SECTION VI

MATHEMATICAL ECONOMICS

(*a*)

ON THE APPLICATION OF MATHEMATICS TO POLITICAL ECONOMY

[THIS is the Presidential Address delivered to Section F of the British Association in 1889. Mathematics are described as a useful, though not an indispensable adjunct to economic studies; a finish to the training of an economist comparable with a knowledge of the Classics as part of a general education. Mathematical conceptions are found to be specially appropriate to the cases, frequent in economics, where there are several mutually dependent variable quantities; and where it is required to determine the maximum value of a quantity dependent on several variables.

If the subjects which would now be described as "quasi-rent" and "external economies" are inadequately treated in this Address, it should be remembered that the Address was delivered before the publication of the *Principles of Economics*.]

At the meeting of the British Association which was held at Cambridge about a quarter of a century ago, Jevons submitted to this section a "general mathematical theory of political economy," which, as he himself records, was "received without a word of interest or belief." I propose to consider the justice of the unfavourable verdict which our predecessors appear to have passed on the mathematical method introduced by Jevons.

There is some difficulty in discussing so abstruse a subject in this place. It is as if one should discourse on the advantages of classical education on an occasion on which it might seem pedantic to cite the learned languages. I shall evade this difficulty by addressing to students some appended notes,[1] which, like the boy of the proverb, are to be seen, not heard.

The cardinal article of Jevons's theory is that the value in

[1] The appended notes are referred to by letters of the alphabet, thus: (*a*).

exchange of a commodity measures, or corresponds to, the utility of the least useful portion of that commodity. What a person pays per month or year for a sack or ton of coal is not what he would be willing to give for the same rather than be without fuel altogether. Rather the price is proportioned to the advantage which the consumer expects from the portion which he could best dispense with—to the "final utility," in Jevons's happy phrase.

I shall not be expected here to dwell on a subject which has been elucidated in treatises of world-wide reputation, such as those of Professors Marshall, Sidgwick, Walker, and I would add Professor Nicholson's article on Value in the *Encyclopædia Britannica*. Those writers seem to present what I may call the economical kernel of Jevons's theory divested of the mathematical shell in which it was originally enclosed; whereas my object is to consider the use of that shell: whether it is to be regarded as a protection or an encumbrance.

I may begin by removing an objection which the mere statement of the question raises. The idea of reducing human actions to mathematical rule may present itself to common sense as absurd. One is reminded of Swift's "Laputa," where the beef was cut into rhomboids and the pudding into a cycloid, and the tailor constructed a very ill-fitting suit by means of rule and compasses. It should be understood, however, that the new method of economical reasoning does not claim more precision than what has long been conceded to another department of science applied to human affairs, namely, Statistics. It is now a commonplace that actions such as suicide and marriage, springing from the most capricious motives, and in respect of which the conduct of individuals most defies prediction, may yet, when taken in the aggregate, be regarded as constant and uniform. The advantage of what has been called the law of large numbers may equally be enjoyed by a theory which deals with markets and combinations.

But, indeed, even the limited degree of arithmetical precision which is proper to statistical generalisations need not be claimed by our mathematical method rightly understood. It is concerned with quantity, indeed, but not necessarily with number. It is not so much a political arithmetic as a sort of economic algebra, in which the problem is not to find x and y in terms of given quantities, but rather to discover loose quantitative relations of the form: x is greater or less than y; and increases or decreases with the increase of z.

Such is the character of what may be called perhaps the leading proposition in this calculus, namely, the mathematical theory of Supply and Demand. The use of a curve introduced by Cournot to represent the amount of a commodity offered, or demanded, at any particular price, supplemented by Jevons's theory of final utility (*a*), does not indeed determine what price will rule in any market. But it assists us in conjecturing the direction and general character of the effect which changes in the condition or requirements of the parties will produce. For example, in the case of international trade the various effects of a tax or other impediment, which most students find it so difficult to trace in Mill's laborious chapters, are visible almost at a glance by the aid of the mathematical instrument (*b*). It takes Professor Sidgwick a good many words to convey by way of a particular instance that it is possible for a nation by a judiciously regulated tariff to benefit itself at the expense of the foreigner. The truth in its generality is more clearly contemplated by the aid of diagrams such as those employed by the eminent mathematical economists Messrs. Auspitz and Lieben (*c*).

There seems to be a natural affinity between the phenomena of supply and demand and some of the fundamental conceptions of mathematics, such as the relation between function and variable,[1] between the ordinate of a curve and the corresponding abscissa,[2] and the first principles of the differential calculus, especially in its application to the determination of *maxima* and *minima*. The principle of Equilibrium is almost as dominant in what Jevons called the mechanics of industry as in natural philosophy itself (*d*). In so many instances does mathematical science supply to political economy what Whewell would have called "appropriate and clear" conceptions. Their use might, perhaps, be illustrated by comparing—however fancifully, and *si parva licet componere magnis*—the advance in economics which Jevons initiated or continued to the advance in mathematics which the higher method invented by Sir William Rowan Hamilton appears to have effected. Algebra and geometry are to ordinary language in political economy somewhat as quaternions are to ordinary algebraic geometry in mathematical physics; if we

[1] The treating as constant what is variable, e.g., *supply, margin, wages-fund*, is the source of most of the fallacies in political economy.

[2] For instance, the two meanings of increased demand—which Mr. Sidgwick has contrasted as the *rise* and the extension of demand—are most easily and with least liability to logomachy distinguished as the variation of an ordinate (1) due to the displacement of the curve, the abscissa not varying, or (2) corresponding to an increment of the abscissa, the curve being undisturbed.

accept the view of the latter relation which has been given by a very competent judge, Clerk-Maxwell. "I am convinced," he says, "that the introduction of the ideas as distinguished from the operations and methods of quaternions will be of great use in the study of all parts of our subject, and especially . . . where we have to deal with a number of physical quantities, the relations of which to each other can be expressed far more simply by a few expressions of Hamilton's than by the ordinary equations." [1] This is the spirit in which the economist should employ mathematics—"the ideas as distinguished from the operations and methods."

In considering the above given, and indeed any concrete, instances, it is hardly possible to keep to what may be called the simplest type of supply and demand, the ideal market in which we contemplate only two groups of competitors and only two articles of exchange: say, gold for corn, or any other *quid pro quo*. In general, and especially when considering what rates of exchange tend to rule in an average of transactions, it is proper to take into account that the dealings in one market will affect those in another. If the entrepreneur has less to pay for machinery, *ceteris paribus*, he will be able to offer more on the labour market. Thus we obtain the idea of a system of markets mutually dependent. In a general view of this correlation it is not necessary to distinguish whether the state of one part is connected as cause or effect with the other parts of the system. As Professor Marshall says: [2] "Just as the motion of every body in the solar system affects and is affected by the motion of every other, so it is with the elements of the problem of political economy" (*e*).

This conception of mutually dependent positions is one in which minds disciplined in mathematical physics seem peculiarly apt to acquiesce. In other quarters there may be observed a restless anxiety to determine which of the variables in a system of markets is to be regarded as determining or regulating the others. In one of the principal economic journals there has lately been a pretty stiff controversy on the question which of the parties in the distribution of the national produce may be regarded as "residual claimants upon the product of industry"; [3] whether it

[1] Clerk-Maxwell, *Electricity and Magnetism*, Art. 10. He says in the context, "As the methods of Descartes are still the most familiar to students of science, and as they are really the most useful for purposes of calculation, we shall express all our results in the Cartesian form." Compare Professor Marshall's dictum with respect to the use of the vulgar tongue in economic reasonings, cited below, p. 287.

[2] In a remarkable review of Jevons's Theory in the *Academy* of 1st April, 1872. [3] *Quarterly Journal of Economics*, 1887, p. 287; 1888, p. 9.

is the working class which occupies this preferential position, or if the "real keystone of the arch" is interest. Such questions certainly admit of a meaning, and possibly of an answer. But they will probably appear of secondary importance to those who accept, as the first approximation to a correct view of the subject, the principle of mutual dependence—what may be called the Copernican theory of distribution, in which one variable is not more determined by another than the other is by that one (*f*).

Among the factors of this economic equilibrium I have not as yet explicitly included cost of production. Rather, the system of markets which so far I have had in view is that which would arise if all the articles of exchange were periodically rained down like manna upon the several proprietors, and each individual sought to maximise his advantage according to the law of final utility. But now we must observe that self-interest does not operate only in this fashion. We must take account of efforts and sacrifices.

Here again the language of symbol and diagram is better suited than the popular terminology to express the general idea that all things are in flux, and that the fluxions are inter-dependent. In Professor Marshall's words, "as a rule, the cost of production of a thing is not fixed; the amount produced and its normal value are to be regarded as determined simultaneously under the action of economic laws. It, then, is incorrect to say, as Ricardo did, that cost of production alone determines values; but it is no less incorrect to make utility alone, as others have done, the basis of value." [1] Among those who may have gone astray in the latter sense, who, in their recoil from Scylla, are at least sailing dangerously near Charybdis, may be placed the important Austrian school, who have rediscovered and restated the theory of final utility without the aid of mathematical expression. To amplify a figure suggested by one them,[2] let us figure the hard conditions of industrial life by the austerity of a schoolmaster who, in order to cultivate patience and fortitude in his scholars, should distribute among them certain rewards—it might be toys and sweets—in return for certain amounts of fatigue and pain endured. Thus the cost of procuring a marble might be writing

[1] *Economics of Industry*, p. 148.

[2] *Cp.* Professor Böhm-Bawerk: "Es kann ein Erzieher einem Knaben, um ihn gegen Wehleidigkeit abzuhärten, für die tapfere, freiwillige Erduldung von Schmerzen ein sehnlich begehrtes Spielzeug in Aussicht stellen. So untergeordnet das Vorkommen solcher Fälle auch sein mag, so wichtig ist es für die Theorie festzustellen, dass Arbeit und Arbeitsplage doch nicht der einzige Umstand ist, auf den sich . . die Wertschätzung gründen kann." *Konrad's Jahrbuch* 1866, p. 43.

out twenty lines, the cost of a top standing half an hour in the stocks. Supposing exchange to be set up among the members of the youthful population, free competition being assumed, there would theoretically arise an equilibrium of trade in which the value of each article would correspond to its final utility. That is, if a top exchanged for ten marbles, it might be expected that each boy would prize the last top about as highly as the last decade of marbles which he thought fit to purchase. So far final utility might be regarded as the regulating principle.

