On Finite Simple Groups and Their Classification

by Ron Solomon

Danny Gorenstein called it the "Thirty Years War," for the Classification battles were fought mostly in the decades 1950–1980, although the dream of a classification of all finite simple groups goes back at least to the 1890s. In this brief article, I shall attempt to give some sense of the mathematical highpoints of the original proof and the ongoing revision project. I shall also give some personal reflections on the sociology of the classification effort, and finally I shall discuss some current and future directions for research in finite group theory. Many thanks are due to Jon Alperin, Michael Aschbacher, George Glauberman, Bill Kantor, Radha Kessar, Richard Lyons, and Steve Smith for valuable critiques of this article.

But first, a word about our sponsors: the finite simple groups themselves. Before there even was a mathematical term "group", Lagrange, Gauss, and their contemporaries were familiar with the cyclic groups Z_p and the alternating groups A_n. Galois, who gave us the term "group" and the concept of a normal subgroup, was also familiar with the fractional linear groups $\mathrm{PSL}(2,p)$ and their connection to p-division points on elliptic curves. Jordan described the classical linear (or matrix) groups over prime fields, and this was extended to all finite fields by Dickson. This yields the projective special linear, unitary, symplectic, and orthogonal groups acting on a finite-dimensional vector space over a finite field, whose derived subgroups are all simple, except in a few cases for very small dimensions and fields. The work of Chevalley, Tits, Steinberg, Suzuki, and Ree in the 1950s provided a systematic description of finite analogues for all of the complex simple Lie groups, some real forms and related "twists". Finally in the period 1965–1974, a great gold rush unearthed twenty-one new simple groups to supplement the five which had been discovered in the 1860s by Mathieu and dubbed "sporadic" by Burnside. Like the elementary particles of physics, sporadic simple groups were often predicted several years before their existence was confirmed. For example, the Monster was predicted in 1973, but not constructed until 1980.

. . . the study of simple groups generated amazing insights into the structure of finite groups and uncovered several of the most fascinating objects in the mathematical firmament.

A vast literature of theorems, most of which were published between 1955 and 1983, combines to yield the following result.

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Galois Stroth Gorenstein Gauss Suzuki Tits

The Classification of the Finite Simple Groups

Let G be a finite simple group. Then G is one of the following**:**

1. a cyclic group of prime order, Z_p;
2. an alternating group, $A_n, n \geq 5$;
3. a classical linear group $\mathrm{PSL}(n,q)$, $\mathrm{PSU}(n,q)$, $\mathrm{PSp}(2n,q)$ or $\mathrm{P}\Omega^{\varepsilon}(n,q)$;
4. an exceptional or twisted group of Lie type ${}^3D_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, ${}^2F_4(2^n)'$, $G_2(q)$, ${}^2G_2(3^n)$ or ${}^2B_2(2^n)$;
5. a sporadic simple group: M_{11}, M_{12}, M_{22}, M_{23}, M_{24} (the Mathieu groups); J_1, J_2, J_3, J_4 (the Janko groups); Co_1, Co_2, Co_3 (the Conway groups); HS, Mc; Suz (Co_1 'babies'), Fi_{22}, Fi_{23}, Fi_{24}' (the Fischer groups); $F_1 = M$ (the Monster), $F_2, F_3, F_5, He (= F_7)$ (Monster 'babies'); Ru; Ly; ON.

The Mathematics of the Classification

The term "Thirty Years War" is apt, inasmuch as the first close approximation to the eventually successful classification strategy was proposed by Richard Brauer at the International Congress of Mathematicians in Amsterdam in 1954. His idea was:

> *In a finite nonabelian simple group G, choose an involution z (an element of order two) and consider its centralizer $C_G(z) = \{g \in G \colon gz = zg\}$. Show that the isomorphism type of $C_G(z)$ determines the possible isomorphism types of G.*

During the period 1950–1965, Brauer and others honed methods for solving the class of problems: Given a specific H, determine up to isomorphism all simple groups G having an involution z with $C_G(z) \equiv H$. Once Feit and Thompson proved Burnside's odd order conjecture in 1963, Brauer's strategy gained further credibility: at least one could find an involution in any nonabelian finite simple group!

The subgroup $C_G(z)$ is an example of a (p-)local subgroup of G, i.e., the normalizer of a nonidentity p-subgroup of G for some prime p. (In this case, $p = 2$.) Brauer's philosophy represented the first version of a type of local-global principle that was to determine the shape of the Classification proof.

