Eady's Model

BAROCLINIC INSTABILITY

from

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1. Introduction

One of the notable features of the motions of both atmosphere and oceans is the presence of fluctuations at time scales *not* directly related to the periodicities of solar heating. Further, the observations of the fluctuations show that they occur erratically in time. This suggests that these variations are amplifications of the small scale disturbances which are inevitably present in any real system. More precisely, if the state of the flow is unstable with respect to small fluctuations, the latter may grow in amplitude with time. Remarkably, it turns out that the study of instability associated with highly simplified atmospheric flows, explains to a great extent observed weather waves in the atmosphere. The time and space scales of modes of instability demonstrated by *Eady* (1949) and *Charney* (1947) are remarkably close to observations.

Most stability theories idealize the initial state of the flow as zonally uniform. Study of such a state of flow reveals in the most straightforward way the underlying mechanisms behind the instability process. If vertical shear is present in such a flow, it implies the presence of horizontal temperature gradients, and therefore the presence of available potential energy. This energy may be released and transferred to the small fluctuations by a process known as *baroclinic instability*. As the instability grows, the center of mass of the fluid is lowered. In growing waves in the atmosphere, cold air moving downwards and equatorwards displaces the warmer air moving polewards and upwards.

The most important feature of baroclinic instability is that it exists even in the situation of rapid rotation (small Rossby number) and strong stable stratification (large Richardson's number) typically observed in the atmosphere. A simple but elegant model demonstrating the baroclinic instability in its purest form was introduced by Eric *Eady* in *1949*. This paper discusses the basic mechanism behind baroclinic instability and Eady's model following *Pedlosky* (1987). Section 2 describes how the problem can be formulated for an arbitrary initial perturbation upon an initial basic state zonal flow. In section 3, Eady's model is presented and the nature of the structure of the growing unstable wave is estimated. Consequently, an estimate is made of the length scale of the unstable waves for the case of the atmosphere. Limitations of the Eady's model are listed in section 4. Finally, the conclusions drawn throughout the preceeding sections are summarized in section 5.

2. Formulation of the problem

As mentioned in section **1**, we can safely assume a zonally uniform initial flow with velocity described by $U_0(y,z)$, where U_0 is the non-dimensional velocity of order unity. A streamfunction Ψ describing the flow can be given by

$$U_0 = -\frac{\partial \Psi(y,z)}{\partial y} \tag{1}$$

In this section, we will formulate the stability problem for the basic state described by the above streamfunction, for the case of the atmosphere. Oceanographic stability problem can be considered as a special case of the atmospheric problem. The density field $\rho_s(z)$ can be considered as a constant and the atmospheric potential-temperature anomaly θ can be replaced by the negative of oceanic density anomaly.

A perturbation ϕ can be imposed over the initial state, such that the streamfunction can now be written as

$$\psi(x,y,z,t) = \Psi(y,z) + \phi(x,y,z,t) \tag{2}$$

Such a flow must satisfy the quasi geostrophic equation of motion given by

$$\left[\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right]\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial \psi}{\partial z}\right) + \beta y\right] = 0 \tag{3}$$

If we substitute (2) in (3), we obtain the following equation which is nonlinear in ϕ

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right) q + \frac{\partial \phi}{\partial x}\frac{\partial \Pi_0}{\partial y} + \left\{\frac{\partial \phi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial q}{\partial x}\right\} = 0 \tag{4}$$

where q and Π_0 are the perturbation and ambient potential vorticities respectively, given by

$$q = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial \phi}{\partial z}\right) \tag{5}$$

and

$$\Pi_0 = \beta y + \frac{\partial^2 \Psi}{\partial y^2} - \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial \Psi}{\partial z}\right) \tag{6}$$

with the gradient of ambient potential vorticity

$$\frac{\partial \Pi_0}{\partial y} = \beta - \frac{\partial^2 U_0}{\partial y^2} - \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial U_0}{\partial z}\right) \tag{7}$$

