Lecture 012 (April 4, 2005)

Cubical sets

Simplicial sets are contravariant set-valued functors defined on the category of $\mathbf{\Delta}$ of finite sets and order preserving maps, and as such are artifacts of the combinatorics of finite sets.

Cubical sets depend on or represent the combinatorics of the power sets of finite ordered sets.

Write

$$\underline{\mathrm{n}} = \{1, 2, \dots, n\},$$

and write

$$\mathbf{1}^n = \mathbf{1}^{\times n}, \quad \mathbf{1} = \{0, 1\}.$$

$\mathbf{1}^0$ is the category consisting of one object and one morphism.

$\mathcal{P}(\underline{\mathrm{n}})$ is the poset of subsets of the set $\underline{\mathrm{n}}$.

Fact: There is an isomorphism of posets

$$\Omega_n : \mathbf{1}^n \xrightarrow{\cong} \mathcal{P}(\underline{\mathrm{n}})$$

$\Omega_n(\epsilon) = \{i \mid \epsilon_i = 1\}$ for $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbf{1}^n$.

The box category $\Box$ consists of certain poset maps $\mathbf{1}^m \to \mathbf{1}^n$.

Every finite totally ordered set A has a unique order-preserving bijection $\underline{n} \to A$, and it is convenient to represent box category morphisms as poset morphisms $\mathcal{P}(A) \to \mathcal{P}(B)$ where A and B are finite ordered sets.

There are two families of poset maps $\mathcal{P}(A) \to \mathcal{P}(B)$ which generate the box category:

1) Suppose $A \subset B \subset C$ (finite ordered sets).

$$[A, B] = \{D \subset C |\ A \subset D \subset B\} \subset \mathcal{P}(C)$$

is called an **interval** of subsets. There is a canonical poset isomorphism

$$\mathcal{P}(B - A) \xrightarrow{\cong} [A, B]$$

defined by $E \mapsto A \cup E$. The composite

$$\mathcal{P}(B - A) \xrightarrow{\cong} [A, B] \subset \mathcal{P}(C)$$

is called a *face functor*, and is also denoted by $[A, B]$.

2) Suppose $\emptyset \neq B \subset C$ (finite ordered sets). There is a poset morphism

$$s_B : \mathcal{P}(C) \to \mathcal{P}(B)$$

defined by $E \mapsto E \cap B$. s_B is called a *degeneracy functor*.

The **box category** $\square$ is the subcategory of the category of poset morphisms $\mathbf{1}^m \to \mathbf{1}^n$ which is generated by the face and degeneracy functors.

Suppose that $A \subset B$ and E are subsets of a finite ordered set C. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{P}(B-A) & \xrightarrow{[A,B]} & \mathcal{P}(C) \\ {\scriptstyle s_{E\cap(B-A)}}\downarrow & & \downarrow{\scriptstyle s_E} \\ \mathcal{P}((B\cap E)-(A\cap E)) & \xrightarrow[{[A\cap E, B\cap E]}]{} & \mathcal{P}(E) \end{array}$$

which allows one to show that all morphisms of the box category $\square$ are composites

$$\mathcal{P}(C) \xrightarrow{s_E} \mathcal{P}(E) \xrightarrow{[A, A\cup E]} \mathcal{P}(D).$$

These decompositions are unique

Examples:

1) Every $i \in C$ determines two intervals in $\mathcal{P}(C)$, namely $[\{i\}, C]$ and $[\emptyset, C - \{i\}]$.

If $C \cong \underline{\mathrm{n}}$, then $[\{i\}, C]$ uniquely determines a functor

$$d^{(i,1)} : \mathbf{1}^{n-1} \to \mathbf{1}^n,$$

while $[\emptyset, C - \{i\}]$ determines a functor

$$d^{(i,0)} : \mathbf{1}^{n-1} \to \mathbf{1}^n.$$

For $\epsilon = 0, 1$ $d^{(i,\epsilon)} : \mathbf{1}^{n-1} \to \mathbf{1}^n$ is defined by

$$d^{(i,\epsilon)}(\gamma_1, \dots, \gamma_{n-1}) = (\gamma_1, \dots, \overset{i}{\epsilon}, \dots, \gamma_{n-1}).$$

2) Every $j \in C$ determines a poset map $s_{C-\{j\}}$: $\mathcal{P}(C) \to \mathcal{P}(C - \{j\})$, or $s^j : \mathbf{1}^n \to \mathbf{1}^{n-1}$. s^j is the projection which drops the j^{th} entry:

$$s^j(\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_n)$$

$s^1 : \mathbf{1} \to \mathbf{1}^0$ is the map to the terminal object $\mathbf{1}^0$.