The battle to restrict the structure of H was intricately interwoven during the decade 1965–1975 with the quest for new sporadic simple groups to create a unique mathematical tapestry. This precis will focus only on the battle. This then was one of the principal remaining challenges of the Classification problem in 1965: *How does one use the hypothesis of simplicity of G to restrict the structure of the centralizer H?*

For example, H could hypothetically have arbitrarily complicated solvable normal subgroups. But in the actual finite simple groups, solvable normal subgroups of centralizers of involutions are of very restricted type. In particular, normal subgroups of odd order in centralizers of involutions are cyclic and "almost" central. There is a suggestive analogy in the theory of Lie algebras: If L is a finite-dimensional semisimple Lie algebra over C and x is a semisimple element of L (i.e., $\rho(x)$ is diagonalizable for any matrix representation ρ of L), then the centralizer $C_L(x)$ is a *reductive* Lie algebra (i.e., any solvable ideal of $C_L(x)$ is central). The achievement of the analogous theorem for finite simple groups (the B_p-Theorem) is the longest chapter in the entire classification and centers around three themes:

1. Signalizer Method

The signalizer method provides the most far-reaching answer to the question:

> *How can one exploit the absence of solvable normal subgroups in G to bound the structure of solvable normal subgroups of centralizers $H = C_G(z)$***?**

The key initial ideas for the study of "A-signalizers" appeared in the work of Thompson, while the concept of a signalizer functor is due to Gorenstein. The crucial Signalizer Functor Theorem (see Appendix) gives conditions under which a collection of A-invariant p'-subgroups of G can be glued together into a single *proper* p'-subgroup of G. (X is a p'*-group* if the prime p does not divide the order of X.) The wished-for conclusion is that this subgroup, $\Theta(G;A)$, is a *normal* subgroup of G, whence $\Theta(G;A) = 1$. This is a lengthy journey, a principal way-station of which is the proof that if $\Theta(G;A)$ is not nor-

Steinberg Dickson

mal in G, then $N_G(\Theta(G;A))$ is a *strongly p-embedded subgroup* of G.

2. Strong p-Embedding

Definition. Let G be a finite group. A proper subgroup M of G is a *strongly p-embedded subgroup* of G if p divides $|M|$, but p does not divide $|M \cap M^g|$ for any $g \in G - M$.

This means that in the transitive permutation action of G on the coset space $M\backslash G$, every element of order p fixes exactly one point. The evolution of this theory passes through the work of Frobenius, Zassenhaus, Feit, Brauer, Suzuki, and others and, for $p = 2$, reaches a very elegant conclusion in the *Strongly Embedded Theorem* of Bender. This identifies all simple groups with a strongly 2-embedded subgroup and, in particular, asserts that *no* simple group has a strongly 2-embedded $2'$-local subgroup (by a *p'-local subgroup* we mean the normalizer of a nonidentity p'-subgroup).

When p is odd, the story is messier. For applications to signalizer theory, the crucial fact is: *No simple group G of* p-rank ≥ 3 *has a strongly p-embedded p'-local subgroup.* This has only been established after the fact of the Classification. However, in the inductive context, the required special case of this result was established by Aschbacher. Because nontrivial signalizer functors lead to strongly p-embedded p'-local subgroups, they in turn do not in general exist. This is the key to establishing the crucial B_p-Theorem whose statement we shall now approach.

3. Semisimple Elements and Components

Definitions. We call a finite group L quasisimple if $L = [L, L]$ and $L/Z(L)$ is simple. A good example is $\mathrm{SL}(n, q)$ for $(n, q) \neq (2, 2)$ or $(2, 3)$.

We call L a *component* of H if L is quasi-simple and L is a subnormal subgroup of H, i.e., there is a normal series (in the sense of Jordan-Holder) from L to H. Distinct components of H commute and the (commuting) product of all components of H is denoted $E(H)$.

Components play a dominating role in the centralizers of *semisimple* elements in classical linear groups (indeed, in all groups of Lie type). For example, if $G = \mathrm{GL}(n, q)$ with q odd, then involutions t in G are semisimple (in fact, diagonalizable) and a typical such t and its centralizer $H = C_G(t)$ are:

