Agreement in circular societies

Christopher S. Hardin

January 9, 2009

Abstract

A common question in mathematics is whether a family of sets has nonempty intersection, and it is useful to have criteria that help answer these questions (for example, any chain of nonempty compact sets in a metric space has nonempty intersection). These questions can be recast in terms of approval voting, in which each voter has an approval set (the set of options that the voter approves of). Having an option that every voter approves of amounts to the approval sets having nonempty intersection. If this fails, one can still ask whether there is an option that a large proportion of the voters approve of.

Francis Su et al. have considered this problem in the case where the approval sets are intervals on a line [1], and showed that if out of every m voters, there are k who can agree on an option, then there is a single option that at least $(k-1)/(m-1)$ of the population approves of. We consider the case where the approval sets are arcs on a circle, and answer a question by Su by showing an analogous result: if out of every m voters, there are k who can agree on an option, then there is a single option that more than $(k-1)/m$ of the population approves of.

1 Introduction

In approval voting systems, each voter indicates approval or disapproval for each candidate or option (we will use the term *platform*), and the platform with the most approval votes wins. What conditions guarantee that there will be a platform with the approval of at least half (or all) the voters? Without introducing any further assumptions, one can only reach trivial conclusions, but in many situations, there is some underlying property of the set of platforms and votes that allows for more interesting results. For example, it is common to imagine a linear political spectrum from liberal to conservative, with platforms lying along this spectrum, and to assume that for each voter, the set of platforms the voter approves of (which we call the voter's *approval set*) forms a subinterval of the spectrum. Or, suppose a town is trying to determine a school budget; the interval $[0, \infty)$ forms the spectrum of possible budgets (in dollars, say), and voters will typically approve of some subinterval of this spectrum. In these *linear societies*, where the spectrum is a line and the approval sets are intervals, Francis Su et al. have shown that if, out of every 3 voters, there is a platform that at least 2 of them approve of, then there must be a platform with the approval of at least 1/2 of all voters [1] (we call this an *agreement proportion* of at least 1/2). More generally, they showed that if among every m voters, k can agree on a platform, then one can obtain an agreement proportion of at least $(k-1)/(m-1)$ (this is stated more precisely as Theorem 1.1 below).

In other situations, one might wish to consider *circular societies*, in which the spectrum of platforms is a circle and the approval sets are arcs. For example, some suggest that a circular political spectrum is a better model than a linear spectrum. Sometimes there is something inherently circular about the set of options at hand: a professional society might be trying to determine what time of year to hold a recurring annual meeting; a committee might want to determine the background hue for the cover of a report; a group of scouts lost in the woods might need to choose a compass bearing; *etc*. Or, one could have a linear spectrum, but allow approval sets to be intervals or their complements (perhaps a voter finds a particular interval unsavory, or just wants to cancel someone else's vote); in this case, one is essentially forming a circular spectrum by wrapping the linear spectrum around a circle, and allowing arcs as approval sets. But most importantly, if one seeks a general understanding of how Theorem 1.1 changes as one changes the form of the spectrum and approval sets, then circular societies are one of the first cases to be considered. During his invited address at the 2006 Joint Mathematics Meetings [4], Su posed the question of what analog of Theorem 1.1 might hold for circular societies.

This paper answers Su's question with Theorem 1.2, an analog of Theorem 1.1 that applies to circular societies and has a strict lower bound of $(k-1)/m$ (instead of $(k-1)/(m-1)$) on the agreeement proportion. Despite a strong resemblance in the statement of the two theorems, the proofs are quite different. Su et al. use chromatic numbers of perfect graphs to prove Theorem 1.1, while we prove Theorem 1.2 by exhibiting a class of easily analyzed societies (the *uniform societies*) and showing that it is possible to transform an arbitrary circular society, in a permissible way, into one of these uniform societies.

In Section 1.1, we make the above notions precise, introduce notation, and state Theorems 1.1 and 1.2. In Section 1.2, we describe the general context where these results reside, and pose some further questions. We prove Theorem 1.2 in Sections 2–4.

1.1 Definitions and results

A *society* is a tuple $S = (V, X, A)$, where V is a finite set of *voters*, X is a set of *platforms*, and $A : V \to \mathcal{P}(X)$ is a function that associates each voter $x \in V$ with a set A_x of platforms in X. We call X the *spectrum* of S, and A_x the *approval set* of voter x. The intention is that the platforms are options (or candidates) that the voters must choose among, and A_x is the set of platforms that voter x approves of.