But it is equally true that the final *dis*-utilities of the exchanged articles will be equal. If a top is worth ten marbles, we are entitled to expect such an adjustment of trade that each and every boy would as soon stand in the stocks half an hour as write out two hundred lines—the cost of ten marbles at twenty lines per marble.

To be sure final utility may be conceived as operating by itself without reference to cost of production, as we tacitly assumed in our first paragraphs. Whereas the converse conception of a traffic in discommodities [1] has less place in real life.

But it is not worth while weighing the two principles against each other, *in vacuo*, so to speak, and abstracting the real circumstances by which each is differently modified. As these are introduced the balance will oscillate now in favour of one side, now of the other; perhaps leaving it ultimately uncertain whether cost of production or final utility is the more helpful in the explanation of economic phenomena.

For instance, in our allegory let us introduce the supposition that there is only one variety of cost—say the common labour of writing out verses. If now the authorities fix twenty lines as the cost of a marble, and two hundred as the cost of a top, it is predictable that a top will be worth ten marbles. It is equally true, indeed, now as before, that the final utility of a top will be equal to the final utility of ten marbles. But the latter proposition, though equally true, is not equally useful. For it does not afford

[1] Suppose our allegorical schoolmaster should discontinue the system of rewards, and prefer to cultivate diligence by requiring each boy from time to time to bring up a certain number of lines, written out—whether by himself or another would not be scrutinised—or to be responsible for the cleaning of a window, after the manner of Mr. Squeers's practical method and so on. In the traffic of discommodities which would be set up on this supposition the (negative) value of each article of exchange would be measured solely by its disutility. However, it must be admitted, I think, that this latter hypothesis is rather more absurd than the former abstraction—with reference to ordinary life at least; for, as it happens, the traffic in impositions more nearly resembles what is said to occur in schools.

the simple and exact method of prediction which is obtained by the Ricardian view upon the supposition made. But then the supposition that there is only one variety of sacrifice is not always appropriate. And even if it were appropriate, it might not be helpful when we introduce the condition that the cost of procuring each article is not fixed definitely, but varies increasingly or decreasingly with the amount procured. Thus the cost of the first marble given out might be twenty lines; of the next marble, twenty-one lines; with an equally varying scale for tops. Upon this supposition the two propositions that value corresponds to final utility and also final disutility might be equally true, but equally useless for the purpose of prediction.

Again, it may be that a man is freer to vary the extent of his expenditure than the duration of his work (*g*). The final disutility experienced by the secretary of the British Association during its meetings must be fearful. For it is not open to him to terminate at pleasure his day's work, as if he were employed by the piece. He would not, however, have accepted the office unless the advantages, less by all the trouble, were at least as great as in any other position open to him. Now this equation of the net advantages in different occupations is—co-ordinately and (in a mathematical sense) *simultaneously* with the equation of final utility for different kinds of expenditure—a condition of normal economic equilibrium (*h*). Yet again, the free play of this tendency is impeded by the existence of "non-competing groups."

I cannot be expected here to enumerate all the conditions of economic equilibrium. For a complete exposition of the complexities, at which I have thought it necessary to glance, I must refer to the second book of Professor Sidgwick's *Political Economy*. It will be evident to his readers [1] that what may be called the

[1] There occurs to me only one point at which the use of mathematical illustrations more complicated than those which I have referred to in my first two headings would conduce to the apprehension of Mr. Sidgwick's theorems. I allude to his repeated statement that, not only in international trade, as Mill pointed out, but also in trade in general, there may be several rates of exchange at which the supply just takes off the demand. This statement, taken without reservation, goes the length of destroying the prestige which is now attached to competition. Professor Marshall in an important passage recommends arbitrators and combinations to imitate the method of a celebrated engineer, who, in order to make a breakwater, first ascertained the slope at which a bank of stones would naturally be arranged under the action of the waves, and then let down stones so as to form such a slope (*Economics of Industry*, p. 215). Now, if gravitation acted sometimes vertically and sometimes at an angle of 45°, if the forces of competition tended to two distinct positions of equilibrium, the construction of the economic breakwater would become arbitrary. It is important, therefore, to show the limits of Professor Sidgwick's theory.

general economic problem of several trading bodies distributing and exchanging *inter se* under the influence of self-interest, and in a regime of competition, is much more hopelessly difficult than the as yet imperfectly solved dynamical problem of several material bodies acting on each other *in vacuo*. When Gossen, the predecessor of Jevons as exponent of the law of final utility, compares that principle to the law of gravitation, and the character of our science to that of astronomy, he betrays a parental partiality. A truer, though still too flattering, comparison would be afforded by some very immature and imperfect specimens of physics, say the theory of fluid motion applied to the problems of house ventilation.

There is a certain resemblance between the uniformity of pressure to which the jostling particles of a gas tend, and the unity of price which is apt to result from the play of competition. As the architect is guided by studying the laws according to which air flows, so it will help the builder of economic theory to have mastered the principle of movement towards equilibrium. But even in the material constructions practice is apt to lag far behind theory, as every reader in the British Museum knows. Much less are we able to predict what currents will flow between the different compartments of the industrial system. We know so imperfectly the coefficient of fluid friction, and the other conditions of the general problem: what compartments may be regarded as completely isolated and hermetically sealed, which partitions are porous and permeable.

Indeed there has been noticed one mode of competition, which it does not seem easy or helpful to represent by physical analogies —the transference from one occupation to another, the equation of net advantages or total utilities in different employments; industrial as distinguished by Cairnes from commercial competition. The latter operation appears to me to admit much better of mathematical expression than the former, which is not so well represented by the equilibrium of a physical system.[1] Accordingly the equation of net advantages has been judiciously omitted by Jevons in his formulation of the cost of production. And the Helvetian Jevons, as we may call Professor Walras,

[1] Commercial competition might be likened to a system of lakes flowing into each other; industrial competition to a system of vessels so communicating by means of valves, that when the level in one exceeded that of another to a certain extent, then *per saltum* a considerable portion of the contents of that one (a finite difference as compared with the differentials of the open system) is discharged into the other.

appears to have altogether made abstraction of the cost of production considered as importing sacrifice and effort.*

Professor Walras, illustrating the operation of a simple market, supposes each dealer, before going to market, to write down his scale of requirements—how much he would be willing to buy or sell at each price. From these data it would be easy to calculate beforehand the rate of exchange which would prevail in the market formed by those individuals. But, when we advance from the simplest type of market to the complexities introduced by division of labour, it is seen to be no longer a straightforward problem in algebra or geometry, given the natures of all the parties, to find the terms to which they will come. Here, even if we imagine ourselves in possession of numerical data for the motives acting on each individual, we could hardly conceive it possible to deduce *a priori* the position of equilibrium towards which a system so complicated tends.

Accordingly it may be doubted whether the direct use of mathematical formulæ extends into the region of concrete phenomena much below the height of abstraction to which Jevons has confined himself. However, the formulation of more complicated problems has still a negative use, as teaching the Socratic lesson that no exact science is attainable. As Dupuit, one of the greatest of Jevons's mathematical predecessors, points out, "Quand on ne peut savoir une chose, c'est déjà beaucoup que de savoir qu'on ne sait rien." [1] If, he says, the early theorists, instead of formulating the balance of trade, had confined themselves to declaring the question above their powers, they would probably have done a greater service than the successors who refuted them. So Cournot, referring to his own mathematical treatment of economics. "Aussi nos modestes prétensions étaient-elles non d'accroître de beaucoup la domaine de la science proprement dite, mais plutôt de montrer (ce qui a bien aussi son utilité) tout ce que nous manque pour donner la solution vraiment scientifique de questions que la polémique quotidienne tranche hardiment." [2] Similarly Jevons says,[3] "one advantage of the theory of economics, carefully studied, will be to make us very careful in our conclusions when the matter is not of the simplest possible nature."

In the vineyard of science to perform the part of a pruning-hook is an honourable function; and a very necessary one in this

* For a defence of this statement see below, p. 310.

[1] *Annales des Ponts et Chaussées*, 1844, p. 372.

[2] *Revue Sommaire.*

[3] *Theory of Political Economy*, p. 157, second edit.

age of luxuriant speculation, when novel theories teem in so many new economic journals. I give in the appended notes an example of this corrective process applied to a theory of great worth and authority, and concerning the most vital interests, such as the relations of employer and employed, and the socialistic attack on capital (*i*). In directing this weapon of criticism against Professor Walker, I act upon the Miltonic rule for selecting an adversary:

> "Best with the best, more glory will be won,
> Or less be lost."

In the preceding remarks I have had in view, as presumably most favourable to computation, the case of bargains in which there is competition on both sides. It is now to be added that the mathematical method is nearly as applicable to a regime of monopoly. Here Cournot, rather than Jevons, is our guide. Cournot's masterly analysis of the dealings between a monopolist seller and a number of buyers competing against each other has been copied out of mathematics into the vulgar tongue by many well known writers, and need not here be repeated (*k*).

It is in this department perhaps that we can best answer Cairnes's challenge to Jevons to produce any proposition discovered by the mathematical method which is not discoverable by ordinary reasoning. Not, indeed, that the economist is bound to answer that challenge; any more than, in order to prove the advantages of international trade, he is concerned to deny that claret may be produced in Scotland.

The following proposition is a particular case of a more general theorem given by Cournot. Let there be a railway and a line of steamers, each forming part of a certain through journey, and separately useless: the fares will be lower when both means of transport belong to a single company than where there is less monopoly, the two services being in the hands of two companies, each seeking its own gain independently of the other.*

The *rationale* of this somewhat paradoxical proposition is not easily discerned without the aid of symbols. Cournot, in a popular [1] redaction of the theories which he first conceived in a mathematical form, suggests, as a generally intelligible explanation, that it is better to be at the mercy of a single master than of several petty tyrants. But this seems to be a commonplace of the sort which, in the absence of rigid reasoning, has so often deceived the amateur economist. Might it not be applied to the case of monopoly in general?

