Notes on Projective Differential Geometry

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These are Very Rough Stream-of-Consciousness Notes for two expository lectures at the IMA in July 2006. The finished product will have references and, hopefully, make sense!

Projective differential geometry was initiated in the 1920s, especially by Élie Cartan and Tracey Thomas. Nowadays, the subject is not so well-known. These notes aim to remedy this deficit and present several reasons why this should be done at this time. The deeper underlying reason is that projective differential geometry provides the most basic application of what has come to be known as the 'Bernstein-Gelfand-Gelfand machinery'. As such, it is completely parallel to conformal differential geometry. On the other hand, there are direct applications within Riemannian differential geometry. We shall soon see, for example, a good geometric reason why the symmetries of the Riemann curvature tensor constitute an irreducible representation of $\mathrm{SL}(n,\mathbb{R})$ (rather than $\mathrm{SO}(n)$ as one might naïvely expect). Projective differential geometry also provides the simplest setting in which overdetermined systems of partial differential equations naturally arise.

Let M be a smooth real manifold of dimension n. There are two ways to define a projective differential geometry on M. One is geometric and intuitive. The other is more operational and useful in practice. Their equivalence is the subject of the following proposition.

Proposition 1 *Two torsion-free connections ∇_a and $\widehat{\nabla}_a$ on M have the same geodesics as unparameterised curves if and only if*

$$\widehat{\nabla}_a\omega_b = \nabla_a\omega_b - \Upsilon_a\omega_b - \Upsilon_b\omega_a \qquad (1)$$

for some 1-form Υ_a.

Proof. Let $\pi : TM \to M$ denote the tangent bundle to M and let V denote the vertical subbundle of $T(TM)$ so that we have the exact sequence

$$0 \to V \to T(TM) \to \pi^*TM \to 0 \qquad (2)$$

of vector bundles on TM. The connection ∇_a may be viewed as defining a splitting of this exact sequence, in other words defining a horizontal subbundle complementary to V. Each element $X^a \in T_pM$ then pulls back to a unique horizontal vector. We obtain a vector field $\mathcal{X}$ on TM whose integral curves define the geodesics spray of ∇_a. To say that the connection $\widehat{\nabla}_a$ has the same unparameterised geodesics as ∇_a is to say that the corresponding vector field $\widehat{\mathcal{X}}$ differs from $\mathcal{X}$ by a multiple of the Euler field along the fibres of V.

Any two torsion-free connections are related by

$$\widehat{\nabla}_a\omega_b = \nabla_a\omega_b - \Gamma_{ab}{}^c\omega_c$$

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for some tensor $\Gamma_{ab}{}^c = \Gamma_{(ab)}{}^c$. This tensor defines the corresponding change of splitting of (2). Specifically, at $X^a \in T_pM$ the change is given by the $X^a\Gamma_{ab}{}^c$, regarded as a homomorphism from T_pM to $T_pM = V_p$. It follows that ∇_a and $\widehat{\nabla}_a$ have the same unparameterised geodesics if and only if $X^aX^b\Gamma_{ab}{}^c$ is a multiple of X^c for all X^a. But it is a matter of linear algebra to check that

$$X^aX^b\Gamma_{ab}{}^{[c}X^{d]} = 0 \text{ for all } X^a \quad \text{ if and only if } \quad \Gamma_{(ab)}{}^c = \Upsilon_a\delta_b{}^c + \Upsilon_b\delta_a{}^c \text{ for some } \Upsilon_a.$$

This completes the proof. □

Definition 1 *We shall say that two torsion-free connections ∇_a and $\widehat{\nabla}_a$ on M are* <u>*projectively equivalent*</u> *if and only if they have the same geodesics as unparameterised curves. A* <u>*projective structure*</u> *on M is a projective equivalence class of torsion-free connections on M.*

Proposition 1 gives an alternative, more operational, definition of projective equivalence according to (1). From this point of view, is it also clear that a projective structure is really a local notion (that can be patched together with a partition of unity).