Some further notation is helpful. Given a platform $p \in X$, the *support* $\mathrm{supp}(p)$ of p is the set of voters that approve of p, and the *agreement number* $a(p)$ of p is the number of voters that approve of p. We can also restrict these notions to a subset $W \subseteq V$ of voters: $\mathrm{supp}_W(p)$ is the set of voters in W that approve of p, and $a_W(p)$ is the number of voters in W that approve of p. The *agreement number* $a(S)$ of society S is the greatest number of voters that can agree on a single platform. More formally:

$$\begin{aligned}
\mathrm{supp}(p) &= \{\, x \in V \mid p \in A_x \,\}, \\
\mathrm{supp}_W(p) &= \{\, x \in W \mid p \in A_x \,\}, \\
a(p) &= |\mathrm{supp}(p)|, \\
a_W(p) &= |\mathrm{supp}_W(p)|, \\
a(S) &= \max_{p \in X} a(p).
\end{aligned}$$

With any of this notation, if there is ambiguity about which society is in question, we can include it as a superscript: A_x^S is the approval set of x in society S, and so on. We will abuse notation and allow S to be treated as its set of voters; in particular, we write $x \in S$ for $x \in V^S$ and $|S|$ for $|V^S|$.

A society S is (k, m)-*agreeable* if $1 \le k \le m \le |S|$ and for every $W \subseteq S$ with $|W| = m$, there exists a platform p such that $a_W(p) \ge k$; in other words, out of every m voters, there must be k among them who can agree on a single platform. The *agreement proportion* of S is the fraction $a(S)/|S|$ of the voters who can agree upon a single platform.

The types of societies of immediate interest are linear societies and circular societies. A *linear society* is a society in which the spectrum is $\mathbb{R}$ (or an interval of $\mathbb{R}$) and approval sets are intervals. A *circular society* is a society in which the spectrum is the circle S^1, and approval sets are connected subsets of the circle (typically arcs). Figure 1 gives an example of

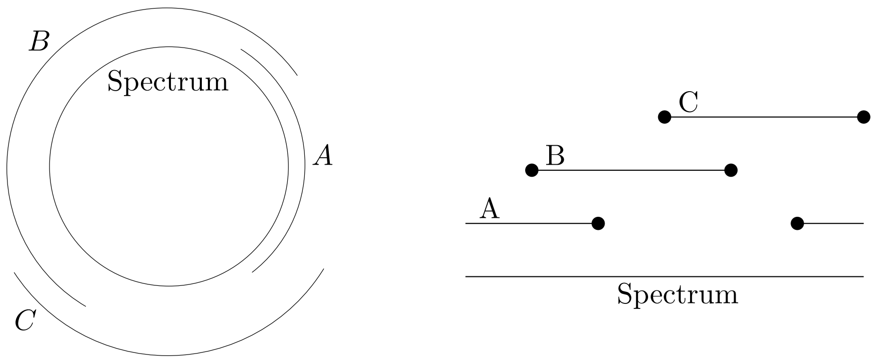

Figure 1: A circular society with voters A, B and C, drawn two ways: with the spectrum as a circle, and with the spectrum as an interval.

a circular society T: we have 3 voters A, B, and C whose approval sets intersect pairwise, but the intersection of the three approval sets is empty. By unwrapping the circle to form an interval, we can present T as it appears on the right in the figure; for clarity, we favor this latter form. Society T is $(2,2)$-agreeable: every pair of voters has overlapping approval sets. (It is also $(3,2)$-agreeable, but not $(3,3)$-agreeable.) The agreement number of T is $a(T) = 2$ (we can get 2 voters to agree upon a single platform, but not 3); this gives T an agreement proportion of $a(T)/|T| = 2/3$.

The following theorem describes how (k,m)-agreeability affects agreement proportion in linear societies.