* For a more exact statement, see E.

[1] *Revue Sommaire.*

It would be hard to say how much this remarkable proposition may add to the arguments in favour of the Government monopolising railways. Nor would I undertake to estimate the practical significance of Cournot's numerous mathematical theorems on the taxation of monopolists. We might perhaps compare the function of the sovereign science with respect to the theory of monopolies to the duty of Government as to their management—to exercise a general supervision without attempting to control details.

We have in the last few paragraphs been supposing monopoly on one side of the market, on the other side a public competing with each other. Let us now consider the bargain between two monopolists, whether individuals, or rather corporate trading bodies, combinations in the most general sense of the term. The mathematical analysis of this case brings very clearly into view the important property, which is not very prominent in writings of the pre-Jevonian era, that the bargain between two self-interested co-contractors is not determinate in the same sense as in a regime of perfect competition.

No doubt, if we take a very simple case—such as that imagined by De Quincey, of a bargain between the owner of a musical box and a colonist already on his way to a distant region where no luxuries can be purchased—it is easy to see that the bargain may settle down at any point between certain limits. But where both the amount of commodity to be sold and money to be paid are variable, as in the momentous case of the bargain between a combination of employers on the one hand and employees on the other, it is a less familiar truth that the terms of the contract are in general to some extent indeterminate. For instance, the bargain may be either all in the interest of the one party, say long hours and small pay, or on the other hand high wages with much leisure.

The significance of this proposition has been missed by many of those who have treated the subject without the aid of the appropriate apparatus. Some fail to see that there is any peculiarity in the bargain between isolated units. Another discerns the indeterminateness of the bargain only in the special case in which the article exchanged is a large indivisible object, like a house. Another limits the difficulty to the case of a single negotiation as distinguished from a contract which, as in the actual labour market, may be modified from time to time. Another tells us that in such a bargain the most anxious party gains least.

All these phrases seem to obscure the cardinal distinction that

perfect competition tends to a determinate settlement, whereas in a regime of combination a principle of adjustment is still to seek. What is that principle?

At a former meeting of the British Association, on the occasion of a discussion on sliding scales, I stated the difficulty which there might be, in the absence of competition, in defining fair wages and reasonable terms, and I asked the eminent professor who introduced the subject in what direction one should look for a solution of this difficulty. His reply imported, as I understood, that no other general rule can be given but this: to obtain a full knowledge of, and bring a candid judgment to bear on, all the circumstances relevant to each case. To which I would add that one circumstance relevant to this whole class of cases is just the fact that there is in the abstract such a marked difference between combination and competition.

Possibly the dry light of abstract science may enable us to see a little further into this difficulty. Analysis strongly suggests that the right solution is what may be called the utilitarian arrangement, that which is productive of the greatest sum-total of advantage for all concerned. The utilitarian determination is clearly discerned to be by no means necessarily coincident with the settlement towards which competition tends. For instance, the "vrai prix," in Condillac's sense, as determined by the play of supply and demand in the labour market, might be such that the entrepreneur class should take the lion's share, leaving the labourer a bare and painful subsistence; but there is no ground to believe that this is the best possible arrangement. From an abstract point of view it is by no means evident that a free labour market is the only way to equity, that any interference with it must involve injustice." [1] Nor need it appear "a great fundamental principle—as inevitable in its action as gravitation—that a fair day's wages for a fair day's labour is determined by the proportion which the supply in the market bears to the demand." [2] It may be true indeed, in a practical sense, that perfect competition is "not less harmonious and beneficent in its operation than gravity;" [3] but theoretically it is tenable that there is an adjustment of contracts more beneficent than that which the mechanical play of competition tends to establish (*l*).

To introduce these philosophical conceptions of utilitarianism will doubtless seem irrelevant to those who are immersed in the details of business. But the practical man should be reminded

[1] Danson, *Wealth of Households.* [2] Rupert Kettle on "Arbitration."
[3] Walker, *Political Economy.*

that in other spheres of action, politics and morals, the principle of utility, however badly received at first, has exercised a great influence—though doubtless not so great as was expected by the theorists of Bentham's school, and needing to be largely tempered with common sense.

Such, I think, are the principal points at which mathematical reasoning is capable of being applied to political economy. [In estimating the use of this method, it is natural to take as our standard the helpfulness of mathematics in other departments of science.

As compared with mathematical physics, the mathematical theory of political economy shows many deficiencies. First, there is the want of numerical data, which has been already noticed. It is true that there is a faint hope of obtaining what Jevons too confidently expected, statistical data for the relations between demand and price. It is true also that in the higher mathematics conclusions which are quantitative without being numerical are more frequent than is usually supposed. Some political economy is as exact as some mathematical physics. The fields cultivated by Section A and Section F may overlap, but it must be admitted that the best part of our domain corresponds to what is the worst part of theirs. If you inquire as to the products of inferior soils, we must confess, if we do not wish to conceal the nakedness of the land, that over a large portion of our territory no crop is produced. We are employed only in rooting out the tares which an enemy has planted. Much of our reasoning is directed to the refutation of fallacies, and a great part of our science only raises us to the zero point of nescience from the negative position of error. "Sapientia prima stultitiâ caruisse." In this introductory portion of political economy we have seen that the mathematical method is likely to be serviceable.[1]

It is not to be supposed, however, that the work of our section is wholly destructive; that like the islanders of whom it is said that they earned a precarious livelihood by washing one another's clothes, so we are occupied only in mangling each other's theories. Like imprudent sectaries, by our mutual recriminations we have obscured the virtues common to our profession. What Jevons said of Cairnes, that his own opinions were much more valuable than his objections against other people's opinions, is true of Jevons himself and other controversial economists. Now, this possibility of mutual misunderstanding by persons who are both in the right is connected with a circumstance which it is not

[1] See above, p. 281.

irrelevant here to notice. It is that in our subject, unlike physics, it is not often clear what is the prime factor, what elements may be omitted in a first approximation. One writer on rent may emphasise distance from the centres of population as the main attribute, and introduce fertility of soil as a perturbation of the abstract result given by the first view. Another fixes attention on the powers of the soil, and allows for other elements, as for friction. So, in the theory of money, the state of credit, or the quantity of metal, have each been regarded as the prime variable.[1] It need not be pointed out how unfavourable to exact science is such a state of the subject matter. Imagine an astronomer hesitating whether in the determination of Jupiter's movements the sun or the planet Saturn played the most important part. That is the condition of many of our speculations.

It will not be expected that from such materials any very elaborate piece of reasoning can be constructed. Accordingly another point of contrast with mathematical physics is the brevity of our calculations. The whole difficulty is in the statement of our problems. The purely computative part of the work is inconsiderable. Scarcely has the powerful engine of symbolic language been applied, when the train of reasoning comes to a stop. The case is like that of the "swell" in *Punch*, who, about to enter a hansom, inquires solicitously of the driver whether he has got a good horse. "Yes, sir; very good 'oss." "Aw—then dwive to next door." However, our road, though short, is so slippery as to require every precaution.

It follows that in economics, unlike physics, the use of symbols may perhaps be dispensed with by native intelligence. It must be admitted that the correct theory of value has been rediscovered by Menger, and restated by his follower, Böhm-Bawerk, without the explicit use of mathematics. Without the law, they have done by nature the things contained under the law. Still, under a higher dispensation, they might have attained greater perfection. Nor can equal accuracy be ascribed to all the followers of Menger. Nor is the terseness which comes of mathematical study a characteristic of this Austrian school (*m*).

Another point of contrast between the mathematical science of the physicist and the economist is that the former appeals to a larger public. Mathematics is as it were the universal language of the physical sciences. It is for physicists what Latin used to

[1] Compare Cournot: "Ce que l'un néglige dans une première approximation comme un fait secondaire et accessoire, un autre le regardera comme le fait principal et dominant."—*Principes*, Book IV. chap. vii.

be for scholars; but it is unfortunately Greek to many economists. Hence the writer who wishes to be widely read—who does not say, with the French author, *J'imprime pour moi*—will do well not to multiply mathematical technicalities beyond the indispensable minimum, which we have seen reason to suppose is not very large. The parsimony of symbols, which is often an elegance in the physicist, is a necessity for the economist. Indeed, it is tenable that our mathematical constructions should be treated as a sort of scaffolding, to be removed when the edifice of science is completed. As Professor Marshall, one of the highest authorities on this subject, says: "When a man has cleared up his mind about a difficult economic question by mathematical reasoning, he generally finds it best to throw aside his mathematics and express what he has to say in language that is understanded of the people." [1] Upon this view mathematical discipline might be compared to grammar or to the study of classical literature, which it is profitable to have learnt thoroughly, while it is pedantic to obtrude one's learning.

From these considerations it may appear that our little branch of learning is of quite a rudimentary form. The solid structure and regular ramifications of the more developed mathematical sciences are wanting. A less unfavourable contrast would be presented if we compared our method, not with applied mathematics generally, but with that particular branch of it which comes nearest to ours in its proximity to human interests—the use of the Calculus of Probabilities in social statistics.

There is really only one theorem in the higher part of the calculus, but it is a very difficult one, the theory of errors, or deviations from an average. The direct applications of this theory to human affairs are not very considerable. Perhaps the most conspicuous example is afforded by an investigation to which, if I had undertaken to review the work done in our subjects during the past year, I ought to have directed particular attention—Mr. Galton's rigid proof of the fact and amount of *regression*, or reversion, in children compared with parents and other relationships.

But, beyond the isolated instances in which the theory of deviations is applied in social statistics with the same strictness and cogency as in physics, there is a wide zone of cases in which the abstract theory is of use as giving us some idea of the value to be attached to statistical results. Mr. Galton justly complains of the statisticians who "limit their inquiries to averages, and do

[1] *Academy*, June 11, 1881.

not revel in the more comprehensive views" of the deviations from averages. "Their souls seem as dull to the charm of variety as that of the native of one of our flat English counties, whose retrospect of Switzerland was that, if its mountains could be thrown into its lakes, two nuisances would be got rid of at once." But great caution is required in transferring the theory of errors to human affairs; and the Calculus of Probabilities may easily be made, in Mill's phrase, the "opprobrium of mathematics."