As the non-linearity in (4) makes it impractical to solve, we have to assume that, at least initially, the amplitude of the perturbations is sufficiently small ($\phi << 1$), such that the terms proportional to ϕ^2 and ϕq are neglected, leading to the linear stability problem

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right) q + \frac{\partial \phi}{\partial x}\frac{\partial \Pi_0}{\partial y} = 0 \tag{8}$$

We need boundary conditions to complete the problem for ϕ. The lateral boundary conditions can be assumed to be rigid walls at $y = \pm 1$, to effectively isolate the region from its surroundings. This will make sure that the source of the instability must lie within the region under consideration. For v_0 to vanish at these boundaries, it implies

$$\frac{\partial \phi}{\partial x} = 0, \qquad y = \pm 1 \tag{9}$$

The upper and lower boundary conditions can be assumed to be

$$w = 0, \qquad z = 0,1 \tag{10}$$

A normal mode approach can be used to solve the above perturbation problem by assuming the solution to be of the form

$$\phi(x, y, z, t) = \mathrm{Re}\{\Phi(y, z)\exp[ik(x - ct)]\} \tag{11}$$

If we substitute this in the linearized equation (8), we obtain the normal mode problem for Φ, viz.

$$(U_0 - c)\left[\frac{1}{\rho_s}\frac{\partial}{\partial z}\frac{\rho_s}{S}\frac{\partial \Phi}{\partial z} + \frac{\partial^2 \Phi}{\partial y^2} - k^2 \Phi\right] + \Phi \frac{\partial \Pi_0}{\partial y} = 0 \tag{12}$$

along with the boundary conditions (10). The latter, in the absence of topography and friction (Pedlosky 1987; pp 524) can be shown to satisfy

$$(U_0 - c)\frac{\partial \Phi}{\partial z} - \frac{\partial U_0}{\partial z}\Phi = 0, \qquad z = 0,1 \tag{13}$$

3. Eady's Model

To isolate the effects of vertical shear, an initial flow independent of y is assumed. In a non dimensional sense, this can be expressed by the relation

$$U_0 = z \tag{14}$$

Using the thermal wind relationship $(\partial U_0/\partial z) = -(\partial \Theta_0/\partial y)$ we recognize that the potential temperature gradient is constant:

$$\frac{\partial \Theta_0}{\partial y} = -1 \tag{15}$$

Further, in Eady's model the stratification parameter S is taken to be constant; Boussinesq approximation is used, i.e. ρ_s is taken as constant; the effect due to sphericity of earth is purposely left out by assuming

$$\beta = 0 \tag{16}$$

Using (14) and (7), we find that the gradient of ambient potential vorticity vanishes, i.e.

$$\frac{\partial \Pi_0}{\partial y} = 0 \tag{17}$$

so that the normal mode problem developed in (12) becomes

$$(z-c)\left\{S^{-1}\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial y^2} - k^2\Phi\right\} = 0 \tag{18}$$

along-with specialized boundary conditions obtained using $U_0 = z$ (14) in (13). These boundary conditions now become

$$\begin{aligned} -c\frac{\partial \Phi}{\partial z} - \Phi = 0, \qquad z = 0 \\ (1-c)\frac{\partial \Phi}{\partial z} - \Phi = 0, \qquad z = 1 \end{aligned} \tag{19}$$

Since solutions must vanish at $y = \pm 1$, it suggests that we can sought solutions of the form

$$\Phi(y,z) = A(z)\cos l_n y \tag{20}$$

where

$$l_n = \left(n + \frac{1}{2}\right)\pi, \qquad n = 0,1,2,... \tag{21}$$

Substituting this form of the solution (18) in the normal mode problem (18) gives

$$(z-c)\left[\frac{d^2 A}{dz^2} - \mu^2 A\right] = 0 \tag{22}$$

where

$$\mu^2 = \left(k^2 + l_n{}^2\right)S \tag{23}$$

with boundary conditions

$$\begin{aligned} c\frac{dA}{dz} + A = 0, \qquad z = 0 \\ (c-1)\frac{dA}{dz} + A = 0, \quad z = 1 \end{aligned} \tag{24}$$