Theorem 1.1 (Su et al.). *If S is a (k,m)-agreeable linear society, then*

$$\frac{a(S)}{|S|} \geq \frac{k-1}{m-1}, \tag{1}$$

and this bound is best possible (in a sense that we do not make precise here). $\square$

It is not hard to see that Theorem 1.1 does not apply to circular societies. For instance, the society T of Figure 1 is $(2,2)$-agreeable, but $a(T)/|T| = 2/3 < (2-1)/(2-1)$. Experimentation with a few societies suggests $(k-1)/m$ as a lower bound on the agreement proportion of a (k,m)-agreeable circular society. This turns out to be correct; the difficulty lies in proving it, and the rest of the paper is devoted to the following theorem.

Theorem 1.2. *(a) If S is a (k,m)-agreeable circular society, then*

$$\frac{a(S)}{|S|} > \frac{k-1}{m}; \tag{2}$$

equivalently, $a(S) \geq \lfloor \frac{k-1}{m} |S| \rfloor + 1$.

(b) For every $N \geq m \geq k \geq 1$*, there is a* (k, m)*-agreeable circular society* S *of size* N *with* $a(S) = \lfloor \frac{k-1}{m} N \rfloor + 1$.

We prove the sharpness condition (b) in Section 2, and establish the lower bound (a) in Section 4. (Section 3 discretizes societies, necessary for Section 4.)

1.2 Further questions

Our interest in results like Theorems 1.1 and 1.2 is not just motivated by an interest in approval voting. Rather, approval voting provides a convenient presentation of a broader mathematical problem: given a family of sets, is their intersection nonempty? Treating each set as the approval set of a voter, this amounts to asking whether there is a single platform with the approval of all voters. Or, even if a family of sets has empty intersection, is there a sense in which *many* of the sets must have nonempty intersection? Studying (k, m)-agreeability and its consequences is one line of approach to these questions.

Let $\mathcal{A}$ be a family of subsets of a set X. An $\mathcal{A}$*-society* is a society with spectrum X and approval sets in $\mathcal{A}$. What is the best lower bound we can put on the agreement proportion of (k, m)-agreeable $\mathcal{A}$-societies? A few cases of interest are the connected subsets of a topological space (particularly graphs), the open balls of a metric space, the convex subsets of an affine space, and solid rectangles in $\mathbb{R}^n$. (Linear societies can be seen as a special case of any of these; circular societies are a special case of the first two.)

Some known set-intersection theorems can be seen in this light. Su et al. [1] point to Helly's theorem [3] as an example, which we reproduce here. An $\mathbb{R}^d$*-convex society* is a society with spectrum $\mathbb{R}^d$ and convex approval sets. Helly's theorem (below) tells us that a $(d+1, d+1)$-agreeable $\mathbb{R}^d$-convex society has agreement proportion 1.

Theorem 1.3 (Helly)**.** *Suppose* $A_1, \ldots, A_n \subseteq \mathbb{R}^d$*,* $d < n$*, are convex such that the intersection of any* $d + 1$ *of these sets is nonempty. Then* $\cap_i A_i \neq \varnothing$*.* □

We would like to draw attention to graphs: given a graph G as spectrum, and allowing connected subsets of G as agreement sets, what can one say about (k, m)-agreeable societies of this form? (By a graph, we mean a 1-dimensional cell complex [2], so that each edge is considered as a copy of the

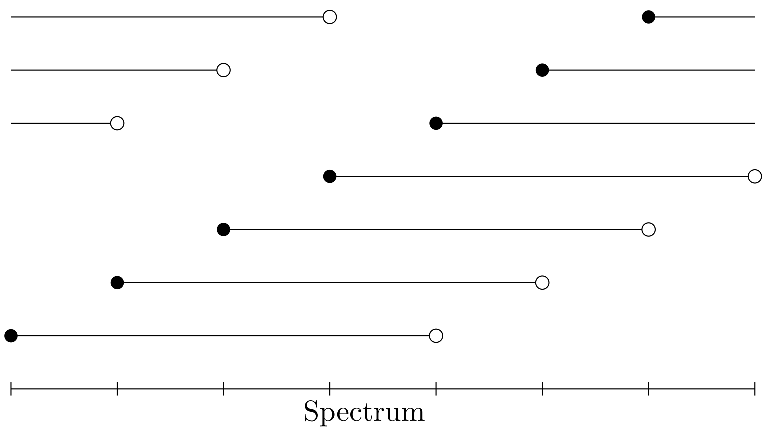

Figure 2: The uniform society $U(7,4)$. (Hollow circles denote open endpoints.)

unit interval.) Linear societies are the special case in which the graph takes the form of a line segment; circular societies are the special case in which the graph consists of a single cycle. The next step is to understand trees; in particular, does Theorem 1.1 apply to trees? This appears to be the case, but it is not known.

