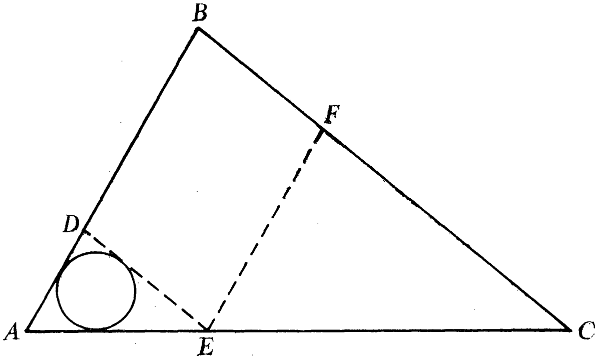

FIG. 12

REMARK 2. If the triangle T in Problem 3 is replaced by any convex polygon, it is clear that the proofs of the properties of γ still hold except that the intersection of γ with a given side of the polygon may be a single point or empty. However, it is still true that γ is composed of circular arcs of equal radii each tangent to each side that it meets together with line segments contained in sides of the given polygon.

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VENN DIAGRAMS AND INDEPENDENT FAMILIES OF SETS

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1. Introduction. Let $\mathscr{A} = \{A_1, \cdots, A_n\}$ be a family of n simple closed curves in the Euclidean plane. We shall say that $\mathscr{A}$ is an *independent family* provided the intersection

$$X_1 \cap X_2 \cap \cdots \cap X_n \text{ is nonempty} \qquad (*)$$

whenever each set X_j is chosen to be either the interior or the exterior of the curve A_j.

For example, the family of 3 circles and the families of 4 or 5 congruent ellipses in Figure 1 are independent.

