Fortgeschrittene Optionspreistheorie

Martingales and Measures

Black‘s Model

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September 2009

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Derivatives Dependent on a Single Underlying Variable

Consider a variable, θ, (not necessarily the price of a traded security) that follows the process

$$\frac{d\,\theta}{\theta} = m\,dt + s\,dz$$

Imagine two derivates dpendent on θ withprices f_1 and f_2. Suppose

$$\frac{d\,f_1}{f_1} = \mu_1\,dt + \sigma_1\,dz$$

$$\frac{df_2}{f_2} = \mu_2\,dt + \sigma_2\,dz$$

Forming a Riskless Portfolio

We can set up a riskless portfolio Π, consisting of

$+\sigma_2 f_2$ of the 1st derivative and

$-\sigma_1 f_1$ of the 2nd derivative

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$$

$$\delta\Pi = (\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2)\delta t$$

Market Price of Risk

$$\text{Since the portfolio is riskless : } \delta\Pi = r\,\Pi\delta t$$

$$\text{This gives : } \mu_1\sigma_2 - \mu_2\sigma_1 = r\,\sigma_2 - r\,\sigma_1$$

$$\text{or} \qquad \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

- This shows that $(\mu - r)/\sigma$ is the same for all derivatives dependent on the same underlying variable, θ
- We refer to $(\mu - r)/\sigma$ as the market price of risk for θ and denote it by λ

Martingales

- A martingale is a stochastic process with zero drift
- A variable following a martingale has the property that its expected future value equals its value today

Alternative Worlds

In the traditional risk - neutral world

$$df = rf\,dt + \sigma f\,dz$$

In a world where the market price of risk is λ

$$df = (r + \lambda\sigma) f\,dt + \sigma f\,dz$$

A Key Result

If we set λ equal to the volatility of a security g, then Ito’s lemma shows that f/g is a martingale for all derivative security prices f

(f and g are assumed to provide no income during the period under consideration).

Forward Risk Neutrality

We refer to a world where the market price of risk is the volatility of g as a world that is forward risk neutral with respect to g.

If E_g denotes a world that is FRN wrt g

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

Alternative Choices for the Numeraire Security g

- Money Market Account
- Zero-coupon bond price
- Annuity factor

Money Market Account as the Numeraire

- The money market account is an account that starts at \$1 and is always invested at the short-term risk-free interest rate
- The process for the value of the account is

$$dg = rg\,dt$$

- This has zero volatility. Using the money market account as the numeraire leads to the traditional risk-neutral world

- Since $g_0 = 1$ and $g_T = \exp\left(\int_0^T r\,dt\right)$, the equation $\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$

  becomes $f_0 = \hat{E}\left[e^{-\int_0^T r dt} f_T\right]$

  where Ê denotes expectations in the traditional risk-neutral world.

Zero-Coupon Bond Maturing at time *T* as Numeraire

The equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = P(0,T)E_T[f_T]$$

where P(0,T) is the zero-coupon bond price and E_T denotes expectations in a world that is forward risk neutral with respect to the bond price.

In a world that is FRN wrt $P(0,T)$, the expected value of a security at time T is its forward price

Annuity Factor as the Numeraire

The equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = A(0)E_A\left[\frac{f_T}{A(T)}\right]$$

Suppose that $s(t)$ is the swap rate corresponding to the annuity factor A.
Then:

$$s(t)=E_A[s(T)]$$

Black's Model & Its Extensions

- Main Approaches to Pricing Interest Rate Options:
  - Use a variant of Black's model
  - Use a no-arbitrage (yield curve based) model
- Black's model is similar to the Black-Scholes model used for valuing stock options.
- It assumes that the value of an interest rate, a bond price, or some other variable at a particular time T in the future has a lognormal distribution.
- No assumptions about the stochastic behavior of interest rates and bond prices.
- Valuation of
  - European bond options
  - Caps/floors
  - European swaptions.
- The model cannot be used for
  - American options,
  - callable bonds
  - Structured notes

Black's Model & its Extensions

- The mean of the probability distribution is the forward value of the variable.
- The standard deviation of the probability distribution of the log of the variable is $\sigma\sqrt{T}$
  where σ is the volatility.
- The expected payoff is discounted at the T-maturity rate observed today.

