COMPARATIVE ECONOMIC CYCLES

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Abstract

The income cycles that have been experienced by six OECD countries over the past 24 years are analysed. The amplitude of the cycles relative to the level of aggregate income varies amongst the countries, as does the degree of the damping that affects the cycles. The study aims to reveal both of these characteristics. It also seeks to determine whether there exists a clear relationship between the degree of damping and the length of the cycles. In order to estimate the parameters of the cycles, the data have been subjected to the processes of detrending, anti-alias filtering and subsampling.

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1 Introduction

This paper compares the economic cycles that have beset a group of six OECD countries, which have experienced similar socio-economic conditions over the past quarter century. The shocks and the disturbances that have impinged on their economies have had differing intensities; and the effects have been mitigated by a wide variety of fiscal and monetary policies. The aim is to determine the extent to which the consequences of such differences are perceptible in the aggregate indices of economic activity.

For the purposes of this assessment, we shall employ a stylised model of the economic cycle, for which we shall attempt to estimate the parameters from the data of each country. These parameters will provide the basis for our comparisons. We shall employ some simple methods of estimation, but we shall need to process the data in novel ways in order to isolate the features that are of interest.

Within the framework of our analysis, there can be two explanations for the strength of an economic cycle. The first explanation lies in the power of the disturbances that are the driving force of the cycle. The second explanation lies in the ability of the economy to dissipate the energy of these disturbances. This would be reflected in the rate of convergence to the steady state of an economy that was somehow relieved of disturbances, but it can also be estimated from the behaviour of an economy experiencing disturbances, on the assumption that they constitute white noise.

The frictional effects of the tax system that are described as fiscal drag will serve to dissipate the energy of the disturbances. An overreactive regime of economic regulation, acting like a stiff spring in a mechanical system, might have the opposite effect of maintaining this energy within the economic system. For example, it was the contention of Dow (1964), in a monograph that was highly influential at the time, that, in the early post-war years, such a regime had increased the frequency and the severity of the fluctuations in the British economy.

The countries that we shall examine have experienced high levels of economic growth throughout the period in question. The economic recessions that have occurred have been characterised more often by diminutions in the rates of growth than by absolute reductions in the levels of output. In other words, the economic cycles have been carried on the backs of rising trends. The problem of separating the trends from the cycles is a difficult one that has generated much debate, and we feel bound to offer our own opinions.

Some of the distinctions that have arisen in the course of this debate, such as the distinction between deterministic and stochastic trends, may have been drawn too firmly. (For discussions of these issues, see King *et al.* 1991, Nelson and Plosser 1982, Pagan 1997 and Perron 1988, 1989.) In practice, the data cannot be relied upon to distinguish unequivocally between such stylised models as the polynomial trend and the unit-root stochastic trend, when both are buried in noise. (The means of discriminating between the two models have been discussed recently by Andreou and Spanos 2003 and by Marriott, Naylor and Tremayne 2003.)

It may also be true that our own epigram concerning the determinants of the economic cycle, which is based on a mechanical analogy, is drawn too simply and that the underlying realities are far more complex than we shall be proposing for the sake of argument. Nevertheless, our model does take us into realms that have not been explored fully by economists.

2 A Schematic Model of the Economic Cycle

2.1 The structural time-series model

In proposing simple macroeconomic models, we are liable to assume that the relative proportions of the economic aggregates are maintained, approximately, despite variations in the levels. Examples are provided by the ratios of consumption and investment to gross national product (GNP), of which the underlying constancy is frequently postulated. It is reasonable to assume that, making allowance for their stochastic nature, the relative amplitude of the economic fluctuations is also maintained throughout the period spanned by the data.

A schematic model of a macroeconomic index might, therefore, set

$$Y(t) = \Xi(t)B(t) \quad \text{with} \quad B(t) = 1 + \sum_j \sigma_j \cos(\omega_j t + \theta_j), \tag{1}$$

where $Y(t)$ stands for an aggregate economic index, such as GNP, and where $\Xi(t)$ is its underlying trend. Modulating this trend is the factor $B(t)$, which comprises a sum of sinusoids. The jth sinusoid has an amplitude of σ_j, a phase angle of θ_j radians and a period of $\tau_j = 2\pi/\omega_j$, where ω_j is an angular velocity, or frequency, measured in radians per period.

For statistical purposes, we might amend this model by replacing $B(t)$ by the factor $1 + \beta(t)$, where $\beta(t)$ is generated by a linear stochastic process of an autoregressive (AR) or autoregressive moving-average (ARMA) variety. Over a finite period, the output of such a process can also be expressed as a sum of sinusoids, the parameters of which are assumed to have been drawn from statistical distributions.

However, there are some problems with this formulation. First, as we have defined it, there is no limit on the range of the stochastic process $\beta(t)$, whereas it is necessary, at least, that its negative deviations should be limited in order to prevent $Y(t)$ from becoming negative. Secondly, it is likely that we should wish to include an additional stochastic factor that is unrelated to the cycles.

The first difficulty is answered by setting

$$B(t) = e^{\beta(t)} = \left(1 + \beta(t) + \frac{\{\beta(t)\}^2}{2!} + \cdots\right), \tag{2}$$

where $\beta(t)$ follows a linear stochastic process of variance σ^2. Then, $\Xi(t)B(t)$ can be multiplied by the additional stochastic factor $\mathcal{E}(t)$, bounded in the same manner as $B(t)$, and the amended model becomes

$$Y(t) = \Xi(t)B(t)\mathcal{E}(t) \quad \text{or} \quad y(t) = \xi(t) + \beta(t) + \varepsilon(t), \tag{3}$$

where $y(t) = \ln Y(t)$, $\xi(t) = \ln \Xi(t)$, $\beta(t) = \ln B(t)$ and $\varepsilon(t) = \ln \mathcal{E}(t)$ are the logarithms of the factors. The equation in logarithms corresponds to a so-called structural time series model or unobserved components model of the sort that has been treated extensively by Harvey (1989).

It transpires that the economic cycle can be represented, within the logarithmic data, by a second-order autoregressive AR(2) process. The equation of an AR(2) process can be compared with a second-order differential equation, which is associated with numerous physical processes that provide good analogies for the cycle. Accessible accounts of linear differential equations and of their application to physical systems have been provided by Gabel and Roberts (1987), Mayne (1984) and Thompson (1983).

2.2 Differential and difference equations

Consider a damped harmonic oscillator driven by a sinusoidal forcing function with a frequency of ω_f. A spring-mass system with viscous damping is an example. The equation of motion is

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + hy(t) = \phi\cos(\omega_f t), \tag{4}$$

where m is the oscillating mass, c is the coefficient of viscous damping, and h is the return force per unit of displacement, which can be described as the stiffness of the spring. (In the terminology of economics, h is described as the strength of the error-correction mechanism.) All of these coefficients are positive. The variable $y(t)$ is the displacement of the system about its stationary point. The solution of this differential equation is

$$y(t) = A\cos(\omega_f t - \theta), \tag{5}$$

where

$$A^2 = \frac{\phi^2}{(h - m\omega_f^2)^2 + (c\omega_f)^2} \quad \text{and} \quad \tan\theta = \frac{c\omega_f}{h - m\omega_f^2} \tag{6}$$

are the expressions that provide the amplitude A and the phase displacement θ. In giving these expressions an interpretation, we may observe that the natural frequency of an undamped system is $\omega_n = \sqrt{h/m}$. This gives $h = m\omega_n^2$. Putting the latter into the expression for A^2 shows that the amplitude gain is greatest when the driving frequency ω_f coincides with the natural frequency ω_n.

The system dissipates energy at an average rate that is equal to the input of power from the driving force. It also stores a quantity of energy. At the maximum amplitude, this consists of potential energy, stored in the form of the strain energy of the spring, whereas, at the point of maximum velocity, where the spring is unstretched, it consists of kinetic energy, stored in the mass by virtue of its velocity.