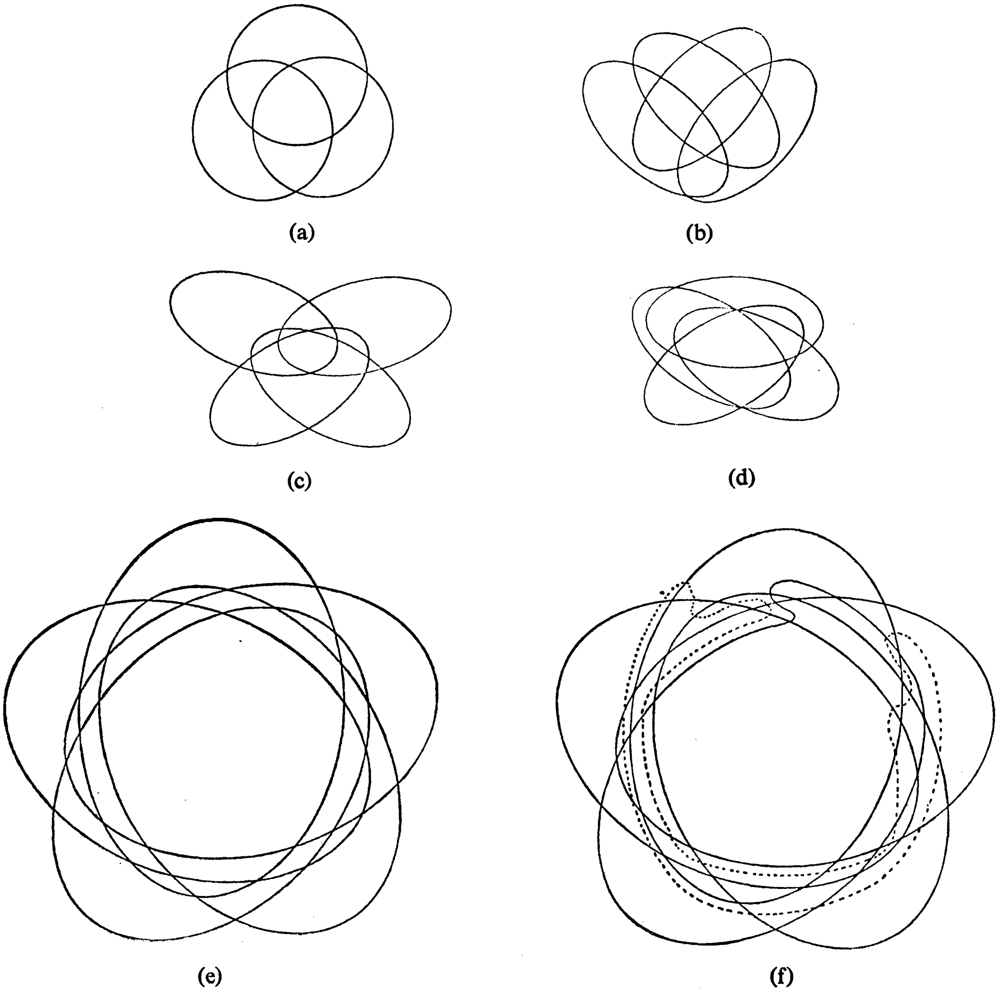


FIG. 1

Independent families and their higher dimensional analogues are interesting from several points of view; for example, Marczewski [1947] considers them in connection with certain extension problems in measure theory. We are more interested in their geometric properties, and in particular with the special types of independent families that are usually called *Venn diagrams*. Introduced by Venn [1880], the Venn diagrams are meant to facilitate and make visually accessible the relations of classes in first-order predicate calculus; various markings, hatching, etc., that are occasionally used for such purposes are of no importance for our discussions. Venn diagrams have been widely popularized by many texts on elementary logic and set theory, starting with Venn [1881]; among more modern ones we may mention Quine [1950], Birkhoff-MacLane [1953], Rosser [1953], Suppes [1957], Kenelly [1967], Gardner [1968], Roethel-Weinstein [1972]. Unfortunately and quite ironically, logicians and set theorists are mostly not interested in geometry and as a consequence their definitions are rather vague as to the precise requirements on an independent family to be considered a Venn diagram. In many of the books the discussion is limited to $n \leqq 3$, and a few even express doubts in the possibility or the feasibility of Venn diagrams with large n. Moreover, insofar as the definitions can be interpreted at all, different authors appear to have different limitations in mind when speaking of Venn diagrams, and even individual authors seem to vary their interpretations from instance to instance. For example, Venn [1880; 1881] seems to require that each A_j be a simple closed curve, but one of the examples he gives for $n = 5$ fails to have that property. In Austin [1971] each of the sets in (*) is required to be connected, but the last two illustrations fail to have that property. While Austin [1971] requires that no point should belong to 3 of the curves A_j, Henderson [1963] definitely allows that.

We find it convenient, and agreeing in spirit with Venn and many other authors, to define a *Venn diagram* of n curves in the plane as an independent family $\mathscr{A} = \{A_1, \cdots, A_n\}$ such that each of the sets $X_1 \cap X_2 \cap \cdots \cap X_n$ in (*) is an open, connected region, called a *cell* of the Venn diagram. It is easy to verify that all the bounded cells (that is, all cells except the single unbounded one) are even simply connected, as is also the complement of the unbounded cell.

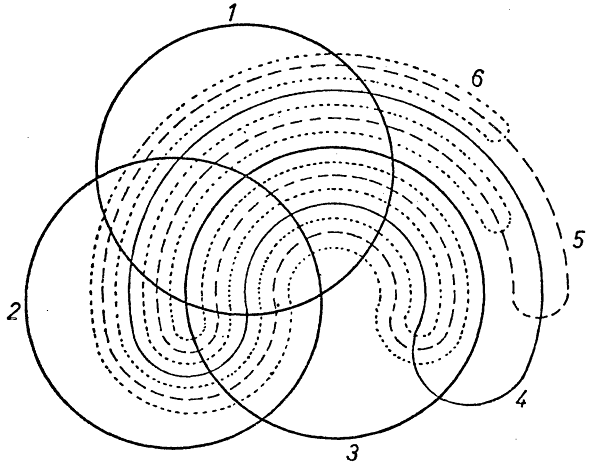


FIG. 2

The independent families in Figure 1 are easily seen to be Venn diagrams, but it is not completely obvious that Venn diagrams exist for arbitrarily large n. However, as observed already by Venn [1880, p. 8] (see also Venn [1881], Baron [1969]) a simple inductive construction may be used to establish the existence of such Venn diagrams. In Figure 2 we show Venn's inductive construction of diagrams with up to 6 curves, which should suffice to indicate the pattern Venn [1880] had in mind. Other constructions with the same purpose have been described by Berkeley [1937], More [1959], and Bowles [1971].

