EINSTEIN'S EQUATIONS AND CLIFFORD ALGEBRA

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Abstract. Einstein's equations of the general theory of relativity are rewritten within a Clifford algebra. This algebra is otherwise isomorphic to a direct product of two quaternion algebras. A multivector calculus is developed within this Clifford algebra which differs from the corresponding complexified algebra used in the standard space-time algebra approach.

1. Introduction

In recent years, applications of Clifford algebras in physics have been strongly developed, in particular in the area of the general theory of relativity [1, 2, 3]. Yet, most of these applications have neglected the close connection of Clifford algebras to quaternions. In the following, we shall give a formulation of Einstein's equations within a Clifford algebra which is isomorphic to a direct product of two quaternion algebras and thus differs from the corresponding complexified algebra used in the space-time algebra approach (STA).

2. Clifford Algebra

2.1 Definitions

Consider the Clifford algebra (over real numbers) having four generators $e_0,\ e_1,\ e_2,\ e_3$, anticommuting when distinct ($e_1 e_2 = -e_2 e_1$) and such that

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${e_0}^2 = -1,\ {e_1}^2 = {e_2}^2 = {e_3}^2 = +1$. A basis of the algebra is constituted by the sixteen elements 1, $e_0(\equiv j),\ e_1,\ e_2,\ e_3,\ e_0e_1,\ e_0e_2,\ e_0e_3,\ e_3e_2(\equiv I),\ e_1e_3(\equiv J),\ e_2e_1(\equiv K),\ e_1e_2e_3(\equiv k),\ e_0e_3e_2,\ e_0e_1e_3,\ e_0e_2e_1,\ e_0e_1e_2e_3(\equiv i)$. The elements $i,\ j,\ k$ satisfy the relations

$$i^2 = j^2 = k^2 = ijk = -1$$

and similarly for $I,\ J,\ K$. Hence, $i,\ j,\ k$ constitute a quaternionic system as well as $I,\ J,\ K$ the latter commuting with the first ($iJ = Ji$, etc.). Consequently, a general element of the algebra or Clifford number can be written as a set of four quaternions $(\alpha_0;\ \alpha_1,\ \alpha_2,\ \alpha_3)$ or $(\alpha_0;\ \alpha)$ and explicitly

$$(a + ib + jc + kd;\ \mathbf{m} + i\mathbf{n} + j\mathbf{r} + k\mathbf{s})$$

with $\mathbf{m} = m_1I + m_2J + m_3K$ and similarly for $\mathbf{n}$, $\mathbf{r}$, and $\mathbf{s}$ [4, 5]. The sixteen dimensions of a Clifford number decompose into scalars a, pseudo-scalars ib, vectors $(jc;\ k\mathbf{s}) \equiv jc + k\mathbf{s}$, bivectors $\mathbf{m} + i\mathbf{n}$ and trivectors $kd + j\mathbf{r}$. Two Clifford numbers multiply according to the rule

$$(\alpha_0;\ \alpha)(\beta_0;\ \beta) = (\alpha_0\beta_0 - \alpha.\beta;\ \alpha_0\beta + \alpha\beta_0 + \alpha \times \beta)$$

where $\alpha.\beta,\ \alpha \times \beta$ stand for the ordinary dot and vector products. Since quaternion multiplication is not commutative, the order of the elements in the formula must be respected. The product of two quaternions is obtained via the same product, the coefficients being real.

The conjugated A_c of a Clifford number A is defined by $(a + ib - jc + kd;\ -\mathbf{m} - i\mathbf{n} + j\mathbf{r} - k\mathbf{s})$ with $(AB)_c = B_cA_c$, where $A,\ B$ are Clifford numbers and the commutator is given by

$$2[A,\ B] = AB - BA.$$

2.2 Interior and Exterior Products

Interior and exterior products of two vectors u and v can be defined quite generally via the formula

$$uv = \lambda u.v + \mu u \wedge v$$

where $\lambda,\ \mu$ are constants [6]; adopting the choice $\lambda = \mu = -1$, one obtains the relations

$$2u.v = -(uv + vu)$$
$$2u \wedge v = -(uv - vu).$$

The exterior product being associative, the interior and exterior products of a vector v with a multivector $A_p = v_1 \wedge v_2 \ldots \wedge v_p$ where v_p are vectors is given by

$$2v.A_p = (-1)^p[vA_p - (-1)^p A_p v]$$
$$2v \wedge A_p = (-1)^p[vA_p + (-1)^p A_p v].$$