2 Sharpness and uniform societies

Before handling arbitrary circular societies, we will consider societies in which the approval sets are spread as evenly as possible over the circle. These societies will be enough to establish Theorem 1.2(b), and they will arise again in our proof of Theorem 1.2(a).

Definition 2.1. For integers $1 \leq h \leq N$, we define the circular society $U(N,h)$ as follows. First, we identify the circle S^1 with the real numbers modulo N. The set of voters in $U(N,h)$ is $\{0, 1, \ldots, N-1\}$, though we will generally think of this as the group $\mathbb{Z}_N$ so that we can speak of voter $1-N$, and so on. For voter i, we let $A_i = [i, i+h)$. Societies of the form $U(N,h)$ are called *uniform societies.* (An example appears in Figure 2.)

The first observation we make about $U(N,h)$ is that every platform appears in exactly h approval sets. In particular, $a(U(N,h)) = h$. The following lemma describes which uniform societies are (k,m)-agreeable.

Lemma 2.2. *For integers $N \geq m \geq k \geq 1$ and $h \geq 1$, the following are equivalent.*

(i) $U(N,h)$ *is* (k,m)*-agreeable.*

(ii) $h \geq \lfloor \frac{k-1}{m} N \rfloor + 1$

(iii) $k \leq \lceil mh/N \rceil$

Proof. The equivalence (ii)⇔(iii) is straightforward:

$$\begin{aligned} h \geq \lfloor \tfrac{k-1}{m} N \rfloor + 1 &\iff h > \tfrac{k-1}{m} N \\ &\iff k < mh/N + 1 \\ &\iff k \leq \lceil mh/N \rceil \end{aligned}$$

Now suppose $k \leq \lceil mh/N \rceil$. Choose any set W of m voters in $U(N,h)$. The average value of $a_W(p)$ is

$$\begin{aligned} \frac{1}{N} \int_0^N a_W(p)\, dp &= \frac{1}{N} \sum_{x \in W} \int_0^N a_{\{x\}}(p)\, dp \\ &= \frac{1}{N} \sum_{x \in W} h \\ &= mh/N. \end{aligned}$$

It follows that $a_W(p) \geq mh/N$ for some platform p; since $a_W(p)$ is an integer, $a_W(p) \geq \lceil mh/N \rceil \geq k$. So $U(N,h)$ is (k,m)-agreeable. This establishes (iii)⇒(i).

It remains to establish (i)⇒(iii). We will do this by taking m voters in $U(N,h)$ who are spaced as evenly as possible, and showing that at most $\lceil mh/N \rceil$ of them can agree on a single platform. Define $v : \mathbb{Z} \to \mathbb{Z}$ by $v(i) = \lfloor iN/m \rfloor$. Observe that for any k,

$$\begin{aligned} v(k + \lceil mh/N \rceil) - v(k) &= \lfloor (k + \lceil mh/N \rceil) N/m \rfloor - \lfloor kN/m \rfloor \\ &\geq \lfloor (k + mh/N) N/m \rfloor - \lfloor kN/m \rfloor \\ &= \lfloor kN/m + h \rfloor - \lfloor kN/m \rfloor \\ &= h. \end{aligned}$$

In particular, intervals of the form $(p-h, p]$ (which arise below) can contain at most $\lceil mh/N \rceil$ points from the range of v. Observe also that $v(i + km) = v(i) + kN$; in particular, the range of v is closed under adding multiples of N. The significance of this latter observation is what it contributes to the robustness of the previous observation: if we project the real numbers down to the real numbers modulo N, an interval $(p-h, p]$ can still contain at most $\lceil mh/N \rceil$ distinct points from the range of v.