$$t = \begin{pmatrix} -I_{m\times m} & 0 \\ 0 & I_{r\times r} \end{pmatrix}$$

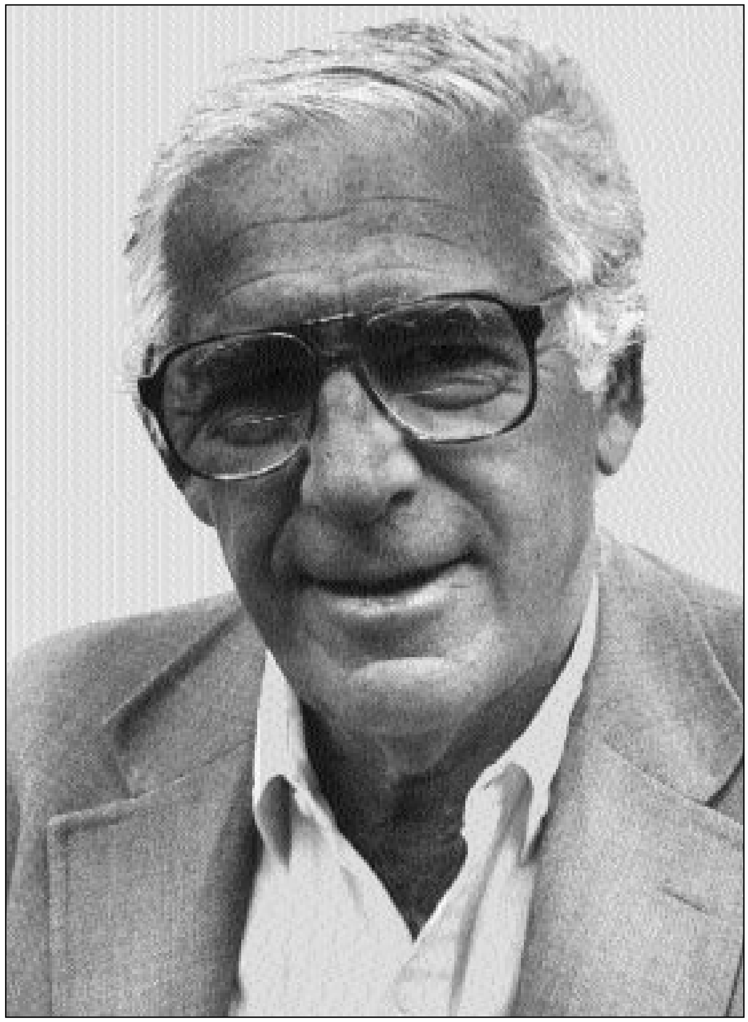
$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathrm{GL}(m, q), B \in \mathrm{GL}(r, q) \right\}$$

$$\equiv \mathrm{GL}(m, q) \times \mathrm{GL}(r, q),$$

$$n = m + r.$$

If $m > 1$ and $(m, q) \neq (2, 3)$, then $\mathrm{SL}(m, q) = E(\mathrm{GL}(m, q))$ is a quasisimple component of H, and likewise for $\mathrm{SL}(r, q)$. Thus, except for the small cases noted, $E(H) \equiv \mathrm{SL}(m, q) \times \mathrm{SL}(r, q) = [H, H]$.

In contrast to this, if $G = \mathrm{GL}(n, 2^m)$, then involutions t in G are unipotent matrices and the



In 1982, Danny Gorenstein (above), Ron Solomon, and Richard Lyons began a "Revision Project" intended to produce a "new and complete proof of the Classification."

centralizer of t has a large "unipotent radical"—a normal 2-subgroup—and has no components.

The early work of Feit, Suzuki, and Thompson dealt exclusively with groups whose local subgroups H were solvable, thus for which $E(H) = 1$. It was Gorenstein and Walter in the mid '60s who first came to grips with the general problem of simple groups with nonsolvable local subgroups.

Definitions. *The generalized Fitting subgroup* of H is $F^*(H) = F(H)E(H)$ where $F(H)$ is the Fitting subgroup of H, i.e., the (unique) largest normal nilpotent subgroup of H. We call a p-element x of G *semisimple* if $E(C_G(x)) \neq 1$. We call a p-element x of G *unipotent* if $F^*(C_G(x))$ is a p-group.

Caveat: If G is a classical linear group, this notion of semisimple roughly corresponds to the classical notion, but there are definite discrepancies. In a *simple* classical linear group over a field of characteristic p, every nonidentity unipotent element (in the classical sense) is unipotent in the above sense.

The principal application of the signalizer method is to establish a slightly weakened version of the following theorem:

Theorem. *If G is simple of p-rank ≥ 3, then either some $x \in G$ of order p is semisimple or every $X \in G$ of order p is unipotent.*

At the heart of this analysis is the B_p-Theorem. (See Appendix.)

In the semisimple setting a second important answer to the question: *How does one exploit the simplicity of G?* is provided by the following theorem of Aschbacher (as refined by Foote):

Aschbacher's Component Theorem: Suppose that $E(G)$ is simple and G contains a semisimple involution. Then there is some semisimple involution x such that $C_G(x)$ has a *normal* subgroup K which is either quasi-simple or isomorphic to $O^+(4, q)'$ and such that $Q = C_G(K)$ is *tightly embedded* (i.e., $|Q \cap Q^g|$ is odd for all $g \in G - N_G(Q)$).

In practice, when K is a known quasi-simple group, tight embedding usually implies that Q has 2-rank 1. In the current revision of the Classification proof, Aschbacher's theorem is extended to (somewhat different) *p-Component Theorems* for all primes p.

With the B_p-Theorem and the p-Component Theorems in hand, Brauer's original strategy can now be vindicated and refined to an inductive algorithm for classifying those finite simple groups that contain a semisimple p-element for some prime p:

1. Choose a semisimple element x of prime order p whose centralizer contains a large component K, as promised by the p-Component Theorem. By induction, K is a *known* quasi-simple group and by the p-Component Theorem, K is almost all of $C_G(x)$.
2. Now for each known quasi-simple group K and each prime p, classify all finite simple groups G having an element x of order p with $C_G(x)$ approximately equal to K.

