POMPEIU'S THEOREM REVISITED

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In 1936, the Romanian mathematician Dimitrie Pompeiu discovered the following simple, yet beautiful, result in Euclidean plane geometry [4], [5]:

Pompeiu's Theorem. *Let* $\triangle ABC$ *be an equilateral triangle, and let* M *be a point in the plane determined by it. Then, the lengths* $MA, MB,$ *and* MC *form the sides of a triangle. The triangle formed by these sides is degenerate if and only if* M *lies on the circumcircle of* $\triangle ABC$*.*

A quick proof, by construction, of the first part of this result goes as follows: consider a rotation of $\pi/3$ radians about the point C that maps point A to point B, and M to M'. Clearly, $\triangle MCM'$ is equilateral and $MM' = MC$. Since $MA = M'B$, we conclude that $\triangle MBM'$ has sides equal to MA, MB, and MC (see Figure 1; all marked angles are equal to $\pi/3$ radians).

The theorem is not classical in that not every geometry textbook mentions it. Nowadays, it is more likely that a student will encounter this theorem as a corollary of the so-called Ptolemy inequality, named after Claudius Ptolemy, who is credited with the proof of the corresponding equality in a cyclic quadrilateral, that is, a quadrilateral with its four vertices on a circle. A slight drawback, however, of this approach

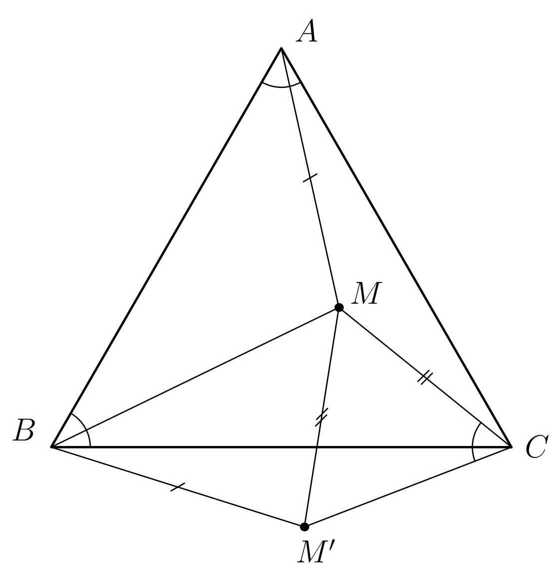

Figure 1.

is that it does not shed much light on further properties of the (non-degenerate) triangle determined by the sides MA, MB, and MC. Such properties require identifying the triangle, which, in turn, points to a constructive proof of Ptolemy's theorem.

Pompeiu established similar results for the square and other regular polygons [3], [7] (see also [1], [2]). Although he was extremely prolific, and published about 150 articles and books in several areas of mathematics, including geometry, Pompeiu principally distinguished himself in the field of analysis. He obtained a Ph.D. degree in mathematics in 1905 with a thesis written under the direction of Henri Poincaré, and, later, became known to the larger mathematical community for a challenging conjecture bearing his name [6] dating to year 1929. Upon his return to Romania, he founded mathematics schools of partial differential equations and mechanics. It is no surprise, then, that Pompeiu's original proof of his theorem was via complex numbers!

In this note, we revisit Pompeiu's theorem in the realm of arbitrary triangles. We generalize the result to what is known as *Ptolemy's inequality*. We also prove a symmetric formula, previously unknown as far as we know, that relates the areas of these triangles. As a tribute to the mathematical legacy of the Romanian mathematician we call it *Pompeiu's area formula*.

We provide two proofs of the Ptolemy inequality, one, in the spirit of Pompeiu's proof, via complex numbers, that seems to be missing from the literature, the other using synthetic (Euclidean) geometry. The latter proof leads to further introspection on the properties of certain triangles determined by four given points. Both our proofs are elementary, are based on simple ideas, and involve straightforward algebraic computations. The analytic proof requires only rudimentary notions about complex numbers. The geometric proof, by inversion, is known. We provide a slight modification of it and include all the details. The interested reader can consult [8], [9], [10] and the references therein for more on this approach. Furthermore, we will use this proof to arrive at our symmetric formula for areas.

Our generalization of Pompeiu's theorem (that is, Ptolemy's inequality) is

Theorem 1. *Given $\triangle ABC$ and a point M in its plane, the lengths $MA \cdot BC$, $MB \cdot CA$, and $MC \cdot AB$ form the sides of a triangle. This triangle is degenerate if and only if M is on the circumcircle of $\triangle ABC$.*

1. Proof via complex numbers.

Given a point $P(x, y)$ in the plane, we use the complex number $p = x + iy$ to denote such a point. Three points are said to be *collinear* if they all sit on the same line. Three distinct points are said to be *concyclic* if there is a circle passing through them.

For the remainder of this proof, we let a, b, c, and m be complex numbers representing the points A, B, C, and M. Furthermore, we choose a system of xy-coordinates such that A is identified with the origin, that is, we assume, without loss of generality, that $a = 0$.

Proof 1. From the obvious equality

$$m(b-c) = -(m-b)c + (m-c)b,$$

by taking absolute values, we obtain

$$|m||b-c| \leq |m-b||c| + |m-c||b|. \tag{1}$$

In other words,

$$MA \cdot BC \leq MB \cdot CA + MC \cdot AB,$$

so that the lengths considered do form a triangle.