To proceed further, let us now consider the consequences of (1) for other tensor fields. For X^a a vector field we have, dual to (1),

$$\widehat{\nabla}_aX^b = \nabla_aX^b + \Upsilon_aX^b + \Upsilon_cX^c\delta_a{}^b.$$

If ω_{ab} is a 2-form (a covariant tensor $\omega_{ab} = \omega_{[ab]}$), then

$$\begin{aligned} \widehat{\nabla}_a\omega_{bc} &= \nabla_a\omega_{bc} - 2\Upsilon_a\omega_{bc} - \Upsilon_b\omega_{ac} - \Upsilon_c\omega_{ba} \\ &= \nabla_a\omega_{bc} - 3\Upsilon_a\omega_{bc} + \Upsilon_a\omega_{bc} + \Upsilon_b\omega_{ca} + \Upsilon_c\omega_{ab} \\ &= \nabla_a\omega_{bc} - 3\Upsilon_a\omega_{bc} + 3\Upsilon_{[a}\omega_{bc]} \end{aligned}$$

and, more generally, for an p-form $\omega_{bc\cdots d}$,

$$\widehat{\nabla}_a\omega_{bc\cdots d} = \nabla_a\omega_{bc\cdots d} - (p+1)\Upsilon_a\omega_{bc\cdots d} + (p+1)\Upsilon_{[a}\omega_{bc\cdots d]}.$$

In particular, for an n-form $\omega_{bc\cdots de}$ we find that

$$\widehat{\nabla}_a\omega_{bc\cdots de} = \nabla_a\omega_{bc\cdots de} - (n+1)\Upsilon_a\omega_{bc\cdots de}$$

or, more succinctly

$$\widehat{\nabla}_a\sigma = \nabla_a\sigma - (n+1)\Upsilon_a\sigma$$

for a volume form σ (for simplicity let us suppose that M is oriented). If we introduce the terminology <u>projective</u> <u>density</u> of <u>weight</u> w for sections of the line bundle $(\Lambda^n)^{-w/(n+1)}$, then we see that

$$\widehat{\nabla}_a\sigma = \nabla_a\sigma + w\Upsilon_a\sigma$$

when σ is such a density. Let us write $\mathcal{E}(w)$ for the bundle of projective densities of weight w (and also for its sheaf of smooth sections). Let us also write $\mathcal{E}_a$ for the bundle of 1-forms and $\mathcal{E}_a(w)$ for the bundle of 1-forms of weight w obtained by tensoring the 1-forms with $\mathcal{E}(w)$. For σ_a such a projectively weighted 1-form we find that

$$\widehat{\nabla}_a\sigma_b = \nabla_a\sigma_b + (w-1)\Upsilon_a\sigma_b - \Upsilon_b\sigma_a.$$

In particular, when $w = 2$ we conclude that

$$\widehat{\nabla}_a\sigma_b = \nabla_a\sigma_b + \Upsilon_a\sigma_b - \Upsilon_b\sigma_a \quad \text{whence} \quad \widehat{\nabla}_{(a}\sigma_{b)} = \nabla_{(a}\sigma_{b)}.$$

In other words

$$\mathcal{E}_a(2) \ni \omega_a \longmapsto \nabla_{(a}\sigma_{b)} \in \mathcal{E}_{(ab)}(2) \tag{3}$$

is projectively invariant. Similarly

$$\mathcal{E}_a \ni \omega_a \longmapsto \nabla_{[a}\sigma_{b]} \in \mathcal{E}_{[ab]}$$

is projectively invariant: this is the familiar exterior derivative $d : \Lambda^1 \to \Lambda^2$.