A **cubical set** X is a contravariant set-valued functor $X : \Box^{op} \to \mathbf{Set}$.

A **morphism of cubical sets** $f : X \to Y$ is a natural transformation of functors, and we have a category $\Box - \mathbf{Set}$ of cubical sets.

Write $X_n = X(\mathbf{1}^n)$, and call this set the set of n**-cells** of X.

Examples:

1) **standard** n**-cell** $\Box^n = \hom_\Box(\ ,\mathbf{1}^n)$ Every $x \in X_n$ is classified by a cubical set map $x : \Box^n \to X$. The faces $d_{(i,\epsilon)}(x)$ of x are the composites

$$\Box^{n-1} \xrightarrow{d^{(i,\epsilon)}} \Box^n \xrightarrow{x} X$$

The degeneracy $s_j(x)$ is represented by

$$\Box^{n+1} \xrightarrow{s^j} \Box^n \xrightarrow{x} X$$

A cell y is **degenerate** if $y = s_j x$; otherwise it is **non-degenerate**.

2) The **boundary** $\partial\square^n$ is the union of the images of the maps $d^{(i,\epsilon)} : \square^{n-1} \to \square^n$. There is a coequalizer

$$\bigsqcup_{\substack{(\epsilon_1,\epsilon_2)\\ 0\leq i<j\leq n}} \square^{n-2} \rightrightarrows \bigsqcup_{(i,\epsilon)} \square^{n-1} \to \partial\square^n$$

where $\epsilon_i \in \{0,1\}$.

3) $\sqcap^n_{(i,\epsilon)}$ is the subobject of $\partial\square^n$ which is generated by all faces except for $d^{(i,\epsilon)} : \square^{n-1} \to \square^n$. There is a coequalizer diagram

$$\bigsqcup \square^{n-2} \rightrightarrows \bigsqcup_{(j,\gamma)\neq(i,\epsilon)} \square^{n-1} \to \sqcap^n_{(\epsilon,i)}$$

where the first disjoint union is indexed over all pairs (j_1,γ_1), (j_2,γ_2) with $0 \leq j_1 < j_2 \leq n$ and $(j_k,\gamma_k) \neq (i,\epsilon)$, $k = 1,2$.

3) The assignment $\mathbf{1}^n \mapsto B(\mathbf{1}^n)$ defines a simplicial set-valued functor $\square \to \mathbf{S}$. If X is a simplicial set, there is an associated **singular** cubical set $S(X)$, with n-cells

$$S(X)_n = \hom(B(\mathbf{1}^n), X).$$

Note that $B(\mathbf{1}^n) \cong (\Delta^1)^{\times n}$. The singular functor $S : \mathbf{S} \to \square-\mathbf{Set}$ has a left adjoint $|\ | : \square-\mathbf{Set} \to \mathbf{S}$, called **triangulation**, which is defined by

$$|Y| = \varinjlim_{\square^n \to Y} B(\mathbf{1}^n).$$

The colimit is indexed by members of the **cell category** $i_{\square}Y$: the objects of the cell category are the cells $\square^m \to Y$ and the morphisms of $i_{\square}Y$ are the incidence relations

$$\begin{array}{ccc} \square^r & \longrightarrow & \square^m \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

NB: there are similarly defined realization and singular functors

$$| \ | : \square - \mathbf{Set} \leftrightarrows \mathbf{Top} : S$$

relating cubical sets and topological spaces; realization is left adjoint to the singular functor.

4) Suppose that C is a small category. The **cubical nerve** $B_{\square}C$ is the cubical set with n-cells

$$B_{\square}C_n = \hom_{cat}(\mathbf{1}^n, C).$$

The cells of $B_{\square}C$ are the hypercube diagrams in C.