With some refinements (in particular, one must choose $p = 2$, if possible), this is the strategy which handles roughly half of the Classification proof, but does not handle:

4. Quasi-Unipotent Groups

Definition. We call G *quasi-unipotent* if *every* element of G of order p is unipotent for *all* primes p such that G has p-rank ≥ 3.

Thus the other half of the Classification problem is the determination of all quasi-unipotent groups. In the context of the classification of finite-dimensional semisimple Lie algebras over C, a roughly analogous problem is quickly resolved by Engel's Theorem: A finite-dimensional Lie algebra L, all of whose elements are ad-nilpotent, is itself a nilpotent Lie algebra (hence, in particular, L is not semisimple). In the classification of the finite simple groups, this problem is considerably thornier. It sits logically at the base of the entire problem, in the sense that any *minimal* simple group is quasi-unipotent.

[The classification of the minimal simple groups was achieved in the monumental Odd Order Theorem of Feit and Thompson and the N-Group Theorem of Thompson.]

There are four principal cases of the general Quasi-unipotent Problem:

A. The Odd Order Case, treated by Feit and Thompson.

B. The 2-Rank 2 Case, treated by Alperin, Brauer, Gorenstein, Walter and Lyons.

The remaining two cases, together with many of the fundamental ideas for their solution, first emerged clearly in Thompson's fundamental work on "N-groups".

C. The Classical Klinger-Mason Case: G is quasi-unipotent of 2-rank ≥ 3 and some 2-local subgroup M has p-rank ≤ 2 for *every* odd prime p. (The "thin" subcase is when the 2-local p-rank of G is 1.)

D. The Classical Quasi-Thin Case: G is quasi-unipotent of 2-rank ≥ 3 but *every* 2-local subgroup P has p-rank ≤ 2 for *every* odd prime p. (The "thin" subcase is when the 2-local p-rank of G is 1.)

The first step in Case D is to establish that G is generated by two 2-locals, say P_1 and P_2, containing a common Sylow 2-subgroup T of G. The existence of at least two maximal 2-locals containing a given Sylow 2-subgroup T is guaranteed by the *Global $C(G;T)$-Theorem*: If a Sylow 2-subgroup T of G lies in a unique maximal 2-local P of G, then P is a strongly embedded subgroup of G and G is known.

The Quasi-thin Theorem asserts that in Case D, if G does not have a strongly embedded subgroup, then G is a group of Lie type in characteristic 2 of Lie rank 2 generated by a pair of (parabolic) subgroups P_1 and P_2 or G is on a short list of exceptions. The strategy is to construct the "building", i.e., the (P_1, P_2)-coset geometry, and then identify this geometry and the associated group. The original proof of the Quasi-thin Theorem by Aschbacher and Geoff Mason (only in preprint, except for [A1]) uses the Weak Closure Method, introduced by Thompson and extended by Aschbacher. Current work towards a new proof uses the Amalgam Method, introduced by Goldschmidt. These methods afford the final deep answer to the question:

How does one exploit the simplicity of G to bound the p-local structure of a (quasi-unipotent) simple group G?

This completes a very brief overview of the strategy for the Classification proof.

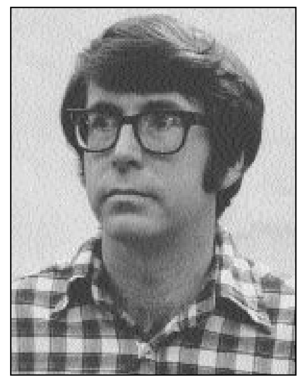
The Sociology of the Classification in the 1970s

> Up to the early 1960s, really nothing of real interest was known about general simple groups of finite order. ... Since *[1962]*, finite group theory simply is not the same any more.
>
> —*Richard Brauer (ICM, 1970)*

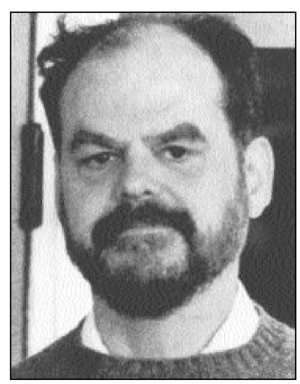
The Odd Order Theorem of Feit and Thompson (followed by Thompson's N-group paper) was a singularity in the evolution of finite group theory. An understanding of the dramatic new ideas and methods introduced in this 255-page paper became almost indispensible for continued participation in the Classification endeavor. Gorenstein wrote the 'Reader's Guide' in 1968: Finite Groups. He also provided the optimism, the organization and, in 1972, a '16-step plan' for the completion of the Classification proof. Although obsolete in several important points within months of its articulation, Gorenstein's program was a critical source of problems and inspiration for the 'young Turks' who attacked the Classification in the '70s. Yet another critical new feature of the '70s, most notably in the work of Timmesfeld and Aschbacher, was the fusion of the geometric methods of Fischer, Hall, and Shult with the architectonic analysis of Thompson, Gorenstein, and Walter.