In the next section we shall investigate a number of questions dealing with independent families and Venn diagrams from the point of view of combinatorial geometry; two of them correct erroneous assertions found in the literature. In Section 3 we shall discuss a number of related results and open problems.

2. Venn diagrams formed by convex curves. We start with a simple observation which has several interesting consequences.

LEMMA. *If an independent family of* n *curves is such that each two curves meet in at most* j *points, then*

$$j \geqq (2^n - 2) \Big/ \binom{n}{2} = 4(2^{n-1} - 1)/n(n-1).$$

Proof. If no 3 curves have a common point, then the curves define in the plane a network with $v \leqq j\binom{n}{2}$ nodes (intersection points) and $e = 2v$ edges (arcs); therefore, by the Euler relation, the number of regions determined by this network is $e - v + 2 \leqq j\binom{n}{2} + 2$. But if the curves form an independent family the number of regions is at least 2^n, and the assertion follows for this case. The general case may be reduced to the one just considered by observing the possibility of deforming the curves so that each two still meet in at most j points, no three have a point in common, and no region is obliterated.

We may remark that the Lemma remains valid even if its assumptions are weakened to require each two curves to have an intersection consisting of at most j connected components.

If C is a convex curve (that is, the boundary of a 2-dimensional compact convex subset of the plane) we shall denote by $s(C)$ the largest number of curves in an independent family consisting of curves similar to C, and by $s^*(C)$ the analogously defined number for Venn diagrams. By $h(C)$ and $h^*(C)$ we shall denote the numbers similarly defined for families consisting of curves positively homothetic to C. Clearly $s(C) \geqq s^*(C) \geqq h^*(C)$ and $s(C) \geqq h(C) \geqq h^*(C)$ for all C.

THEOREM 1. (i) *If* C *is a circle then* $s(C) = s^*(C) = 3$. (ii) *For every ellipse* E *we have* $s(E) = s^*(E) = 5$. (iii) *For every convex curve* C *we have* $h(C) = h^*(C) = 3$.

Proof. Two circles intersect in at most 2 points, two ellipses in at most 4 points, and the intersection of two positively homothetic convex curves is well known to have at most 2 connected components. Therefore, we may apply the Lemma with $j = 2$, 4, and 2, respectively, and find that $s(C) < 4$, $s(E) < 6$, and $h(C) < 4$. Figure 1(a) shows that $s^*(C) \geqq 3$ for a circle C, and it is easy to construct similar examples that establish (iii). The Venn diagram in Figure 1(e) shows $s^*(E) \geqq 5$; similar diagrams may be constructed using five copies of any noncircular ellipse. This completes the proof of Theorem 1.

Remarks. The first part of Theorem 1 has been well known since Venn [1880]. However, the assertion that $s^*(E) = 4$ for ellipses E,—which is contradicted by part (ii) of Theorem 1 and by the diagram in Figure 1(d)—was also made by Venn

[1880] and repeated by many authors; among others in the article *Logic Diagrams* in Edwards [1967]. For a weak version of part (iii) see Austin [1971].

Let *k-gon* designate any convex polygon with at most k sides. We shall denote by $n(k)$ the maximal number of members in any independent family of k-gons in the plane, and by $k(n)$ the minimal k such that there exists an independent family of n k-gons. The similarly defined numbers dealing with Venn diagrams shall be denoted $n^*(k)$ and $k^*(n)$. We have

THEOREM 2. $$\lim_{k\to\infty}\frac{n(k)}{\log_2 k} = \lim_{k\to\infty}\frac{n^*(k)}{\log_2 k} = 1.$$

