Linear (Airy) wave theory

We begin with the incompressible Navier-Stokes equations –

$$\frac{D\vec{u}}{Dt} = \vec{F} - \frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u} \tag{1}$$

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

where $D\vec{u}/Dt$ is the **total derivative**, or **material derivative**.

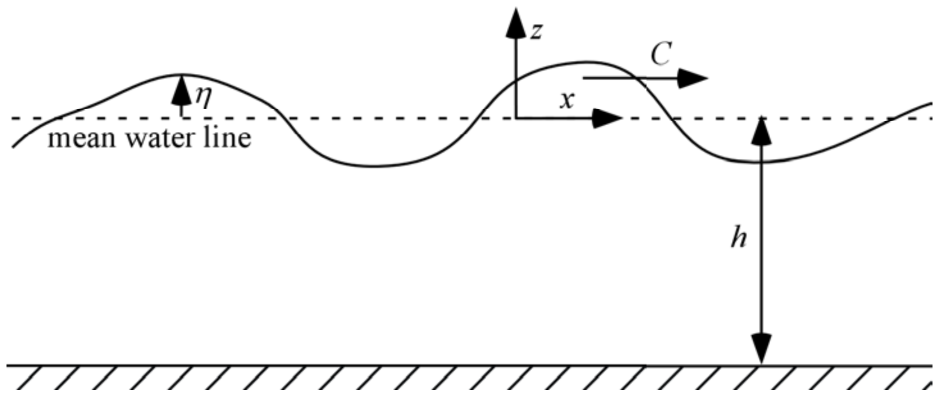
$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \tag{3}$$

Assuming inviscid, irrotational, two-dimensional flow and no other motions (i.e., no currents) a sinusoidal wave field produces the potential (ideal) flow solution to the flow and pressure field –

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{4}$$

where ϕ is the velocity potential and ψ is the stream function. These quantities also have the property of being orthogonal everywhere –

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial z}, \quad \frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial z} \tag{5}$$



Like most fluid-mechanical problems, the difficulty lies in the imposition of boundary conditions. The boundary conditions take the form of three equations. We define η to be the perturbation from mean sea level.

The first boundary condition is the **kinematic condition**, which states that a parcel of fluid at the surface remains at the surface (Currie).

$$-\frac{\partial\eta}{\partial t} - u\frac{\partial\eta}{\partial x} + w = 0 \tag{6}$$

Converting the velocity components u and w into velocity potential leaves

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial z} \qquad \text{at } z = \eta \qquad (7)$$

The second boundary condition is that the pressure on the surface is constant (presumably in the case of the ocean – the atmospheric pressure). We will use the Bernoulli equation to find

$$\frac{\partial \phi}{\partial t} + \frac{\Theta}{\rho} + \frac{1}{2}\nabla \phi \cdot \nabla \phi + g\eta = F(t) \qquad \text{at } z = \eta \qquad (8)$$

where the pressure $p = \Theta(x,t)$ on $z = \eta$ and ρ is the fluid density. Finally, there is no flow through the bottom boundary, which means

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = -h \qquad (9)$$

Small-amplitude approximation

We intend to examine small-amplitude plane waves. That is, waves that are small with respect to the depth of the water and have small water surface angles. They are sometimes called **short** or **Stokes waves**. They are also two-dimensional.

Stokes (1847, 1880) assumed that the solution of Equation (4) with boundary conditions defined by Equations (7-9) (i.e., the wave-field defined above) could be expressed by a Fourier series. Linear (Airy) wave theory uses only the first term of this series. Without dwelling on details, that term is

$$\phi(x,z,t)=\frac{gH}{2\omega}\frac{\cosh k(z+h)}{\cosh kh}\sin(kx-\omega t) \tag{10}$$

This result requires the water surface elevation to be in the form of

$$\eta(x,t)=\frac{H}{2}\sin\frac{2\pi}{L}(kx-\omega t) \tag{11}$$

where $k=2\pi/L$ is the **wavenumber**, $\omega=2\pi/T$ is **radian frequency** (T is the period). The **celerity** (or wave speed) $C=\omega/k$ has the form

$$C^2=\frac{gL}{2\pi}\tanh\frac{2\pi h}{L} \tag{12}$$

or alternatively…

$$\omega^2=gk\tanh kh \tag{13}$$

Equation (13) is often called the dispersion relation (discussed below).

