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#### Abstract

We show that the two sequencs $a(n)$ and $b(n)$ (defined below) have no common element. That is, for all positive integers $n$ and $m$, we never have $a(n)=b(m)$. The proof depends on the fact that the two Chebyshev sequences $T_{n}(x), x=2,3$ can be used to aproximate $a$ and $b$ and it is easy to prove that $\left|T_{n}(2)-T_{m}(3)\right| \geq 1$ for all positive integers $n \neq m$.


Recall that $t(n)=\frac{n(n+1)}{2}$ is the $n$-th trianglular number.
Theorem 1. For a given positive integer e, there never exist positive integers $n$ and $m$ such that

$$
2 t(e)=t(n) \text { and } 3 t(e)=t(m)
$$

The proof follows after we develop some facts.
The sequence of positive integers $a(e)$ for which there exists a positive integer $n$ such that $2 t(a(e))=t(n)$ is item A053141 in EIS. ${ }^{1}$.

The sequence of positive integers $b(e)$ for which there exits a positive integer $m$ such that $3 t(b(e))=t(m)$ is item $A 061278$ in EIS. .

The basic formulas we will need need are:

1. $p=\frac{1}{4}-\frac{1}{4 \sqrt{2}}$ and $q=\frac{1}{4}+\frac{1}{4 \sqrt{2}}$
2. $r=\frac{1}{4}-\frac{1}{4 \sqrt{3}}$ and $s=\frac{1}{4}+\frac{1}{4 \sqrt{3}}$
3. $\alpha=3-2 \sqrt{2}$ and $\beta=3+2 \sqrt{2}$
4. $\gamma=2-\sqrt{3}$ and $\delta=2+\sqrt{3}$

Using standard matrix methods we can develop the following explicit formulae for the sequences $a$ and $b$

$$
a(n)=-\frac{1}{2}+(3-2 \sqrt{2})^{n}\left(\frac{1}{4}-\frac{1}{4 \sqrt{2}}\right)+(3+2 \sqrt{2})^{n}\left(\frac{1}{4}+\frac{1}{4 \sqrt{2}}\right)
$$

[^0]and
$$
b(n)=-\frac{1}{2}+(2-\sqrt{3})^{n}\left(\frac{1}{4}-\frac{1}{4 \sqrt{3}}\right)+(2+\sqrt{3})^{n}\left(\frac{1}{4}+\frac{1}{4 \sqrt{3}}\right) .
$$

This reminds us of an explicit formula for the ChebyshevT polynomials of the first kind:

$$
T_{n}(x)=\frac{1}{2}\left(\left(x-\sqrt{x^{2}-1}\right)^{n}+\left(x+\sqrt{x^{2}-1}\right)^{n}\right)
$$

which yields for $x=2,3$

$$
T_{n}(2)=\frac{1}{2}\left(\gamma^{n}+\delta^{n}\right) \text { and } T_{n}(3)=\frac{1}{2}\left(\alpha^{n}+\beta^{n}\right),
$$

where $\alpha, \beta, \gamma$ and $\delta$ are defined above.
Then we can write:

1. $a(n)=p \alpha^{n}+q \beta^{n}-\frac{1}{2}$
2. $b(n)=r \gamma^{n}+s \delta^{n}-\frac{1}{2}$

Lemma 1. As $n \rightarrow \infty, p \alpha^{n} \rightarrow 0$ and $r \gamma^{n} \rightarrow 0$. Therefore as $n \rightarrow \infty$,

$$
q \beta^{n} \rightarrow a(n)+\frac{1}{2} \text { and } s \delta^{n} \rightarrow b(n)+\frac{1}{2}
$$

Summing up, using the equations preceeding this lemma, we have this result: As $n \rightarrow \infty$,

$$
2 T_{n}(3) \rightarrow \beta^{n} \text { and } 2 T_{n}(2) \rightarrow \delta^{n},
$$

and this leads finally to: as $n \rightarrow \infty$

$$
2 q T_{n}(3) \rightarrow a(n)+\frac{1}{2} \text { and } 2 s T_{n}(2) \rightarrow b(n)+\frac{1}{2} .
$$

Our theorem at the beginning of this article requires us to show that no $a(n)$ can equal a $b(m)$. First we show this is true for the sequencs $T_{n}(2)$ and $T_{n}(3)$. That is we show $T_{n}(2)=T_{m}(3)$ is impossible for any two integers $n$ and $m$. In fact this is quite simple because these two sequences satisfy Pell equations. That is, for any positive integer $n$ there always exist two positive integers $x$ and $y$ so that the following equations are true:

$$
T_{n}(2)^{2}-3 x^{2}=1 \text { and } T_{n}(3)^{2}-8 y^{2}=1
$$

So suppose for two positive integers $n$ and $m$ there exists an integer $k$ such that $k=T_{n}(2)=T_{m}(3)$. Then

$$
k^{2}-3 x^{2}=1 \text { and } k^{2}-8 y^{2}=1
$$

for positive integers $x$ and $y$. This leads to $\left(\frac{x}{y}\right)^{2}=\frac{8}{3}$ which is impossible.
Here is what we know. First, for all positive integers $n$ and $m$,

$$
\left|T_{n}(2)-T_{m}(3)\right| \geq 1
$$

To continue: For all positive integers $n \geq m$ we have $T_{n}(3)>T_{m}(2)$ and hence:

$$
\left|q T_{n}(3)-s T_{m}(2)\right|=s\left(\frac{q}{s} T_{n}(3)-T_{m}(2)\right)>s\left|T_{n}(3)-T_{m}(2)\right|
$$

because $q>s$.
Proof. We have proven that

$$
2 q T_{n}(3) \rightarrow a(n)+\frac{1}{2} \text { and } 2 s T_{n}(2) \rightarrow b(n)+\frac{1}{2}
$$

and that $\left|q T_{n}(3)-s T_{m}(2)\right|$ can never approach zero which means $|a(n)-b(m)|$ can never be zero.

## References

1. I. G. Bashmakova. Diophantus and Diophantine Equations. The M.A.A. Dolciana Mathematical Expositions-No. 20., (1997)
2. Graham Everest et al Reccurence Sequences. Mathematical Surveys and Monographs Vol. 104
3. Encyclopedia of Integer Sequences, available at http://www.research.att.com/~njas/sequences/Seis.html.

[^0]:    ${ }^{1}$ EIS is The Encyclopedia of Integer Sequences