Proof. Since two k-gons intersect in at most $2k$ connected components, we may apply the Lemma with $j = 2k$ and deduce $\lim_{k\to\infty} n(k)/\log_2 k \leqq 1$. In order to complete the proof we shall describe the construction of a Venn diagram $\mathscr{A}(n)$ of n k-gons showing that $n^*(k) \geqq n = 2+\log_2 k$. For $n = 4$ we take the diagram $\mathscr{A}(4)$ consisting of four quadrangles indicated by solid lines in Figure 3; we denote by $P(4)$ the quadrangle $ABCD$. For $n > 4$ we obtain $\mathscr{A}(n)$ by adding a polygon $P(n)$ to $\mathscr{A}(n-1)$. The polygon $P(n)$ is a 2^{n-2}-gon formed by crossing twice each of the 2^{n-3} sides of $P(n-1)$, with vertices situated on the polygons crossing $P(n-1)$ and sufficiently close to $P(n-1)$ to make $P(n)$ a convex 2^{n-2}-gon. (Compare Figure 3 in which the dashed polygon illustrates $P(5)$, the dotted one $P(6)$.) This completes the proof of Theorem 2.

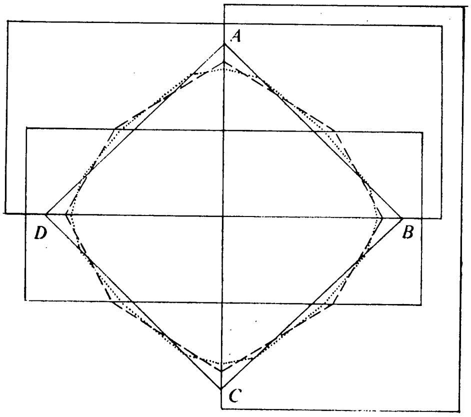


FIG. 3

Remarks. Theorem 2 appears in Rényi-Rényi-Surányi [1951]; their proof of the second part uses a different construction than the one just described, and yields a poorer bound. More importantly, however, Rényi-Rényi-Surányi [1951] base their proof of the upper bound on the statement obtained from our lemma by replacing the inequality of its conclusion by the stronger relation

(**) $$j \geqq \frac{2^{n-1}}{n-1}.$$

Unfortunately, the proof they give for (**) is fallacious, and the assertion itself is false. A counterexample to (**) with $j = n = 6$ is shown in Figure 1(f). However, the Rényi-Rényi-Surányi proof of Theorem 2 could be salvaged if relation (**) is true or independent families of n k-gons, with $j = 2k$. If this more restricted variant of (**) were established, equality could be substituted for the inequalities in the following result; but we expect that equality holds in Theorem 3 in any case.

THEOREM 3. $k(3)=k^*(3) = 3$; $k(4) = k^*(4)=3$; $k(5) = k^*(5) = 3$; $k(6) \leqq k^*(6) \leqq 4$; $k(7) \leqq 6$.

Proof. The Venn diagrams in Figures 6(a), 8(b), and 4 establish the first three parts of the theorem. A Venn diagram of six quadrangles is shown in Figure 5, while a drawing of an independent family of 7 hexagons will be supplied b the author to any interested reader.

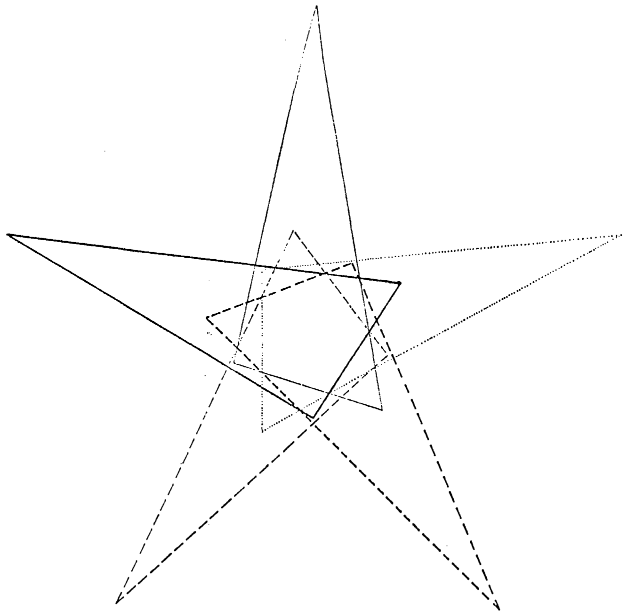


FIG. 4

3. Variations and open problems. (1) The construction we used in the proof of Theorem 2 is a modification to convex polygons of the method described in More [1959]. It may be adapted to obtain other variants of the result, such as the following:

(i) If the construction is not required to proceed beyond $\mathscr{A}(n)$, then it is easily seen that $P(n)$ may even be chosen as a 2^{n-3}-gon; therefore we actually have $n(k) \geqq n^*(k) \geqq 3 + \log_2 k$.