The pace of the Classification in the '70s was exhilarating. Not a single leading group theorist besides Gorenstein believed in 1972 that the Classification would be completed this century. By 1976, almost everyone believed that the Classification problem was "busted". The principal reason was Michael Aschbacher's lightning assaults on the B-Conjecture, the Thin Group Problem, and the Strongly p-embedded 2-local problem. Also, in 1976 Timmesfeld announced a

Richard Brauer

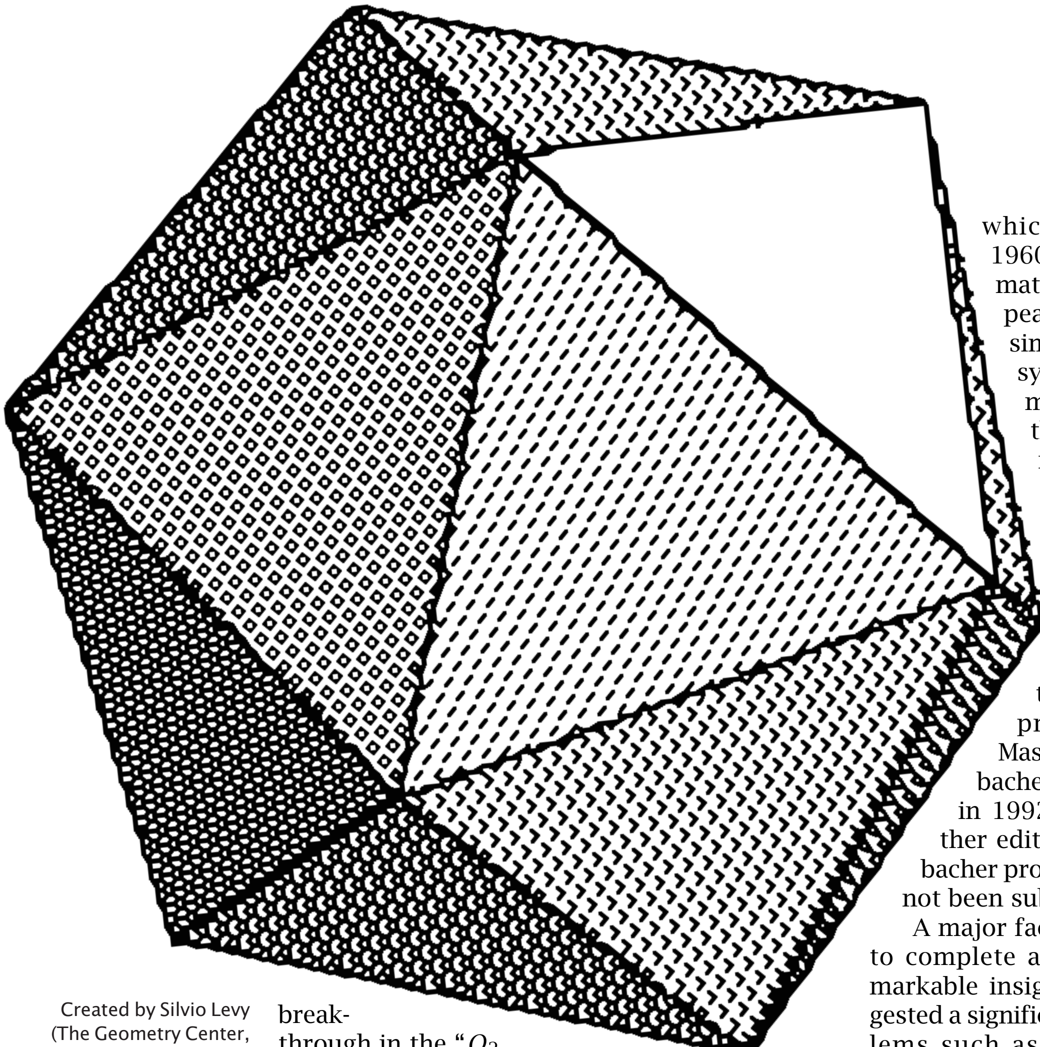


Michael Aschbacher



John Thompson

Numerous mathematicians have been directly or indirectly involved in this research, many of their names appear in gray along the tops of these pages.



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break-through in the "O_2 extraspecial" problem. (Although major cases remained to be handled by Smith, Stroth, and others, Timmesfeld "broke the back" of the problem.) This had a profound psychological impact because sixteen of the twenty-six sporadic simple groups (including the Monster and its babies, as well as Co_1, Co_2, Fi'_{24} and J_4) satisfy the O_2 extraspecial hypothesis. Psychologically, Timmesfeld's theorem was a finiteness theorem for sporadic simple groups. The endless frontier was closing, and by the 1976 Duluth Conference, Aschbacher, Gorenstein and Lyons, Gilman and Griess, and Geoff Mason had staked their claims to all of the remaining major pieces of the Classification.