Now let us suppose that the projective structure on M arises from a Riemannian metric g_{ab}. A vector field X^a on M said to be a Killing field if and only if $\mathcal{L}_X g_{ab} = 0$ where $\mathcal{L}_X$ is the Lie derivative along X^a. Equivalently,

$$\nabla_{(a}X_{b)} = 0, \tag{4}$$

where ∇_a is the Levi Civita connection associated to g_{ab} and the vector field X^a is identified with the 1-form X_a by means of the metric. Geometrically, the Killing fields are the infinitesimal isometries of M. In the presence of the metric g_{ab}, the bundle of volume forms Λ^n is canonically trivialised and so we may regard X_a as having projective weight 2 if we so wish. In this sense we have shown:–

Proposition 2 *The Killing operator $X^a \mapsto \nabla_{(a}X_{b)}$ is projectively invariant.*

This observation may seem a little contrived but it acquires more significance when it is realised that (3) is the first in a natural sequence of projectively invariant differential operators. In order the describe this sequence we need firstly to develop a little more basic projective differential geometry and we start with some curvature conventions.

For any torsion-free connection ∇_a on TM, define its curvature tensor $R_{ab}{}^c{}_d$ by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)X^c = R_{ab}{}^c{}_d X^d$$

or, equivalently,

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_d = -R_{ab}{}^c{}_d\omega_c. \tag{5}$$

It satisfies the Bianchi symmetry $R_{[ab}{}^c{}_{d]} = 0$ and may be uniquely and conveniently written as

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + 2\delta_{[a}{}^c\mathrm{P}_{b]d} + \beta_{ab}\delta^c{}_d, \tag{6}$$

where

$$W_{[ab}{}^c{}_{d]} = 0, \quad W_{ab}{}^c{}_d \text{ is totally trace-free}, \quad \beta_{ab} = -2\mathrm{P}_{[ab]}.$$

If we replace the connection ∇_a by $\widehat{\nabla}_a$ is accordance with (1), then we find that

$$\widehat{W}_{ab}{}^c{}_d = W_{ab}{}^c{}_d, \quad \widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b, \quad \widehat{\beta}_{ab} = \beta_{ab} + 2\nabla_{[a}\Upsilon_{b]}. \tag{7}$$

In particular, the Weyl curvature $W_{ab}{}^c{}_d$ is projectively invariant. The Bianchi identity $\nabla_{[a}R_{bc]}{}^d{}_e = 0$ may be rewritten as

$$4\nabla_{[a}\mathrm{P}_{b][c}\delta_{d]}{}^e - \nabla_a W_{cd}{}^e{}_b + \nabla_b W_{cd}{}^e{}_a = 4\nabla_{[c}\mathrm{P}_{d][a}\delta_{b]}{}^e - \nabla_c W_{ab}{}^e{}_d + \nabla_d W_{ab}{}^e{}_c. \tag{8}$$

It has the following consequences

$$\nabla_c W_{ab}{}^c{}_d = 2(n-2)\nabla_{[a}\mathrm{P}_{b]d} \quad \text{and} \quad \nabla_{[a}\beta_{bc]} = 0. \tag{9}$$

Therefore, the cohomology class $[\beta] \in H^2(M, \mathbb{R})$ is a global invariant of the projective structure and the obstruction to choosing a connection in the projective class with symmetric Schouten tensor P_{ab}. The tensor β_{ab} also finds a geometric interpretation as the curvature on densities. Specifically,

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\sigma = w\beta_{ab}\sigma \quad \text{for } \sigma \in \mathcal{E}(w). \tag{10}$$

If there is a connection in the projective class with $\beta_{ab} = 0$ (and this is always the case locally), then we may work exclusively with such connections to obtain a more restricted notion of equivalence. Specifically, we allow only closed 1-forms in (1). The resulting structure is called special projective or equi-projective. The analogy with the conformal case is stronger and many of the formulae below are simpler for special projective structures.