In particular, we shall frequently use the relation $2B.v = Bv - vB$ where B is a bivector. The interior and exterior products of two multivectors A_p and B_q are defined with Casanova [3] by

$$A_p.B_q = (v_1 \wedge \ldots \wedge v_{p-1}).(v_p.B_q)$$
$$A_p \wedge B_q = v_1 \wedge (v_2 \wedge \ldots v_p \wedge B_q).$$

3. Application to Einstein's Equations

3.1 Pseudo-euclidean Space

Consider the pseudo-euclidean four-dimensional vector space $(jx^0; k\mathbf{x})$ and its interior product

$$x.x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2;$$

according to whether the latter is positive, negative or nil, the vector is said to be time like, space like or isotropic. A unit time like vector (space like) has an interior product equal to 1 (resp. -1). The vectors $e_0,\ e_1,\ e_2$ and e_3 are orthogonal, unit vectors and constitute a basis of the pseudo-euclidean space. The reciprocal basis e^α is defined by $e^\mu.e_\nu = \delta^\mu{}_\nu (e^0 = e_0, e^1 = -e_1, e^2 = -e_2, e^3 = -e_3)$.

An orthochronous Lorentz transformation is expressed by

$$x' = axa_c$$

where a is an even multivector (constituted by an even product of e_α) such that $aa_c = 1$; the same relation $A' = aAa_c$ applies to any multivector A and yields the relativistic invariant AA_c. A pure rotation of angle θ around the unit (space-like) vector $k\mathbf{u}$ is given by $a = \cos\theta/2 + \mathbf{u}\sin\theta/2$; a pure Lorentz transformation is given by $a = \text{ ch } \varphi/2 + i\mathbf{u} \text{ sh } \varphi/2$ (with $\text{ th } \varphi = v/c$) and transforms (actively) a point at rest into a point moving with the velocity v in the direction of the unit vector $k\mathbf{u}$.

3.2 Riemannian Space

Consider a space with an elementary four-dimensional displacement $DM = \omega^i e_i$ and an affined connection $De_i = \omega^j{}_i e_j$. A covariant differentiation of a vector A is defined by

$$DA = dA + d\omega.A$$

with $2d\omega = \omega^{ij} e_i \wedge e_j$ where $d\omega$ is a bivector and $\omega^{ij} = \omega^i{}_k (e^k.e^j)$; dA is the ordinary differential with respect to the components. Under a Lorentz transformation $A' = fAf_c$, DA and $d\omega$ transform respectively according to $DA' = fDAf_c$ and

$$d\omega' = fd\omega f_c - 2df\ f_c.$$

The space is said to be riemannian if it is without torsion and has a curvature, two conditions which are expressed by

$$D(\Delta M) - \Delta(DM) = 0 \tag{1}$$

$$(D\Delta - \Delta D)A = \Omega.A \tag{2}$$

where Ω is the bivector $d\delta\omega - \delta d\omega + [d\omega,\ \delta\omega]$; $D,\ \Delta$ designate covariant differentiations in two independent directions. Equation (1) entails that the affined connection bivector $d\omega = I_k du^k$ is determined by the elementary displacement $DM = \sigma_m du^m$ via the linear equations

$$I_k\ .\sigma_m - I_m.\sigma_k = \frac{\partial \sigma_k}{\partial u^m} - \frac{\partial \sigma_m}{\partial u^k}$$

where the partial derivatives are taken only with respect to the components and $\sigma_i,\ I_k,\ u^k$ are respectively vectors, bivectors and the parameters. More explicitly, writing

$$2d\omega = L^{km}{}_j\ \omega^j(d) e_k \wedge e_m$$

$$d\omega^k(\delta) - \delta\omega^k(d) = -C^k{}_{ij}\omega^i(d)\omega^j(\delta),$$

