Author: Dipl. Ing. Andrija Radović, http://www.andrijar.com

A REPRESENTATION OF FACTORIAL FUNCTION, THE NATURE OF γ CONSTAT AND A WAY FOR SOLVING OF FUNCTIONAL EQUATION $F(x) = x \cdot F(x - 1)$

The Γ function is given as recursive relation by the definition:

$$F(x) = x \cdot F(x - 1) \tag{1}$$

Derivative of the recursive relation is given as:

$$F'(x) = F(x - 1) + x \cdot F'(x - 1), \tag{2}$$

$$n!' = (n - 1)! + n \cdot (n - 1)!'$$

$$(n - 1)!' = (n - 2)! + (n - 1) \cdot (n - 2)!' \tag{3}$$

$\Rightarrow$

$$n!' = (n - 1)! + n \cdot ((n - 2)! + (n - 1) \cdot (n - 2)!') \tag{4}$$

$$(n - 2)!' = (n - 3)! + (n - 2) \cdot (n - 3)!' \tag{5}$$

$\Rightarrow$

$$n!' = (n - 1)! + n \cdot ((n - 2)! + (n - 1) \cdot ((n - 3)! + (n - 2) \cdot (n - 3)!')) \tag{6}$$

$$n!' = \frac{n!}{n} + \frac{n!}{n - 1} + \frac{n!}{n - 2} + n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3)!' \tag{7}$$

If the recursive process is continued to (n-n)!' it will be obtained:

$$n!' = n! \cdot \left(\sum_{i = 1}^{n} \frac{1}{i} \right) + n! \cdot C_f \tag{8}$$

Where the C_f constant is given as $C_f = \left. \frac{\partial(n!)}{\partial n} \right|_{n = 0} = -\gamma$. Now it is possible to be obtained a series of equations that every single one is derived by a previous:

$\Rightarrow$

$$n!' = n! \cdot \left(C_f + \sum_{i = 1}^{n} \frac{1}{i} \right) \tag{9}$$

$\Rightarrow$

$$\frac{\partial}{\partial n} LOG_e(n!) = C_f + \sum_{i=1}^{n} \frac{1}{i} \tag{10}$$

⇒

$$LOG_e(n!) = \int_0^n \left(C_f + \sum_{i=1}^{n} \frac{1}{i} \right) \cdot dn \tag{11}$$

This sum can be summarized regarding the next formula that defines the sum of geometric series:

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1} \tag{12}$$

Now, the next equations is given that may be applied for the summarization of the series:

$$\int_0^x \left(\sum_{i=0}^{n-1} x^i \right) \cdot dx = \int_0^x \frac{x^n - 1}{x - 1} \cdot dx \tag{13}$$

⇒

$$\sum_{i=0}^{n-1} \frac{x^{i+1}}{i+1} = \int_0^x \frac{x^n - 1}{x - 1} \cdot dx \tag{14}$$

⇒

$$\sum_{i=1}^{n} \frac{x^i}{i} = \int_0^x \frac{x^n - 1}{x - 1} \cdot dx \tag{15}$$

The final form of the formula for the series summarization is:

$$\sum_{i=1}^{n} \frac{1}{i} = \lim_{x \to 1} \sum_{i=1}^{n} \frac{x^i}{i} = \int_0^1 \frac{x^n - 1}{x - 1} \cdot dx \qquad (16)$$

Now It can be derived the formula of the factorial function:

$$n! = EXP\left(\int_0^n \left(C_f + \sum_{i=1}^{n} \frac{1}{i} \right) \cdot dn \right) = EXP\left(C_f \cdot n + \int_0^n \left(\int_0^1 \frac{x^n - 1}{x - 1} dx \right) \cdot dn \right) \qquad (17)$$

⇒

$$n! = EXP(C_f \cdot n) \cdot EXP\left(\int_0^n \left(\int_0^1 \frac{x^n - 1}{x - 1} \cdot dx \right) \cdot dn \right) \qquad (18)$$