We are now in a position to construct the claimed next operator in the sequence. There is a general theory to be explained later. Here, we shall construct it 'by hand'. It will be a second order operator acting on $\mathcal{E}_{(ab)}(2)$ so let us consider now $\nabla_a\nabla_c h_{bd}$ for a symmetric covariant tensor h_{bd} of projective weight 2. Under a projective change of connection, we find

$$\begin{aligned}
\widehat{\nabla}_c h_{bd} &= \nabla_c h_{bd} - \Upsilon_b h_{cd} - \Upsilon_d h_{bc} \\
\Longrightarrow \widehat{\nabla}_a\widehat{\nabla}_c h_{bd} &= \nabla_a(\nabla_c h_{bd} - \Upsilon_b h_{cd} - \Upsilon_d h_{bc}) - \Upsilon_a(\nabla_c h_{bd} - \Upsilon_b h_{cd} - \Upsilon_d h_{bc}) \\
&\quad - \Upsilon_c(\nabla_a h_{bd} - \Upsilon_b h_{ad} - \Upsilon_d h_{ba}) - \Upsilon_b(\nabla_c h_{ad} - \Upsilon_a h_{cd} - \Upsilon_d h_{ac}) \\
&\quad - \Upsilon_d(\nabla_c h_{ba} - \Upsilon_b h_{ca} - \Upsilon_a h_{bc}) \\
&= \nabla_a\nabla_c h_{bd} - 2\Upsilon_b\nabla_{(a}h_{c)d} - 2\Upsilon_d\nabla_{(a}h_{c)b} - 2\Upsilon_{(a}\nabla_{c)}h_{bd} \\
&\quad - (\nabla_a\Upsilon_b - \Upsilon_a\Upsilon_b)h_{cd} - (\nabla_a\Upsilon_d - \Upsilon_a\Upsilon_d)h_{bc} \\
&\quad + 2\Upsilon_b\Upsilon_{(a}h_{c)d} + 2\Upsilon_d\Upsilon_{(a}h_{c)b} + 2\Upsilon_b\Upsilon_d h_{ac},
\end{aligned}$$

which we may rewrite using (7) as

$$\begin{aligned}
\widehat{\nabla}_a\widehat{\nabla}_c h_{bd} - 2\widehat{\mathrm{P}}_{a(b}h_{d)c} &= \nabla_a\nabla_c h_{bd} - 2\mathrm{P}_{a(b}h_{d)c} \\
&\quad - 2\Upsilon_b\nabla_{(a}h_{c)d} - 2\Upsilon_d\nabla_{(a}h_{c)b} - 2\Upsilon_{(a}\nabla_{c)}h_{bd} \\
&\quad + 2\Upsilon_b\Upsilon_{(a}h_{c)d} + 2\Upsilon_d\Upsilon_{(a}h_{c)b} + 2\Upsilon_b\Upsilon_d h_{ac}.
\end{aligned}$$

It follows straightforwardly that

$$(\nabla_{(a}\nabla_{c)} + \mathrm{P}_{(ac)})h_{bd} - (\nabla_{(b}\nabla_{c)} + \mathrm{P}_{(bc)})h_{ad} - (\nabla_{(a}\nabla_{d)} + \mathrm{P}_{(ad)})h_{bc} + (\nabla_{(b}\nabla_{d)} + \mathrm{P}_{(bd)})h_{ac}$$

is projectively invariant. Notice that the resulting tensor r_{abcd} has the symmetries of the usual Riemann tensor:–

$$r_{abcd} = -r_{bacd}, \quad r_{abcd} = -r_{abdc}, \quad r_{abcd} + r_{bcad} + r_{cabd} = 0.$$

Next, it is easily verified that if r_{abcd} has Riemann tensor symmetries and is of projective weight 2, then

$$r_{abcd} \longmapsto \nabla_{[a}r_{bc]de}$$

is projectively invariant (cf. Bianchi identity).