(ii) Let a Venn diagram $\mathscr{A} = \{A_1, \cdots, A_n\}$ be called *convex* provided all the A_j are convex curves and all the bounded cells of $\mathscr{A}$ are convex sets as is the complement of the unbounded cell. We shall denote by $n^{**}(k)$ and $k^{**}(n)$ the numbers corre-

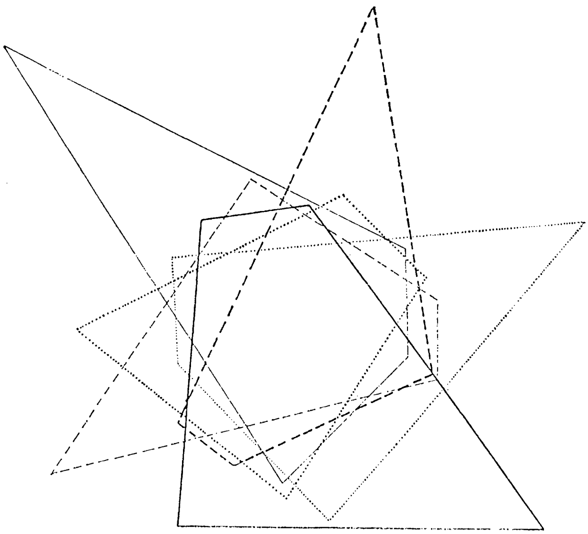
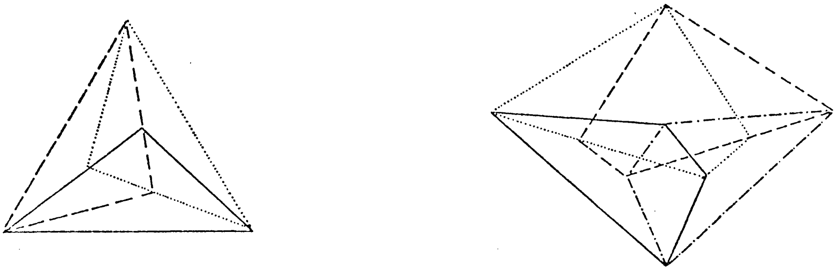


FIG. 5



(a) (b)

FIG. 6

sponding to $n^*(k)$ and $k^*(n)$ but referring to convex Venn diagrams. (It should be noted that the existence of $k^{**}(n)$ and of convex diagrams with arbitrarily large numbers of curves is not self-evident. While the existence of Venn diagrams for arbitrary n in which each cell is convex may easily be deduced from Steinitz's theorem on convex polyhedra (see, for example, Grünbaum [1967, Chapter 13]), the satisfaction of the additional requirement of convexity of the curves A_j cannot be guaranteed by that theorem.) But an easy modification of our construction may be used to establish $n^{**}(k) \geqq 2 + \log_2(k-1)$, so that Theorem 2 may be extended to $\lim_{k\to\infty} n^{**}(k)/\log_2 k = 1$. The diagrams in Figure 6 prove that $k^{**}(3) = 3$ and $k^{**}(4) \leqq 4$; actually it may be shown that $k^{**}(4) = 4$, and it is probable that $k^{**}(5) = 5$.