We seek the functional form of $A(z)$ for every value of k and n. In general c in (22) will be a complex number. Let us first consider the case for non-singular solutions for (22) such that

$$\frac{d^2 A}{dz^2} - \mu^2 A = 0 \tag{25}$$

This differential equation has the general solution

$$A(z) = a\cosh \mu z + b\sinh \mu z \tag{26}$$

where a and b are arbitrary constants. Boundary conditions (24) must be used to relate them, i.e.

$$a + \mu c b = 0 \tag{27}$$

$$a[(c-1)\mu\sinh \mu + \cosh \mu] + b[(c-1)\mu\cosh \mu + \sinh \mu] = 0 \tag{28}$$

Nontrivial solutions for a and b imply determinant of coefficients a and b vanishes, which yields a quadratic equation in c

$$c^2 - c + \frac{1}{\mu}\left(\coth \mu - \frac{1}{\mu}\right) = 0 \tag{29}$$

with the solution

$$c = \frac{1}{2} \pm \frac{1}{\mu}\left[\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right]^{1/2} \tag{30}$$

Since $\mu/2 \geq \tanh(\mu/2)$ for all values of μ, roots are real if $\mu/2 > \coth(\mu/2)$ and complex if otherwise. The critical value μ_c such that $\mu_c/2 = \coth(\mu_c/2)$ numerically equals $\mu_c = 2.3994$

If $\mu > \mu_c$, for each k and n we get two real roots for phase speed (fig. **1**). As μ approaches μ_c from above, the phase speeds for two roots coalesce. For very large wave numbers (23) implies that μ will be very large. So one of the roots in (30) goes to zero (i.e. equal to the velocity at the lower boundary (14)) and the other root goes to 1 (i.e. equal to the velocity at upper boundary).

However, if $\mu < \mu_c$, we get two complex roots from (30) which are complex conjugates with the real part $c_r = 0.5$, which is actually equal to the mean velocity of the basic flow. (11) shows that the complex part of c can give rise to a growing mode with growth rate kc_i.

The condition $\mu < \mu_c$ also implies a condition on the stability parameter S using (23) such that

$$S < \frac{\mu_c^{\ 2}}{l_n^{\ 2}} = 4\frac{\mu_c^{\ 2}}{\pi^2(2n+1)^2} = \frac{2.333}{(2n+1)^2} \tag{31}$$

Thus $n = 0$ or the least wiggly mode is the most unstable mode since even relatively higher values of S will lead to growing modes. Fig. **2** shows the growth rate as a function of k for

$S = 0.25$ for the most unstable mode $n = 0$. Although c_i is largest as μ approaches zero (fig **1**) i.e. for longer waves (high k), the growth rate kc_i in fig. 2 shows a maximum at

$$k_m = 3.1277 \tag{32}$$

and as expected vanishes for $\mu > \mu_c$. The corresponding wavelength can be obtained using $\lambda_* = (2\pi/k_m)L$ where L depends upon S through

$$S = \frac{N_s{}^2 D^2}{f_0{}^2 L^2} = \frac{L_D{}^2}{L^2} \tag{33}$$

where L_D is the Rossby radius of deformation ($L_D = N_s D/f$). For the atmosphere, if L_D is 10^3 km, we obtain $\lambda_* \approx 4000$ km. In other words, half the distance from a crest to trough is $O(L_D)$ which is in excellent agreement with the observed scale of mid latitude synoptic disturbances (baroclinic waves) in the atmosphere. This immediately validates the mechanism of baroclinic instability for explaining the existence of observed transient long waves in the atmosphere.

Since c is known, we can obtain the structure of the unstable wave by calculating b in terms of a. Notice that we are unable to predict the amplitude of the disturbance from the linear theory. However, the form of $A(z)$ (from (26)) and thus Φ (from (20)) and can be calculated. Fig **2**(b) shows the amplitude and phase angle for the most unstable wave. Some distinct features can be noted

- Note that the phase angle tilts westward, i.e. against the current.
- The lower level wave leads the upper level wave by nearly 90^o.

- The amplitude is nearly symmetric about its minimum value attained at the $z = 0.5$.