Tractors

We define a canonical rank $n+1$ vector bundle $\mathcal{E}_A$ on M as follows. For each choice of connection in the projective class $\mathcal{E}_A$ is identified as a direct sum

$$\mathcal{E}_A = \mathcal{E}(1) \oplus \mathcal{E}_A(1).$$

Under change of connection (1), however, this splitting changes accordingly

$$\widehat{\begin{pmatrix} \sigma \\ \mu_a \end{pmatrix}} = \begin{pmatrix} \sigma \\ \mu_a + \Upsilon_a \sigma \end{pmatrix}.$$

Notice that there is a canonical exact sequence

$$0 \to \mathcal{E}_a(1) \to \mathcal{E}_A \to \mathcal{E}(1) \to 0.$$

The bundle $\mathcal{E}_A$ is called a <u>tractor</u> bundle and comes equipped with an invariantly defined connection. For a particular connection ∇_a in the projective class define

$$\nabla_a \begin{pmatrix} \sigma \\ \mu_b \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathrm{P}_{ab}\sigma \end{pmatrix}.$$

It is straightforward to check that this definition is projectively invariant:–

$$\begin{aligned}
\widehat{\nabla}_a \widehat{\begin{pmatrix} \sigma \\ \mu_a \end{pmatrix}} &= \widehat{\nabla}_a \begin{pmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \end{pmatrix} \\
&= \begin{pmatrix} \widehat{\nabla}_a \sigma - (\mu_a + \Upsilon_a \sigma) \\ \widehat{\nabla}_a(\mu_b + \Upsilon_b \sigma) + \widehat{\mathrm{P}}_{ab}\sigma \end{pmatrix} \\
&= \begin{pmatrix} \nabla_a \sigma + \Upsilon_a \sigma - (\mu_a + \Upsilon_a \sigma) \\ \nabla_a(\mu_b + \Upsilon_b \sigma) - \Upsilon_b(\mu_a + \Upsilon_a \sigma) + (\mathrm{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b)\sigma \end{pmatrix} \\
&= \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \Upsilon_b \nabla_a \sigma - \Upsilon_b \mu_a + \mathrm{P}_{ab}\sigma \end{pmatrix} \\
&= \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathrm{P}_{ab}\sigma + \Upsilon_b(\nabla_a \sigma - \mu_a) \end{pmatrix} \\
&= \widehat{\begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathrm{P}_{ab}\sigma \end{pmatrix}} = \widehat{\nabla_a \begin{pmatrix} \sigma \\ \mu_b \end{pmatrix}}.
\end{aligned}$$

The curvature of the tractor connection is easily calculated:–

$$\begin{aligned}
\nabla_a\nabla_b \begin{pmatrix} \sigma \\ \mu_c \end{pmatrix} &= \nabla_a \begin{pmatrix} \nabla_b\sigma - \mu_b \\ \nabla_b\mu_c + \mathrm{P}_{bc}\sigma \end{pmatrix} \\
&= \begin{pmatrix} \nabla_a(\nabla_b\sigma - \mu_b) - (\nabla_b\mu_a + \mathrm{P}_{ba}\sigma) \\ \nabla_a(\nabla_b\mu_c + \mathrm{P}_{bc}\sigma) + \mathrm{P}_{ac}(\nabla_b\sigma - \mu_b) \end{pmatrix} \\
\Longrightarrow\ (\nabla_a\nabla_b - \nabla_b\nabla_a) \begin{pmatrix} \sigma \\ \mu_c \end{pmatrix} &= \begin{pmatrix} (\nabla_a\nabla_b - \nabla_b\nabla_a)\sigma - \mathrm{P}_{ba}\sigma + \mathrm{P}_{ab}\sigma \\ (\nabla_a\nabla_b - \nabla_b\nabla_a)\mu_c + 2(\nabla_{[a}\mathrm{P}_{b]c})\sigma - 2\mu_{[b}\mathrm{P}_{a]c} \end{pmatrix} \\
&= \begin{pmatrix} \beta_{ab}\sigma + 2\mathrm{P}_{[ab]}\sigma \\ -R_{ab}{}^d{}_c\mu_d + \beta_{ab}\mu_c + 2(\nabla_{[a}\mathrm{P}_{b]c})\sigma - 2\mu_{[b}\mathrm{P}_{a]c} \end{pmatrix}.
\end{aligned}$$