A measure that combines both the damping and the stiffness of the system is provided by its loss ratio, which is defined as the energy W dissipated per radian, averaged over the cycle, divided by the peak potential energy U of the system. For a second-order system, there is

$$W = \frac{c\omega_f A^2}{2} \quad \text{and} \quad U = \frac{hA^2}{2}. \tag{7}$$

Therefore, the loss ratio is

$$\frac{W}{U} = \frac{c}{h}\omega_f = \Lambda\omega_f. \tag{8}$$

The loss coefficient $\Lambda = c/h$ is a characteristic of the system that is independent of the frequency ω_f at which it is driven. It can be expressed in terms of the solution of the following auxiliary equation:

$$\begin{aligned} 0 &= ms^2 + cs + h = m(s-\kappa)(s-\kappa^*) \\ &= m\{s^2 - (\kappa+\kappa^*)s + \kappa\kappa^*\} = m\{s^2 - 2\gamma + (\gamma^2+\omega^2)\}, \end{aligned} \tag{9}$$

wherein $\kappa = \gamma + i\omega$ and $\kappa^* = \gamma - i\omega$ are conjugate complex roots. We observe that $c = -2\gamma m$ and that $h = m(\gamma^2 + \omega^2)$. Therefore,

$$\Lambda = \frac{-2\gamma}{\gamma^2 + \omega^2}. \tag{10}$$

It is necessary to express the coefficients γ and ω in terms of the parameters of the second-order difference equation that will be fitted to the data. This may be achieved by comparing the solution of the homogeneous differential equation, derived from (4) by setting the driving force to zero, via $\phi = 0$, with the solution of the corresponding difference equation. The solutions must agree at the points in discrete time over which the difference equation is defined.

The solution of this differential equation is

$$y(t) = \sigma e^{\gamma t}\cos(\omega t - \theta), \tag{11}$$

where σ relates to the initial amplitude at time $t = 0$, ω is the resonant frequency of the system, which is the frequency of the oscillations of the system when the driving force is removed, and $\gamma = -c/2m$ is the negative exponential rate at which, in such circumstances, the system converges to its point of rest.

The homogeneous second-order difference equation takes the form of

$$\begin{aligned} 0 &= (\alpha_0 + \alpha_1 L + \alpha_2 L^2)y(t) \\ &= \alpha_0\{1 - 2\rho\cos(\omega)L + \rho^2 L^2\}y(t), \end{aligned} \tag{12}$$

where L denotes the lag operator such that $Ly(t) = y(t-1)$, and where $\rho \in (0,1)$ is the damping factor. The auxiliary equation is

$$\begin{aligned} 0 &= \alpha_0 z^2 + \alpha_1 z + \alpha_2 = \alpha_0(z-\mu)(z-\mu^*) \\ &= \alpha_0\{z^2 - (\mu+\mu^*)z + \mu\mu^*\} = \alpha_0\{z^2 - 2\rho\cos(\omega)z + \rho^2\}, \end{aligned} \tag{13}$$

wherein $\mu = \beta + i\delta$ and $\mu^* = \beta - i\delta$ are conjugate complex roots such that $\rho^2 = \beta^2 + \delta^2$ and $\omega = \tan^{-1}(\delta/\beta)$. Then, the solution of the homogeneous equation can be denoted by

$$y(t) = \sigma\rho^t\cos(\omega t - \theta). \tag{14}$$

The comparison of (11) and (14) indicates that they agree when

$$e^{\gamma t} = \rho^t, \quad \text{which is when} \quad \gamma = \ln \rho. \tag{15}$$

Therefore, if we know ρ from the difference equation, we know $c/m = -2\gamma$. Also, the parameter ω is common to both equations so, knowing this, we can proceed to find $h/m = \gamma^2 + \omega^2$. Then, the loss parameter $\Lambda = c/h$ can be calculated.

The resonant frequency of an oscillating system, free from a driving force, is related to the coefficients c and h via the following equation:

$$\omega^2 = \frac{4mh - c^2}{4m^2} = \omega_n^2 - \gamma^2. \tag{16}$$

Here, $\omega_n = \sqrt{h/m}$ is the natural frequency of the undamped system, whereas $\gamma = -c/2m$. Increasing the coefficient of friction c has the effect of reducing ω, thereby increasing the length of the cycles. Increasing the stiffness parameter h has the opposite effect of reducing the length of the cycles. It will be interesting to discover which of these will be the dominant relationship in the comparison of the national economies.

In comparing the economies, we shall deal with the relative amplitude of the cycles. Therefore, the absolute size of the economy is not a factor in the analysis. For the differential equation, this is tantamount to setting $m = 1$, whereas, for the difference equation, the corresponding normalisation is to set $\alpha_0 = 1$.

2.3 The process that drives the cycles

The forcing function that drives the cycles is unobservable, and we can only make the assumption that it is a species of white-noise process, which has uniform power across a specified range of frequencies. Its strength will be reflected in an estimate of its variance.

The exogenous disturbances that cross the boundary of the system that represents the national economy might not follow a white-noise process. Nevertheless, we can postulate that they have their origin in a *primum mobile* that is white noise. We can also postulate that the transfer function that links this *primum mobile* to the boundary-crossing disturbances is linear and time invariant. In that case, it can be regarded as a property of the economy, which will be subsumed in the autoregressive process with which we intend to characterise the leading index of the economy.

For the successful analysis of a linear dynamic system, it is necessary that its natural frequency ω_n should not exceed the so-called Nyquist frequency of π radians per period of observation, which corresponds to the highest frequency that is detectable in regularly sampled data. However, the analysis would remain viable even if the frequencies within the forcing function were to exceed this level. In that case, the data generated by a 2nd-order stochastic differential equation should be represented in discrete time by an ARMA$(2, 1)$ model, in which the moving-average parameter is a function of the autoregressive parameters.

This result has been established in two separate but related contexts. Telser (1967) has shown that, if the discrete data generated by an AR(p) process are subsampled, or 'skip sampled' in his terminology, then it is appropriate to describe them by an ARMA$(p, p-1)$ model. Here, we may imagine that the original AR(p) process stands for a valid discrete-time representation of a stochastic differential equation in which the forcing function is a white-noise process bounded by the Nyquist frequency associated with the original sampling rate.

The result has also emerged from the analysis of stochastic differential equations powered by a stream of infinitesimal impulses that constitute the increments of a Wiener process. Thus, by following the arguments of Bartlett (1946), Phadke and Wu (1974) have demonstrated that a stochastic differential equation of order p, powered in this manner, can be represented in discrete time by an ARMA($p, p-1$) process. Pandit and Wu (1975) have considered, in the same manner, the discrete representation of a second-order differential equation.

Our empirical analysis will suggest that the stochastic process driving the business cycles is bounded by a frequency that is considerably less than the Nyquist frequency associated with the quarterly data at our disposal. The business cycle is isolated by detrending the data and by purging it of its seasonal fluctuations. The frequencies of the sinusoidal elements within the resulting sequence are bounded by an upper limit of $\pi/8$ radians per sample period.

The concept of a band-limited white-noise sequence is common in engineering studies; and there are several commercially available computer programs for generating such sequences. Nevertheless, it appears that no clear concept is available of a continuous-time band-limited white-noise process, such as might be used to represent the forcing function of the model of the business cycle.

Such a process could be constituted from a stream of sinc function wave packets distributed along the time axis at intervals of 8 sample periods:

$$\{\psi(t-8k); k = 0, \pm 1, \pm 2, \ldots\} \quad \text{with} \quad \psi(t) = \frac{\sin(\pi t/8)}{\pi t}. \tag{17}$$

These wave packets provide an orthonormal basis for all continuous functions bounded in frequency by $\pi/8$. To obtain the continuous band-limited white noise, the wave packets are multiplied by a corresponding sequence of independently and identically distributed random variables and added together.

An ordinary white-noise sequence will be obtained by sampling the resulting function at the rate of once in every 8 sample periods. An ordinary AR(2) process with a spectral density function extending over the full frequency interval $[-\pi, \pi]$ will be obtained, likewise, by sub-sampling the band-limited AR(2) process at the rate of one in eight.