Suppose $U(N,h)$ is (k,m)-agreeable, and let $W = \{\, v(i) \mid 0 \le i < m \,\}$. Note that W, considered as a set of voters in $U(N,h)$, contains m voters that are spaced as evenly as possible. By the (k,m)-agreeability of $U(N,h)$, let p be a platform that at least k voters in W approve of. Each voter i in $U(N,h)$ has approval set $[i, i+h)$, and $p \in [i, i+h) \Leftrightarrow i \in (p-h, p]$, so the voters of $U(N,h)$ that approve of p are those integers i that lie in $(p-h, p]$. As observed above, this interval can contain at most $\lceil mh/N \rceil$ points in the range of v, so of the m voters in W, at most $\lceil mh/N \rceil$ approve of p. So, $k \le \lceil mh/N \rceil$. This establishes (i)$\Rightarrow$(iii). $\square$

Proof of Theorem 1.2(b). Let $h = \lfloor \frac{k-1}{m} N \rfloor + 1$. Then by Lemma 2.2, $U(N,h)$ is a (k,m)-agreeable society (of size N), and $a(U(N,h)) = h = \lfloor \frac{k-1}{m} N \rfloor + 1$. $\square$

3 Discretization

For the purposes of proving Theorem 1.2(a), we find it convenient to work with discrete representations of circular societies. Before switching to discrete representations, though, we make a few simplifying assumptions.

First, we may restrict our attention to societies in which no voter has an empty approval set. For, suppose Theorem 1.2 applies to such societies, and consider a (k,m)-agreeable circular society S in which i voters have empty approval sets (call them *implacable*). Form the society S' by throwing away these voters; S' must be $(k, m-i)$-agreeable: given $m-i$ voters in S', combine them with the i implacable voters in S; by the (k,m)-agreeability of S, there must be a platform that k of these voters approve of, and each of these voters is in S'. (In particular, this shows that $k \le m-i$.) Since S' is $(k, m-i)$-agreeable, and noting that $a(S) = a(S')$, we have

$$\begin{aligned}
\frac{a(S)}{|S|} &= \frac{a(S')}{|S'|} \cdot \frac{|S'|}{|S|} \\
&> \frac{k-1}{m-i} \cdot \frac{|S'|}{|S|} \\
&= \frac{k-1}{m} \cdot \frac{m}{m-i} \cdot \frac{|S|-i}{|S|} \\
&\ge \frac{k-1}{m} && \text{(since } |S| \ge m\text{)}.
\end{aligned}$$

We may also assume that no voter's approval set is the entire circle. For any such voter, we could find a closed arc disjoint from the boundaries of the approval sets, and remove it from that voter's approval set without influencing $a(S)$ or the society's (k,m)-agreeability.

Now, each voter's approval set is an arc along the circle. Making small perturbations as necessary, we can assume that the endpoints of all these arcs are distinct. (First, at any open endpoint, perturb it to make the interval slightly shorter; this avoids introducing new overlaps between approval sets without losing any existing overlaps. Then, at any closed endpoint, perturb it to make the interval slightly longer; this preserves existing overlaps, and does not introduce any new overlaps since the open endpoints were already perturbed.) With all endpoints distinct, we may further assume that all the approval sets are closed.

Thinking of the circle as an interval $[0,N]$, with 0 and N identified (and $N>0$), we can treat each approval set as an interval with a left and right endpoint. Because of the circular nature, the left endpoint might nominally be to the right of the right endpoint; for example, $[2/3,1/3]$ represents the set $[0,1/3]\cup[2/3,N]$.

For our purposes, a society is sufficiently described by the order in which the left and right endpoints of the voters are encountered as one moves around the circle (or, more precisely, as one moves from 0 up to N). So, each society may be represented by a string of symbols, where each symbol represents a left or right endpoint of a particular voter's approval set. For a voter x, we will use the symbols $[x\rangle$ and $\langle x]$ to denote the left and right endpoints of A_x, respectively. For example, the society from Figure 1 would have a string representation of $[B\rangle\langle A][C\rangle\langle B][A\rangle\langle C]$. Of course, any string representation can be rotated cyclically without changing the society.