Black's Model

$$c = P(0,T)[F_0 N(d_1) - KN(d_2)]$$

$$p = P(0,T)[KN(-d_2) - F_0 N(-d_1)]$$

$$d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma\sqrt{T}}; d_2 = d_1 - \sigma\sqrt{T}$$

- K : strike price
- F_0 : forward value of variable
- T : option maturity
- σ : volatility

The Black's Model: Payoff Later Than Variable Being Observed

$$c = P(0,T^*)[F_0 N(d_1) - KN(d_2)]$$

$$p = P(0,T^*)[KN(-d_2) - F_0 N(-d_1)]$$

$$d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma \sqrt{T}}; d_2 = d_1 - \sigma \sqrt{T}$$

- K : strike price
- F_0 : forward value of variable
- σ : volatility
- T : time when variable is observed
- T* : time of payoff

Validity of Black's Model

Black's model appears to make two approximations:

1. The expected value of the underlying variable is assumed to be its forward price.
2. Interest rates are assumed to be constant for discounting.

We will see that these assumptions offset each other

European Bond Options

- When valuing European bond options it is usual to assume that the future bond price is lognormal
- We can then use Black’s model
- Both the bond price and the strike price should be cash prices not quoted prices

Yield Vols vs. Price Vols

- The change in forward bond price is related to the change in forward bond yield by

$$\frac{\delta B}{B} \approx -D\ \delta y \quad \text{or} \quad \frac{\delta B}{B} \approx -Dy\ \frac{\delta y}{y}$$

  where D is the (modified) duration of the forward bond at option maturity.

- This relationship implies the following approximation $\sigma = Dy_0\sigma_y$ where σ_y is the yield volatility and σ is the price volatility, y_0 is today's forward yield

- Often σ_y is quoted with the understanding that this relationship will be used to calculate σ.

Theoretical Justification for Bond Option Model

Working in a world that is FRN with respect to a zero-coupon bond maturing at time T, the option price is

$$P(0,T)E_T[\max(B_T - K,0)]$$

Also

$$E_T[B_T] = F_0$$

This leads to Black‘s model.

Caps

- A cap is a portfolio of caplets.
- Each caplet can be regarded as a call option on a future interest rate with the payoff occurring in arrears.
- When using Black’s model we assume that the interest rate underlying each caplet is lognormal.

Black's Model for Caps

- The value of a caplet, for period $[t_k, t_{k+1}]$ is

$$L\delta_k P(0, t_{k+1})[F_k N(d_1) - R_K N(d_2)]$$

where $d_1 = \frac{\ln(F_k / R_K) + \sigma_k^2 t_k / 2}{\sigma_k \sqrt{t_k}}$ and $d_2 = d_1 - \sigma\sqrt{t_k}$

- F_k : forward interest rate for (t_k, t_{k+1})
- σ_k : interest rate volatility
- L : principal
- R_K : cap rate
- $\delta_k = t_{k+1} - t_k$

When applying Black’s Model to Caps we must ...

- EITHER
  - Use forward volatilities
  - Volatility different for each caplet
- OR
  - Use flat volatilities
  - Volatility same for each caplet within a particular cap but varies according to life of cap

Theoretical Justification for Cap Model

Working in a world that is FRN wrt a zero - coupon bond maturing at time t_{k+1} the option price is

$$P(0, t_{k+1}) E_{k+1}[\max(R_k - R_X, 0)].$$

Also

$$E_{k+1}[R_k] = F_k$$

This leads to Black's model.

European Swaptions

- When valuing European swap options it is usual to assume that the swap rate is lognormal
- Consider a swaption which gives the right to pay s_K on an *n* -year swap starting at time T . The payoff on each swap payment date is

$$\frac{L}{m}\max\left(s_T - s_K, 0\right)$$

where L is principal, m is payment frequency and s_T is market swap rate at time T

European Swaptions

The value of the swaption is

$$LA\,[s_0 N(d_1) - s_K N(d_2)]$$

where $d_1 = \dfrac{\ln(s_0/s_K) + \sigma^2 T/2}{\sigma\sqrt{T}}; \; d_2 = d_1 - \sigma\sqrt{T}$

s_0 is the forward swap rate; σ is the swap rate volatility; t_i is the time from today until the i th swap payment; and

$$A = \frac{1}{m}\sum_{i=1}^{mn} P(0, t_i)$$

Theoretical Justification for Swap Option Model

Working in a world that is FRN wrt the annuity underlying the swap, the option price is

$$LAE_A[\max(s_T - s_K, 0)]$$

Also

$$E_A[s_T] = s_0$$

This leads to Black's model.