Now, in all these respects there is a considerable resemblance between the higher parts of the two branches of science which are cultivated in this section. It may be said that in pure economics there is only one fundamental theorem, but that is a very difficult one: the theory of bargain in a wide sense. The direct application of mathematical reasoning is, as we have seen, limited—more limited, I think, than the corresponding function of the higher statistics. But, on the other hand, the regulative effect, the educational influence, of studies like those of Cournot and Jevons are probably very extensive.

How extensive, it would be difficult to decide without defining the limits of a province within which our special subject is included—the use of abstract reasoning in political economy. Now, on this vexed question, and with reference to the heated controversy between the historical and the deductive schools, the mathematical economist as such is not committed to any side. It may be dangerous to take wide general views; it may be better to creep from one particular to another rather than ascend to speculative heights. Our only question here is whether, if that ascent is to be made, it is better to ascend by the steep but solid steps of mathematical reasoning, or to beguile the severity of the ascent by the zigzag windings of the flowery path of literature. It is tenable that the former course is safest, as not allowing us to forget at what a dangerous height of abstraction we proceed. As Professor Foxwell has well said,[1] with reference to the mathematical methods in the hands of Jevons and Professor Marshall, "It has made it impossible for the educated economist to mistake the limits of theory and practice, or to repeat the confusions which brought the study into discredit and almost arrested its growth."

I trust that I have succeeded in distinguishing the question what is the worth of abstract reasoning in political economy from the much more easily answered question whether, if it is

[1] In his important letter on "The Economic Movement in England" in the *Quarterly Journal of Economics* for October, 1888.

worth doing, it is worth doing well.[1] The mathematical economist is concerned to separate his method from that mathematical and metaphysical reasoning which Burke repudiates as inapplicable to human affairs; from the abstract method which he has in view when he says:—"The geometricians . . . bring from the dry bones of their diagrams . . . dispositions that make them worse than indifferent about those feelings and habitudes which are the supports of the moral world."[2] Burke is referring to the Jacobin philosophers; but our withers are unwrung, if similar words should be applied to some of the "sophisters and calculators" of a later generation. Just as a political party, if popularly suspected of complicity with crime, would do well to take every opportunity of clearing themselves from that imputation, so the mathematical economist is called on to disown emphatically all sympathy with the flagrant abuses to which the injudicious use of abstract reasoning is undoubtedly liable.

To continue the comparison which I was instituting between the mathematical theory of economics and the Calculus of Probabilities, they have one very unpleasant property in common—a liability to slips. As De Morgan says,[3] "everybody makes errors in probabilities at times, and big ones." He goes on to mention a mistake committed by both Laplace and Poisson, the ineptitude of which he can only parallel by the reasoning of a little girl whom he had called a "daughter of Eve"; to which she retorted, "then you must be a daughter of Adam." It is not to be concealed that economic reasoning, even in its severest form, is sometimes equally inconsistent. I should have hesitated to assert that Cournot has made some serious mistakes in mathematics applied to political economy, but that the authority of the eminent mathematician Bertrand[4] may be cited in support of that assertion.

Again, the more abstract theories of value and of probabilities seem to resemble each other in their distance from the beaten curriculum. Each forms, as it were, a little isolated field on the rarely crossed frontier and almost inaccessible watershed between the moral and the physical sciences.

[1] *Cp.* Professor Foxwell, *loc. cit.* "What the new school protest against is first the unscientific and meagre way in which deduction was used. In their view, though it is worth while to study, and therefore worth while to study accurately, the workings of private interest under a system of competition, yet human nature is not all self-interest. . . ." [2] *Letter to a Noble Lord.*

[3] Writing to Sir W. R. Hamilton (*Life of Hamilton*, by R. Graves, vol. iii.).

[4] *Journal des Savants*, 1883. I hope to show on some future occasion that M. Bertrand's censures of Cournot and Professor Walras are far too severe.

The same character of remoteness belongs perhaps to another province, which is also comparable with ours—the mathematical side of formal logic, the symbolic laws of thought which Boole formulated. There was a certain congruity between Jevons's interest in his logical machine and in what he called the "mechanics of industry." But I venture to regard the latter pursuit as much more liberal and useful than any species of syllogism-grinding.

If you accept these parallels, you will perhaps come to the conclusion that the mathematical theory of political economy is a study much more important than many of the curious refinements which have occupied the ingenuity of scientific men; that as compared with a great part of logic and metaphysics it has an intimate relation to life and practice; that, as a means of discovering truth and an educational discipline, it is on a level with the more theoretical part of statistics; while it falls far short of mixed mathematics in general in respect of that sort of pre-established harmony between the subject matter and the reasoning which makes mathematical physics the most perfect type of applied science.

But we must remember—and the mention of the theory of probabilities may remind us—that any such judgment is liable to considerable error. We cannot hope to measure the utility of a study with precision, but rather to indicate the estimate on either side of which competent judges would diverge—a central point, which will be found, if I mistake not, equally removed from the position of Gossen, who compares the new science to astronomy, and the attitude of Dr. Ingram towards the researches which he regards as nothing more than "academic playthings, and which involve the very real evil of restoring the metaphysical entities previously discarded." [1]

One more general caution is suggested by another of the technical terms which we have employed. What we are concerned to discover is not so much whether mathematical reasoning is useful, but what is its "final utility" as compared with other means of research. It is likely that a certain amount of mathematical discipline—say as much as Mr. Wicksteed imparts in his excellent *Alphabet of Economic Science*—is a more valuable acquisition to a mind already stored with facts than the addition of a little more historical knowledge.

But, in reverting to the subject of final utility, I am reminded

[1] See the passage relating to Jevons in the article on Political Economy in the *Encyclopædia Britannica*, 9th edit.

that presidential addresses, like other things, are subject to this law; and that a discourse on method prolonged beyond the patience of the hearers is apt to become what Jevons called a *discommodity*.

Notes

(*a*) Simple Exchange.—The simplest case of exchange is where there are two large groups of uncombined individuals dealing respectively in two commodities, *e.g.*, corn and money. To represent the play of demand and supply, let any abscissa, Ox in Fig. 1, represent a certain price, and let the quantity of commodity demanded at that price be xp. The locus of p may be called the demand-curve. Similarly, xq represents the quantity offered at any price, Ox; and the locus of q is called the

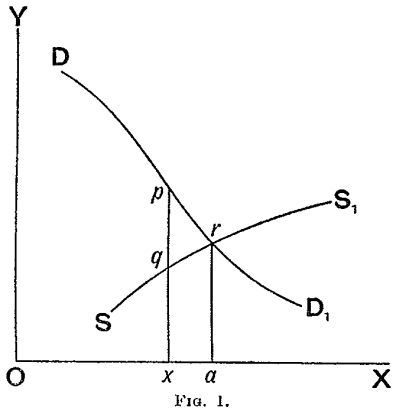

Fig. 1.

supply-curve. The price Oa, at which the demand is just equalled by the supply, is determined by the intersection of these curves. This is Cournot's construction. The converse construction, in which the abscissa stands for quantity of commodity, the ordinate for price, is employed by Mr. Wicksteed in his excellent *Alphabet of Economic Science*.

The diagrammatic representation which most closely corresponds to Jevons's formulæ is that which the present writer, after Professor Marshall, and Messrs. Auspitz and Lieben, independently, have adopted. In this construction the two co-ordinates respectively and symmetrically represent the quantities of the

two commodities exchanged, the *quid* and the *pro quo*. For instance, Fig. 2 may represent the state of supply and demand in the international market between Germany and England. The curve OE denotes that in exchange for any amount of "linen," Oy, England is prepared to supply the quantity of "cloth" yp $(= Ox)$; or, in other words, that in exchange for the quantity Ox of cloth England demands xp $(= Oy)$ of linen. The curve OG is similarly related to Germany's supply and demand. The position of equilibrium is determined by the intersection of these curves.

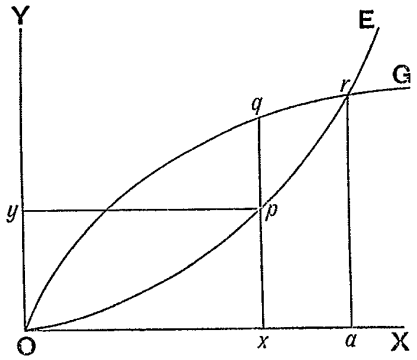

FIG. 2.

(*b*) VARIATIONS IN SUPPLY.—Suppose, as Mill supposes (*Political Economy*, Book III. chap. xviii. § 5), that there has occurred an improvement in the art of producing Germany's export, linen. The altered conditions of supply may be represented by the displaced curve OG', Fig. 3, indicating that whereas before the improvement Germany in exchange for any quantity, Ox, of cloth offered only xq, she now offers xq'. The effect of the improvement on the rate of exchange will depend upon the form of the curve OE beyond the point r. If the intersection of the curve OE is at r_2, vertically above r, we have the case where, as Mill rather awkwardly says, the demand of England for linen increases "in the same proportion with the cheapness." The other cases in which the demand for linen—and accordingly the price, so to speak, of cloth in linen—are increased more or less than the cheapness, are represented by the points of intersection r_1, r_3.

Again the same construction may be used to facilitate the comprehension of the theory of international trade which Professor Sidgwick has recently proposed. Let the curves OE and OG′ represent the conditions of supply and demand, on the hypothesis that cost of transport is annihilated, that England and Germany are in juxtaposition. Now restore the abstracted sea, and the altered conditions of supply and demand *in a market on the English shore* will be represented by the change of OG′ to OG. According to the form of the curve OE the different effects on the rate of exchange are visible at a glance. (*Cp.* Sidgwick, *Political Economy*, Book II. chap. ii. § 3.)

