi](\mathbf{x}_1, \ldots, \mathbf{x}_k) = \Psi(\mathbf{x}_1 - \mathbf{a}, \ldots, \mathbf{x}_k - \mathbf{a}).$$

If $E^+(3)$ is a symmetry group of the system then the $\mathbf{T}$-operators commute with the Hamiltonian $\mathbf{H}$:

$$(11.3)\qquad \mathbf{T}(\mathbf{a}, \mathbf{O})\mathbf{H} = \mathbf{H}\mathbf{T}(\mathbf{a}, \mathbf{O}).$$

For $\mathbf{a} = \boldsymbol{\theta}$ we have seen that (11.3) signifies the conservation of angular momentum. On the other hand, if we set $\mathbf{O} = \mathbf{E}$ in (11.3), differentiate both sides of the equation with respect to a_j, and set $\mathbf{a} = \boldsymbol{\theta}$ we find $\mathbf{P}_j\mathbf{H} = \mathbf{H}\mathbf{P}_j$, where, (10.18),

$$(11.4)\qquad \mathbf{P}_j = -i\Big(\sum_{h=1}^{k} \partial/\partial x_{jh}\Big), \qquad j = 1, 2, 3,$$

is a **linear momentum** operator. Thus, $E^+(3)$ symmetry of a system implies conservation of angular and linear momentum. Conversely, conservation of angular and linear momentum implies $E^+(3)$ symmetry. (In the standard quantum mechanics texts it is shown that conservation of linear momentum implies the Schrödinger wave functions can be factored into two parts. One part describes the motion of the center of mass as a free particle and the other describes the relative motion of the system with respect to the center of mass.)

If the system contains particles with spin the proper symmetry group is $\mathcal{E}^+(3)$, consisting of pairs $\{\mathbf{a}, A\}$, $\mathbf{a} \in R_3$, $A \in SU(2)$, such that

$$\{\mathbf{a}_1, A_1\}\{\mathbf{a}_2, A_2\} = \{\mathbf{a}_1 + R(A_1)\mathbf{a}_2, A_1A_2\}, \tag{11.5}$$

where $R(A_1) \in SO(3)$ is given by (1.20). Here, $\mathcal{E}^+(3)$ and $E^+(3)$ are six-dimensional locally isomorphic groups (they have isomorphic Lie algebras). The map

$$\{\mathbf{a}, A\} \longrightarrow \{\mathbf{a}, R(A)\}$$

is a homomorphism of $\mathcal{E}^+(3)$ onto $E^+(3)$ which covers each element of $E^+(3)$ exactly twice.

The elements of $\mathcal{H}$ are spinor-valued functions $\mathbf{\Psi} = \{\Psi_\mu(\mathbf{x}_1, \ldots, \mathbf{x}_k)\}$, $\mu = 1, \ldots, q$. (If there are several spin indices we combine them into one index of larger domain.) The action of $\mathcal{E}^+(3)$ on $\mathcal{H}$ is

$$\begin{aligned}[\mathbf{T}(\mathbf{a}, A)\mathbf{\Psi}]_\mu&(\mathbf{x}_1, \ldots, \mathbf{x}_k)\\ &= \sum_{\nu=1}^{q} T_{\mu\nu}(A)\Psi_\nu(R(A^{-1})(\mathbf{x}_1 - \mathbf{a}), \ldots, R(A^{-1})(\mathbf{x}_k - \mathbf{a})),\end{aligned} \tag{11.6}$$

where the matrices $T(A)$ define a unitary rep of $SU(2)$, not necessarily irred. It is straightforward to check that $\mathbf{T}$ is a unitary rep of $\mathcal{E}^+(3)$ with respect to the **inner product**

$$(\mathbf{\Psi}, \mathbf{\Phi}) = \int_{R_3{}^k} \sum_{\mu=1}^{q} \Psi_\mu(\mathbf{x}_1, \ldots, \mathbf{x}_k)\bar{\Phi}_\mu(\mathbf{x}_1, \ldots, \mathbf{x}_k)\, d\mathbf{x}.$$

As before, if the $\mathbf{T}(\mathbf{a}, A)$ commute with $\mathbf{H}$ then total angular momentum and linear momentum are conserved.