Meridional heat flux for a perturbation field ϕ is given by $\rho_s \overline{(\partial\phi/\partial x)(\partial\phi/\partial z)}$. It turns out that the increase of phase angle with height (as shown in fig **2**(b)) leads to northward value of heat flux.

Recall that the above analysis is true only for done for non-singular solutions. However, if c is real, it is possible that (26) holds true except for $z = c$. Since $0 \le z \le 1$, and a real value of c lies anywhere in the range $(0,1)$, the appropriate solution must be achieved through

$$\frac{d^2 A}{dz^2} - \mu^2 A = B\delta(z - c) \tag{34}$$

where B is any constant and δ is the Dirac delta function. There are an infinite number of such solutions, each of which corresponds to a real value of c within the range $(0,1)$. Any arbitrary initial disturbance can be represented as a sum of Eady modes and these singular solutions.

4. Limitations of the Eady's model

- Setting $\beta = 0$ is highly unrealistic at the scales $L \sim L_D$
- Presence of a rigid boundary at $z = 1$ is essential for Eady instability.
- Eady model is capable of demonstrating only the essential features of baroclinic instability. The detailed structure is expected to alter considerably in the presence of ambient potential-vorticity gradient.

5. Conclusion

We have found that Eady modes are unstable if $\mu < \mu_c$. Based on this we found a condition on S (equation 29), which depicts that the least wiggly mode of the wave is the most unstable. This condtion allows us to estimate that scales of the order of Rossby radius of deformation are preferred by baroclinic instability, a very insightful conclusion. The complex part of the phase speed was found to be largest if $\mu \to 0$, i.e. for large waves. However the product kc_i becomes zero if $\mu > \mu_c$ and attains a maximum at an intermediate value. The amplitude of the unstable mode attains a minimum at a middle level and its phase angle increases with height, which implies a tilt against the current. This in turn implies the release of the potential energy by the disturbance.

References:

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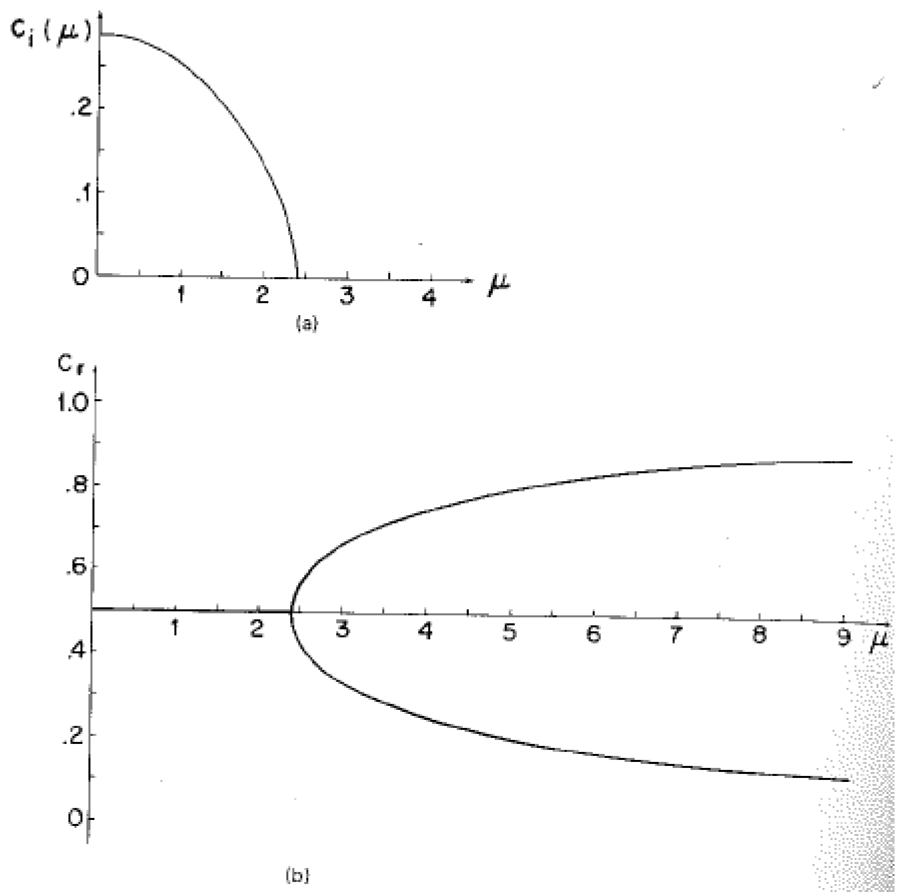