We can disregard the platforms occurring at endpoints, because their support will coincide with the support of platforms either immediately to the left or right. Also, between two consecutive endpoints, any two platforms will have the same support, so we only need to consider one platform between two consecutive endpoints. So, between any two consecutive endpoints, we fix one platform that we call a *representative platform* (r.p.), and we will restrict our attention to r.p.'s. As a matter of notation, when we list a string of endpoints, we may want to name any r.p.'s at hand, and we will do so by allowing r.p.'s to be listed among endpoints; for example, the string $[x\rangle p\langle y]q\langle x]r$ represents a portion of a society in which $[x\rangle$, $\langle y]$, and $\langle x]$ appear consecutively, p is the r.p. between $[x\rangle$ and $\langle y]$, q is the r.p. between $\langle y]$ and $\langle x]$, and r is the r.p. immediately to the right of $\langle x]$.

For a voter x in society S, we will let $|A_x^S|$ denote the number of r.p.'s in A_x^S; we call this the *approval length* of x (and if we think of consecutive endpoints as being spaced one unit apart, $|A_x^S|$ coincides with the actual measure of A_x).

4 The lower bound

Lemma 2.2 shows that Theorem 1.2(a) applies to uniform societies. Knowing this, it should not be surprising that it would also apply to arbitrary societies: non-uniform societies are going to be more "bunched up" in places, which would tend to yield higher agreement numbers. Our proof will give rigor to this intuition; we will start with an arbitrary (k, m)-agreeable society, and smooth it out through transformations that do not break (k, m)-agreeability and do not increase the agreement number, until we reach a uniform society, to which we can apply Lemma 2.2.

Definition 4.1. For societies S and T, we say S *validly transforms to* T, and write $S \rightharpoonup T$, if $|S| = |T|$, $a(S) \geq a(T)$, and (S is (k, m)-agreeable)$\Rightarrow$(T is (k, m)-agreeable) for all k, m. Observe that the relation $\rightharpoonup$ on societies is reflexive and transitive (a preorder). An operation on societies is a *valid transformation* if applying the operation to a society S always yields a society S' with $S \rightharpoonup S'$.

The transformation of an arbitrary society into a uniform society will occur in two phases. First, in Section 4.1, we will transform to a society in which no voter's approval set contains another's. Then, in Section 4.2, we will transform such a society into a uniform society.

4.1 Eliminating containment

Definition 4.2. A society S is *containment-free* if no voter's approval set contains another voter's approval set.

Lemma 4.3. *For any society S, there is a containment-free society T such that $S \rightharpoonup T$.*

Proof. If S is containment-free, then we are done, so suppose that S is not containment-free. Then $A_y \subseteq A_x$ for some $x \neq y$, and (after rotating if necessary) the string representation of S contains $[x\rangle\rho[y\rangle\sigma\langle y]\tau\langle x]$, where ρ, σ, and τ are (possibly empty) strings of intervening endpoints. Let R be the society that results from swapping $\langle x]$ and $\langle y]$; that is, we replace $[x\rangle\rho[y\rangle\sigma\langle y]\tau\langle x]$ with $[x\rangle\rho[y\rangle\sigma\langle x]\tau\langle y]$. (See Figure 3.) Note that this only affects the approval sets of x and y. We wish to show $S \rightharpoonup R$ (although R will not necessarily be the promised society T).

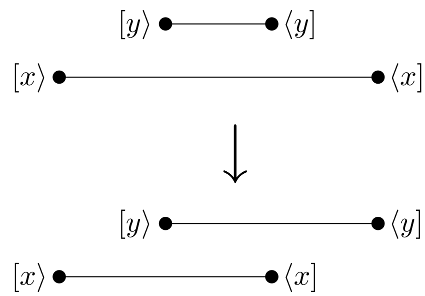

Figure 3: Swapping endpoints to eliminate a containment

For any r.p. q, either the support of q is unchanged by this transformation, or x leaves the support and y enters the support; in either case, $a^R(q) = a^S(q)$. So $a(R) = a(S)$.

Supposing that S is (k,m)-agreeable, we claim that R is (k,m)-agreeable. Let $M \subseteq V^S = V^R$ be any set of m voters. We must show $a^R_M(p) \geq k$ for some r.p. p. There is some tedium in considering all possible cases, but the argument will stand on the following facts: $a^R_{\{x,y\}}(p) = a^S_{\{x,y\}}(p)$ for all p (used in case 2), $A^S_y \subseteq A^R_y$ (used in case 1), and $A^S_y \subseteq A^R_x$ (used in case 3).