The literature of the Classification was always challenging, coming in massive 200-page papers. Nevertheless, there were always individuals and seminar groups that made serious efforts to read and digest most of the papers which appeared during the years 1960–1975. At least 3,000 pages of mathematically dense preprints appeared in the years 1976–1980 and simply overwhelmed the digestive system of the group theory community. Mason's 800-page quasithin typescript has achieved some notoriety, inasmuch as it has never been published. More accurately, it is an extreme point on the spectrum of incompletely assimilated manuscripts from the latter years of the Classification. Indeed, it was not until 1989 that it was noticed that certain small subcases of the problem remained untreated in Mason's typescript, a gap which Aschbacher filled in a typescript distributed in 1992. The manuscript requires further editing, and as yet the Mason-Aschbacher proof of the Quasi-thin Theorem has not been submitted for publication.

A major factor in Mason's initial reluctance to complete and publish his work was a remarkable insight by Goldschmidt, which suggested a significantly different approach to problems such as the quasi-thin problem—the Amalgam Method, mentioned above. Mason believed that this method was superior to the approach that he had been using. Furthermore, several people—notably Goldschmidt, Stroth, and Stellmacher—produced beautiful papers exploiting this method. (Two major papers of Stellmacher—on "thin groups" and on N-groups—also remain unpublished.) Mason believed that a new proof of the Quasi-thin Theorem via the Amalgam Method was inevitable in the near future and abandoned his almost-completed paper.

The necessity of bringing this entire Classification endeavor to a more coherent and compelling resolution was grasped immediately by Danny Gorenstein and led him to spearhead:

The Revision Project

The process of "revision" of the Classification was for years inextricably associated with the name of Helmut Bender, who began the creative reworking of the Odd Order Theorem and the Abelian and Dihedral Sylow 2-Subgroup Theorems in the late ’60s. In so doing, he enriched all of group theory with fundamental new concepts—e.g., $F^*(G)$—and new theorems. Later, several others, notably Glauberman, Peterfalvl, and Enguehard undertook various “revision” projects. This effort has produced an almost complete revision of both the Odd Order Theorem and the identification of the split BN-pairs of rank 1 (i.e., $\mathrm{PSL}(2,q)$ $\mathrm{PSU}(3,q)$, ${}^2B_2(2^n)$ and ${}^2G_2(3^n)$).

In a somewhat different vein, Goldschmidt introduced the Amalgam Method with the principal intention of “revising” the weak closure arguments at the heart of Thompson’s N-group paper and of the thin and quasi-thin group work of Aschbacher and Mason. This program has had major successes, notably Stellmacher’s revisions of the core of Thompson’s N-group paper and Aschbacher’s thin group paper. It too has had a fruitful influence on the related fields of finite geometry and geometric group theory.

In 1982, Danny Gorenstein launched a “revision project” in which he was joined by Richard Lyons and myself. This project is intended to complement the work of the other revision efforts to yield a new and complete proof of the Classification. Here is a brief discussion of the status of this (GLS) project and its interfaces with the other revision efforts.

The work of GLS rests on a foundation of background results. In addition to the contents of standard textbooks and monographs, these consist principally of:

1. The Odd Order Theorem and the identification of the split BN-pairs of rank 1. (As discussed above, these theorems have already been subjected to extensive review and revision.), and

2. The existence, uniqueness, Schur multipliers, and other basic properties of the twenty-six sporadic simple groups. (These properties are stated without proof in [GL] and in the Atlas [Co]. Many have published proofs in journals. Aschbacher has taken a first major step towards codifying this knowledge in [A2]. There is also a book on the sporadic groups in preparation by Griess.)

Sir W. R. Hamilton observed in 1856 that the icosahedral group (left) may be defined abstractly as the group generated by two substitutions of orders 2 and 3, respectively, whose product is of order 5.

The GLS work itself will appear in a series of approximately twelve volumes to be published by the American Mathematical Society. It is subdivided into five parts.

Part I consists of: overview and outline, general group theory, and properties of the known simple groups. In particular, this will complement the existing literature on properties of the finite groups of Lie type and will enumerate the assumed results on sporadic groups. The overview and outline volume has been sent to the publisher. The remainder of Part I should be completed within the next year.

Part II presents the fundamental “uniqueness theorems” on which the Classification rests: the Suzuki-Bender Strongly Embedded Theorem (with extensions), the Strongly p-embedded Theorem (work of Gernot Stroth), the Global $C(G;T)$-Theorem (Joint work with Richard Foote) and tje p-Component Uniqueness Theorems. The main mathematical body of Part II is complete, although much remains to be done in terms of preparation of preliminary sections.