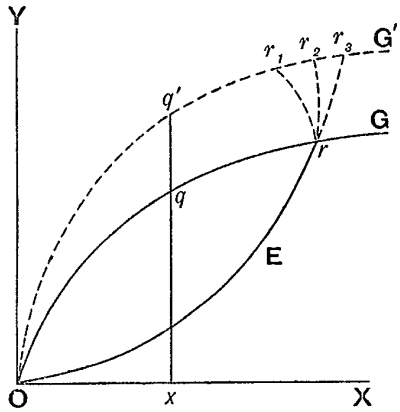

FIG. 3.

(*c*) GAIN OF TRADE.—To measure the variations in the advantage accruing from trade by the variations of price—or more generally rate of exchange—is a confusion which could hardly have occurred to the mathematical economist. The simplest method of illustrating the gain of trade is that proposed by Messrs. Auspitz and Lieben. In Fig. 4, let O*n* be the locus of a point *t*, such that a certain individual in exchange for the quantity O*x* of one commodity will just be willing to give the quantity *tx* of another commodity, will neither gain nor lose by that bargain. Then, if he obtain O*x* in return for only *rx*, he is a gainer by that bargain to the extent of *tr*. The curve thus defined is called the *utility-curve*.

By combining properly the utility-curves for all the individuals of a community, we obtain what may be called a

collective utility-curve. There is a peculiar propriety in taking one axis, say the ordinate, to stand for money. Let ON then in Fig. 5 be the collective utility-curve, in this sense, for the

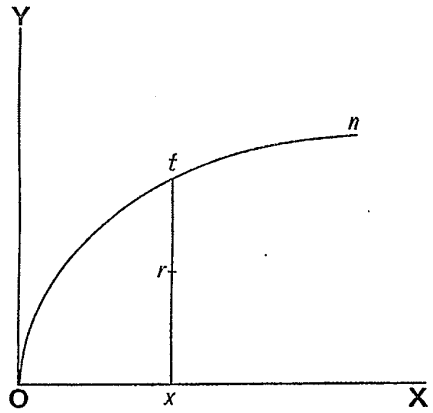

FIG. 4.

German community with respect to cloth. Let OG represent the demand of Germany for cloth, as before, except that the ordinate now stands for money, not linen. And let OE represent

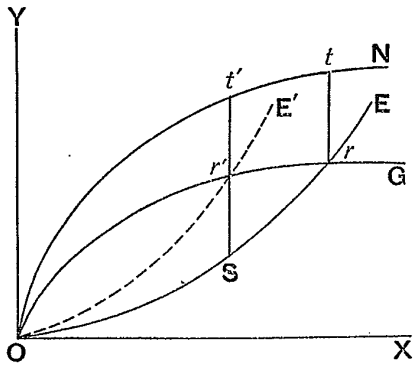

FIG. 5.

the supply of cloth in exchange for money on the part of England. Then the gain to Germany of the trade with England is represented by a vertical distance *tr*.

Now let Germany impose a tax on the import of cloth. The effect of the tax will be to displace the supply-curve OE in the manner indicated by the dotted curve OE′. Let r' be the new point of intersection between the demand—and (displaced) supply-curve. The advantage to Germany in the way of trade is now $t'r'$. To which is to be added the tax r'S—accruing to Germany. Since t'S may very well be greater than tr, Germany may gain by the imposition of the tax.[1]

What difficulties the reader may feel about the proposition will disappear on reference to Messrs. Auspitz and Lieben's beautiful and original reasoning (*Theorie der Preise*, §§ 80–82). In the light of their constructions it will be at once seen what conditions of supply and demand are favourable to the endeavour of one nation to gain by taxing the imports from (or exports to) another. It will be noticed that the particular supposition entertained by Professor Sidgwick (Book III. chap. v. § 2)—that the quantity consumed of the taxed import is constant—is not essential.

It may be observed that the *utility-curve* is a particular case of the "*indifference-curve*" employed by the present writer (*Mathematical Physics*, p. 21). Also the lines tr and t'S are particular cases of the "preference-curve" (*ibid.*). If these more general conceptions are employed, the demonstration will not require that we should put the ordinate for money, regarded as a constant measure of utility. The interpretation assigned to the curves OG and OE in our second and third figures may still stand.

(*d*) Economic Equilibrium.—By analogy with well-known physical principles, economic equilibrium may be regarded as determined by the condition that the advantage of all parties concerned, the integrated utility of the whole economic system, should be a *maximum*. This *maximum* is in general subject, or in technical phrase *relative*, to certain conditions; in particular what Jevons called the "law of indifference," that in a market all portions of a commodity shall be exchanged at the same rate. But occasionally this condition is suspended: as often as we take what may be called a socialistic or utilitarian view as distinguished from that incommensurability of pleasures appertaining to different persons, which Jevons in a remarkable passage of his *Theory* (p. 15) has postulated. It will be found that this postulate

[1] The construction only applies accurately to trade on a scale so small that the opening of the trade does not alter the value of money in the country.

must be abandoned when we consider the gain of trade, as in our note (*c*), or the theory of combinations, as in note (*l*), and on other occasions.

In general, the first condition of a maximum, that the first term of variation should vanish, gives the Jevonian equations of exchange the demand curves of other writers.

The second condition of a maximum, that the second term of variation should be negative, finds its fulfilment in certain well known propositions which involve the conception of a decreasing rate of increase, viz. the law of diminishing returns, the law, or laws, of diminishing utility and increasing fatigue.

For some propositions it is proper to take account not only of the sign, but also the magnitude, of the second differential of utility. Thus when Professor Walker is contending that in case of "any increase of product resulting from the introduction of any new force into industry," the whole increment will fall to be added to the share of the working class; he argues, quite correctly upon his premises, that if the improvement does not "increase the amount of tools and supplies required in production"—since there is no greater demand for capital in the case supposed—"... there can be no increase in the rate or amount of interest" (*Quarterly Journal of Economics*, 1887, pp. 283 and 284). Analytically we should find that the variation in the rate of interest due to the disturbance of equilibrium, say $\Delta\iota$, was indefinitely small as compared with the variation in the rate of wages, say Δw, because the decrease in the rate at which the utility of capital increases is indefinitely great. The argument requires that this second differential should be immense at the position of equilibrium.

(*e*) COMPLEX EXCHANGE is the general case of simplex exchange above analysed. We have now several, instead of two, categories of dealers and commodities. In both cases equilibrium is determined upon the principle that each individual seeks to maximise his own advantage, subject to the conditions (1) that in a market there is only one price for any article, and (2) that all which is bought is sold, and all which is sold is bought. Let there be m dealers and n articles. And the first article being taken as the measure of value, let the prices of the remaining articles be $p_2, p_3, \ldots p_n$. Let the quantities of commodities bought or sold by any individual, say No. r, be $x_{r1}, x_{r2}, \ldots . x_{rn}$; each variable with its sign, *plus*, if bought, *minus*, if sold. Let the advantage of the individual, regarded as a function of his purchases and sales, be $\psi_r (x_{r1}, x_{r2} \ldots . x_{rn})$. There is

sought the system of values assigned to the variables for which this function is a maximum, subject (α) to the condition which follows from the first assumption above made: $x_{r1} + p_2 x_{r2} +$ *etc.* $+ p_n x_{rn} = 0$. In order to determine the maximum of ψ_r subject to this condition, we obtain (β) by the Calculus of Variations $(n - 1)$ equations of the form—

$$\left(\frac{d\psi_r}{dx_{r1}}\right) = \frac{1}{p_2}\left(\frac{d\psi_r}{dx_{r2}}\right) = \ldots = \frac{1}{p_n}\left(\frac{d\psi_r}{dx_{rn}}\right)$$

(with certain conditions as to the second term of variation). To which is to be added the equation (α). We have thus n equations relating to the r^{th} individual. The same being true of each of the m individuals, we have in all m n equations of the forms (α) and (β). We have also (γ), from the condition that everything which is bought is sold, and conversely, n equations of the following form: $x_{1s} + x_{2s} +$ *etc.* $+ x_{ms} = 0$.

But of the $(m + n)$ equations of the forms (α) and (γ) only $(m + n - 1)$ are independent. For adding the m equations of the form (α) we have:

$$\left.\begin{array}{rcccc} & (x_{11} & + x_{21} & + \ldots & + x_{m1} \\ + p_2(& x_{12} & + x_{22} & + \ldots & + x_{m2} \\ + & — & — & — & — \\ + p_n(& x_{1n} & + x_{2n} & + \ldots & + x_{mn} \end{array}\right\} = 0.$$

Now, if any $(n - 1)$ of the equations of the form γ, say all but the first, are given, then in the last written equation the coefficients of $p_2 \ldots p_n$ vanish. Therefore the first equation of the form (γ), viz. $x_{11} + x_{21} +$ *etc.* $+ x_{m1}$, is also given. We have thus $mn + (n - 1)$ equations to determine $mn + (n - 1)$ quantities, viz. the x variables, which are mn in number, and the $(n - 1)$ p's.

The great lesson to be learnt is this. The equations are *simultaneous*, and their solution *determinate*. That the factors of economic equilibrium are simultaneously determined is a conception which few of the literary school have received. The reader is referred for fuller statements to Professor Walras's *Économie Politique*, 2nd edit., and to Messrs. Auspitz and Lieben's Appendix IV.

(*l*) Commercial Competition.—Abstracting that change of occupations which Cairnes ascribed to "Industrial" as distinguished from "Commercial," competition (comp. Sidgwick's *Political Economy*, Book II. chap. i.), let us suppose that the x's of the last note, which primarily denoted commodities ready for immediate consumption, include also agencies of production, (the use of) land, labour, and capital. We may conceive entrepreneurs buying these agencies from landlords, labourers, and

capitalists, and selling finished products to the public. We have thus the appropriate idea of rent, wages, interest, and (normal) prices determined *simultaneously* (in the mathematical sense).