3 The Trend Component

3.1 Polynomial detrending

We shall now explore two of the alternative methods that are available for extracting the trend from the data. What remains will contain the economic cycle as well some other motions that will need to be removed and discarded.

In seeking to define the trend, we need not regard it as a wholly objective entity. Its definition can be adapted to the analytic purposes of the study as well as to the circumstances of the economy over the period in question. In our case, a quadratic trend within the logarithmic data provides a firm benchmark against which the cyclical activities of the economy can be measured. Whereas a linear trend in the logarithms would signify constant exponential growth, a quadratic trend can accommodate a rate of growth that is changing over time. We shall also investigate the effects of using the more flexible filtering method of Hodrick and Prescott (1980, 1997) for generating the trend.

We begin by considering the matrix version of the difference operator and its inverse, which is the cumulation operator. These will be useful in portraying both the method of polynomial regression and the method of Hodrick and Prescott. Indeed, our purpose is to depict the linear trend as a limiting case of the Hodrick–Prescott trend.

Consider, therefore, the identity matrix of order T defined by

$$I_T = [e_0, e_1, \ldots, e_{T-1}], \tag{18}$$

where e_j represents a column vector that contains a single unit preceded by j zeros and followed by $T-j-1$ zeros. Then, the finite-sample lag operator is the matrix

$$L_T = [e_1, \ldots, e_{T-1}, 0], \tag{19}$$

which has units on the first subdiagonal and zeros elsewhere. This is obtained from the identity matrix by deleting the leading column and by appending a column of zeros to the end of the array.

The matrix that takes the d-th difference of a vector of order T is given by

$$\nabla_T^d = (I - L_T)^d. \tag{20}$$

The matrix may be partitioned such that $\nabla_T^d = [Q_*, Q]'$, where Q_*' has d rows. The inverse matrix is partitioned conformably to give $\nabla_T^{-d} = [S_*, S]$. We may observe that

$$\begin{bmatrix} S_* & S \end{bmatrix} \begin{bmatrix} Q_*' \\ Q' \end{bmatrix} = S_*Q_*' + SQ' = I_T, \tag{21}$$

and that

$$\begin{bmatrix} Q_*' \\ Q' \end{bmatrix} \begin{bmatrix} S_* & S \end{bmatrix} = \begin{bmatrix} Q_*'S_* & Q_*'S \\ Q'S_* & Q'S \end{bmatrix} = \begin{bmatrix} I_d & 0 \\ 0 & I_{T-d} \end{bmatrix}. \tag{22}$$

The matrix $\nabla_T^{-d} = [S_*, S]$ is a lower-triangular Toeplitz matrix, which is characterised completely by its leading column. The elements of that column are the ordinates of a polynomial of degree $d-1$ of which the argument is the row index $t = 0, 1, \ldots, T-1$. Moreover, the leading d columns of the matrix ∇_T^{-d}, which constitute the submatrix S_*, provide a basis for all polynomials of degree $d-1$ that are defined on the integer points $t = 0, 1, \ldots, T-1$.

The ordinates of a polynomial of degree $d-1$ defined over the integers $t = 0, 1, \ldots, T-1$ are given by

$$p = S_* r_*, \quad \text{where} \quad r_* = Q_*' p. \tag{23}$$

Since the polynomial is fully determined by the elements of the starting-value vector r_*, fitting it to the data in the vector $y = [y_0, \ldots, y_{T-1}]'$ according to the least-squares criterion is a matter of minimising

$$(y-p)'(y-p) = (y - S_* r_*)'(y - S_* r_*) \tag{24}$$

with respect to r_*. The resulting values are

$$r_* = (S_*'S_*)^{-1}S_*'y \quad \text{and} \quad p = S_*(S_*'S_*)^{-1}S_*'y. \tag{25}$$

For an alternative expression, we may use the identity

$$S_*(S_*'S_*)^{-1}S_*' = I - Q(Q'Q)^{-1}Q', \tag{26}$$

which follows from the fact that Q and S_* are complementary matrices such that $Q'S_* = 0$ and $\text{Rank}[Q, S_*] = T$.

Using (26) in (25) gives the following expression for the vector of polynomial ordinates:

$$p = y - Q(Q'Q)^{-1}Q'y. \tag{27}$$

3.2 The spectral structure of the data

Figure 1 shows the quarterly sequence of the logarithms of aggregate income in the U.K. for the period 1964 to 2003, through which a quadratic trend has been interpolated by a weighted least squares regression. The weights, which are increasing towards the beginning and the end of the sample, have the effect of ensuring that, in these vicinities at least, the trend adheres closely to the data. This is so as to avoid any disjunctions in the periodic extension of the data at the points where the end of one replication of the sample joins the beginning of the next. The quadratic trend is virtually a linear trend. Figure 2 shows the periodogram of the residuals, which are the deviations of the sequence from this trend.

The spectral signature of the low-frequency cycles that surround the trend is clearly represented in the periodogram. It occupies a range of frequencies extending from zero to $\pi/8$ radians. Centred on the frequencies of $\pi/2$ and π are the spikes that are the spectral signature of the seasonal variations that affect the income sequence. The remainder of the periodogram may be described as dead space punctuated by small elements of noise.

The periodogram of the original trended data is of little use in discerning the low-frequency structure. The latter is concealed within the slew of spectral power that is attributable to the disjunctions that occur in the periodic extension of the data.

It is also the case that none of the low-frequency spectral structure will be evident in the periodograms of either the first or the second differences of the data. The gain factor of the second-differencing operator at $\pi/10$, for example, is 0.00958, which means that the low-frequency ordinates of the periodogram are so severely attenuated by the differencing operation as to become invisible.

It is evident from equation (27) that the residuals from fitting a polynomial of degree $d-1$, which are found in the vector $y-p$, contain exactly the same information as the differences of order d within the vector $Q'y$. Nevertheless, they serve to reveal the spectral structure over the entire range of frequencies. It should be emphasised that the use of polynomial residuals as a means of revealing the spectral structure does not imply any decision to model the trend via a polynomial function.

3.3 The Hodrick–Prescott filter

The second method of trend extraction entails the notion of a stochastic trend. This represents the cumulative effects of stochastic elements that impart an upward drift to the economy. The usual statistical model of such a trend is a second-order, or integrated, random walk, which may be subject to drift. A common device for extracting such trends is the Hodrick–Prescott (1980, 1997) filter.

The Hodrick–Prescott (H–P) filter may be derived in reference to an equation

$$y(t) = \xi(t) + \eta(t), \tag{28}$$

in which the doubly-infinite data sequence $y(t)$ is expressed as the sum of a trend component $\xi(t)$, which follows a second-order random walk, and a residual component $\eta(t)$, which is white noise. (We shall use the corresponding roman letters $x(t)$ and $h(t)$ to denote the estimates of $\xi(t)$ and $\eta(t)$, respectively.) The random walk is described by the equation $(1-L)^2\xi(t) = \zeta(t)$, where $\zeta(t)$ is a white-noise process that is independent of $\eta(t)$, and where L is the lag operator, such that $Ly(t) = y(t-1)$. Therefore, the differenced data sequence

$$\begin{aligned} (1-L)^2 y(t) &= (1-L)^2\xi(t) + (1-L)^2\eta(t) \\ &= \zeta(t) + \kappa(t) \end{aligned} \tag{29}$$

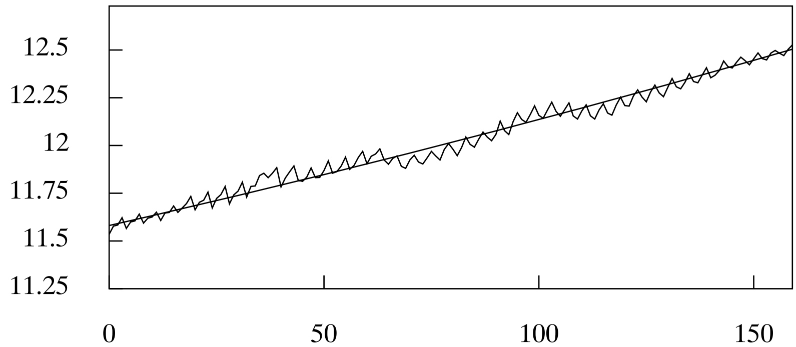

Figure 1: The quarterly sequence of the logarithms of income in the U.K. for the years 1964 to 2003, together with a quadratic trend interpolated by a weighted least-squares regression.