Although we have been led to expression (11.6) through Hilbert-space considerations, this expression makes sense independent of Hilbert space. In general any spinor-valued function which transforms under $\mathcal{E}^+(3)$ by (11.6) is called a **spinor field**. If the matrices $T(A)$ satisfy $T(A) = T(-A)$ then the operators $\mathbf{T}$ define a single-valued rep of $E^+(3)$. In this case the function $\mathbf{\Psi}$ is usually called a **tensor field**. Tensor fields abound in classical physics. For example the electromagnetic field $E_j(\mathbf{x})$, $j = 1, 2, 3$, transforms under $E^+(3)$ as a tensor field of rank one, i.e., the matrices $T(A)$ define a rep equivalent to $\mathbf{D}^{(1)}$. Similarly, magnetic fields, elasticity tensors, current tensors, and moment-of-intertia tensors all transform as tensor fields under $E^+(3)$. True spinor fields occur primarily in quantum mechanics and relativistic physics. The best known example is the Dirac electron field where $q = 4$ and $T(A)$ defines a rep equivalent to $\mathbf{D}^{(1/2)} \oplus \mathbf{D}^{(1/2)}$.

Let $\Psi_\mu(\mathbf{x}, t)$ be a spinor field transforming according to (11.6) with $k = 1$. Suppose $\Psi_\mu(\mathbf{x}, t)$ describes some physical quantity which is a solution of a system of q linear differential equations

$$\sum_{jshp} B^{jshp}(\mathbf{x}, t)\frac{\partial^{j+s+h+p}\mathbf{\Psi}(\mathbf{x}, t)}{\partial x^j\,\partial y^s\,\partial z^h\,\partial t^p} = \boldsymbol{\theta}, \tag{11.7}$$

where the B^{jshp} are $q \times q$ matrix functions, $\mathbf{\Psi}(\mathbf{x}, t)$ is a $1 \times q$ column vector, and $\boldsymbol{\theta}$ is the zero vector. Assuming the isotropy of space–time we see that Eq. (11.7) can be physically meaningful only if they assume the same form in every cartesian coordinate system: If we replace $\mathbf{x}$ by $\mathbf{x}' = R(A)\mathbf{x} + \mathbf{a}$, t by $t' = t + c$, and $\Psi_\mu(\mathbf{x}, t)$ by $\Psi_\mu'(\mathbf{x}', t') = \sum T_{\mu\nu}(A)\Psi_\nu(\mathbf{x}, t)$ in (11.7), then the resulting system of equations should be equivalent to (11.7), i.e., the primed equations should be linear combinations of the unprimed equations and conversely. We shall classify all such Euclidean invariant equations (under certain restrictions). The dynamical equations of *any* physical theory which admits $\mathcal{E}^+(3)$ as a symmetry group, via the rep (11.6), will be found in our classification. Our analysis will provide a group-theoretic framework within which all Euclidean invariant physical theories can be described and compared.

Note first that Eq. (11.7) are invariant under all translations in space and time if and only if the matrices B^{jshp} are independent of $\mathbf{x}$ and t. Now we dispense with translation invariance and restrict our attention to invariance under the operators $\mathbf{T}(A) = \mathbf{T}(\boldsymbol{\theta}, A)$, which form a rep of $SU(2)$. Furthermore, we can eliminate dependence on t in (11.7) by considering only solutions of the form $\mathbf{\Psi}(\mathbf{x}, t) = \mathbf{\Psi}(\mathbf{x})e^{i\omega t}$. Then $\partial/\partial t$ is replaced by $i\omega$. (This amounts to taking the Fourier transform in t.)

We can always write (11.7) as a system of first-order differential equations by introducing new components $\Psi_\mu(\mathbf{x})$, $\mu > q$. This will be shown later when we consider specific examples. Thus, we can reduce (11.7) to a system of l equations

$$\left(B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z} + C\right)\mathbf{\Psi}(\mathbf{x}) = \boldsymbol{\theta}, \tag{11.8}$$

where B_1, B_2, B_3, C are constant $l \times r$ matrices, $\mathbf{\Psi}(\mathbf{x}) = (\Psi_\mu(\mathbf{x}))$ is a $1 \times r$ column vector and the action of $SU(2)$ on $\mathbf{\Psi}(\mathbf{x})$ is

$$[\mathbf{T}(A)\mathbf{\Psi}]_\mu(\mathbf{x}) = \sum_{\nu=1}^{r} S_{\mu\nu}(A)\Psi_\nu(R(A^{-1})\mathbf{x}), \qquad \mu = 1, \ldots, r. \tag{11.9}$$

Here $r \geq q$ and $S(A)$ is a matrix rep of $SU(2)$.