Part III presents the proof of the “generic case”. This constitutes the classification (subject to the inductive assumption that all proper simple sections are isomorphic to known simple groups) of most of the finite simple groups, including A_n for $n \geq 13$ and the groups of Lie type of rank ≥ 4 (except for a few defined over F_2). Operationally, the generic case treats those local configurations to which, for some prime p, the Signalizer Method can be effectively applied to verify the B_p-Property for a large number of semisimple p-elements of G. More than half of the mathematical body of Part III is complete.

The “special” (nongeneric) part of the proof divides into “odd” and “even” cases, according to whether involutions in G behave like semisimple or unipotent elements. The groups to be identified include A_n for $n \geq 12$, all of the groups of Lie type of rank ≤ 2, and all of the sporadic groups. (Groups of Lie rank 3 are split between the generic and special cases.) Because the Signalizer Method breaks down in these "narrow" cases and, even more, because available recognition theorems require very detailed internal information, the analysis of the special odd and even cases is extremely lengthy and delicate. Nevertheless, the main mathematical body of the proof for the Special Odd Case (Part IV) is almost complete. The Special Even Case (Part V)—primarily the extended Klinger-Mason and Quasithin Cases—on the other hand, is in by far the sketchiest state of all the parts of the revised proof, because of the Quasi-thin Case.

The classification of the finite simple groups is an ongoing organic process, whose progress in the three decades since the Odd Order Theorem has been extraordinary. An excellent detailed overview of the original proof is provided in [G1] with references for all the major theorems stated above. An outline of the revised proof appears in [G2] and a more detailed introduction will constitute volume I of the GLS series.

Speculations on Current and Future Directions for Finite Group Theory

Quite a bit of recent research in finite group theory has developed in response to problems from other areas of mathematics—e.g., field theory, model theory, graph theory, finite geometry. Many of these problems had been around for quite a while, but suddenly became accessible thanks to the completion of the Classification and some of its immediate corollaries, for instance, the classification of finite 2-transitive permutation groups. Surveys of some of this work are available in [G1], [G2], and in Kantor's article in [M]. No doubt there will be an ongoing flow of such questions and answers. Here I will briefly mention some of the active areas of research.

Representation Theory

In recent decades the work of Steinberg, Curtis, Lusztig, and many others has shed considerable light on the natural (i.e., characteristic p) representation theory of finite groups of Lie type in characteristic p, (as well as on the complex representation theory). Some of this work already plays a crucial role in the Classification proof. It would be valuable to develop a unified and efficient treatment of those "small" characteristic p modules (quadratic, failure of factorization, small centralizer, etc.) which are critical to this work.

In another vein, several mathematicians—notably Alperin, Broué, Puig, and Robinson—have been exploring extensions of Brauer's theory of modular representations, which might yield major simplifications of the Classification proof. In particular, the B_p-Conjecture is a natural candidate for proof by modular methods. It is also tempting to hope for a modular proof of an odd prime analogue of the Z'-Theorem. A good reference for these and related themes is [Ca].

Maximal Subgroups and Primitive Permutation Representations

Extending works of Steinberg, Seitz, and Testerman have shown how to lift many embedding questions for finite Lie type groups to questions for algebraic groups. Also, they have extended the work of Dynkin on maximal subgroups of simple Lie groups to simple algebraic groups in characteristic p. Unfortunately, their lifting result has prime restrictions that cry out to be removed or at least significantly weakened. Numerous other questions in this context remain open. A good introduction with references is available in Seitz's article in [LS].

Geometry and Topology

There are numerous geometries or simplicial complexes naturally associated with finite groups in general and simple groups in particular. Examples include the p-group complexes studied by Brown, Quillen, and others. This generalizes the buildings of Tits, associated with the p-local structure of Lie type groups in characteristic p. These complexes are useful in describing the p-local structure of the sporadic groups and shed light on their p-modular representations, a viewpoint poineered by Ronan and Smith. Many are simply connected, and this fact can be used to establish uniqueness results for sporadic simple groups (Aschbacher and Segev) and give presentations for simple and near-simple groups; e.g., the Y-presentation for the Bimonster (see below) as well as the classical Steinberg-Curtis-Tits presentations for the groups of Lie type.

On the other hand, certain natural geometries are far from simply connected, e.g., the 5-local geometry for the Lyons simple group Ly and the sporadic affine building covering it, investigated by Kantor. To date, the connectivity results remain somewhat mysterious and the nature of the "exotic" universal covers of some of the naturally-occurring nonsimply connected geometries remains totally obscure.

The complexes alluded to above are in general of rank at least 3. The rank 2 case dovetails with the theory of amalgams. Here the geometries are never simply connected. Indeed the infinite amalgam can be studied independent of the knowledge of any faithful finite completion. One particularly intriguing infinite amalgam (of rank 3) associated both with the Lie group Spin(7) and the sporadic simple group Co_3 was studied by Chermak and has recently been shown by Benson to be related to a 2-completed finite loop space discovered by Dwyer and Wilkerson. Such spaces seem to be rare (like finite simple groups) and Benson speculates that their classification might be a fruitful endeavor.