In a primary view of complex exchange it is proper with Jevons to regard each portion of commodity sold, each negative variable, say, $-x_{rs}$, as a deduction from an initial store, say ξ_{rs}. But when we consider production, we regard ξ as a function of the outlay of the entrepreneur. Supposing that the entrepreneur confines himself to the production of a single article, let the gross produce, in money, after replacing capital, be $f_r(c_r, l_r)$, where f_r is a function depending on the individual's skill, energy, opportunities, etc., c_r is the amount of capital borrowed by him, and l_r the number of acres of a certain quality which he rents. The net produce is obtained by deducting from this quantity the payments $c_r\iota + l_r\rho$, where ι is the rate of interest and ρ is the rent per acre. Thus the advantage which the entrepreneur seeks to maximise is of the form

$$\psi_r(x_{r1}, x_{r2} \ldots [f_r(c_r,l_r) - c_r\iota - l_r\rho] - x_{rr}p_{rr}, \ldots);$$

whence $\frac{df_r}{dc_r} = \iota$ and $\frac{df_r}{dl_r} = \rho$. The first of these equations expresses a well-known proposition regarding the final utility of capital. The second equation expresses a less familiar condition with respect to the number of acres which will be rented on an ideal supposition of the homogeneity and divisibility of land above the margin of cultivation.

What then, and where, is the Ricardian theory of rent? Its symbolic statement is $l_r\rho = f(c_r,l_r) - f(c_r, o) = f_r(c_r, l_r) - c_r \times \iota$; where $f(c_r,o)$ is the gross produce of c_r capital laid out by the individual numbered r, on land below the margin obtainable for nothing in as large quantities as desired. It will be found that these equations postulate that the quantity of land above the margin is small as compared with the number of applicants, and that $f(c_r, o)$ is identical with $c_r \times \iota$, which are the common Ricardian assumptions. The validity of these assumptions as a first approximation, the need of correction where greater accuracy is required (truths which some minds seem incapable of holding together), have been admirably pointed out by Mr. Sidgwick (*Political Economy*, Book II. chap. vii. § 2). The second approximations made by him may be usefully expressed in the symbols which have been proposed, or rather in those which the student may construct for himself. I do not put forward those which occur to me as the best—if, indeed, there is any absolutely

best in the matter of expression. For some purposes it would have been proper to take account of the various qualities of land (as I have elsewhere done—*British Association Report*, 1886). For other purposes it would be well to put labour hired by the entrepreneur as an independent variable. When this or any other variable is omitted, we are to understand that there is implied the best possible arrangements with respect to the variables which are not expressed. The nature of this implication is shown in the following note.

(*g*) So far we have been taking for granted that the entrepreneur does his best, without reference to the motives acting upon him, the pleasures procurable by the sale of his product. Formally it would be proper to take account that the utility-function ψ_r involves the *effort*, say e_r, explicitly, as fatigue diminishes advantage, and implicitly, as exertion increases production. Corresponding to the new variable we have a new equation, the complete differential of ψ_r, with reference to e_r, say $\left(\frac{d\psi_r}{de_r}\right)+\left(\frac{d\psi_r}{df_r}\right)\frac{df}{de_r}=0$. It is a nice question how far effort should be regarded as an independent variable; how far the essential principle of piece-work prevails in modern industry.

(*h*) INDUSTRIAL COMPETITION.—The condition that net advantages should be equal in industries between which there is mobility may thus be contemplated. Let us put the advantage of an individual, say No. r, engaged in the occupation s as a function of his net income, the price of the articles on which his expenditure is made, and the disutility of effort. Say ϕ_{rs} (f_{rs} (π_1, π_2, ... e_{rs}), $p_1, p_2 \ldots . - e_r$); where ϕ_{rs} is a utility-function, not necessarily the same for the same individual in different occupations, since his indulgences may vary with the nature of his employment; f_{rs} — a symbol not identical with the f of the last but one note—is the individual's net earnings in the business s, involving prices π_1, π_2, etc., of all manner of agents of production, involving also as stated in note (*g*) the effort e_{rs}; p_1, p_2, etc., are prices of articles of consumption as a function of which the individual's advantage may be obtained by means of the equations (α) and (β) in note (*e*)—eliminating the quantities consumed. The last variable in the function ϕ_{rs}, the explicit e_{rs}, has a negative sign prefixed, to indicate that the direct effect of increased fatigue is diminished advantage.

The equation of Net Advantages imports that the advantage, ϕ_{rr}, of the occupation of which the individual chooses is not less than ϕ_{rs}, the advantage of any other occupation open to him. It

is important to observe that for all occupations the complete differential with regard to e is zero; in symbols $\left(\frac{d\phi}{d\mathfrak{f}}\right)\frac{d\mathfrak{f}}{de} + \left(\frac{d\phi}{de}\right) = 0$. But this equation conveys no presumption that the final disutility in different occupations is the same that $\left(\frac{d\phi_{rs}}{de_{rs}}\right) = \left(\frac{d\phi_{rt}}{de_{rt}}\right)$. The equation of final disutility holds only where efforts and sacrifices are capable of being applied in "doses" to any number of occupations. The latter is the only case, I think, contemplated by Jevons in his analysis of Cost of Production (*Theory*, chap. v.). The inquiry, what is meant in general by saying that the cost of production of two articles is equal, must start from right conceptions about Final and Total Utility. But this is not the place to follow up the difficult investigation. I do not attempt here to discuss any matter fully, but only to illustrate the suitability of the subtle language of mathematics to economical discussions.

(*i*) PROFESSOR WALKER'S THEORY OF BUSINESS PROFITS.—Professor Walker's theory, as stated in the *Quarterly Journal of Economics* for April, 1887, involves the proposition that the remuneration of the lowest, the least gifted employers, is on a level with that of the labouring class. Concerned as we are here with methods rather than results, it is allowable to posit this premiss without expressing an opinion as to its accuracy. It is fortunate not to have to take side on an issue concerning which the highest authorities are divided, and statistical demonstration is hardly possible.

But, though the expositor of method is not called upon to dispute the truth of this proposition, he has something to say against the evidence which has been adduced in proof of it. He must enter a protest against the form of the following argument:—

"Let our hypothesis be clearly understood. We assume, first, that there is in a given community a number of employers, more or fewer, who alone are, by law or by custom, permitted to do the business of that community, . . . or else who are so exceptionally gifted and endowed by nature for performing this industrial function that no one not of that class would aspire thereto, or would be conceded any credit or patronage should he so aspire. Secondly, we assume that neither in point of ability nor opportunity has any one member of this class an advantage as against another . . . all being, we might say, the exact copies of the type taken, whether that should involve a very high or a comparatively low order of industrial power.

" Now, in the case assumed, what would be true of business profits, the remuneration of the employing class? I answer that if the members of this class were few, they might conceivably effect a combination among themselves, and . . . fix a standard for their own remuneration. . . . If, however, the community were a large one, and if the business class . . . were numerous, such a combination . . . would be impracticable, . . . the members of the business class would begin to compete with each other. From the moment competition set in it would find no natural stopping place until it had reduced profits to that minimum which, for the purposes of the present discussion, we call *nil*.

" What, in the case supposed, would be the minimum of profits? I answer: This would depend upon an element not yet introduced into our problem. The ultimate minimum would be the amount of profits necessary to keep alive a sufficient number of the employing class to transact the business of the community. Whether, however, competition would force profits down to this low point would depend on the ability or inability of the employing class to escape into the labouring class. We have supposed that labourers could not become employers; but it does not follow that employers might not become labourers and earn the wages of labourers. . . ."—(*Quarterly Journal of Economics*, 1887, p. 270 and context.)

This reasoning will puzzle those who have received the abstract theory of supply and demand as formulated by the mathematical school [above, notes (*a*) and (*d*)]. Because the dealers on one side of a market, as the employers in the labour market, compete against each other without combination, it does not follow that the advantage which they obtain from their bargains is *nil*. The *minimum* to which the play of competition tends is not necessarily small in the sense of a bare subsistence. It is a *minimum* only in the mathematical sense in which every position of equilibrium is a minimum (of potential energy in physics; in psychics, may we say, of potential utility. See note (*d*)).

Representing the entrepreneur's demand for work by the curve OG (Fig. 5), where the abscissa measures work done, and the ordinate money payable out of the wages and profit fund, and putting OE for the offer of the workmen, we see that the point r may be distant to any extent from the utility-curve ON, which indicates the advantage of a transaction (see note (*c*)). As far as abstract theory, without specific data, carries us, the

competing entrepreneurs may make very good bargains. They may be ever so prosperous; they may be, in Burke's fine phrase, "gambolling in an ocean of superfluity."

So far, on the hypothesis that neither in point of ability nor opportunity has any one member of this class an advantage as against another. The heterogeneity of faculty will, of course, introduce a graduation of gain. But in this flight of steps it is not necessary that the lowest should be on a level with the grade of common labour. The scale of profits may be a sort of Jacob's ladder, culminating in a paradise of luxury, and having its lowest rung suspended high above the plain of ordinary wages.

Let us suppose, however, that the writer has tacitly made some assumption as to the numbers of the "numerous" business class relatively to the "large" community (compare the parallel passages in his *Political Economy*, pars. 280, 236). Still what does the consideration of business profits as rent do more than the received principle of supply and demand? If the workmen, believing that in the distribution regulated by competition too much has been assigned to brain and too little to muscle, determine to reduce profits by means of a combination, should they stay their hand because they are told that profits (above the lowest grade) are of the nature of rent? The terms "rent" and "margin" may indeed suggest that the extra profits of the abler entrepreneurs exactly corresponds to their greater ability. It might seem that if, so to speak, we pushed down all the higher faculties to the level of the lowest grade of business power, the diminution of the total distributed, of the wages and profits fund, would exactly correspond to the subtraction from the earnings of the degraded entrepreneurs, while everything else remained constant. Conversely it might be argued that the increment of produce due to the existence of superior ability may justly be assigned as extra profit.