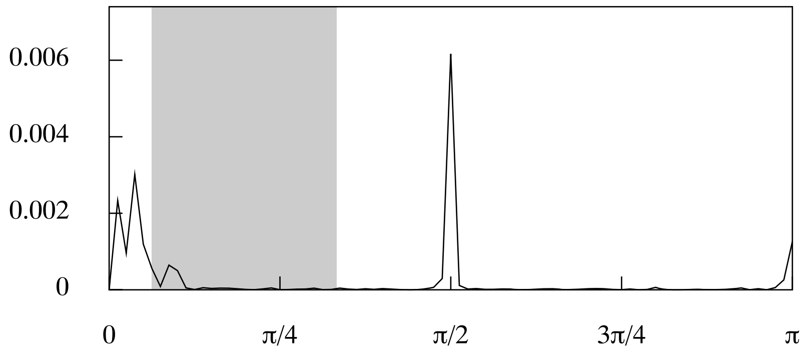

Figure 2: The periodogram of the residuals obtained by fitting a quadratic trend through the logarithmic sequence of U.K. income. A band, with a lower bound of $\pi/16$ radians and an upper bound of $\pi/3$ radians, is masking the periodogram.

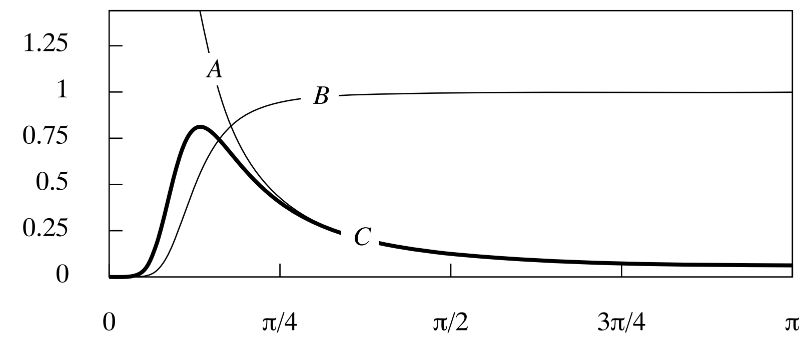

Figure 3: The effect of applying the Hodrick–Prescott filter to a random walk. The smoothing parameter is $\lambda = 100$ and the variance of the white-noise process driving the random walk is $\sigma_\varepsilon^2 = 0.25$

constitutes a stationary process; and the autocovariance generating functions of the differenced components are

$$\gamma_\zeta(z) = \sigma_\zeta^2 \quad \text{and} \quad \gamma_\kappa(z) = \sigma_\eta^2(1-z)^2(1-z^{-1})^2. \tag{30}$$

According to the Wiener–Kolmogorov principle, the detrending highpass filter is derived by setting $z = L$ in the following ratio of autocovariance generating functions:

$$\psi(z) = \frac{\gamma_\kappa(z)}{\gamma_\kappa(z) + \gamma_\zeta(z)} = \frac{\sigma_\eta^2(1-z)^2(1-z^{-1})^2}{\sigma_\eta^2(1-z)^2(1-z^{-1})^2 + \sigma_\zeta^2}. \tag{31}$$

The residual component is estimated by $h(t) = \psi(L)y(t)$. The complementary lowpass filter, which estimates the trend, is derived by setting $z = L$ within the function $1 - \psi(z)$.

Setting $z = e^{-\mathrm{i}\omega}$ in $\psi(z)$ gives the frequency response of the filter which, in this instance, is a real-valued function on account of the symmetry of the filter in respect of z and z^{-1}. The squared modulus of the frequency response function, which, in this case, is just the square, constitutes its squared gain. This is plotted in Figure 3, for a particular value of $\lambda = \sigma_\eta^2/\sigma_\zeta^2$, as the curve that is labelled B. Also plotted on the diagram is the pseudo spectrum of a first-order random walk labelled A.

The curve labelled C in the diagram, is the spectral density function of a detrended sequence derived from the random walk by applying a filter with a smoothing parameter of $\lambda = 100$. In place of this single curve, one can imagine a family of curves generated by varying the value of λ. In that case, one would discern that the functions associated with lower values of λ have peaks of lesser height located at higher frequency values. The inference is that the lower the value of the smoothing parameter λ the shorter are the durations of the cycles in the detrended series and the less is their amplitude.

3.4 The finite-sample filter

In practice, the data are available as a finite sequence, which constitutes a vector $y = \xi + \eta$, where ξ is the trend and η is the noise. Therefore, filters must be derived that operate on finite sequences. Recall that Q', defined by (20)–(22), denotes the matrix version of the second-difference operator. Then

$$\begin{aligned} Q'y &= Q'\xi + Q'\eta \\ &= \zeta + Q'\eta, \end{aligned} \tag{32}$$

where

$$\begin{gathered} E(\zeta) = 0, \qquad D(\zeta) = \sigma_\zeta^2 I_{T-2}, \\ E(\eta) = 0, \qquad D(\eta) = \sigma_\eta^2 I_T, \\ \text{and} \quad C(\zeta, Q'\eta) = 0. \end{gathered} \tag{33}$$

The independence of ξ and η implies that $D(Q'y) = \sigma_\eta^2 Q'Q + \sigma_\zeta^2 I$.

On the assumption that the components have a normal distribution, there is the following joint density function:

$$N(\zeta, \eta) = (2\pi)^{1-T}\sigma_\zeta^{2-T}\sigma_\eta^{-T}\exp\{-\frac{1}{2}(\sigma_\zeta^{-2}\xi'QQ'\xi + \sigma_\eta^{-2}\eta'\eta)\}. \tag{34}$$

The maximum-likelihood estimate x of the trend component ξ is found by minimising the following criterion function, which is derived from the quadratic exponent of the density function by setting $\eta = y - \xi$:

$$S(\xi) = \sigma_\zeta^{-2}\xi'QQ'\xi + \sigma_\eta^{-2}(y-\xi)'(y-\xi). \tag{35}$$

The minimising value of ξ is

$$x = \sigma_\eta^{-2}(\sigma_\zeta^{-2}QQ' + \sigma_\eta^{-2}I)^{-1}y. \tag{36}$$

According to the matrix inversion lemma, there is

$$(\sigma_\zeta^{-2}QQ' + \sigma_\eta^{-2}I)^{-1} = \sigma_\eta^2\Big\{I - Q(Q'Q + [\sigma_\zeta^2/\sigma_\eta^2]I)^{-1}Q'\Big\}. \tag{37}$$

Using this in (36) and writing $\sigma_\zeta^2/\sigma_\eta^2 = \lambda^{-1}$, we get

$$x = y - Q(Q'Q + \lambda^{-1}I)^{-1}Q'y. \tag{38}$$

This is the appropriate finite-sample version of the H–P trend estimation filter.

We should make two observations in respect of this equation. First, if y were to follow a linear trend, then the equation would deliver $x = y$ since, in that case, $Q'y = 0$. Secondly, as $\lambda \to \infty$, the equation will tend to that of the least-squares estimator of a linear trend, which is represented by (27). We note that λ, which is conventionally described as the smoothing parameter, is also a noise/signal variance ratio. When the noise is strong relative to the signal, the Hodrick–Prescott filter is also liable to deliver a trend that is approximately linear within wide neighbourhoods.

We should note that the filter is also appropriate to the case where there is stochastic drift in the signal process. To see this, let $y = \xi + \phi + \eta$, where ϕ is a quadratic function. An analysis of equation (38) shows that ϕ will be fully incorporated in x, which is the estimate of the trend component. Whereas it will remain intact within y, it will be virtually nullified by the high pass operator $Q(Q'Q + \lambda^{-1}I)^{-1}Q'$.