(2) A Venn diagram with n curves is called *symmetric* (Henderson [1963]) if all the curves are congruent and obtained from each other by rotations through multiples of $2\pi/n$ about a fixed point. The diagram of $n = 5$ triangles in Figure 4 is symmetric. Henderson [1963] gives two examples of symmetric diagrams with

$n = 5$, one formed by pentagons, the other by quadrangles; it is easy to modify them so as to obtain symmetric Venn diagrams consisting of triangles, combinatorially distinct from each other and from the one in Figure 4. (It is not known whether every symmetric Venn diagram of 5 triangles is isomorphic with one of these three.) As observed by Henderson [1963], if a symmetric Venn diagram has n curves then n is necessarily a prime number. Henderson [1963] mentions that he has found a symmetric Venn diagram of 7 hexagons. The present author's search for such a diagram has been unsuccessful, as have been attempts to clarify Henderson's claim; at present it seems likely that no such diagram exists. Concerning the analogously defined concept of a *symmetric independent family* of n curves there is no restriction that n be prime. In Figure 8(a) we have a symmetric independent family of 4 triangles (compare Austin [1971]), while Figure 7 shows a symmetric independent family of 6 quadrangles. The independent family of 7 hexagons mentioned in the proof of Theorem 3 is also symmetric. It is not known whether for every n there exists a symmetric independent family of n $k(n)$-gons; we conjecture that that is the case. But it should be stressed that the existence of symmetric Venn diagrams is still open for each prime $n \geqq 7$.

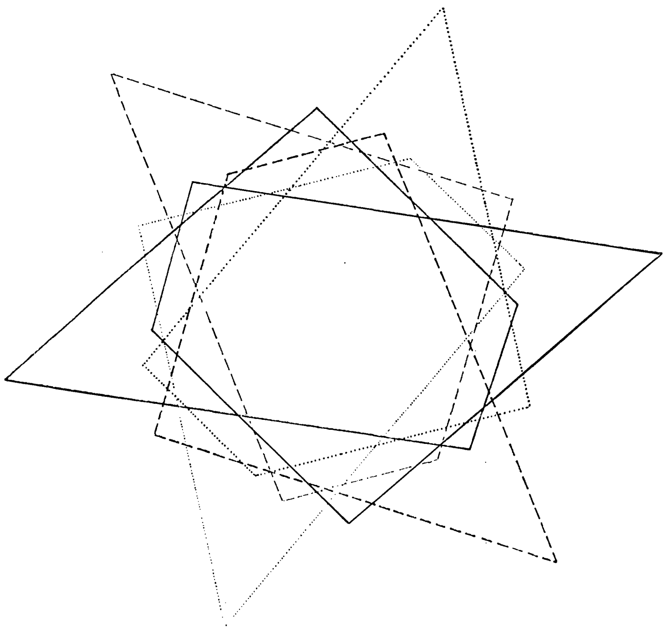
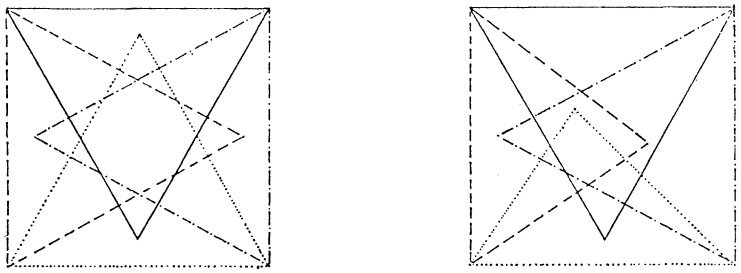


FIG. 7

(3) An independent family, or a Venn diagram, is called *simple* provided no point belongs to the boundary of three of the sets. Let $k_s(n)$ [or $k_s^*(n)$] be the smallest k for which there exists a simple independent family [or a simple Venn diagram] with n k-gons. The example in Figure 6 shows that $k_s^*(3) = 3$, and Figure 8(a) establishes $k_s(4) = 3$; the example in Figure 8(b) proves even $k_s^*(4) = 3$. No other values seem to be known.



(a) (b)

FIG. 8

(4) Concerning the functions $k(n)$ and $k^*(n)$ we venture two guesses:

(i) $k(n) = k^*(n)$ for all n.

(ii) $k(n) =]2^{n-2}/(n-1)[$ for all n, where $]x[$ denotes the smallest integer $\geqq x$.