There is a good notion of **skeleta** for cubical sets: $\operatorname{sk}_n X$ is the subobject of X which is generated by the cells X_k, $0 \leq k \leq n$. Clearly, $\operatorname{sk}_{n-1} X \subset \operatorname{sk}_n X$, and one can show that there is a pushout

$$\begin{array}{ccc} \bigsqcup_{x \in NX_n} \partial\Box^n & \longrightarrow & \operatorname{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \bigsqcup_{x \in NX_n} \Box^n & \longrightarrow & \operatorname{sk}_n X \end{array}$$

where NX_n denotes the non-degenerate part of X_n. Proving this requires showing that if x, y are degenerate n-cells with the same boundary, then $x = y$ — see Lemma 18 of "Categorical homotopy theory"

A **cofibration** of cubical sets is a monomorphism, a **weak equivalence** of cubical sets is a map $f : X \to Y$ which induces a weak equivalence $|X| \to |Y|$ of triangulations (equivalently a weak equivalence $Bi_{\square}X \to Bi_{\square}Y$). A fibration of cubical sets is a map which has the RLP wrt to all inclusions $\sqcap^n_{(i,\epsilon)} \subset \square^n$.

Theorem: (Cisinski)

1) With these definitions $\square - \mathbf{Set}$ satisfies the axioms for a proper closed model category.

2) The cubical singular and triangulation functors induce a Quillen equivalence

$$| \ | : \mathrm{Ho}(\square - \mathbf{Set}) \simeq \mathrm{Ho}(\mathbf{S}) : S_{\square}.$$

Cisinski's theorem is perhaps the deepest result in abstract homotopy theory. It is mentioned for cultural reasons here, and will not be needed in the sequel. It is proved in Cisinski's thesis, and again in "Categorical homotopy theory".

One can use standard techniques to show that there is a model structure on cubical sets for which the cofibrations are monomorphisms and weak equivalences are those maps which induce weak equivalences of triangulations, and that the resulting homotopy category is Quillen equivalent to the standard homotopy category for simplicial sets. Showing that the fibrations are as described is the interesting part.

There's one little problem: categorical products of cubical sets are very badly behaved.

Example: An n-cell $(\sigma, \tau) : \square^n \to \square^1 \times \square^1$ is a pair of n-cells of $\square^1$.

$\square^1 \times \square^1$ has two distinct non-degenerate 2-cells, namely the identity on $\mathbf{1}^2$ and the twist automorphism $\tau : \mathbf{1}^2 \to \mathbf{1}^2$. These 2-cells have the common boundary that one expects, namely $\partial\square^2$ (up to a twist), but there is an additional non-degenerate 1-cell $\Delta : \mathbf{1} \to \mathbf{1}^2$ given by the diagonal map. It follows that $|\square^1 \times \square^1|$ has the homotopy type of $S^2 \vee S^1$.

The problem is fixed as follows: define

$$\square^n \otimes \square^m = \square^{n+m},$$

and more generally set

$$X \otimes Y = \varinjlim_{\square^n \to X,\ \square^m \to Y} \square^n \otimes \square^m.$$

Then one can show that there is a natural isomorphism

$$|X \otimes Y| \cong |X| \times |Y|.$$

Remark: I did not say that $\square - \mathbf{Set}$ has a simplicial model structure, because it doesn't. It has, instead, a cubical model structure, with function complex object $\mathbf{hom}(X,Y)$ specified by

$$\mathbf{hom}(X,Y)_n = \hom(X \otimes \square^n, Y).$$

Note that if $i : A \to B$ and $j : K \to L$ are cofibrations of cubical sets, then the induced map

$$(B \otimes K) \cup_{(A \otimes K)} (A \otimes L) \to (B \otimes L)$$

is a cofibration (for this you need to know that triangulation reflects monomorophisms — Cor. 23 of "Categorical homotopy theory") which is trivial if either i or j is trivial. It follows that if $p : X \to Y$ is a fibration and if $i : A \to B$ is a cofibration as above, then the map

$$\mathbf{hom}(B,X) \to \mathbf{hom}(A,X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B,Y)$$

is a fibration which is a weak equivalence if either i or p is trivial. This is the cubical analogue of Quillen's axiom **SM7** and so one could say that cubical sets has a **cubical model structure**