The H–P filter has been used as a lowpass smoothing filter in numerous macroeconomic investigations (see, for example, Hartley et. al. 1998), where it has been customary to set the smoothing parameter to certain conventional values. Thus, for example, the econometric computer package *Eviews 4.0* (2000) imposes the following default values:

$$\lambda = \begin{cases} 100 & \text{for annual data,} \\ 1,600 & \text{for quarterly data,} \\ 14,400 & \text{for monthly data.} \end{cases} \tag{39}$$

An alternative to specifying the smoothing parameter λ is to specify a frequency value ω_c such that $\psi(\omega_c) = 0.5$. This frequency corresponds to the midpoint in the transition of the gain of the lowpass H–P filter from the value unity, which is attained when $\omega = 0$ to the value of zero, which is attained when $\omega = \pi$. The closer is ω_c to 0, the higher is the value of λ. One might be tempted to describe ω_c as the nominal cut-off point of the filter, but, in view of the gradual transition of the gain of the H–P filter from unity to zero, this could be regarded as a misnomer.

The correspondence between ω_c and λ is as follows:

$$\lambda = 1/4\{1 - \cos(\omega_c)\}^2 \quad \text{and} \quad \omega_c = \cos^{-1}(1 - 1/\sqrt{4\lambda}). \tag{40}$$

Instead of specifying ω_c directly, it may be easier to specify the duration of the cycles of this frequency. For a duration of τ years, the frequency is $\omega_c = 2\pi/(\tau s)$, where s is the number of observations per year.

It has become customary to define the business cycle, in the manner of Burns and Mitchell (1946), as a composite of sinusoidal motions of durations not exceeding 8 years and not less than one-and-a-half years. (For examples, see Baxter and King 1999 and Christiano and Fitzgerald

2003.) When the nominal value of the limiting duration is set at 8 years, the frequency response of the lowpass H–P filter has considerable leakage across the boundary, with the effect that large proportions of some of the business-cycle components are removed from the residue. Therefore, in order to achieve a good representation of the business cycle, the smoothing parameter of the H–P filter should greatly exceed the value that corresponds to a duration of 8 years.

4 The Issue of Spurious Cycles

4.1 The gain of the Hodrick–Prescott filter

The H–P filter has been subject to an oft-repeated aspersion that it is liable to induce spurious cycles in the detrended data. The argument has been made, for example, by Cogley and Nason (1995) and it has been supported by Harvey and Jaeger (1993), amongst others. Contrary opinions have been offered by Pollock (1997, 2000) by Pedersen (2001) and by Valle e Azevedo, (2002). There is a semantic issue at the root of these differences of opinion, but there is also evidence of a widespread misunderstanding.

The highpass H–P filter $\psi(L)$ has a frequency response function for which the gain never exceeds unity. (An example of the squared gain of the filter is given by the curve labelled B in Figure 3.) This means that its effect is either to preserve or to attenuate the sinusoidal elements of which a data sequence is composed. The filter never amplifies any sinusoidal elements and it never introduces any. If a sinusoid is present in the processed data, then it must also be present to no lesser extent in the original data. Therefore, such cyclical components cannot be induced or accentuated by the filter.

The proposal of Cogley and Nason that the H–P filter can generate business cycle dynamics is based upon an analysis of the frequency response of a filter that results from conflating the H–P filter with the unit-root summation operator belonging to the model of a random walk. The random walk $y(t)$ is modelled by the equation $(1 - L)y(t) = \varepsilon(t)$, where $\varepsilon(t) = \{\varepsilon_t; t = 0, \pm 1, \pm 2, \ldots\}$ stands for a white-noise process. The filtered sequence is

$$h(t) = y(t) - x(t) = \{\psi(L)/(1 - L)\}\varepsilon(t). \tag{41}$$

Cogley and Nason attribute to the H–P filter the gain of the filter $\psi(L)/(1 - L)$, instead of the gain of $\psi(L)$, which is the true gain of the H–P filter.

An example of the squared gain of the filter $\psi(L)/(1 - L)$ is given by the curve labelled C in Figure 3. This curve also represents the spectrum of the filtered sequence $h(t)$. It certainly manifests a strong spectral peak which indicates the presence of cyclical motions within $h(t)$. However, such motions are attenuated versions of those that are present in the data process $y(t)$; and they are not induced by the filter.

Notwithstanding the nature of its Fourier decomposition, which comprises sinusoidal motions at all frequencies, there is some justification for the assertion that a true random walk contains no genuine cycles. The process is the product of an accumulation of statistically independent increments; and it has no inherent central tendency.

On the other hand, genuine cyclical motions are commonly regarded as the products of centralising forces that increase in proportion to the distance of an object from the point to which it is tethered. It is on this basis that the cycles that are generated by filtering a random walk might be regarded as illusory artefacts.

Economic trends are often modelled as random-walk processes. Nevertheless, such models need not be interpreted in a literal manner. Whereas a random walk evolves in an unbridled manner, economic trends are subject to evident constraints. They are driven by the buoyant forces of entrepreneurial endeavour and consumer aspirations, and they are constrained by the more-or-less pliable limits of productive capacity and resource availability. In a thriving economy, they alternately press against the constraints and rebound from them in a manner that is undeniably cyclical.

The econometric practice of modelling aggregate economic activity as a cumulation of stochastic increments is in marked contrast to the emphasis that has been given to centripetal mechanisms, such as the error-correction mechanism that is at the heart of co-integration analysis.

4.2 Finite and infinite random walks

There may be doubts about the applicability to economic circumstances of an analysis, such as the one that underlies Figure 3, that postulates a random walk defined over a doubly infinite sequence of integers. Such a process is unbounded in mean, and it does not have a finite variance. It is expected, at any point in time, to be infinitely remote from the origin. By contrast, the random walks that are postulated in applied econometrics are defined over a finite interval, and they have starting values at a finite distance from the origin.

A measure of the difficulties in interpreting an infinite random walk is provided by the limiting case of the infinite-sample Hodrick–Prescott filter, where $\lambda \to \infty$. This is when the finite-sample filter delivers a linear trend. In the limit, the corresponding infinite-sample filter has a frequency response function with a unit gain everywhere except at zero frequency. Therefore, it is virtually an allpass filter; which suggests that the character of an infinite random walk should be unaffected by a process of linear detrending.

This is in contrast to what we expect from the linear detrending of a finite random-walk sequence. Fitting a straight line by least-squares regression to a finite segment of a random walk will result in a residual sequence of mean zero that will inevitably show a reversion to the mean. From a global perspective, which views the sample as whole, this central tendency will not be affected by a growing sample duration. There will be the same number of crossings of the trend line on average, regardless of the length of the sample; and the number of crossings will be few.

On the other hand, if we look myopically at the sampled sequence, then the effect of increasing the sample size will be to reduce the rate of mean reversion, as measured from one point to the next. Therefore, eventually, the linearly detrended random walk will become a random walk itself. Thus, we are able to reconcile the behaviour of the limiting case of the infinite-sample H–P filter with the behaviour of the finite-sample filter.

We may consider the random walk to be the product of a regular process of sampling applied to a continuous Wiener process. The Wiener process is self-similar in the sense that short segments, viewed in detail, have the same appearance as longer segments, viewed more distantly. Therefore, allowing for a change of scale, the effect of increasing the size of the sample by allowing a growing number of points to accumulate with the passage of time is no different from the effect of increasing their number by increasing the rate at which a finite segment of the process is sampled. Indeed, it may be easier to see these things in the small—such as when a Wiener process is defined on a unit interval—than in the large.

The effect of fitting a linear trend by least-squares to a finite segment of a random walk has been analysed by Chan, Hayya and Ord (1971). They have found a formula for the autocovariances of the residuals. They have also provided a formula for the expected value of the estimated autocovariances of the residuals. This has been corrected by Nelson and Kang (1981), who have extended the analysis. The latter have revealed that the spectral density function, derived from the expected values of the sample autocovariances, has a peak at a frequency that corresponds to a cycle of a duration that is 0.83 times the number of sample periods.