Case 1: $x \notin M$. We can choose p with $a^S_M(p) \geq k$ (since S is (k,m)-agreeable), and $a^R_M(p) \geq a^S_M(p) \geq k$, since $A^R_z \supseteq A^S_z$ for all $z \neq x$.

Case 2: $y \in M$. Again, choose p with $a^S_M(p) \geq k$. If p is one of the platforms whose support was unchanged, then $a^R_M(p) = a^S_M(p) \geq k$; otherwise, p is a platform whose support lost x but gained y, and we have $a^R_M(p) \geq a^S_M(p) \geq k$.

Case 3: $x \in M$, $y \notin M$. We will use the fact that $A^S_y \subseteq A^R_x$, by swapping x and y before applying the (k,m)-agreeability of S, and then swapping back. Let $\sigma : V \to V$ be the involution that swaps voters x and y (and is otherwise the identity); σ induces an involution on the power set of V by pointwise application. Since $|\sigma M| = |M| = m$, we can choose p such that $a^S_{\sigma M}(p) \geq k$. Let $K = \mathrm{supp}^S_{\sigma M}(p)$; $|K| \geq k$. Then $\sigma K \subseteq \mathrm{supp}^R_M(p)$, so $a^R_M(p) \geq k$.

We have now established $S \rightharpoonup R$. Of course, R is not necessarily containment-free, but we can show that if we repeat the above transformation enough times, we must eventually reach a containment-free society. Essentially, this is because the standard deviation of the approval lengths decreases each time we apply the transformation, so we cannot keep applying the transformation forever; the following argument makes this precise.

Define the *square sum* of a society Q to be

$$\sum_{x \in Q} |A_x^Q|^2.$$

In S, let i, j, and k be the number of r.p.'s, respectively, between $[x\rangle$ and $[y\rangle$, $[y\rangle$ and $\langle y]$, and $\langle y]$ and $\langle x]$. Then we have $i, j, k > 0$, $|A_x^S| = i + j + k$, $|A_y^S| = j$, $|A_x^R| = i + j$, and $|A_y^R| = j + k$. It is easy to verify that $|A_x^S|^2 + |A_y^S|^2 > |A_x^R|^2 + |A_y^R|^2$. All other terms in the square sum are the same for S and R, because $A_z^S = A_z^R$ for $z \neq x, y$. Thus, the above transformation decreases the square sum of a society. Noting that the square sum is a positive integer, we see that the transformation can only be performed a finite number of times before we reach a containment-free society T. $\square$

4.2 Left-right alternation

Definition 4.4. A society is *left-right alternating* if its sequence of endpoints alternates between left and right endpoints.

Lemma 4.5. *For any containment-free society S, there is a left-right alternating containment-free society T such that $S \rightharpoonup T$.*

Proof. Suppose S is not left-right alternating. Then, since there is an equal number of left and right endpoints, there must be a pair of consecutive right endpoints somewhere. Continuing to the right, one must eventually reach a left endpoint. Thus, there must be a sequence of endpoints of the form $p\langle x]q\langle y]r[z\rangle$ somewhere in the society. (Note that $x \neq y$, since each voter has only one right endpoint. We also have $y \neq z$; otherwise, we would have a containment $A_x \subseteq A_y$.) Note that $a^S(p) = a^S(q) + 1$, because $\mathrm{supp}^S(p) = \mathrm{supp}^S(q) \cup \{x\}$ (and $x \notin \mathrm{supp}^S(q)$); continuing this reasoning, we have $a^S(p) = a^S(q) + 1 = a^S(r) + 2$.

Let S' be the society that results from swapping $\langle y]$ and $[z\rangle$, so that $p\langle x]q\langle y]r[z\rangle$ becomes $p\langle x]q[z\rangle r\langle y]$. (See Figure 4.) Since we are expanding A_y and A_z without altering any other agreement sets, this preserves (k, m)-agreeability. This also does not introduce any containments: when we moved $[z\rangle$ to the left, it did not pass any other left endpoints; similarly, when we moved $\langle y]$ to the right, it did not pass any other right endpoints.