Much remains to be understood about these tantalizing "objects at infinity" that seem to complete the space of finite simple groups. A good starting point for further reading is [LS].

Monsterology

The Monster remains the single most tantalizing simple group, with apparent (but as yet mysterious) connections to Kac-Moody Lie theory, quantum field theory, modular functions, and congruence subgroups of $SL(2, Z)$.

It is well known how to show that a finite group generated by involutions acts analytically on a Riemann surface. It would be interesting to understand higher-dimensional analytic representations of finite (simple) groups. In particular one would hope for a natural analytic representation of the Monster in dimension 24 (more or less), which would clarify the connections between the Monster (and its subgroups) and classical elliptic modular functions. Recent work of Ivanov and Norton has established that the Bimonster (i.e., the wreathed product of the Monster by Z_2) is the quotient of a certain infinite Coxeter group (presented by the "$Y_{5,5,5}$-diagram") by a single additional relation. Is this a clue to right analytic object on which the Monster acts?

Perhaps this will lead to a new unified approach to all of the finite simple groups, perhaps only to a deeper level of insight into the Monster and its babies. Further material and references are available in [M] and in [LS].

Algebraic Combinatorics

The Classification has permitted the solution of numerous classification problems concerning combinatorial structures with large automorphism groups. More ambitiously, Bannai, T. Ito, and others (See [Bl]) have championed the application of generalized hypergeometric functions to the study of certain association schemes. For large rank, all such known schemes arise naturally from some (almost) simple group. On the other hand, the classification of all such association schemes would subsume the classification of the finite simple groups!

The eruption of mathematics during the heyday of the study of simple groups generated amazing insights into the structure of finite groups and uncovered several of the most fascinating objects in the mathematical firmament. Naturally, our understanding remains incomplete, some loose ends remain dangling and the future of research in finite group theory promises as many insights and surprises as the past.

References

[A1] M. Aschbacher, *Thin finite simple groups*, J. Algebra **46** (1978), 50–152.

[A2] M. Aschbacher, *Sporadic groups*, Cambridge University Press, Cambridge, 1994.

[Bl] E. Bannai and T. Ito, *Algebraic combinatorics* I: *Association schemes*, Benjamin/Cummings, Menlo Park, 1984.

[Ca] M. Cabanes, ed., *Representations lineaires des groupes finis*, Asterisque, 181–182 (1990).

[Co] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[G1] D. Gorenstein, *Finite simple groups: An introduction to their classification*, Plenum Press, New York, 1982.

[G2] D. Gorenstein, *Classifying the finite simple groups*, Bull. A.M.S. **14** (1986), 1–98.

[GL] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic 2 type*, Memoirs Amer. Math. Soc. **276** (1983).

[LS] M. W. Liebeck and J. Saxl, ed., *Groups, combinatorics and geometry*, L.M.S. Lecture Notes Series **165**, Cambridge University Press, Cambridge, 1992.

[M] J. McKay, ed., *Finite groups—Coming of age*, A.M.S. Contemporary Mathematics, Vol. **45**, Providence, 1982.

Appendix of Technical Definitions and Theorems

Definition. Let A be an elementary abelian p-subgroup of a finite group G of p-rank ≥ 3. Suppose that θ is a function which assigns to each $a \in A^{\#}$ an A-invariant solvable p'-subgroup of $C_G(a)$. We call θ an A-signalizer functor if the balance condition holds: $\theta(a) \cap C_G(a') = C_G(a) \cap \theta(a')$ for all $a, a' \in A - \{1\}$.

The Signalizer Functor Theorem (Goldschmidt, Glauberman): $\Theta(G; A) = \langle \theta(a) \colon a \in A - \{1\} \rangle$ is an A-invariant solvable p'-subgroup of G. (There is an important extension of this result to non-solvable signalizer functors due to Patrick McBride.)

Definition. The p-layer of $H, L_{p'}(H)$, is the (unique) minimal normal subgroup of H which maps onto $E(H/O_{p'}(H))$.

$L_{p'}$-**Balance Theorem** (Gorenstein-Walter): If every component L of $X/O_{p'}(X)$ satisfies the "Schreler property" (i.e. $\mathrm{Aut}\, L/\mathit{lnn}\, L$ is solvable), then $L_{p'}(Y) \leq L_{p'}(X)$ for every p-local subgroup Y of X. (For $p = 2$, a sufficient "weak Schreier property" was established by Glauberman as a corollary of his Z'-Theorem. The full Schreier property is established only as an a posteriori consequence of the Classification.)

The B_p-Theorem. If $O_{p'}(G) = 1$ and if x is a p-element of G, then $L_{p'}(C_G(x) \leq E(C_G(x))$.