Nelson and Kang have emphasised the risk of finding spurious dynamics in the residuals from the inappropriate detrending of a random walk. A myopic analysis that looks only at the values of the parameters of an estimated autoregressive model is clearly at risk of drawing false conclusions. However, an analysis that measures the rate of mean reversion relative to the length of the sample can avoid the risk.

The frequencies of the fluctuations around their interpolated linear trends of the sequences that we shall analyse are greater than the frequency that is characteristic of a random walk. Therefore, the sequences appear to manifest a genuine cyclicality. Figure 4 provides such evidence.

5 The Empirical Results

In the empirical analysis of this section, we shall detrend the logarithmic data by interpolating a quadratic function, and we shall also use the H–P filter. A quadratic function is used instead of a linear function in order to accommodate cases where there is some indication of a gradual increase or decrease, over the sample period, in the rate of the underlying growth of GDP; but we have found that, in most cases, the quadratic function is virtually a linear function. We shall pursue the method of estimating the trend via the H–P filter, mainly for comparative purposes; and we shall be interested to see the effect of varying the smoothing parameter.

It is a notable circumstance when the data are amenable to a linear or a quadratic detrending in several countries over the same protracted period, as is the case for each of the six OECD countries throughout our sample period. Figure 4 shows the evidence of this. In different eras and over longer periods, we would expect to resort, instead, to a flexible method of trend estimation that employs the H–P filter and that accommodates structural breaks via local variations in the smoothing parameter.

Some of the data sequences in Figure 4 contain seasonal fluctuations, whereas others have been deseasonalised. These differences are of no account in the analysis of business cycles, which comprise components that are of much lower frequencies than the seasonal fluctuations. The later will be removed automatically in the process of anti-alias filtering, which has the effect of deseasonalising the data.

5.1 The method of estimation

The results from fitting an AR(2) to the data of each of the six OECD countries, for the period 1980Q1–2003Q4, are displayed in Table 1. These estimates are the outcome of a multistep procedure.

For a start, a quadratic trend—which serves as a benchmark against which the cyclical activity of the national economies can be measured—has been fitted to the logarithms of the data. The residuals from the extracted trend, which are liable to be examined for their low-frequency content, contain seasonal and irregular components at higher frequencies, which are not directly attributable to the business cycle.

To remove these components, the detrended data from each country is subjected to a process of lowpass filtering and subsampling, which serves to discard all except the information that lies in the frequency interval $[0, \pi/8]$ in the spectrum of the original data. The filter is implemented in the frequency domain by selecting the appropriate Fourier coefficients of the data. The filtered data is reconstructed via a Fourier synthesis.

The filtering is also effective in overcoming any aliasing that could arise from the process of subsampling that selects every eighth data point. In the absence of an anti-aliasing filter, subsampling by a factor of 8 would serve to map the information content in the upper seven octaves of the frequency range into its lowest octave, wherafter the latter would be expanded by a factor of 8 to occupy the interval $[0, \pi]$. With proper anti aliasing, the effect of the subsampling

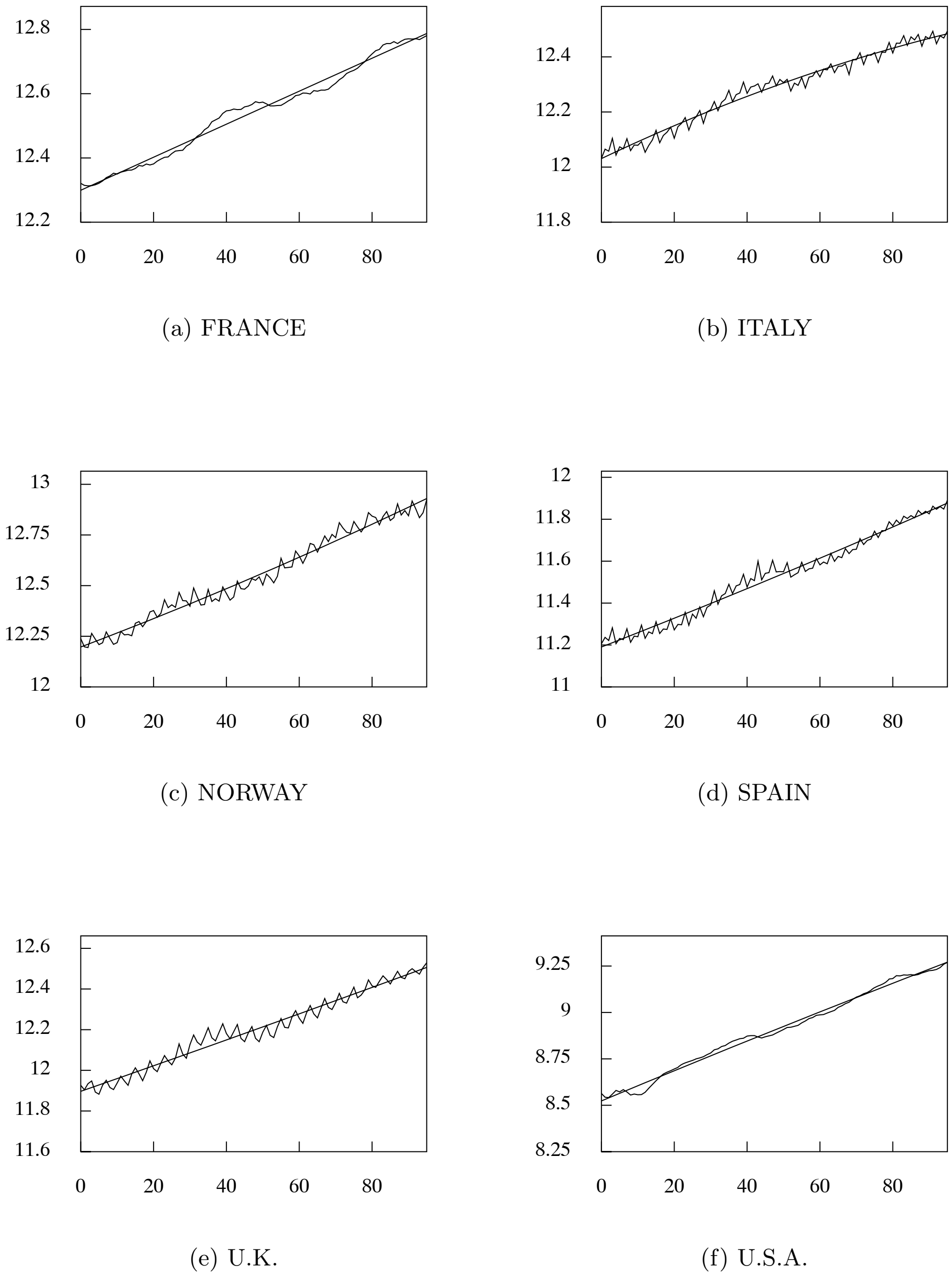

Figure 4: The logarithms of the aggregate incomes of six OECD countries for the period 1980Q1–2003Q4 with interpolated quadratic trends.

Table 1: The business-cycle parameters obtained via quadratic detrending

	α_1	α_2	ρ	ω	σ_x	σ_ϵ	b
SPAIN	−0.6072	0.7592	0.8713	69.61	0.0204	0.0126	0.0272
FRANCE	−0.5838	0.7178	0.8473	69.85	0.0152	0.0100	0.0204
U.K.	−0.5270	0.6749	0.8215	71.29	0.0197	0.0138	0.0252
NORWAY	−0.4899	0.5519	0.7429	70.75	0.0202	0.0179	0.0272
ITALY	−0.3904	0.5504	0.7419	74.74	0.0122	0.0949	0.0252
U.S.A.	−0.1318	0.2446	0.4946	82.34	0.0161	0.0176	0.0328

is to map the unimpaired contents of the original data in the frequency interval $[0, \pi/8]$ onto the wider interval $[0, \pi]$, which corresponds to the full range of frequencies within any sampled data sequence.

