4. Triangular matrices

- terminology
- forward and backward substitution
- inverse

Definitions

a square matrix A is *lower triangular* if $a_{ij} = 0$ for $j > i$

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & 0 \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}$$

A is *upper triangular* if $a_{ij} = 0$ for $j < i$ (A^T is lower triangular)

a triangular matrix is *unit* upper/lower triangular if $a_{ii} = 1$ for all i

a triangular matrix is *nonsingular* if the diagonal elements are nonzero

Forward substitution

solve $Ax = b$ with A lower triangular and nonsingular

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Algorithm:

$$\begin{aligned} x_1 &:= b_1/a_{11} \\ x_2 &:= (b_2 - a_{21}x_1)/a_{22} \\ x_3 &:= (b_3 - a_{31}x_1 - a_{32}x_2)/a_{33} \\ &\vdots \\ x_n &:= (b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})/a_{nn} \end{aligned}$$

Cost: $1 + 3 + 5 + \cdots + (2n-1) = n^2$ flops

Recursive formulation

Block matrix formulation (for $n \times n$ matrix A)

$$\begin{bmatrix} a_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ B_2 \end{bmatrix}$$

- a_{11} is scalar, A_{21} is $(n-1) \times 1$, A_{22} is $n \times n$
- x_1 is scalar, X_2 is an $(n-1)$-vector, b_1 is scalar, B_2 is an $(n-1)$-vector
- $a_{11} \neq 0$, A_{22} is nonsingular and lower triangular

Forward substitution

1. $x_1 := b_1/a_{11}$
2. solve $A_{22}X_2 = B_2 - A_{21}x_1$ by forward substitution

Back substitution

solve $Ax = b$ with A upper triangular and nonsingular

$$\begin{bmatrix} a_{11} & \cdots & a_{1,n-1} & a_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

Algorithm:

$$\begin{aligned} x_n &:= b_n/a_{nn} \\ x_{n-1} &:= (b_{n-1} - a_{n-1,n}x_n)/a_{n-1,n-1} \\ x_{n-2} &:= (b_{n-2} - a_{n-2,n-1}x_{n-1} - a_{n-2,n}x_n)/a_{n-2,n-2} \\ &\vdots \\ x_1 &:= (b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n)/a_{11} \end{aligned}$$

Cost: n^2 flops

Right inverse of a triangular matrix

to compute a right inverse, write $AX = I$ as

$$A \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$$

X_k is column k of X; e_k is kth unit vector (of size n)

then compute the X_k's by solving n sets of linear equations

$$AX_1 = e_1, \qquad AX_2 = e_2, \qquad \ldots, \qquad AX_n = e_n$$

using forward or backward substitution

Conclusion: if A is triangular and nonsingular, then it has a right inverse

Left inverse of a triangular matrix

to compute a left inverse, write $YA = I$ as $A^TY^T = I$, *i.e.*,

$$A^T \left[\begin{array}{cccc} Y_1 & Y_2 & \cdots & Y_n \end{array} \right] = \left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \right]$$

Y_k is column k of Y^T; e_k is kth unit vector (of size n)

then compute the Y_k's by solving n sets of linear equations

$$A^TY_1 = e_1, \qquad A^TY_2 = e_2, \qquad \ldots, \qquad A^TY_n = e_n$$

using forward or backward substitution

Conclusion: if A is triangular and nonsingular, then it has a left inverse

Inverse of a triangular matrix

if right and left inverse exist, they must be equal:

$$Y = Y(AX) = (YA)X = X$$

Conclusion: if A is triangular and nonsingular, then it has an inverse A^{-1}

$$AA^{-1} = A^{-1}A = I$$

- A^{-1} is lower triangular if A is lower triangular
- A^{-1} is upper triangular if A is upper triangular
- $(A^T)^{-1} = (A^{-1})^T$
- solution of $Ax = b$ can be expressed as $x = A^{-1}b$

Summary

if A is triangular and nonsingular (has nonzero diagonal elements), then:

- $Ax = b$ or $A^T x = b$ can be solved in n^2 flops
- A and A^T have inverses
- A has a full range: $Ax = b$ is solvable for all b
- A has a zero nullspace: unique solution of $Ax = 0$ is $x = 0$
- A^T has a full range: $A^T x = b$ is solvable for all b
- A^T has a zero nullspace: unique solution of $A^T x = 0$ is $x = 0$