However, from (6) we obtain

$$R_{ab}{}^d{}_c\mu_d = (W_{ab}{}^d{}_c + 2\delta_{[a}{}^d\mathrm{P}_{b]c} + \beta_{ab}\delta^d{}_c)\mu_d = W_{ab}{}^d{}_c\mu_d + 2\mu_{[a}\mathrm{P}_{b]c} + \beta_{ab}\mu_c.$$

Also, recall that $\beta_{ab} = -2\mathrm{P}_{ab}$. Therefore,

$$(\nabla_a\nabla_b - \nabla_b\nabla_a) \begin{pmatrix} \sigma \\ \mu_c \end{pmatrix} = \begin{pmatrix} 0 \\ -W_{ab}{}^d{}_c\mu_d + 2(\nabla_{[a}\mathrm{P}_{b]c})\sigma \end{pmatrix}.$$

Of course, this curvature must be projectively invariant: it is easily verified that

$$\widehat{\nabla}_{[a}\widehat{\mathrm{P}}_{b]c} = \nabla_{[a}\mathrm{P}_{b]c} + \tfrac{1}{2}W_{ab}{}^d{}_c\Upsilon_d. \tag{11}$$

Proposition 3 *The tractor connection is flat if and only if*

$$\begin{aligned} W_{ab}{}^c{}_d &= 0 \quad \textit{if } n \geq 3 \\ \nabla_{[a}\mathrm{P}_{b]c} &= 0 \quad \textit{if } n = 2. \end{aligned}$$

Proof. If $n \geq 3$ and $W_{ab}{}^c{}_d = 0$, then (9) implies that $\nabla_{[a}\mathrm{P}_{b]c}$ also vanishes. When $n = 2$, however, the Weyl curvature automatically vanishes by symmetry considerations and (11) says that the Cotton-York tensor $\nabla_{[a}\mathrm{P}_{b]c}$ is projectively invariant. □

RELATE TO Cartan connection and other tractor bundles.

There are induced connections on all tractor bundles. For example

$$\Lambda^2\mathcal{E}_A = \mathcal{E}_{[AB]} = \mathcal{E}_a(2) + \mathcal{E}_{[ab]}(2)$$

transforms by

$$\widehat{\begin{pmatrix} \sigma_a \\ \mu_{ab} \end{pmatrix}} = \begin{pmatrix} \sigma_a \\ \mu_{ab} + 2\Upsilon_{[a}\sigma_{b]} \end{pmatrix}$$

and inherits a canonical tractor connection defined by

$$\nabla_a \begin{pmatrix} \sigma_b \\ \mu_{bc} \end{pmatrix} = \begin{pmatrix} \nabla_a\sigma_b - \mu_{ab} \\ \nabla_a\mu_{bc} + 2\mathrm{P}_{a[b}\sigma_{c]} \end{pmatrix}$$

with curvature given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a) \begin{pmatrix} \sigma_c \\ \mu_{cd} \end{pmatrix} = \begin{pmatrix} -W_{ab}{}^d{}_c\sigma_d \\ 2W_{ab}{}^e{}_{[c}\mu_{d]e} + 4(\nabla_{[a}\mathrm{P}_{b][c})\sigma_{d]} \end{pmatrix}. \tag{12}$$

BLURB about PROLONGATION and OVERDETERMINED stuff.