Pagan (1997) has remarked that AR(2) models that are fitted to detrended quarterly logarithmic output data typically possess real roots, whereas they might be expected to possess complex roots reflecting the dynamics of the business cycles. Within the context of quarterly data, the business cycle is a low frequency phenomenon; and it is not surprising that the AR roots do not reflect its cyclicality.

Within the context of annual data, the business cycle has a considerably higher frequency; and an AR(2) model fitted to such data would almost certainly capture its cyclicality. Compared with quarterly data, annual data is liable to suffer from the effects of aliasing and phase distortion. By applying the processes of anti aliasing and subsampling to the quarterly data, we are ensuring that the information that is extracted is appropriate to the purpose of estimating the business cycle.

The processes of filtering and subsampling the data are illustrated in Figures 5 and 6 which relate to the U.K. over an extended data period of 40 years running form 1964Q1 to 2003Q4. Figure 6 displays the periodogram of the detrended data over the interval $[0, \pi/8]$, and it corresponds to a segment of Figure 2. Figure 7 is an Argand diagram that indicates the location in the complex plane of the roots of the autoregressive operator of the estimated AR(2)model.

The estimates have been derived by maximising the likelihood function of Whittle (1951), which entails the assumption that the data are the product of a circular process. (It is equivalent to propose that the sample represents one cycle of a periodic function.)

To sustain this assumption, some attention has to be paid to the problem of the disjunction that can occur at the point where the end of the sample is joined to its beginning. Whereas the problem can be ignored if the data sequence is a lengthy one, such as the extended U.K. sequence, it needs to be addressed in the case of the shorter sequences from the six OECD countries.

We believe that an appropriate way of dealing with this matter is to extend the sample at both the ends by forecasting and backcasting the data. In particular, our procedure uses the preliminary estimates of an AR(2) model to lengthen the sample by 25 per cent. Then, the extrapolated sample is subjected to a tapering operation based on a split cosine bell. The latter is nothing but a cosine bell with an inserted stretch of units.

5.2 The dynamics of the economies

Our first concern is to assess the degree of damping to which each economy has been subject over the sample period. This is revealed by the estimated value of the damping factor ρ, which,

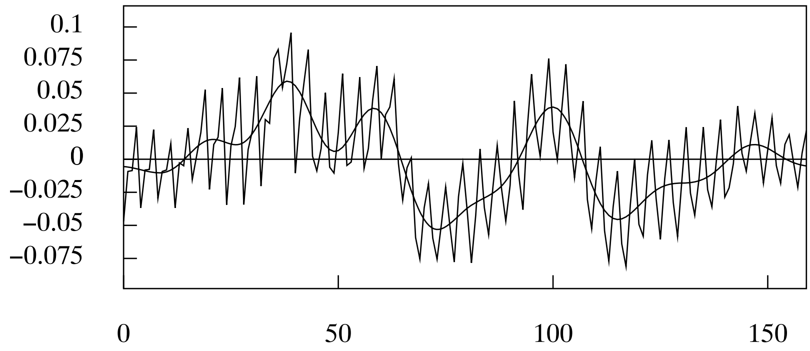

Figure 5: The residual sequence from fitting a quadratic trend to the income data of Figure 1. The interpolated line, which represents the business cycle, has been obtained from the Fourier ordinates that generate the periodogram of Figure 6.

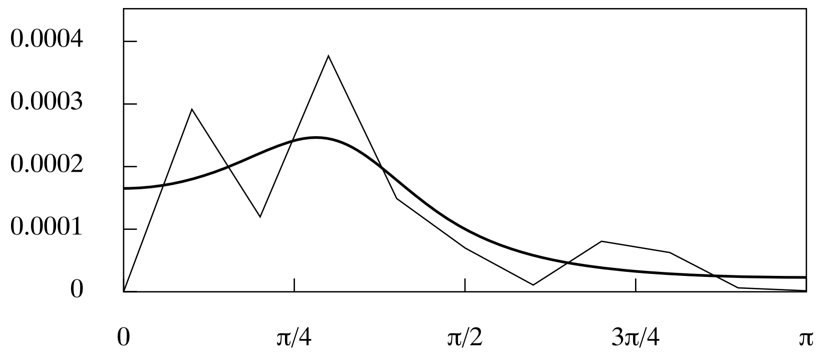

Figure 6: The periodogram of the sub sampled anti-aliased data with the parametric spectrum or an estimated AR(2) model superimposed.

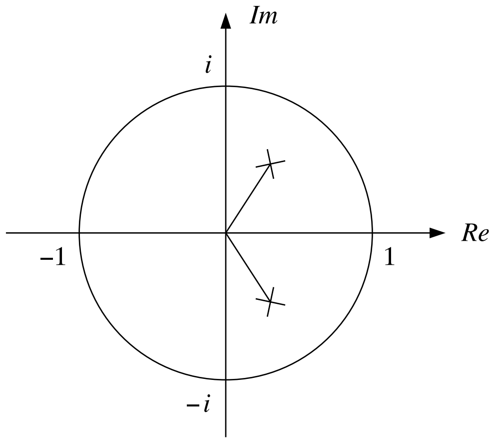

Figure 7: An Argand diagram indicating the location in the complex plane of the roots of the autoregressive operator of the estimated AR(2)model.

Table 2: The structural business-cycle parameters

	c	h	Λ	$\sigma_x/\sigma_\varepsilon$
SPAIN	0.2755	1.4948	0.1843	1.6202
FRANCE	0.3315	1.5136	0.2190	1.5166
U.K.	0.3932	1.5868	0.2478	1.4327
NORWAY	0.5944	1.6130	0.3684	1.1319
ITALY	0.5972	1.7909	0.3334	1.2909
U.S.A.	1.4081	2.5610	0.5498	0.9158

together with the angular velocity ω, is entailed in the expression for the homogeneous second-order difference equation under (12) and in its analytic solution under (14).

A question that was posed at the end of section 2 was how the frequency value ω, or equivalently the length of the cycle, that is implied by the estimated AR(2) equation is related to the degree of damping. Table 1 shows a high degree of inverse correlation between the rankings of the values of ρ and ω, which is the angular velocity or frequency measured in degrees per biennium—the lower the damping, i.e. the closer ρ is to unity, the lower is ω and the longer is the cycle.

Table 1 also indicates, via the values of σ_x, the relative amplitude of the economic cycles for each of the six countries. This is measured as the standard deviation from the interpolated trend line of the logarithmic income series. The values of σ_ε represent estimates of the magnitudes of the disturbances that drive the cycles.

The final column of the Table 1 gives the values of b which is the slope parameter of a linear trend interpolated through the logarithmic series by least-squares regression. These values represent the average rates of growth of the countries for the period in question.

The parameters ρ and ω provide a complete characterisation of the dynamic properties of a second-order system. However, it is also insightful to characterise the system in terms of the fundamental structural parameters c and h, which are, respectively, the coefficients of friction and of stiffness, and in terms of the loss parameter $\Lambda = c/h$, which is their ratio. These are displayed in Table 2, which also gives the ratio $\sigma_x/\sigma_\varepsilon$. The relationship of the two sets of parameters is via the following equations, which are indicated by (9) and (15),

$$\text{(i)} \quad \gamma = \ln(\rho), \qquad \text{(ii)} \quad c = -2m\gamma \quad \text{and} \quad \text{(iii)} \quad h = m(\gamma^2 + \omega^2), \tag{42}$$

wherein $m = 1$ in consequence of a normalisation. To make matters more intelligible, we display, in Table 3, the rankings of the various measures.

The coincidence of the ranking of ρ with the inverse ranking of c follows necessarily from their analytic relationship, which is indicated by (i) and (ii) of (42). The the close relationship of the ranking of ω and h, there is no such necessity, since γ is also present in (iii). Given the coincidence of the rankings of ρ and c and the closeness of those of ω and h, it follows that c and h display the same high degree of rank correlation as do ρ and ω, albeit in the direct rather than the inverse sense.