equation (1) yields the relation

$$2L_{ijk} = (C_{kij} - C_{ijk} - C_{jki})$$

with the standard raising or lowering of indexes $C_{kij} = (e_k.e_m)C^m{}_{ij}$. Ω can be expressed as

$$\begin{aligned}\Omega &= -\Omega_{km}\omega^k(d)\omega^m(\delta)\\ &= -\left\{\frac{\partial I_k}{\partial u^m} - \frac{\partial I_m}{\partial u^k} + [I_k,\ I_m]\right\}du^k\delta u^m\end{aligned}$$

with $2\Omega_{km} = R^{ij}_{\ \ km} e_i \wedge e_j$ where $R^{ij}_{\ \ km}$ is the Riemann tensor. Bianchi's first identity is given by relation (3) entailing (4)

$$\Omega_{ij}.e_k + \Omega_{jk}.e_i + \Omega_{ki}.e_j = 0 \tag{3}$$

$$\Omega_{ij}.(e_k \wedge e_m) = \Omega_{km}.(e_i \wedge e_j). \tag{4}$$

Bianchi's second identity is given by

$$\Omega_{ij\,;\,k} + \Omega_{jk\,;\,i} + \Omega_{ki\,;\,j} = 0 \quad \text{with}$$

$$D'\Omega(d,\delta) = d'\Omega + [d'\omega, \Omega] = \Omega_{ij;\,k}\ \omega^i(d)\omega^j(\delta)\omega^k(d').$$

Finally, the Ricci tensor $R_{ik} = R^h_{\ ihk}$ and the curvature $R = R^k_{\ k}$ are obtained via the relations

$$R_k = -\Omega_{ik}.e^i = R_{ik}e^i\ ,$$

$$R = -(\Omega_{ik}.e^i).e^k = -\Omega_{ik}.(e^i \wedge e^k).$$

3.3 Einstein's Equations

Einstein's field equations [7-9] can be written in the form

$$\frac{1}{2}\Omega_{ij}.(e^i \wedge e^j \wedge e^k) = \kappa\ T^k \qquad (i,\, j = 0,\, 1,\, 2,\, 3) \tag{5}$$

with $T^k = T^{ik}e_{i,}$ $\kappa = 8\pi G/c^4$, and $2B.T = -(BT + TB)$, where T^{ik} is the energy-momentum tensor and $B,\ T$ are respectively a bivector and a trivector. The left member of equation (5) is identical to the standard expression $(R^{ik} - \delta^{ik}R/2)e_i$. The proof goes as follows; using the relation

$$(B.T).V = (B \wedge V).T$$

where $B,\ T,\ V$ are respectively a bivector, a trivector and a vector, one has

$$\begin{aligned}\left[\frac{1}{2}\Omega_{ij}.(e^i \wedge e^j \wedge e^k)\right].e_\mu &= \frac{1}{2}R^{\alpha\beta}_{\ \ \ ij}(e_\alpha \wedge e_\beta \wedge e_\mu).(e^i \wedge e^j \wedge e^k)\\ &= -\frac{1}{2}R^{\alpha\beta}_{\ \ \ ij}\ \delta^{ijk}_{\alpha\beta\mu} = R^{\beta k}_{\ \ \ \beta\mu} - \frac{1}{2}\delta^k_\mu\ R.\end{aligned}$$

Equation (5) presents a close formal resemblance to the equation obtained within the differential calculus despite a distinct algebraic context [10]. Finally, calling u the four-velocity the equation of motion is given by

$$Du = du + d\omega.u = 0.$$

As an application, in the case of the Schwarzschild metric, the metric and the bivector $d\omega$ are respectively given by (with $m = GM/c^2$)

$$ds^2 = (1 - \frac{2m}{r})c^2dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$d\omega = \cos\theta d\varphi I + (1 - \frac{2m}{r})^{1/2}(-\sin\theta \; d\varphi \, J + d\theta K) + iImcdt/r^2.$$

4. Conclusion

Einstein's equations have been formulated within a Clifford algebra revealing a formal analogy with the equations derived from the exterior differential calculus. A multivector calculus based on this algebra, directly linked to quaternions, and distinct from the complexified algebra used in STA has been developed with the hope of showing the usefulness of quaternions in physics.

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