Assumptions in the derivation:

- incompressibility

- irrotational flow
- inviscid (viscous/drag/friction terms are negligible)
- two-dimensional wave field
- no ambient velocity (i.e., no current)
- small-amplitude waves

It is worthwhile to examine the behavior of the hyperbolic functions in Equations (10-13) in the limit of certain water depths to obtain simpler relationships.

Deep-water waves

In the limit of $x \to \infty$, $\tanh x \to 1$. In this case, we can solve explicitly for the deep-water wavelength L_∞ using (12). For Airy waves –

$$L_\infty = \frac{gT^2}{2\pi} \tag{14}$$

In actuality, $\tanh x > 3$ is ~ unity, so the assumption of deep-water is generally appropriate for $h/L_\infty > 0.5$. It follows that

$$C_\infty = \sqrt{g/k} \tag{15}$$

and the dispersion relation becomes

$$\omega = \sqrt{gk} \tag{16}$$

Shallow-water waves

In the other limit, when $x \to 0$, $\tanh x \to x$. This yields the shallow-water wavelength L_s

$$L_s = T\sqrt{gh} \tag{17}$$

The shallow-water celerity C_s yields the familiar result –

$$C_s = \sqrt{gh} \tag{18}$$

which is not dependent on wave properties, but merely the total flow depth.

Intermediate depths

When $0.25 > h/L_\infty > 0.05$, you need to use Equation (12). As you can see, Equation (12) is implicit in *L*. The problem is normally solved graphically. An alternative is given by the *Shore Protection Manual* (1984) suggests

$$L = L_\infty \sqrt{\tanh\left(2\pi h/L_\infty\right)} \tag{19}$$

Other properties of waves

Dispersion

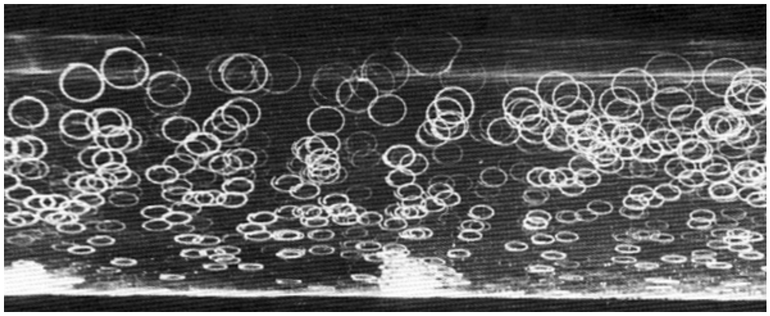
Dispersion refers to the sorting of waves of different sizes with time. If wave speeds are dependent on the wavenumber (e.g., deep-water Airy waves), the wave-field is said to be **dispersive**. If the wave speed is independent of wavenumber (e.g., shallow-water Airy waves), the wave-field is **non-dispersive**.