For the present we assume C is a nonsingular $r \times r$ matrix. Then multiplying (11.8) on the left by C^{-1} we see that this system of equations is equivalent to a system of the form

$$\left(L_1\frac{\partial}{\partial x} + L_2\frac{\partial}{\partial y} + L_3\frac{\partial}{\partial z}\right)\mathbf{\Psi}(\mathbf{x}) = \kappa\mathbf{\Psi}(\mathbf{x}), \tag{11.10}$$

where the L_j are $r \times r$ matrices and $\kappa \neq 0$ is a constant. (We could take $\kappa = -1$ but it is convenient to leave it arbitrary.)

By passing to a new basis if necessary we can assume that the matrices $S(A)$ take the form

$$(11.11)\qquad S(A) = \begin{pmatrix} \overbrace{T^{(0)}(A) \quad \cdot \quad T^{(0)}(A)}^{\alpha_0} & & & Z \\ & \cdot & & \\ & & \overbrace{T^{(u)}(A) \quad \cdot \quad T^{(u)}(A)}^{\alpha_u} & \\ Z & & & \cdot \end{pmatrix},$$

where $T^{(u)}(A)$ is a matrix realization of $\mathbf{D}^{(u)}$ and α_u is the multiplicity of $\mathbf{D}^{(u)}$ in $S(A)$. In other words we have decomposed the action S of $SU(2)$ on the components of $\mathbf{\Psi}$ into a direct sum of irred reps. In this new basis we relabel the components of $\mathbf{\Psi}$ as Ψ^m_{un}, the component of the mth canonical basis vector in the nth occurrence of $\mathbf{D}^{(u)}$ in (11.11). Here $-u \leq m \leq u$ and $1 \leq n \leq \alpha_u$. In terms of the new basis, the system of equations still takes the form (11.10).

We can express the partial derivatives on the left-hand side of our equations as linear combinations of $\partial_1, \partial_0, \partial_{-1}$, (10.12), which form a canonical basis for $\mathbf{D}^{(1)}$. Thus, the left-hand side is a linear combination of terms $\partial_1 \Psi^p_{vn}, \partial_0 \Psi^p_{vn}, \partial_{-1} \Psi^p_{vn}$. For fixed v and n, and p ranging over $-v, -v+1, \ldots, v$ these $3(2v+1)$ quantities transform according to $\mathbf{D}^{(1)} \otimes \mathbf{D}^{(v)} \cong \mathbf{D}^{(v+1)} \oplus \mathbf{D}^{(v)} \oplus \mathbf{D}^{(v-1)}$. Thus the new basis functions

$$h^{m'}_{u'vn} = \sum_{pq} C(1, j; v, p \mid u', m')\, \partial_j \Psi^p_{vn}$$

$$(11.12)\qquad u' = \begin{cases} v+1, v, v-1, & \text{if } \quad v \geq 1, \\ \frac{3}{2}, \frac{1}{2}, & \text{if } \quad v = \frac{1}{2}, \\ 1, & \text{if } \quad v = 0, \end{cases} \quad -u' \leq m' \leq u',$$

transform irreducibly under $\mathbf{D}^{(u')}$. Since the CG coefficients are unitary we can express each of the terms $\partial_j \Psi^p_{vn'}$ on the left-hand side in (11.10) as a linear combination of the $h^{m'}_{u'vn'}$ and rewrite (11.10) as

$$(11.13)\qquad \sum_{vn'm'u'} B^{um}_{m'u'vn} h^{m'}_{u'vn'} = \kappa \Psi^m_{un}.$$

Consider the subsystem of $2u+1$ equations (11.13) for which u and n are fixed, and $-u \leq m \leq u$. Now $\Psi'^m_{un}(\mathbf{x}') = \sum T^u_{km}(A)\Psi^k_{un}(\mathbf{x})$ and $h'^{m'}_{u'vn'}(\mathbf{x}') = \sum T^{u'}_{jm'}(A) h^j_{u'vn'}(\mathbf{x})$ so this subsystem will be invariant under $SU(2)$ if and only if the left-hand side of the subsystem transforms like a canonical basis for $\mathbf{D}^{(u)}$. From (11.12) we see that any invariant system must take the form

$$\sum_{u'n'} B^{un}_{u'n'} \sum_{pj} C(1, j; u' \, m' \mid u, m) \partial_j \Psi^{m'}_{u'n'} = \kappa \Psi^m_{un}, \tag{11.14}$$

where the sum is taken over $u' = u + 1, u, u - 1$ and $n' = 1, \ldots, \alpha_{u'}$. The constants $B^{un}_{u'n'}$ are completely arbitrary and there is one equation for each component Ψ^m_{un} of $\mathbf{\Psi}$. Note that integral values of u are never coupled with half-integral values of u in (11.14). If both values occur, the system breaks up into two independent subsystems, one coupling integral and the other coupling half-integral values.