There is also a close rank correlation between the damping factor ρ and the variance ratio $\sigma_x/\sigma_\varepsilon$. Both of these are readily expressed in terms of the parameters of the difference equation. Whereas $\rho^2 = \alpha_2$, there is

$$\frac{\sigma_x^2}{\sigma_\varepsilon^2} = \frac{(1+\alpha_2)}{(1-\alpha_2)(1+\alpha_2+\alpha_1)(1+\alpha_2-\alpha_1)}. \tag{43}$$

Table 3: The rankings of the parameters of Tables 3.1 and 3.2 in descending order of magnitude and in ascending order—via the numbers in parenthesis

	ρ	c	ω	h	Λ	σ_x	σ_ε	$\sigma_x/\sigma_\varepsilon$	b
SPAIN	1(6)	6(1)	6(1)	6(1)	6(1)	1(6)	5(2)	1(6)	2(4)
FRANCE	2(5)	5(2)	5(2)	5(2)	5(2)	5(2)	6(1)	2(5)	6(1)
U.K.	3(4)	4(3)	3(4)	4(3)	4(3)	3(4)	4(3)	3(4)	4(2)
NORWAY	4(3)	3(4)	4(3)	3(4)	3(4)	2(5)	2(5)	5(2)	2(4)
ITALY	5(2)	2(5)	2(5)	2(5)	2(5)	6(1)	1(6)	4(3)	4(2)
U.S.A.	6(1)	1(6)	1(6)	1(6)	1(6)	4(3)	3(4)	6(1)	1(6)

(See, for example, Pollock 1999 p. 533.) This variance ratio increases as $\alpha_2 = \rho^2$ increases towards unity. Also, within the range of variation of α_1, in which $1 + \alpha_2 + \alpha_1 > 0$, the variance ratio increases as α_1 declines. In Table 1, there is a perfect inverse rank correlation between α_1 and α_2. This gives rise to the almost perfect rank correlation between ρ and $\sigma_x/\sigma_\varepsilon$.

To understand the implications of these various relationships, one should make reference to equation (16) which expresses ω in terms of the structural coefficients c and h. The equation indicates that, within a given system with a fixed stiffness parameter h, increasing the damping coefficient c will lead to a lengthening of the duration of the cycle. On the other hand, for fixed c, increasing h will shorten the duration of the cycle.

Since c and h are free to vary independently across the economies, there should be no firm expectation concerning the nature of their relationship. It transpires that h and c tend to vary together with a positive correlation, which means that their variations have offsetting effects. However, for the sample of 6 OECD countries, the effects of the variations in h heavily outweigh the offsetting effects of the variations in c.

When these effects are discerned through the autoregressive parameters ρ and ω, it is found that economies with greater damping tend to have shorter cycles. This is the opposite of a relationship that would be observed in a single system governed by second-order dynamics and subject to variable damping. In that case, increasing the damping would lengthen the cycles.

There is a supposition that the lower the relative amplitude of the business cycle the better are the prospects for the growth of the economy. This idea was famously propounded in a pamphlet published in the U.K. by the National Economic Development Council (1963) under the title of *Conditions Favourable to Faster Growth.* It was proposed that the economic stop-go policies of the United Kingdom in the preceding decade had created a series of booms and slumps that had inhibited the growth of the economy.

Table 3 gives no clear indication of a negative relationship between the rates of growth of the countries and the relative amplitude of their economic cycles. However, the question remains of whether such a relationship could be found by observing the same economy in different epochs.

5.3 The effects of the Hodrick–Prescott filter

We now proceed to examine the effect of using the lowpass Hodrick–Prescott filter, instead of a quadratic function, to remove the trend from the data, in the attempt to reveal the business cycle. We shall attribute a range of alternative values to the smoothing parameter λ. The higher the value of the smoothing parameter, the more rigid is the estimated trend. The analysis of section 3 has shown that, as λ increases, the resulting trend tends to a linear function that is interpolated through the data by a least-squares regression.

Table 4: Parameters for smoothing parameter equal to 8000

	α_1	α_2	ρ	ω	Λ	σ_x	σ_ε
SPAIN	−0.40128	0.70107	0.83730	76.14	0.20	0.01516	0.01049
FRANCE	−0.28380	0.64995	0.80620	79.86	0.22	0.01140	0.00836
UK	−0.24384	0.65370	0.80852	81.33	0.21	0.01414	0.01049
NORWAY	−0.25612	0.57652	0.75929	80.29	0.27	0.01414	0.01140
ITALY	−0.10582	0.40901	0.63954	85.25	0.37	0.00894	0.00774
U.S.A.	0.07377	0.36670	0.60556	93.49	0.34	0.01265	0.01183

Table 5: Parameters for smoothing parameter equal to 1600

	α_1	α_2	ρ	ω	Λ	σ_x	σ_ε
SPAIN	−0.07488	0.52742	0.72623	87.04	0.27	0.00894	0.00775
FRANCE	0.05076	0.57258	0.73669	91.92	0.21	0.00707	0.00632
UK	0.11612	0.54694	0.73956	94.50	0.21	0.00949	0.00775
NORWAY	0.15596	0.56620	0.74980	95.97	0.20	0.00894	0.00707
ITALY	0.27894	0.23539	0.50536	106.02	0.35	0.00548	0.00548
U.S.A.	0.50569	0.27336	0.52284	118.92	0.27	0.00894	0.00775

Intuition suggests that the more flexible is the interpolated trend the lower will be the amplitudes of the fluctuations in the residual sequence and the shorter will be the duration of its cycles. This intuition has been supported by an analysis in the frequency domain of the effect of applying the H–P filter to a random walk. Figure 3 is relevant to that analysis.

We begin by setting $\lambda = 8000$. This value corresponds nominally to the frequency value of $\omega = \pi/30$ radians per quarter and to a duration of 15 years. In fact, the resulting trends that are interpolated through the data are virtually linear. When the AR(2) model is fitted to the detrended data, we obtain the parameter values that are recorded in Table 4.

The next value to be assigned to the smoothing parameter is $\lambda = 1600$, which is the value that is commonly used in extracting macroeconomic trends from quarterly data, and it has been proposed by Hodrick and Prescott (1980, 1997). This value produces a trend that strongly reflects the cyclical pattern of the original time series. For this reason, it provides an attenuated version of the business cycle. It will be observed from the comparison of Tables 4 and 5 that the relative amplitudes of the business cycles measured by σ_x are considerably reduced and the duration of the cycles, as reflected in the angular velocity ω, is systematically reduced. This is in accordance with our presuppositions.

The final value to be investigated is $\lambda = 677.13$, which corresponds nominally to an angular velocity of $\pi/16$ radians per quarter and to a duration of 8 years. These values correspond to the upper limit of the business cycles duration according to the definition of Baxter and King (1999). These authors have also attributed a minimum duration of 1.5 years to the business cycles. (The resulting frequency band, which runs from $\pi/16$ radians per period up to $\pi/3$ radians per period, has been imposed on the periodogram of Figure 2.) Artis, Marcellino and Proietti (2004), have implemented the definition of Baxter and King via a bandpass filter that comprises a lowpass H–P filter with $\lambda = 0.52$ followed by a highpass filter with $\lambda = 677.13$. The initial lowpass filter may have little effect upon deseasonalised data for the reasons that there is liable to be very little spectral power within the corresponding stop band.

Table 6: Parameters for smoothing parameter equal to 677.13

	α_1	α_2	ρ	ω	Λ	σ_x	σ_ε
SPAIN	0.23331	0.31921	0.56499	101.92	0.33	0.00548	0.00548
FRANCE	0.30219	0.55054	0.74199	101.75	0.18	0.00548	0.01414
UK	0.41685	0.51692	0.71897	106.85	0.18	0.00707	0.00548
NORWAY	0.40081	0.58485	0.76475	105.19	0.16	0.00632	0.00548
ITALY	0.58644	0.26694	0.51666	124.58	0.26	0.01414	0.00316
U.S.A.	0.81471	0.36210	0.60175	132.61	0.18	0.00707	0.00548

The parameters derived by fitting an AR(2) to these data are given in table 6. We are disinclined to give much credence to these results for the reason that the 8-year limit on the duration of the cycles is an artificial one that does not appear to correspond to any evident feature in the spectral structure of the data.

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