We may prolong the Killing equation (4). To compare with tractors, let us write σ_a instead of X_a. Then (4) may be written as

$$\nabla_a \sigma_b = \mu_{ab}$$

for some skew tensor μ_{ab}. Since σ_a has weight 2, from (5) and (10) we find

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\sigma_d = -R_{ab}{}^c{}_d \sigma_c + 2\beta_{ab}\sigma_d$$

whence

$$\nabla_{[a}\mu_{bc]} = \nabla_{[a}\nabla_b \sigma_{c]} = \beta_{[ab}\sigma_{c]},$$

which we may rewrite as

$$\begin{aligned}
\nabla_a \mu_{bc} &= \nabla_c \mu_{ba} - \nabla_b \mu_{ca} + 3\beta_{[ab}\sigma_{c]} \\
&= \nabla_c \nabla_b \sigma_a - \nabla_b \nabla_c \sigma_a + 3\beta_{[ab}\sigma_{c]} \\
&= R_{bc}{}^d{}_a \sigma_d - 2\beta_{bc}\sigma_a + 3\beta_{[ab}\sigma_{c]} \\
&= R_{bc}{}^d{}_a \sigma_d - \beta_{bc}\sigma_a + 2\beta_{a[b}\sigma_{c]}.
\end{aligned}$$

However, from (6) we find that

$$\begin{aligned}
R_{bc}{}^d{}_a \sigma_d &= (W_{bc}{}^d{}_a + 2\delta_{[b}{}^d \mathrm{P}_{c]a} + \beta_{bc}\delta^d{}_a)\sigma_d \\
&= W_{bc}{}^d{}_a \sigma_d + 2\sigma_{[b}\mathrm{P}_{c]a} + \beta_{bc}\sigma_a \\
&= W_{bc}{}^d{}_a \sigma_d - 2\mathrm{P}_{a[b}\sigma_{c]} - 2\beta_{a[b}\sigma_{c]} + \beta_{bc}\sigma_a.
\end{aligned}$$

Therefore,

$$\nabla_a \mu_{bc} = W_{bc}{}^d{}_a \sigma_d - 2\mathrm{P}_{a[b}\sigma_{c]}$$

and we conclude that Killing fields are equivalent to parallel sections of the connection

$$D_a \begin{pmatrix} \sigma_b \\ \mu_{bc} \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} + 2\mathrm{P}_{a[b}\sigma_{c]} - W_{bc}{}^d{}_a \sigma_d \end{pmatrix}.$$

Somewhat unexpectedly, this is not the tractor connection on $\mathcal{E}_{[BC]}$. Specifically,

$$D_a \begin{pmatrix} \sigma_b \\ \mu_{bc} \end{pmatrix} = \nabla_a \begin{pmatrix} \sigma_b \\ \mu_{bc} \end{pmatrix} - \begin{pmatrix} 0 \\ W_{bc}{}^d{}_a \sigma_d \end{pmatrix}.$$

The curvature of the connection D_a is

$$\begin{aligned}
D_a D_b \begin{pmatrix} \sigma_c \\ \mu_{cd} \end{pmatrix} &= D_a \nabla_b \begin{pmatrix} \sigma_c \\ \mu_{cd} \end{pmatrix} - D_a \begin{pmatrix} 0 \\ W_{cd}{}^e{}_b \sigma_e \end{pmatrix} \\
&= \nabla_a \nabla_b \begin{pmatrix} \sigma_c \\ \mu_{cd} \end{pmatrix} - \begin{pmatrix} 0 \\ W_{cd}{}^e{}_a(\nabla_b \sigma_e - \mu_{be}) \end{pmatrix} - \nabla_a \begin{pmatrix} 0 \\ W_{cd}{}^e{}_b \sigma_e \end{pmatrix} \\
&= \nabla_a \nabla_b \begin{pmatrix} \sigma_c \\ \mu_{cd} \end{pmatrix} - \begin{pmatrix} 0 \\ W_{cd}{}^e{}_a(\nabla_b \sigma_e - \mu_{be}) \end{pmatrix} - \begin{pmatrix} -W_{ac}{}^e{}_b \sigma_e \\ \nabla_a(W_{cd}{}^e{}_b \sigma_e) \end{pmatrix} \\
&= \nabla_a \nabla_b \begin{pmatrix} \sigma_c \\ \mu_{cd} \end{pmatrix} - \begin{pmatrix} -W_{ac}{}^e{}_b \sigma_e \\ 2W_{cd}{}^e{}_{(a}\nabla_{b)}\sigma_e - W_{cd}{}^e{}_a \mu_{be} + (\nabla_a W_{cd}{}^e{}_b)\sigma_e \end{pmatrix}.
\end{aligned}$$