The case where the matrix C is singular or not square is more complicated. Suppose $C = Z$. Equations (11.14) with $\kappa = 0$ clearly fall under this case and in general all invariant equations take roughly this form. However, it is not easy to decide if two systems of equations are equivalent, i.e., there is no simple canonical form for such equations. For $\kappa \neq 0$ this difficulty does not occur: Two systems of equations for the Ψ^m_{un} are equivalent if and only if the constants $B^{un}_{u'n'}$ agree for the two systems.

If C is a singular matrix or is not square then the system of equations can be put in the general form (11.14) where $\kappa \neq 0$ for some equations and $\kappa = 0$ for others. The number of equations is not necessarily equal to the number of components of $\mathbf{\Psi}$ and there is no simple canonical form. Fortunately, in the equations of mathematical physics it is usually true that C is nonsingular.

Our analysis of invariant equations follows Liubarskii [1]. There is another approach to this theory, due to Gel'fand and Shapiro, which is based on Lie algebras. The Lie-algebraic method is much more complicated than that given above but it extends rather easily to the case where the matrices L_1, L_2, L_3 in (11.10) act on infinite-dimensional spaces (Gel'fand et al. [1], Naimark [2]).

For equations invariant under the full orthogonal group $O(3)$ these results have to be slightly modified. The components of $\mathbf{\Psi}$ are labeled $\Psi^m_{u\pm,n}$ corresponding to the reps $\mathbf{D}^{(u)}_{\pm}$, u an integer. The differential operators $\partial_{\pm 1}, \partial_0$ form a canonical basis for $\mathbf{D}^{(1)}_{-}$. It follows from the identities

$$\mathbf{D}^{(1)}_{-} \otimes \mathbf{D}^{(u)}_{\pm} \cong \mathbf{D}^{(u+1)}_{\mp} \oplus \mathbf{D}^{(u)}_{\mp} \oplus \mathbf{D}^{(u-1)}_{\mp} \tag{11.15}$$

that the invariant equations take the form (11.14) except that the components of $\mathbf{\Psi}$ on the left- and right-hand sides of these equations have opposite parity.

We consider some examples. The simplest $\mathcal{E}^+(3)$-invariant equations are those in which the components of $\mathbf{\Psi}$ transform according to the single irred rep $\mathbf{D}^{(u)}$. Denoting the $2u + 1$ components of $\mathbf{\Psi}$ by $\Psi_u{}^m$ we obtain the system of equations

$$a \sum_{j+m'=m} C(1, j; u, m' \mid u, m) \, \partial_j \Psi_u{}^{m'} = \kappa \Psi_u{}^m, \qquad -u \leq m \leq u. \tag{11.16}$$

There is a single arbitrary constant a. This system is not $E(3)$-invariant since the parity of the left-hand side is opposite that on the right. For $E(3)$-invariant equations the action of $O(3)$ on the indices of $\mathbf{\Psi}$ must be reducible.

Consider the manifestly $E(3)$-invariant equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V(\mathbf{x})=\kappa V(\mathbf{x}), \tag{11.17}$$

where $V(\mathbf{x})$ is a scalar $\mathbf{D}_+^{(0)}$. We shall write (11.17) as a system of first-order equations by introducing three new components $V_j(\mathbf{x})=\partial_j V(\mathbf{x}), j=\pm 1, 0$. Clearly the $V_j(\mathbf{x})$ form a canonical basis for $\mathbf{D}_-^{(1)}$. The system (11.17) is equivalent to

$$-\partial_1 V_{-1}+\partial_0 V_0-\partial_{-1}V_1=\kappa V,\qquad \partial_j V=V_j,\qquad j=\pm 1, 0. \tag{11.18}$$

Without loss of generality we can assume $\kappa=1$. The indices of the column vector (V, V_1, V_0, V_{-1}) transform according to $\mathbf{D}_+^{(0)}\oplus\mathbf{D}_-^{(1)}$. By our theory the most general $E(3)$-invariant system with these transformation properties is

$$a\sum_{j=-1}^{1} C(1,j;1,-j\,|\,0,0)\,\partial_j V_{-j}=V,\qquad bC(1,l;0,0\,|\,1,l)\,\partial_l V=V_l, \tag{11.19}$$
$$l=0,\pm 1.$$

It follows from the table (7.28) that (11.19) is identical with (11.18) provided $a=-\sqrt{3}, b=1$.