(5) Notions of independent families and Venn diagrams may easily be extended to higher dimensions in several ways (see, for example, Weglorz [1964], Collings [1972]). We prefer the following definitions, although for $d = 2$ they are somewhat more general than the definitions we adopted in Section 1 (all the results of Section 2 remain valid with these definitions). A family $\mathscr{A} = \{A_1, \cdots, A_n\}$ of n sets in the d-dimensional Euclidean space E^d is *independent* provided (*) holds whenever X_j is chosen to be either A_j or the complement $\sim A_j$ of A_j in E^d. The independent family $\mathscr{A}$ is a *Venn diagram* provided each of the sets int $(X_1 \cap X_2 \cap \cdots \cap X_n)$ is an open d-cell (that is, is homeomorphic to the open d-ball), except that int $(\sim A_1 \cap \sim A_2 \cap \cdots \cap \sim A_n)$ is homeomorphic to the complement of a closed d-ball B^d.

The following generalization of part (i) of our Theorem 1 holds: Each independent family of d-balls has at most $d+1$ members. More precisely, using the obvious adaptation of the notation introduced in Section 2, we have $s(B^d) = s^*(B^d) = d+1$. This was established by Rényi-Rényi-Surányi [1951] (see also Anusiak [1965]), together with the theorem: *An independent family of boxes in E^d, with edges parallel to the coordinate axes, contains at most $2d$ members.*

(6) Let $k(n,d)$ denote the least k such that there exists an independent family of n sets in E^d, each set of which is a d-polytope with at most k facets ($(d-1)$-dimensional faces). Theorem 3 shows that $k(5,2) = 3$, $k(6,2) \leqq 4$ and $k(7,2) \leqq 6$, while the result on boxes in E^d mentioned above shows $k(2d,d) \leqq 2d$. Starting from the Venn diagram of 5 triangles shown in Figure 4 it is easy to deduce that $k(2d+1,d) = d+1$. If we denote by n_d the largest n such that $k(n,d) = d+1$ then by the remark just made $n_d \geqq 2d+1$. On the other hand, crude upper bounds on n_d may be obtained by using the well-known result (see Grünbaum [1971] for details and references) that j hyperplanes in E^d determine at most $\binom{j-1}{d}$ bounded cells. Therefore n_d must satisfy the inequality

$$\binom{(d+1)n_d - 1}{d} \geqq 2^n - 1.$$

Since $\binom{23}{2} = 253 < 255 = 2^8 - 1$, this implies that $n_2 \leqq 7$ (compared to the conjectured value $n_2 = 5$). Similarly, since $\binom{59}{3} = 32{,}509 < 32{,}767 = 2^{15} - 1$, we have $7 \leqq n_3 \leqq 14$, and analogously $9 \leqq n_4 \leqq 22$. It would be interesting to learn more about n_d and $k(n,d)$.