But how little appropriate is this precise conception will at once appear from the annexed diagram. Let OE in Fig. 6 represent the entrepreneur's demand, OW the workmen's offer of the labour, the abscissa representing work done, and the ordinate wages payable out of the wages and profits funds (abstraction being made of interest and rent for land). Let OE be formed by the composition of Oe_0, the collective demand curve for the lowest entrepreneurs, and one or more curves, such as Oe_1, appertaining to the entrepreneurs of higher ability. Now let us shrink these higher natures to the zero of business ability. The individual demand curve for each degraded entrepreneur

will become identical with that from which Oe_0 was formed (by the combination of all the demand-curves for the lowest grade). The new demand-curve will therefore be of the form OE′ intersecting with OW as at the point r'. (Whether the disturbance will stop there will depend upon the nature of the communication between the departments of employer and workman; whether the mobility is one-sided, like that of fluid allowed by a valve to escape from one vessel to another, but not back again—see the end of the passage cited on our p. 301 from the *Quarterly Journal of Economics*—or whether the permeation is perfect.) If Oe_0 is small, if the part played in production by the marginal employers is insignificant, it is probable that the annihilation of

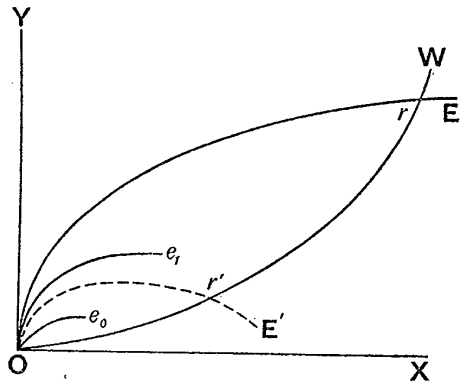

FIG. 6.

the higher grades will result in the destruction of the greater part not only of profits, but also of wages.

Accordingly it appears in general inexact to say that the "surplus which is left in the hands of the higher grades of employers . . . is of their own creation" (*Quarterly Journal of Economics*, April, 1887, pp. 274 and 275); if we define their own creation as the difference between the actual produce and that which would have existed if their superior faculties had not been exercised. In that sense (and what other sense is there?) the surplus of the higher grades is likely to be much *less* than their own creation (especially in the case where the marginal employers are relatively few). We seem to have proved too much. But may we not deduce the *quod est demonstrandum*, that actual profits are deserved, from the larger proposition that the entrepreneurs "own

creation" is by a certain amount greater than their profits? No; for that larger proposition is blocked by the antinomy that the workmen (or the higher grades of them) may by parity claim the greater part of the produce as *their* "own creation"—what would not have existed but for the exertion of their faculties.

In short, we know no more than we knew at first—viz. that the distribuend is produced jointly by the owners of brain and muscle, that the terms of the distribution are determined by supply and demand, and that in this, as in every other market, each more favoured nature enjoys a *rent*, or differential advantage [the nature of which is well illustrated by Messrs. Auspitz and Lieben's construction indicated in our note (c)]. That the surplus earning of the superior entrepreneur is his own creation is true of the individual, but not of the class; in division, but not in composition.

However, Professor Walker may have tacitly made some specific assumptions as to the quantities involved (*e.g.*, the proportion of produce with which the marginal entrepreneurs are concerned); or I may have misinterpreted his statements. Even so, the liability to such misconstruction is a defect in the purely literary method.

It would be easy in the case of less eminent writers to exemplify the part which the mathematical *organon* may play in lopping the excrescences of verbal dialectics. But I must content myself with briefly adverting to one of Professor Walker's critics, Mr. Sidney Webb. His able paper on the *Rate of Interest and the Laws of Distribution*, appears to me to contain several points deserving of attention; with respect to which mathematical conceptions may assist the reader in distinguishing the original from the familiar, and the true from the misleading.

(1) Mr. Webb restates the theory formulated by Jevons, that capital is ideally distributed according to the law of "equal returns to the last increments." (*Rate of Interest and Laws of Distribution*, by S. Webb, pp. 10, 11, and 21 of paper reprinted from *Quarterly Journal of Economics*, January, 1888.)

In symbols (see above, note (*f*), p. 562) let the net earning of any individual be $f_r(c_r) - \iota c_r$; where f_r is a function differing for different individuals according to their faculties and opportunities, c_r is the amount of borrowed capital employed by the individual; ι is the rate of interest; land and labour are not expressed. In equilibrium

$$\frac{df_r(c_r)}{dc_r} = \iota = \frac{df_s(c_s)}{dc_s} = \frac{df_t(c_t)}{dc_t} = \ldots$$

(2) Again, Mr. Webb discerns that the "law of diminishing returns" is applicable to capital as well as to land (*ibid.*, pp. 9 and 20, etc.). This is probably a new truth to the literary economist, who will have some difficulty in reconciling it with the *law of increasing returns* received into the text-books. To the mathematician it is evident that, in order to maximise the net earnings $f(c) - \iota c$, not only must the first differential of this expression vanish, but also the second differential $\frac{d_2 f}{dc^2}$ must be negative, which is the *law of diminishing returns*. It is quite consistent with the supposition that for certain values of the variable, not admissible as a solution of the problem, $\frac{df}{dc}$ should be positive, agreeably to the *law of increasing returns*.

(3) Mr. Webb dwells much on "the special industrial advantages not due to superiority of site or skill" which are enjoyed by some individuals. The use of an expression for the product like our f_r may serve at least to keep in mind the existence of such specialities. It also brings into view a difficulty which has not been sufficiently noticed by those who use *rent* in its metaphorical or secondary sense.

Suppose the extra produce is a function involving several variables (or parameters) like land, ability, opportunity. Say $f_r = \mathrm{F}(\lambda, \alpha, \omega \ . \ . \ .)$, where F is a form common to the community, and λ, α, ω denote the quality of land, ability, and opportunity peculiar to the individual. If the extra produce is $\mathrm{F}(\lambda\ \alpha\ \omega) - \mathrm{F}(o, o, o)$; is $\mathrm{F}(\lambda, o, o) - \mathrm{F}(o, o, o)$ the "economic rent" of land, $\mathrm{F}(o\ \alpha\ o) - \mathrm{F}(o, o, o)$ the rent of ability, $\mathrm{F}(o\ o\ \omega) - \mathrm{F}(o, o, o)$ the extra produce due to opportunity? (*Ibid.*, pp. 16 and 17.) If so, the three parts do not make up the whole!

(4) Anyway, to call the third extra produce *interest* is very unhappy. Its affinities are evidently with rent. (*Cp.* Sidgwick, *Political Economy*, Book II. chap. vii. § 4.)

(5) I should not have complained about the use of a term, but that it is connected with Mr. Webb's main contention against Professor Walker, to which I am unable to attach significance: that "this, not the 'rent of ability,' is the real keystone of the arch." (*Ibid.*, p. 17.) From the point of view here taken (above, p. 276) this search for the "keystone" among the factors of distribution is nearly as hopeless as the speculation of the ancients about the real *up* or *down*.*

* There is here omitted a note (designated "J" in the original) dealing with the phenomenon of "increasing returns," upon the supposition that the collective

(*k*) ONE-SIDED MONOPOLY.—In Fig. 7 let the curve DD′ represent the demand of the public for a monopolised article, the abscissa denoting price, the ordinate quantity. Then, as Cournot shows, if there are no expenses of production the rectangle O*yrx* should be a *maximum* (*Recherches*, art. 25); or rather the *greatest possible*. The solution is not likely to be indeterminate, except in the particular case where the demand-curve is an equilateral hyperbola. Indeterminateness is similarly exceptional when there are expenses of production (*cp.* Sidgwick, *Political Economy*, Book II. chap. ii. § 4).

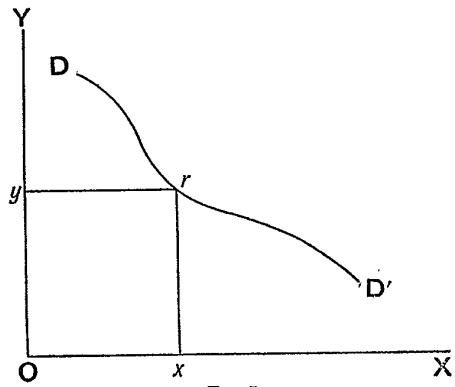

FIG. 7.

(*l*) TWO-SIDED MONOPOLY.—In Fig. 8 let O*p* and O*q* represent the curves of constant satisfaction, or indifference curves (above note (*c*); *Theorie des Preises*, Appendix II.; *Mathematical Psychics*, p. 21) drawn through O for two individuals or combinations respectively. Then the locus of bargains which it is not the interest of both parties to disturb is the *contract-curve*, *pq* (*Mathematical Psychics*, loc. cit.). At what point then on this curve will the transaction settle down? If we assume that the conditions of a market are retained, the required point is at *r* the intersection of the supply- and demand-curves which is on the contract-curve. That is the solution of Messrs. Auspitz and

supply-curve is formed by simple addition from curves of which each represents the offer of an individual at an assigned price, and at one and the same time. This treatment is now obsolete, not taking into account the possibility of "external economies" explained by Dr. Marshall subsequently to the publication of the present paper.

Lieben (*Theorie*, p. 381). It corresponds to the principle laid down by Professor Marshall for the action of arbitrators (referred to above in note to p. 279). But Professor Menger, who has a numerical scheme equivalent to a rudimentary contract-curve (*Grundsätze*, pp. 176–8), and Professor Böhm-Bawerk, referring to the "Spielraum" afforded by the indeterminateness of bargain, recommend to "split the difference." Instead of "equal," "*equitable*" division has been proposed by the present writer, namely, that adjustment which produces the maximum of utility to all concerned; not subject to the conditions of a market, but irrespectively thereof (equations (β) and (γ), without equation (α)

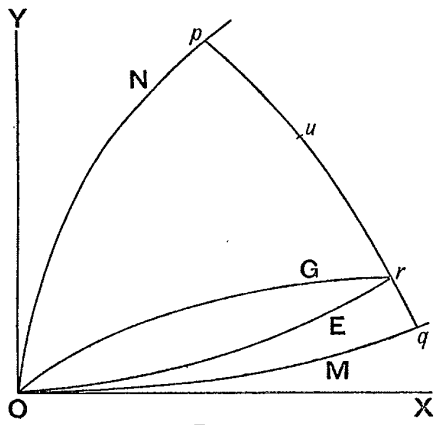

FIG. 8.

in note (*e*) above), the utilitarian arrangement, which also is represented by a point in the contract-curve, say u in Fig. 8. Such might seem to be the ideally most desirable arrangement; but very likely the practically best, the $\pi\rho\alpha\kappa\tau\grave{o}\nu\ \grave{\alpha}\gamma\alpha\theta\acute{o}\nu$, is in the neighbourhood indicated by Professor Marshall and Messrs. Auspitz and Lieben.