Another important example is given by two of Maxwell's equations for an electromagnetic field in a vacuum:

$$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}=\mathbf{0},\qquad \nabla\times\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\mathbf{0}. \tag{11.20}$$

Here $\mathbf{E}(\mathbf{x},t)=(E_x, E_y, E_z)$ is a vector field transforming according to the rep $\mathbf{D}_-^{(1)}$ of $O(3)$ and $\mathbf{H}(\mathbf{x},t)$ is a vector field transforming according to $\mathbf{D}_+^{(1)}$. We are using Gaussian units. If we consider solutions of frequency ω, $\mathbf{E}(\mathbf{x},t)=\mathbf{E}(x)e^{i\omega t}$, $\mathbf{H}(\mathbf{x},t)=\mathbf{H}(\mathbf{x})e^{i\omega t}$, then the equations become

$$(ic/\omega)\,\nabla\times\mathbf{E}=\mathbf{H},\qquad -(ic/\omega)\,\nabla\times\mathbf{H}=\mathbf{E}. \tag{11.21}$$

Expressed in terms of canonical basis vectors $\partial_{\pm 1}, \partial_0, E_{\pm 1}=2^{-1/2}(\pm E_x - iE_y)$, $E_0=E_z$, $H_{\pm 1}=2^{-1/2}(\pm H_x - iH_y)$, and $H_0=H_z$, (11.21) reads

$$\begin{aligned}
&(c/\omega)(\partial_0 E_1-\partial_1 E_0)=H_1, && -(c/\omega)(\partial_0 H_1-\partial_1 H_0)=E_1,\\
&(c/\omega)(\partial_{-1}E_1-\partial_1 E_{-1})=H_0, && -(c/\omega)(\partial_{-1}H_1-\partial_1 H_{-1})=E_0,\\
&(c/\omega)(\partial_{-1}E_0-\partial_0 E_{-1})=H_{-1}, && -(c/\omega)(\partial_{-1}H_0-\partial_0 H_{-1})=E_{-1}.
\end{aligned} \tag{11.22}$$

By our theory the most general $O(3)$-invariant system of equations with C nonsingular and indices transforming according to $\mathbf{D}_-^{(1)}\oplus\mathbf{D}_+^{(1)}$ is

$$\begin{aligned}
&a\sum_{j=-1}^{1} C(1,j;1,m-j\,|\,1,m)\,\partial_j E_{m-j}=H_m,\\
&b\sum_{j=-1}^{1} C(1,j;1,m-j\,|\,1,m)\,\partial_j H_{m-j}=E_m,\qquad m=1,0,-1.
\end{aligned} \tag{11.23}$$

It follows from (7.28) that (11.22) is the special case of (11.23) such that $a = -\sqrt{2}\, c/\omega$, $b = \sqrt{2}\, c/\omega$. The other two Maxwell equations $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{H} = 0$, correspond to the case where C is singular.

Problems

7.1 Compute the Clebsch–Gordan coefficients for all tensor products of irred reps of C_{3v}.

7.2 Determine how the energy levels of an $SO(3)$-symmetric quantum mechanical system split under the influence of a perturbation with D_6 symmetry.

7.3 Compute the level splitting of an $O(3)$-symmetric system under a perturbation with D_{3d} symmetry.

7.4 Prove identity (7.16).

7.5 Determine the double-valued irred reps of the point groups C_3 and D_3.

7.6 Compute the double-valued irred reps of D_6.

7.7 Compute the splitting of levels transforming according to double-valued irred reps of $SO(3)$ under a perturbation with D_3 symmetry.

7.8 Show that the prescription $R(\tau, \mathbf{u}) = \exp(\tau \mathbf{u} \cdot \mathbf{L}) \in SO(3)$ defines a system of coordinates on $SO(3)$ and determine the geometrical significance of these coordinates. Here $\mathbf{u}$ is a unit vector and $\mathbf{u} \cdot \mathbf{L} = u_1 L_1 + u_2 L_2 + u_3 L_3$. Compute the invariant measure in $(\tau, \mathbf{u})$ coordinates and verify explicitly that the simple characters of $SO(3)$ form an orthogonal set.

7.9 Consider a spherical tensor of rank one which transforms as a polar vector under $O(3)$. Determine the selection rules for matrix elements of the tensor between states transforming as irred reps of D_{4h}.

7.10 Repeat the previous problem for a tensor transforming as an axial vector under $O(3)$.