This tranformation does not affect the agreement number of any representative platform except for r. The agreement number of r increases by two, because y and z have entered its support. So, we have $a^{S'}(r) = a^S(r) + 2 = a^S(p)$. This shows that $a(S') = a(S)$: the only opportunity

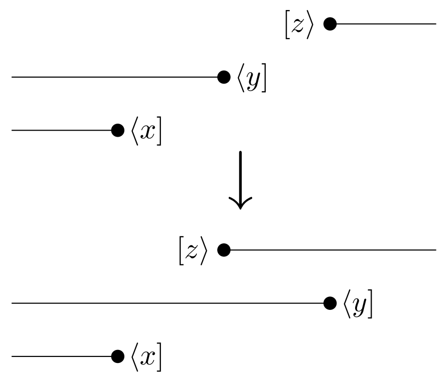

Figure 4: Swapping endpoints to achieve left-right alternation

for $a(S')$ to exceed $a(S)$ would be from what happens at r, but we have $a^{S'}(r) = a^S(p) \leq a(S)$. So, this transformation is valid. Furthermore, each time it is applied, the sum of the lengths of the approval sets goes up. Because this sum is bounded and is always an integer, the transformation can only be applied finitely many times before we reach a left-right alternating society T, which will be containment-free since the transformation does not introduce containments. $\square$

4.3 Uniform societies, again

Our definition of the uniform societies $U(N,h)$ did not conform to the simplifications made in Section 3; namely, the endpoints are not unique. So, we define $U'(N,h)$ to be like $U(N,h)$, except that we use $[i, i+h-\varepsilon]$ in place of $[i, i+h)$, with $\varepsilon = 1/2$, say. This difference is minor enough that we immediately have $a(U'(N,h)) = a(U(N,h)) = h$, and $U'(N,h)$ is (k,m)-agreeable if and only if $U(N,h)$ is, so Lemma 2.2 applies to $U'(N,h)$ as well as $U(N,h)$. The society $U'(N,h)$ can be discretized, and has string representation

$$[0\rangle\, \langle 1-h]\, [1\rangle\, \langle 2-h] \cdots [N-1\rangle\, \langle N-h].$$

(To see this, the reader might find it helpful to consider the cases $h = 1$ and $h = 2$.) In particular, $U'(N,h)$ is left-right alternating, the left endpoints appear in the order $0, 1, \ldots, N-1$, and the right endpoints appear in the same order, but rotated by $1-h$: $1-h, 2-h, \ldots, N-h$. We are now ready to finish.

Proof of Theorem 1.2(a). Let S_0 be any (k,m)-agreeable circular society. By Lemmas 4.3 and 4.5, we can validly transform S_0 into a (k,m)-agreeable society S that is left-right alternating and containment-free. Letting $N = |S|$, we may assume the voters of S are $\{0, 1, \ldots, N-1\}$. Renaming voters as necessary, we may assume that the left endpoints of their approval sets occur in the order $[0\rangle, [1\rangle, \ldots, [N-1\rangle$.

We claim that, up to a cyclic rotation, the right endpoints of the approval sets occur in the same order: $\langle 0], \langle 1], \ldots, \langle N-1]$. Suppose not. Then there must be distinct voters x, y, and z such that their left endpoints occur in the order xyz while their right endpoints occur in the order xzy. Breaking into cases, it is straightforward to show that there must be a containment, a contradiction. So the claim holds.

We have established that S is a left-right alternating society whose left endpoints occur in the order $[0\rangle, [1\rangle, \ldots, [N-1\rangle$ and whose right endpoints occur in the same order, up to a cyclic rotation. Considering the discretization of uniform societies, we can observe that S must have the same discretization as $U'(N, h)$ for some h. This yields $a(S) = a(U'(N, h)) = h$, and by Lemma 2.2, $h \geq \lfloor \frac{k-1}{m} N \rfloor + 1$. We now have $a(S_0) = a(S) = h \geq \lfloor \frac{k-1}{m} N \rfloor + 1$. $\square$

5 Acknowledgments

The author owes a great debt to Francis Su, and would also like to thank Bill Zwicker.

References

[1] Deborah E. Berg, Serguei Norine, Francis Edward Su, Robin Thomas, and Paul Wollan. Set intersections, perfect graphs, and voting in agreeable societies. Preprint.

[2] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[3] Eduard Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresbericht der Deutschen Mathematiker Vereinigung*, 32:175–176, 1923.

[4] Francis Edward Su. Preference sets, graphs, and voting in agreeable societies. MAA Invited Address, Joint Mathematics Meetings, San Antonio, January 2006.