Therefore, bearing (12) and some Bianchi symmetry in mind,

$$(D_aD_b - D_bD_a)\begin{pmatrix}\sigma_c\\ \mu_{cd}\end{pmatrix} = \begin{pmatrix} 0 \\ 2W_{ab}{}^e{}_{[c}\mu_{d]e} + 2W_{cd}{}^e{}_{[a}\mu_{b]e} \\ +\,4(\nabla_{[a}\mathrm{P}_{b][c})\sigma_{d]} - (\nabla_a W_{cd}{}^e{}_b)\sigma_e + (\nabla_b W_{cd}{}^e{}_a)\sigma_e \end{pmatrix}.$$

Notice that the Bianchi identity (8) implies that the tensor

$$r_{abcd} \equiv 2W_{ab}{}^e{}_{[c}\mu_{d]e} + 2W_{cd}{}^e{}_{[a}\mu_{b]e} + 4(\nabla_{[a}\mathrm{P}_{b][c})\sigma_{d]} - (\nabla_a W_{cd}{}^e{}_b)\sigma_e + (\nabla_b W_{cd}{}^e{}_a)\sigma_e$$

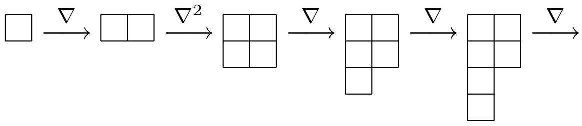
has the symmetries of the usual Riemann tensor.

BGG in flat projective case

The one we've been discovering by hand is

$$\begin{array}{llllllll} X_a & \mapsto & \nabla_{(a}X_{b)} \\ & & h_{ab} & \mapsto & \nabla_c\nabla_{[a}h_{b]d} - \nabla_d\nabla_{[a}h_{b]c} \\ & & & & r_{abcd} & \mapsto & \nabla_{[a}r_{bc]de} \\ & & & & & & B_{abcde} & \mapsto \nabla_{[a}B_{bcd]ef} \\ & & & & & & & \ddots \end{array}$$

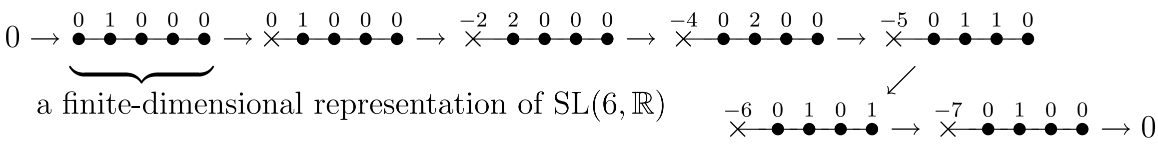
and here are their homes as Young tableau



where these tableau denote representations of $\mathrm{SL}(n,\mathbb{R})$. But better are the bundles on

$$\mathbb{R}^n \hookrightarrow \mathbb{RP}_n = \mathrm{SL}(n+1,\mathbb{R})\Big/\begin{bmatrix} * & * \\ 0 & \mathrm{GL}(n,\mathbb{R})\end{bmatrix}$$

namely (case $n=5$):–



(= linear elasticity complex when $n=3$) and, in general (for $n=5$),