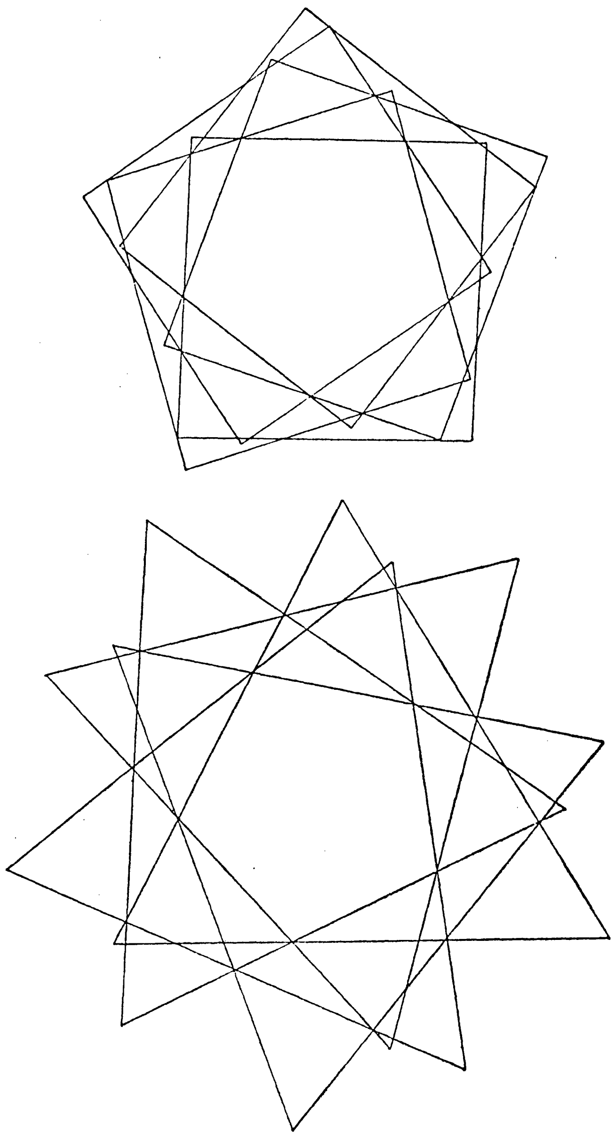


FIG. 9

(7) We denote by $h(C)$ and $h^*(C)$ the maximal number of sets in independent families or Venn diagrams consisting of sets homothetic to the convex set C. As a generalization of part (iii) of Theorem 1, and of the result of Rényi-Rényi-Surányi [1951] on balls, we conjecture that $h(C) = h^*(C) = d + 1$ for every d-dimensional

compact convex set C. (If negative ratios of homothecy are permitted, part (iii) of Theorem 1 does not hold; it would be interesting to determine the bounds in that case as well.)

(8) Another particularly attractive type of Venn diagrams or independent families comprises those consisting of congruent sets. We denote by $c(C)$ [or $c^*(C)$] the maximal number of sets in an independent family [or Venn diagram] consisting of sets congruent to C. From Theorem 1 and Figure 1(d) it follows that $c^*(E) = c(E) = 5$ for each ellipse E. It is easily seen that there is no fixed bound on $c(C)$ when C varies over planar convex sets. It would be of interest to determine whether—as seems likely—there exist planar convex sets C with $c(C) = \infty$, or even such with $c^*(C) = \infty$. For each polygon C the finiteness of $c(C)$ follows from the Lemma of Section 2. The examples in Figure 9 show that $c(T) \geqq 5$ and $c(S) \geqq 5$ if T is an equilateral triangle and S is a square. Probably equality holds in both cases. Other open problems concern the values of $c^*(T)$ and $c^*(S)$, as well as their analogues in higher dimensions.

(9) The author hopes that the facts and the spirit of the above discussion will be of some assistance to those contending with the unfortunate—though fashionable—opinions that "geometry is dead," that it is "reduced to linear algebra," or that one needs years of postgraduate study to understand nontrivial open problems in geometry. Many questions of combinatorial geometry provide not only opportunities to exercise the geometric intuition and a variety of techniques, but also afford considerable aesthetic gratification.

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LAGUERRE'S AXIAL TRANSFORMATION

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This transformation, as we remarked in a recent paper (Pedoe, [3]) has been mostly forgotten by geometers. It is as powerful a tranformation as inversion (transformation by reciprocal radii), being, in a sense, the dual of inversion, and was named by Laguerre *transformation par les semi-droites réciproques*. We shall call Laguerre's half-lines *rays*. They are oriented lines, and a ray touches a *cycle*, an oriented circle, at a point P if the direction of the ray at P coincides with that assigned to the cycle.

The geometry of cycles was largely developed during the 19th century, but never met with the enthusiasm accorded to the geometry of circles, and has been almost completely forgotten, although the theory contains many beautiful theorems. In this paper we have another look at Laguerre's fundamental theorem:

Given three cycles whose axis of similitude intersects none of them, a Laguerre axial transformation can be found which simultaneously maps the three cycles into three points.

In Pedoe [3] we investigated Laguerre's geometrical proof of this theorem and remarked that neither Blaschke [1], Coolidge [2] or Yaglom [5] mention it specifically in their respective discussions of axial transformations. Blaschke comes very near, as we shall see (§8), and it is his discussion of Sophus Lie's approach to the subject which has motivated this paper, in which we connect Lie's work with Laguerre's. Some unexpected views of both 2-dimensional and 3-dimensional geometry will appear during this excursion in Euclidean 3-space.