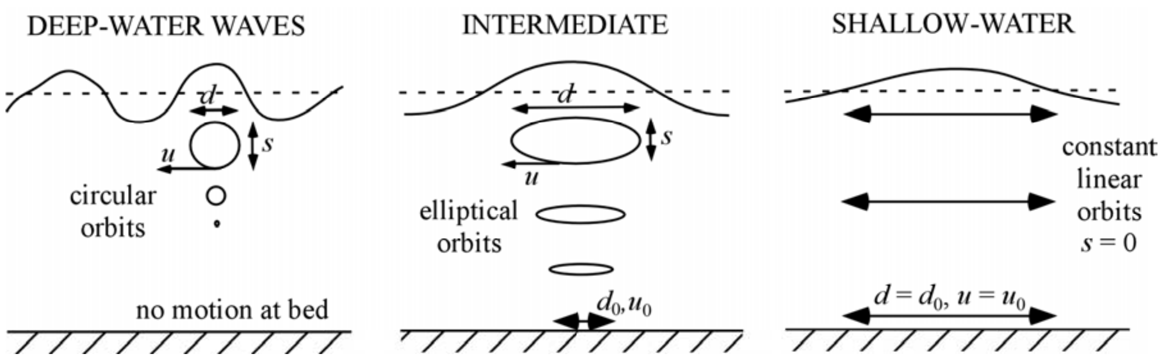
One result of dispersion in deep-water waves is **swell**. Dispersion explains why swell can be so monochromatic (possessing a single wavelength) and so sinusoidal. Smaller wavelengths are dissipated out at sea and larger wavelengths remain and segregate with distance from their source.

Raglan beach, New Zealand

Wave orbital characteristics

The flow paths of parcels within the fluid column have the general form shown below (photograph from Van Dyke's *Album of Fluid Motion*, diagram from Komar).





Considering that Equation (10) defines the paths of all fluid parcels, we can derive an expression for both the diameter and the velocity of the orbital motions.

In deep water, there is no motion at the bed. The diameter of the orbital motion (which is circular) within the water column is

$$d = s = H\exp(kz) = H\exp(2\pi z/L_\infty) \qquad (20a)$$

The orbital velocity also varies exponentially in the vertical

$$u = \frac{\pi H}{T}\exp(kz)\cos(kx - \omega t) \qquad (20b)$$

In shallow water, the motions are not elliptical, but linear. The orbital diameter and velocity (each of which only has a horizontal component) become

$$d = d_0 = H/kh; \qquad u_0 = \frac{H}{2}\sqrt{\frac{g}{h}}\cos(kx - \omega t) \tag{21a,b}$$

Finally, in intermediate depths, the hyperbolic functions remain to find orbital diameters and velocities at the bed –

$$d_0 = H/\sinh kh; \qquad u_0 = \frac{\pi H}{T\sinh kh}\cos(kx - \omega t) \tag{22a,b}$$

Wave energy

In addition to the motion of fluid parcels, Airy theory allows us a simple way to express the energy within a train of gravity waves. You can integrate the potential and kinetic energy for a single wavelength and find that energy per unit length E

$$E = \rho g H^2/8 \tag{23}$$

E is sometimes called the **energy density**.

As we will see in upcoming lectures, the energy density leads to the calculation of the **energy flux** P, a particularly powerful quantity. The energy flux is typically defined

$$P = EC_g \tag{24}$$

where C_g is the **group velocity**.

Group velocity

Contrary to simple intuition, waves of a particular celerity do not travel collectively at that speed, if they are dispersive.

To identify the relationship $n = C_g / C$, we notice

$$\partial\alpha/\partial x = -k, \quad \partial\alpha/\partial t = \omega \tag{25a,b}$$

where α is the 'local phase' of the waves comprising the group.

It follows that

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{26}$$

Lighthill calls (26) the "equation of continuity for phase." Knowing that $\omega = kC$, we find

$$\frac{\partial k}{\partial t} + C_g \frac{\partial k}{\partial x} = 0 \tag{27}$$

where $C_g = d\omega/dk$. Now we simply differentiate the dispersion equation (13) to find

$$2\omega\frac{\partial\omega}{\partial k} = g\left[\tanh kh + \frac{kh}{\cosh^2 kh}\right] \tag{26}$$

which is equivalent to

$$n = \frac{1}{2}\left[1 + \frac{2kh}{\sinh(2kh)}\right] \tag{27}$$

It turns out that for shallow waves, $n \to 1$ and for deep waves $n \to 1/2$.