Integrals may swap the places and thus the next formula is obtained:

$$n! = EXP(C_f \cdot n) \cdot EXP\left(\int_0^1 \frac{x^n - LOG_e(x^n) - 1}{(x - 1) \cdot LOG_e(x)} \cdot dx \right) \qquad (19)$$

Anent,

$$LOG_e(n!) = C_f \cdot n + \int_0^1 \frac{x^n - LOG_e(x^n) - 1}{(x - 1) \cdot LOG_e(x)} \cdot dx \qquad (20)$$

Now, $\Psi(x)$ can be defined on the next way; while the next equation is valid:

$$\frac{1}{x!} \cdot \frac{\partial(x!)}{\partial x} = C_f + \sum_{i=1}^{x} \frac{1}{i} = \Psi(x+1) = C_f + \int_0^1 \frac{y^x - 1}{y - 1} \cdot dy \qquad (21)$$

Where is:

$$C_f = -\gamma \qquad (22)$$

⇒

$$\boxed{\Psi(x) = -\gamma + \int_0^1 \frac{y^{x-1} - 1}{y - 1} \cdot dy} \qquad (23)$$

Let we observe the next expression:

(24)

$$LOG_e(n!) = C_f \cdot n + \int_0^1 \frac{x^n - n \cdot LOG_e(x) - 1}{(x-1) \cdot LOG_e(x)} \cdot dx = C_f \cdot n + \int_0^1 \frac{\frac{x^n - 1}{LOG_e(x)} - n}{x - 1} \cdot dx$$

With respect to the upper expression for factorial, it can be exanimate the nature of the γ constant. If we assume that is $n = 1$, the solution of proper integral is irrational number. Its exponent is irrational number, too. The total solution for the situation is integer number, i.e. zero. Consequently the C_f constant must be irrational number. See the formula:

$$LOG_e(n!) = C_f \cdot n + \int_0^n \left(\int_0^1 \frac{x^n - 1}{x - 1} dx \right) \cdot dn \qquad (25)$$

Let is $n = 1$:

(26)

$$C_f = -\int_0^1 \left(\int_0^1 \frac{x^n - 1}{x - 1} dx \right) \cdot dn = -\int_0^1 \frac{x - LOG_e(x) - 1}{(x-1) \cdot LOG_e(x)} \cdot dx = -\int_0^1 \left(\frac{1}{LOG_e(x)} - \frac{1}{x - 1} \right) \cdot dx$$

⇒

$$\gamma = \int_0^1 \left(\frac{1}{LOG_e(x)} - \frac{1}{x - 1} \right) \cdot dx \qquad (27)$$

⇒

$$C_f = \Psi(1) = \lim_{x \to 0} \left(Ei(LOG_e(x)) - Ei(LOG_e(1 + x)) - LOG_e\left(1 - \frac{1}{x}\right)\right) \qquad (28)$$

⇒

$$\gamma = \lim_{x \to 0} \left(Ei(LOG_e(x)) - Ei(LOG_e(1 + x)) - LOG_e\left(1 - \frac{1}{x}\right)\right) \qquad (29)$$

The next numeric formula for factorial computing is derived by the equations (19) and (26):

$$n! = EXP\left(\int_0^1 \frac{x^n - n \cdot (x - 1) - 1}{(x - 1) \cdot LOG_e(x)} \cdot dx \right) \qquad (30)$$

The next formula for computing of the Ψ is obtained with respect to the formulas (21) and (30):

$$\Psi(n) = \int_0^1 \left(\frac{x^{n-1}}{x - 1} - \frac{1}{LOG_e(x)} \right) \cdot dx \qquad (31)$$

By the formula (30) it can be computed the numerical solution of $(\sqrt{-1})!$:

$$i! = 0.4980156681183560427\,1 - 0.1549498283018106851\,3 \cdot i \qquad (32)$$

Author:
Dipl. Ing. Andrija S. Radović

Address:
Nike Strugara 13ª
11030 Beograd
Telephone: +381 11 2507837
Mobile: +381 64 1404075
E-mails: andrijar@andrijar.com