(*m*) THE AUSTRIAN SCHOOL.—Professor Menger and his followers have expressed the leading propositions of the economic calculus—the law of diminishing utility, the law of demand and supply, and so forth—by means of numerical examples, supplemented with copious verbal explanation. Their success is such as to confirm the opinion that the mathematical method is neither quite indispensable nor wholly useless, *nec nihil nec*

omnia, like most scientific appliances. Conceding that in the main they impart a saving knowledge of the true theory of value, it may still be maintained that they occasionally emphasise the accidents of a particular example as if they formed the essence of the general rule; that their explanations are excessively lengthy; and yet their meaning sometimes is obscure. For instance Professor Böhm-Bawerk may seem to attach undue importance to his conception of the *Grenzpaar*. He illustrates the play of demand and supply by supposing a market in which on the one hand there are a number of dealers each with a horse to sell, and on the other hand a number of would-be buyers (Konrad's *Jahrbuch*, 1886; *Kapital* . . ., p. 211. *Cp.* Mr. James Bonar's excellent article in the *Quarterly Journal of Economics*, October, 1888). The latter are arranged in the order of their strength: first, the one who is prepared to give most for a horse, the highest price which the second can afford is less, and so on. Parallel to this arrangement is that of the would-be sellers: first, he who can sell cheapest; and so on. Upon this hypothesis it might happen that the fifth would-be buyer is willing to give a little more than the lowest figure which the fifth would-be seller will take; while the sixth on the side of the buyers is not willing to give quite as much as the sixth horse-dealer stands out for. In this case five horses only will be sold; and the couple who are the last between whom a bargain is possible—buyer No. 5 and seller No. 5—enjoy a mighty distinction as the *Grenzpaar;* an honour which is to some extent shared with No. 6, the first couple between whom a bargain is impossible.

Now this attention to a particular couple is not always appropriate. How if the weakest actual buyer should prove to be, not buyer No. 5 but buyer No. 1, as to a *second* horse? Professor Böhm-Bawerk, indeed, has thought of this case, and called attention to it in a note to his later redaction (*Kapital* . . ., p. 218).* So far—although the whole simplicity of the scheme is destroyed when we permit second and third horses to the different buyers and sellers—the conception of a "limiting couple" may still be retained. It will be found, however, that this idea is not appropriate to the general case of a divisible commodity, which a single individual on one side of the market may buy from or sell to a large number on the other side. That general case is much more clearly represented by a diagram like Fig. 9, where the inner broken curves represent the dispositions of the individual dealer, the outer thick curves collective supply and

* But see note to B, Vol. I. p. 37 below.

demand (*cp.* Auspitz and Lieben, *Theorie des Preises*). No doubt Professor Böhm-Bawerk's conception is appropriate to a particular case, that in which the *Kleinste Marktübliche Mengeneinheit*, in the phrase of Messrs. Auspitz and Lieben (*ibid.*, p. 123), is considerable. But it is better with those eminent theorists to begin with the general or, at least, the simple case.

As an instance of the excessive circumlocution to which the purely literary method is liable, we may notice the doctrine of objective and subjective value, which occupies many pages in one of the works to which we have referred. Is there really much

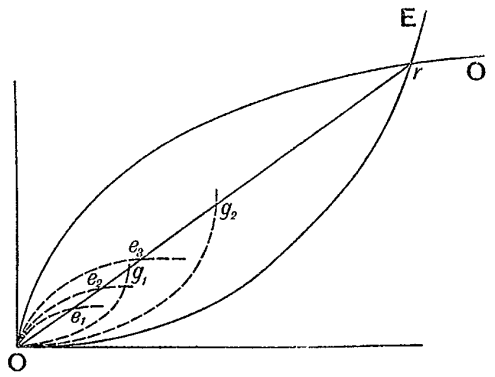

FIG. 9.

more in the distinction than what is visible on a glance at the appropriate diagram? The individual's subjective estimate of worth is expressed by his particular demand- or supply-curve Oe_1, Oe_2, Og_1, Og_2, etc., in Fig. 9. The proper combination of those individual curves gives the collective demand- and supply-curves, of which the intersection represents the "objective" value.

Moreover, verbal circumlocutions are so little adapted to express mathematical conceptions that we are sometimes left in uncertainty as to our author's meaning. When Professor Böhm-Bawerk remarks that there is something special in the labour market, in that the buyer will vary his arrangements according to the price of the article, the rate of interest (*Kapital . . .*, p. 407), does he specify the property which Messrs. Auspitz and Lieben have stated as general; that the utility function (our ψ, note (*e*) above) is discontinuous, being different for large and small values of the variable under consideration?

These deficiencies are more conspicuous in other writings of the Austrian school. A glance at Fig. 9, an intuition of the corresponding algebraic formulæ, will show that the notion of an *average* imported into the doctrine of value by Dr. Emil Sax (*Staatswirthschaft*) is not quite appropriate. As an instance in which great abridgment would be effected by mathematical expression, we might notice the last three chapters of Dr. Zuckerkandl's *Theorie des Preises*. Again the difficulty of conveying technical propositions without the proper phraseology may be illustrated by Professor Wieser's *Der natürliche Werth*, when he speaks of value and final utility having place in a communistic or socialistic state (page 26 note and *passim*). May his meaning thus be formulated? In an economical regime distribution and exchange are regulated by the condition that the final utility of all concerned should be zero, the total utility a maximum, *subject to the law that there should be only one rate of exchange in a market.* In a communistic or utilitarian regime the limitation which the last italicised clause expresses is removed. In terms employed in our note (*e*) the economical adjustment is determined by the equation (α), (β), and (γ); the utilitarian adjustment is determined by (β) and (γ) only—in short, there is the distinction between the points r and u in our Fig. 8 referring to note (*e*).

In offering these trenchant criticisms I regret that my limits impose a curtness which is hardly consistent with courtesy.

(Note referring to p. 281.)

[A passage in this paper relating to Walras was severely criticised by Professor Bortkevitch in the *Revue d'Économie Politique*, January–February, 1890. He rightly connected the passage with, and interpreted it by, a review of Walras's *Éléments d'Économie Politique Pure* (deuxième édition) which I had contributed to *Nature* (September, 1889). I replied to Professor Bortkevitch's criticisms in the *Revue d'Économie Politique*, January 1891. Looking back at these controversial writings—the Address here reprinted and the contemporary review in *Nature*, Professor Bortkevitch's criticism, and my rejoinder—I desire to withdraw any passage which suggests that Walras was at a disadvantage, as compared with Jevons, in that he did not analyse the final disutility of labour so fully and exactly as Jevons did. Professor Bortkevitch (*loc. cit.*) and Professor Pareto (*Giornale degli Economisti Cambi Forestieri*, 1894) are right about that. It was not necessary for Walras's purpose that he should do in that matter exactly as Jevons had done. There is not the connection, which I incidentally suggested but now disown, between Walras's difference from Jevons in this respect and his agreement with Jevons

in another respect which constituted the gravamen of my criticism. But I do not retract the remarks which apply equally to Jevons and Walras: namely, that while they elaborated "final"—or, as we now say, "marginal"—conditions, they left untouched the all-important condition that the *total* utility—the "net advantages," in the phrase introduced by Marshall—in different occupations should be equal (above, p. 280). It was not made a matter of reproach that this condition had not been treated mathematically. Rather, it was pointed out that it would have been very difficult to do so. In my rejoinder of 1891 I illustrate the difficulty by reference to a problem in Todhunter's *Calculus of Variations*, which has for *quæsitum* not simply a *maximum* (or *minimum*), but the *greatest* (or *least*) *possible* value; to determine the course of a ship between two given points, so that the voyage may be accomplished in the shortest possible time. A stroke of the pen gives us the *marginal* condition, from which it follows that the course must be rectilinear. But a series of tentatives may be required to determine what combination of right lines affords the quickest passage. (On the difference between a *maximum* and the *greatest possible* value as the *quæsitum* compare ECONOMIC JOURNAL, 1922, p. 435, and references there given.)

It is not a matter of reproach to mathematical economists to have stopped short of these complexities. But economic theory, as distinguished from mathematical expression, does require the recognition of the important principle designated by Cairnes as "Industrial Competition" and its opposite, "Non-competing Groups." Now Walras's peculiar doctrine of an entrepreneur who makes neither gain nor loss cuts him off from this essential principle. Nothing must everywhere be equal to nothing; and it is difficult to see how the equality—or it may be inequality—of profits in different occupations can be reconciled with this favourite tenet of the Lausanne School. Of course it may be tolerated as an extreme abstraction; a simplification permissible to a path-breaker (*cp.* Pareto, *loc. cit.*). But it seems to deserve pardon rather than praise.

In two of the writings mentioned, the review in *Nature* and the article in the *Revue d'Économie Politique*, criticism is directed against another part of Walras's *Éléments:* namely, his description of the process by which the price in a market is evolved. He describes *a* way rather than *the* way by which economic equilibrium is reached. For we have no general *dynamical* theory determining the path of the economic system from any point assigned at random to a position of equilibrium. We know only the statical properties of the position; as Jevons's analogy of the lever implies (*Theory*, p. 110 *et seq.*). Walras's laboured description of prices set up or "cried" in the market is calculated to divert attention from a sort of higgling which may be regarded as more fundamental than his conception, the process of *recontract* as described in these pages and in an earlier

essay (see β). It is believed to be a more elementary manifestation of the propensity to truck than even the effort to buy in the cheapest and sell in the dearest market. The proposition that there is only one price in a perfect market may be regarded as *deducible* from the more axiomatic principle of recontract (*Mathematical Psychics*, p. 40 and context).]