Fig 1. (a) Imaginary part of c_i as a function of μ. Only unstable mode ($c_i > 0$) is shown. The positive part of phase speed is just a mirror image of the one shown here about the the μ-axis. (b) Real part of c as a function of μ.

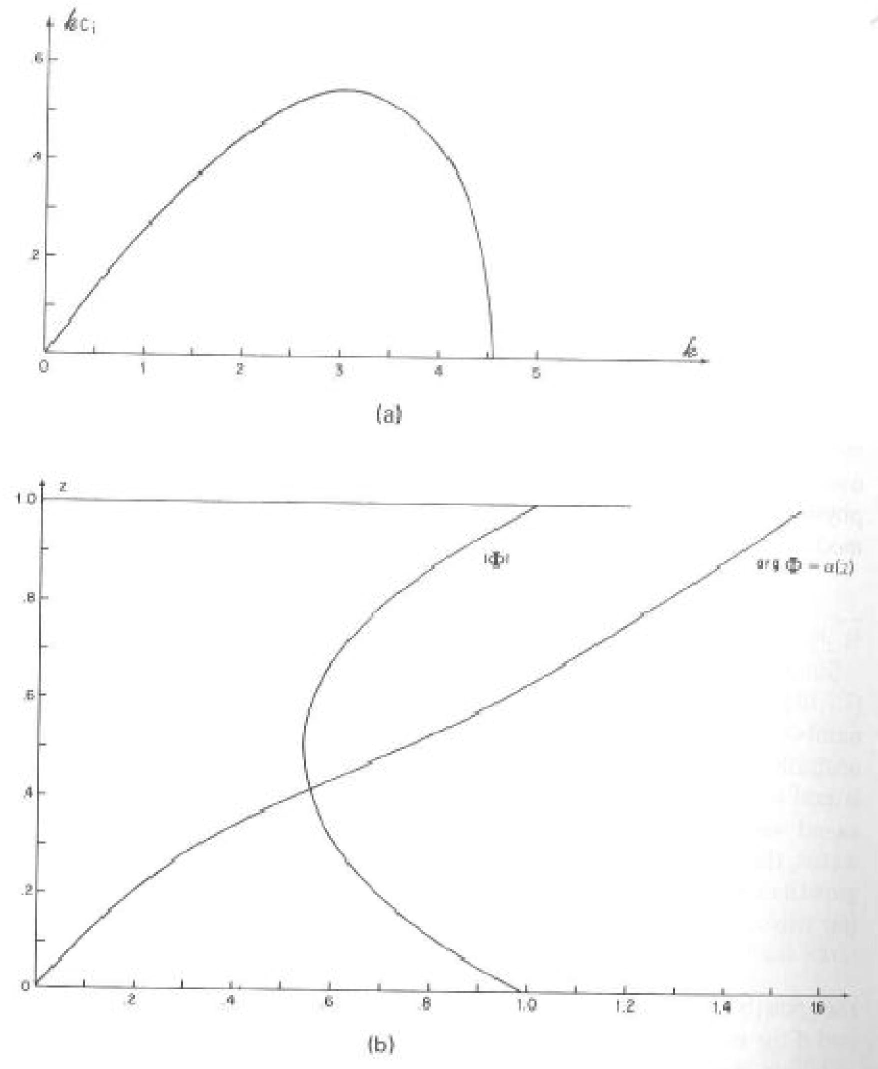

Fig 2. (a) Growth rate as a function of wavenumber for the most unstable mode, with $S = 0.25$. (b) Amplitude and phase angle as a function of height.