Relationship Between Swaptions and Bond Options

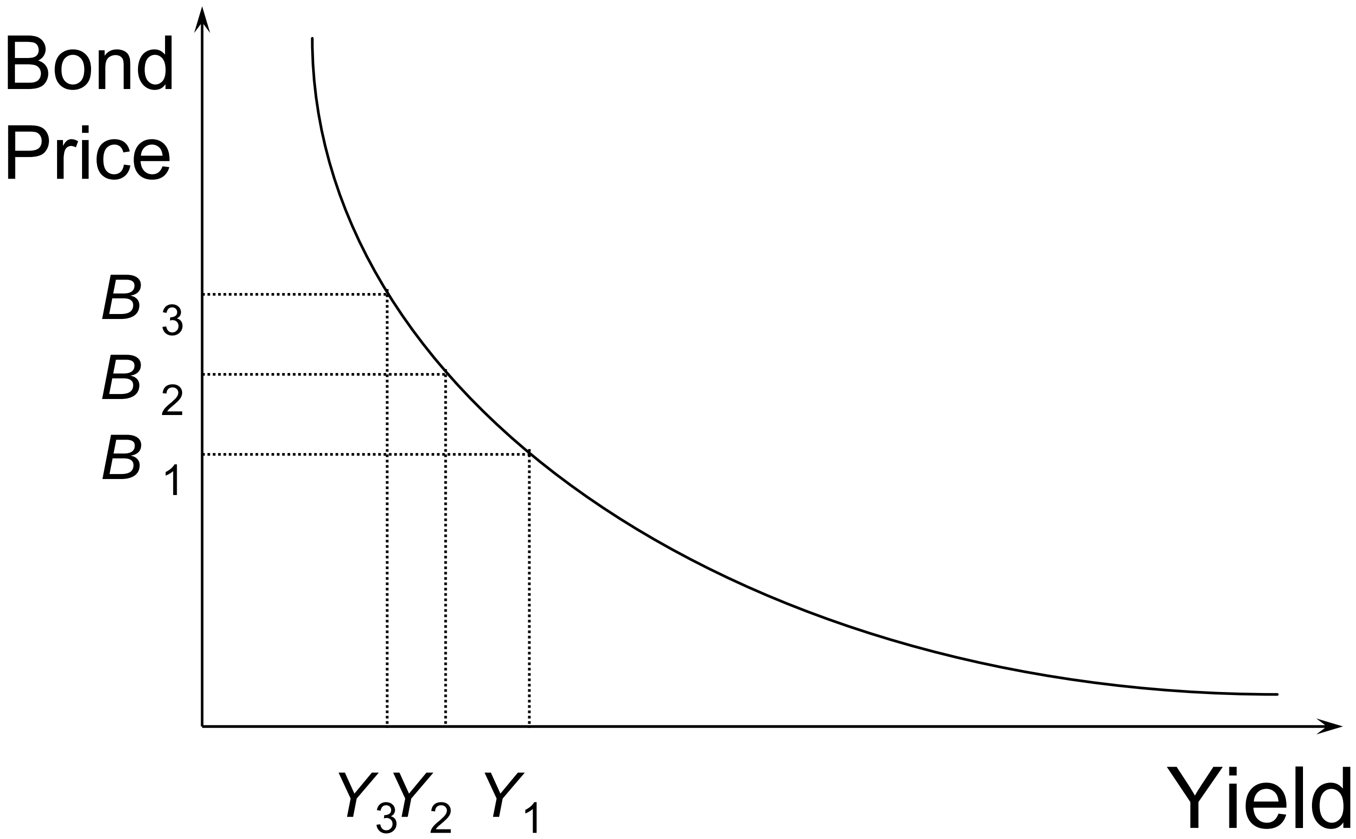
- An interest rate swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond
- A swaption or swap option is therefore an option to exchange a fixed-rate bond for a floating-rate bond

- At the start of the swap the floating-rate bond is worth par so that the swaption can be viewed as an option to exchange a fixed-rate bond for par
- An option on a swap where fixed is paid and floating is received is a put option on the bond with a strike price of par
- When floating is paid and fixed is received, it is a call option on the bond with a strike price of par

Convexity Adjustments

- We define the forward yield on a bond as the yield calculated from the forward bond price
- There is a non-linear relation between bond yields and bond prices
- It follows that when the forward bond price equals the expected future bond price, the forward yield does not necessarily equal the expected future yield
- What is known as a convexity adjustment may be necessary to convert a forward yield to the appropriate expected future yield

Relationship Between Bond Yields and Prices



Analytic Approximation for Convexity Adjustment

- Suppose a derivative depends on a bond yield, y_T observed at time T . Define:
- $G(y_T)$: price of the bond as a function of its yield

  y_0 : forward bond yield at time zero

  σ_y : forward yield volatility
- The convexity adjustment that should be made to the forward bond yield is

$$-\frac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

When is a Convexity Adjustment Necessary

- A convexity or timing adjustment is necessary when the payoff from a derivative does not incorporate the natural time lags between an interest rate being set and the interest payments being made
- They are not necessary for a vanilla swap, a cap or a swap option

References

- Black, F. (1976): „The Pricing of Commodity Contracts“, Journal of Financial Economics 3, March, pp. 167-179.