The p-Version of Finite Element Method

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Questions like "What is the p-version of the finite element method, what are its advantages, why is it important?" are often asked.

To answer these questions, first it must be understood that the finite element method (FEM) is a method by which an approximate solution is obtained to the exact solution of some problem. For example, in linear elasticity, the solution domain, the material properties, the loading conditions and constraint conditions define a problem that has a unique exact solution $\mathbf{u}_{EX}$. A finite element solution $\mathbf{u}_{FE}$ is an approximation to $\mathbf{u}_{EX}$. The quality of approximation depends on the finite element mesh and the polynomial degree of the elements.

In the first implementations of the finite element method the polynomial degree of the elements (denoted by p) was fixed at a low value, typically p=1 or p=2, and the error of approximation was controlled by mesh refinement such that the size of the largest element in the mesh, denoted by h, was reduced. This is the h-version of the finite element method.

In later years research indicated that keeping the finite element mesh fixed and increasing the polynomial degree of elements, denoted by p, has important advantages. This is the p-version. Note that any implementation of the p-version can be operated in the "h-mode", meaning that p can be fixed and h reduced, but the reverse is not true. In this sense, implementations of the h-version are less general than those of the p-version.

Quality assurance

One of the important advantages of the p-version is that it makes assurance of the quality of the computed information more efficient and more convenient than the h-version. There are various measures of quality of the approximate solution. One is the energy norm measure.

By definition, the energy norm is the square root of the strain energy. Denoting the strain energy by U, the energy norm measure of the error is $(U(\mathbf{u}_{EX} - \mathbf{u}_{FE}))^{1/2}$ which is virtually the same as the root-mean-square measure of error in stresses [1].

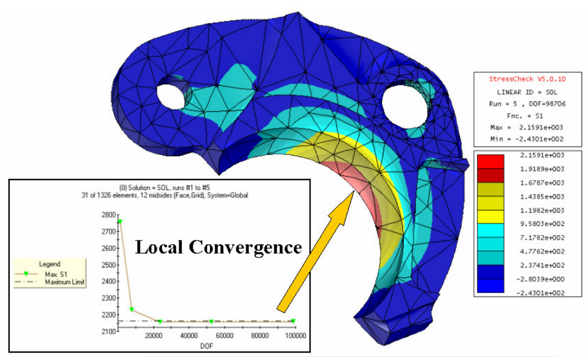
In engineering computations we are interested in data, such as the maximum normal stress, the maximum von Mises stress, the maximum displacement, the first few natural frequencies, etc. These are numbers computed from the finite element solution: $\Phi_i(\mathbf{u}_{FE})$, i=1,2,… It is important to know whether $\Phi_i(\mathbf{u}_{FE})$ is sufficiently close to $\Phi_i(\mathbf{u}_{EX})$. More precisely, one would like to have

$$| \Phi_i(\mathbf{u}_{EX}) - \Phi_i(\mathbf{u}_{FE}) | < \tau|\Phi_i(\mathbf{u}_{EX})|, \quad i=1,2,\dots$$

where τ is a tolerance.

Since generally we do not know $\mathbf{u}_{EX}$, this appears to be an unsolvable problem. We have to remember, however, that $\mathbf{u}_{EX}$ is independent of the mesh and the polynomial degree. Therefore $\Phi_i(\mathbf{u}_{FE})$ cannot be close to $\Phi_i(\mathbf{u}_{EX})$ if $\Phi_i(\mathbf{u}_{FE})$ changes significantly when the mesh is refined or the polynomial degree is increased. To show that $\Phi_i(\mathbf{u}_{FE})$ is virtually independent of h or p, it is necessary to obtain a *sequence* of finite element solutions. With the p-version this is easy to do because the mesh does not have to be changed. In StresssCheck® the user needs to specify a range of p-values and the program automatically computes the corresponding solutions [2]. Any computed data can be displayed as a function of the number of degrees of freedom (N).

Quality assurance in FEM is a process, three steps of which are illustrated in Fig. 1: First, the error in energy norm is estimated from a sequence of solutions. This is a global measure of error and related the root-mean-square error in stresses. Second, it is shown that the data of interest, in this case the maximum principal stress, is virtually independent of the number of degrees of freedom N. This ensures that the local error is small. Third, the stress (or strain) is plotted *without smoothening*. Significant jumps in the contour lines at element interfaces are indicators of error caused by inadequacies in meshing. For further reading on QA we refer to [3].



Global Error

Error Estimate

(0) Solution = SOL, runs #1 to #5 (ALL 1326 Elements)

Run #	DOF	Total Potential Energy	Convergence Rate	% Error
1	1362	-7.5808205688e-002	0.00	63.13
2	7938	-1.2429168751e-001	0.95	11.77
3	23702	-1.2564094034e-001	0.68	5.61
4	52632	-1.2588311726e-001	0.59	3.50
5	98706	-1.2596420639e-001	0.59	2.42

Estimated Limit -1.260378357684640e-001

Figure 1: Illustration of QA procedures

Rate of convergence

An important advantage of the p-version is that the rate of reduction of the error of approximation with respect to N is faster than that of the h-version. This is illustrated by a model problem of two-dimensional elasticity, so constructed that the exact solution is known [1]. Therefore the errors of approximation can be determined exactly.

