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## Abstract

We show that the two sequences a(n) and b(n) (defined below) have no common element. That is, for all positive integers n and m, we never have a(n) = b(m). The proof depends on the fact that the two Chebyshev sequences  $T_n(x)$ , x = 2, 3 can be used to approximate a and b and it is easy to prove that  $|T_n(2) - T_m(3)| \ge 1$  for all positive integers  $n \ne m$ .

Recall that  $t(n) = \frac{n(n+1)}{2}$  is the *n*-th trianglular number.

**Theorem 1.** For a given positive integer e, there never exist positive integers n and m such that

$$2t(e) = t(n)$$
 and  $3t(e) = t(m)$ .

The proof follows after we develop some facts.

The sequence of positive integers a(e) for which there exists a positive integer n such that 2t(a(e)) = t(n) is item A053141 in EIS.<sup>1</sup>.

The sequence of positive integers b(e) for which there exits a positive integer m such that 3t(b(e)) = t(m) is item A061278 in EIS.

The basic formulas we will need need are:

1.  $p = \frac{1}{4} - \frac{1}{4\sqrt{2}}$  and  $q = \frac{1}{4} + \frac{1}{4\sqrt{2}}$ 2.  $r = \frac{1}{4} - \frac{1}{4\sqrt{3}}$  and  $s = \frac{1}{4} + \frac{1}{4\sqrt{3}}$ 3.  $\alpha = 3 - 2\sqrt{2}$  and  $\beta = 3 + 2\sqrt{2}$ 4.  $\gamma = 2 - \sqrt{3}$  and  $\delta = 2 + \sqrt{3}$ 

Using standard matrix methods we can develop the following explicit formulae for the sequences a and b

$$a(n) = -\frac{1}{2} + (3 - 2\sqrt{2})^n \left(\frac{1}{4} - \frac{1}{4\sqrt{2}}\right) + (3 + 2\sqrt{2})^n \left(\frac{1}{4} + \frac{1}{4\sqrt{2}}\right)$$

<sup>&</sup>lt;sup>1</sup>EIS is The Encyclopedia of Integer Sequences

and

$$b(n) = -\frac{1}{2} + (2 - \sqrt{3})^n \left(\frac{1}{4} - \frac{1}{4\sqrt{3}}\right) + (2 + \sqrt{3})^n \left(\frac{1}{4} + \frac{1}{4\sqrt{3}}\right)$$

This reminds us of an explicit formula for the ChebyshevT polynomials of the first kind:

$$T_n(x) = \frac{1}{2} \left( (x - \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^n \right)$$

which yields for x = 2, 3

$$T_n(2) = \frac{1}{2} (\gamma^n + \delta^n) \text{ and } T_n(3) = \frac{1}{2} (\alpha^n + \beta^n),$$

where  $\alpha, \beta, \gamma$  and  $\delta$  are defined above.

Then we can write:

1.  $a(n) = p\alpha^n + q\beta^n - \frac{1}{2}$ 2.  $b(n) = r\gamma^n + s\delta^n - \frac{1}{2}$ 

**Lemma 1.** As  $n \to \infty$ ,  $p\alpha^n \to 0$  and  $r\gamma^n \to 0$ . Therefore as  $n \to \infty$ ,

$$q\beta^n \to a(n) + \frac{1}{2} \text{ and } s\delta^n \to b(n) + \frac{1}{2}.$$

Summing up, using the equations preceeding this lemma, we have this result: As  $n \to \infty$ ,

$$2T_n(3) \to \beta^n \text{ and } 2T_n(2) \to \delta^n,$$

and this leads finally to: as  $n \to \infty$ 

$$2qT_n(3) \to a(n) + \frac{1}{2} \text{ and } 2sT_n(2) \to b(n) + \frac{1}{2}.$$

Our theorem at the beginning of this article requires us to show that no a(n) can equal a b(m). First we show this is true for the sequences  $T_n(2)$  and  $T_n(3)$ . That is we show  $T_n(2) = T_m(3)$  is impossible for any two integers n and m. In fact this is quite simple because these two sequences satisfy Pell equations. That is, for any positive integer n there always exist two positive integers x and y so that the following equations are true:

$$T_n(2)^2 - 3x^2 = 1$$
 and  $T_n(3)^2 - 8y^2 = 1$ 

So suppose for two positive integers n and m there exists an integer ksuch that  $k = T_n(2) = T_m(3)$ . Then

$$k^2 - 3x^2 = 1$$
 and  $k^2 - 8y^2 = 1$ 

for positive integers x and y. This leads to  $(\frac{x}{y})^2 = \frac{8}{3}$  which is impossible. Here is what we know. First, for all positive integers n and m,

$$|T_n(2) - T_m(3)| \ge 1.$$

To continue: For all positive integers  $n \ge m$  we have  $T_n(3) > T_m(2)$  and hence:

$$|qT_n(3) - sT_m(2)| = s(\frac{q}{s}T_n(3) - T_m(2)) > s|T_n(3) - T_m(2)|$$

because q > s.

*Proof.* We have proven that

$$2qT_n(3) \rightarrow a(n) + \frac{1}{2}$$
 and  $2sT_n(2) \rightarrow b(n) + \frac{1}{2}$ 

and that  $|qT_n(3)-sT_m(2)|$  can never approach zero which means |a(n)-b(m)|can never be zero. 

## References

- 1. I. G. Bashmakova. Diophantus and Diophantine Equations. The M.A.A. Dolciana Mathematical Expositions-No. 20., (1997)
- 2. Graham Everest et al Reccurence Sequences. Mathematical Surveys and Monographs Vol. 104
- 3. Encyclopedia of Integer Sequences, available at http://www.research.att.com/~njas/sequences/Seis.html.