The solution domain and the finite element meshes are shown in the insets in Fig. 2. The exact solution has a singularity at the re-entrant corner. The theoretical relationship between the error in energy norm and the number of degrees of freedom is given by the formula:

$$(U(\mathbf{u}_{EX} - \mathbf{u}_{FE}))^{1/2} \approx kN^{-\beta}$$

where k and β are positive constants, independent of h and p. Plotting the error in energy norm vs. N on log-log scale, β is the slope of the line. It is called the rate of convergence. For this problem, using uniform or nearly uniform meshes and p fixed at p=2, the rate of convergence is β=0.272, as seen in Fig. 2. If, on the other hand, we fix the mesh at h=a/2 and let p=2,3,…,8 then β=0.544, twice that of the h-version.

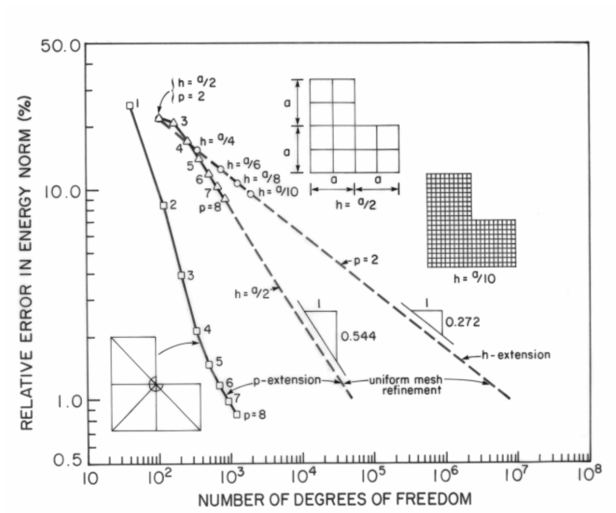


Figure 2: Rates of error reduction**.**

An important development in the 1980's was that with proper selection of the finite element mesh, very fast rates of convergence can be realized. Interestingly, definition of a "proper mesh" for the p-version is very simple: Whenever singular points are present, the mesh should be graded in geometric progression toward the singular point with the common factor of approximately 0.15. Such a mesh is illustrated in Fig. 2. Using this mesh and letting p=1,2,…,8, the error curve looks like an inverted S. This is because for low p values the mesh is overrefined at the singular point and the primary source of error is the set of elements away from the singular point where $\mathbf{u}_{EX}$ is smooth and hence the rate of convergence of the p-version is very strong. In fact, the theoretical rate of convergence of the p-version for smooth solutions is *exponential* [1]. For higher p-values the source of error is the set of elements that have a vertex on the singular point, hence the rate of convergence slows to β=0.544, as before. If we would keep adding layers of geometrically graded elements convergence would remain exponential. This is the hp-version of the finite element method. However, to achieve levels of accuracy usually expected in engineering practice, one or two layers of geometrically graded elements are usually sufficient.

Some analysts have objected that this comparison is not fair because singularities do not exist in "real problems". This objection is flawed, however: The finite element method operates on the input data, not on the "real problem". Sharp corners and edges may be consequences of simplifications introduced in the description of the geometry, nevertheless they do influence the accuracy of the solution. *It is the analyst's responsibility to ensure that the input data are consistent with the goals of computation and the errors of approximation are within acceptable bounds.*

Robustness

Another important advantage of the p-version is that it is much more robust than the h-version. In other words, the performance of the p-version is much less sensitive to input data than that of the h-version. For example, if Poisson's ratio is close to 0.5 then the h-version exhibits a highly undesirable property, known as Poisson's ratio locking. Similar problems occur when the thickness of a plate or shell is small. The corresponding reduction in the rate of convergence is called "shear locking". The remedy has been to use "reduced integration" techniques, arguing that since the elements are "overly stiff", one should "under-integrate" the elements, that is, use fewer integration points in the computation of the stiffness matrix than necessary. This ill advised measure leads to other undesirable consequences known as "hourglassing" or "zero energy modes" that are treated by various ad hoc procedures, making quality assurance impossible.

The p-version also tolerates large aspect ratio elements. This is especially important when analyzing laminated composites where ply-by-ply representation is necessary for resolving local stress and strain distributions. Elements with large aspect ratios are also necessary in finite element analyses of plate and shell problems where boundary layers are present [4], [5].

References

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[2] StressCheck® is trademark of Engineering Software Research and Development , Inc., 111 West Port Plaza, Suite 825, St. Louis, MO 63146, (314) 744-5021, www.esrd.com

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