This triangle is degenerate if and only if inequality (1) is an equality. On the other hand, this happens if and only if

$$\arg((m-b)c) = \arg((m-c)b),$$

or if

$$\frac{(m-b)c}{(m-c)b} \in \mathbb{R}.$$

Setting

$$\frac{(m-b)c}{(m-c)b} = \frac{(\bar{m}-\bar{b})\bar{c}}{(\bar{m}-\bar{c})\bar{b}},$$

and clearing fractions, yields

$$m\bar{m}(c\bar{b} - b\bar{c}) - (|c|^2\bar{b} - |b|^2\bar{c})m + (|c|^2 b - |b|^2 c)\bar{m} = 0.$$

Note that, since the points $0, b$, and c are not collinear, the leading coefficient on the left-hand side cannot be zero. Therefore,

$$m\bar{m} - \alpha m - \bar{\alpha}\bar{m} = 0, \tag{2}$$

where

$$\alpha = \frac{|c|^2\bar{b} - |b|^2\bar{c}}{c\bar{b} - b\bar{c}}.$$

Consider now the complex equation

$$z\bar{z} - \alpha z - \bar{\alpha}\bar{z} = 0,$$

or equivalently,

$$|z-\alpha|^2 = |\alpha|^2.$$

This is the equation of a circle, passing through the origin and with radius $|\alpha|$. Substituting and expanding reveals that, in this case, the circle also passes through the points b and c, and therefore is the circumcircle of $\triangle ABC$. Equation (2) states precisely that M is on this circle. This completes the proof of Theorem 1. $\square$

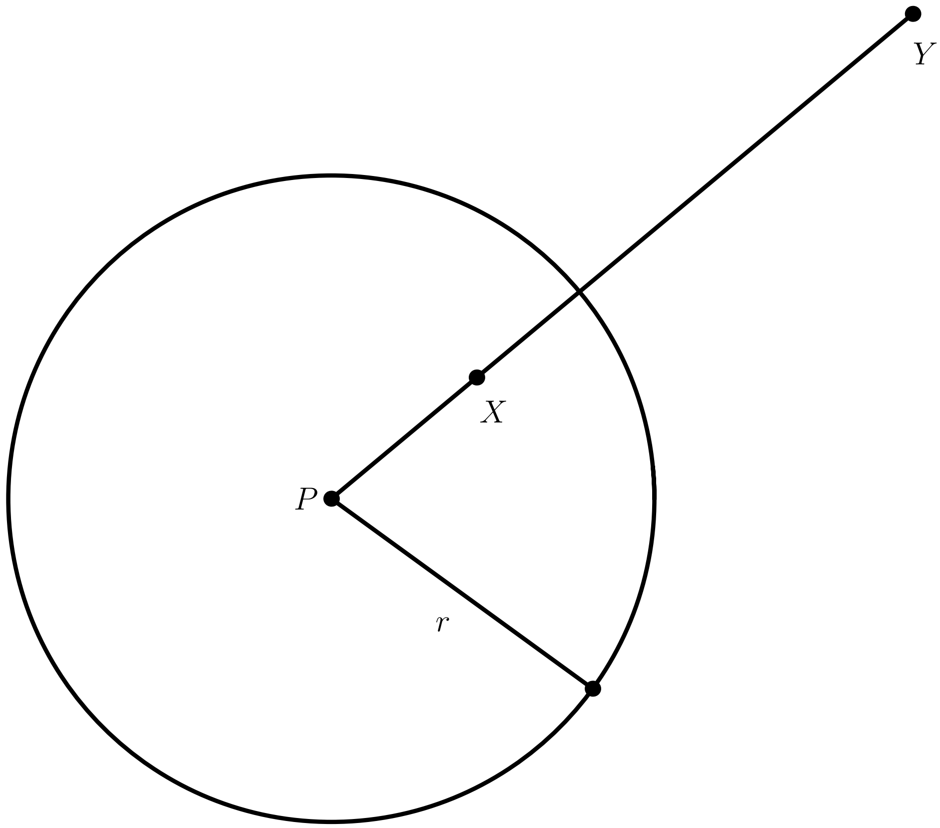

Figure 2. Circle inversion

2. Proof via inversion.

Before giving our second proof, recall that the *circle inversion* centered at the point P and of radius r (of power r^2) is a transformation that maps the plane into itself and sends a point X to its *inverse point* Y on the half-line PX such that $PX \cdot PY = r^2$ (see Figure 2).

Proof 2. We assume that M is in the interior of $\triangle ABC$; the case when M is not in the interior can be proved similarly. We perform the circle inversion centered at A and of radius 1, and denote by B_1, C_1, and M_1 the inverses of points B, C, and M (see Figure 3). Since A, B, C, and M are not concyclic, the points B_1, C_1, and M_1 cannot be collinear.

Therefore, $AB \cdot AB_1 = AM \cdot AM_1 = AC \cdot AC_1 = 1$. Since the angle at vertex A is common, this implies that we have the following pairs of similar triangles: $\triangle ABM \sim$

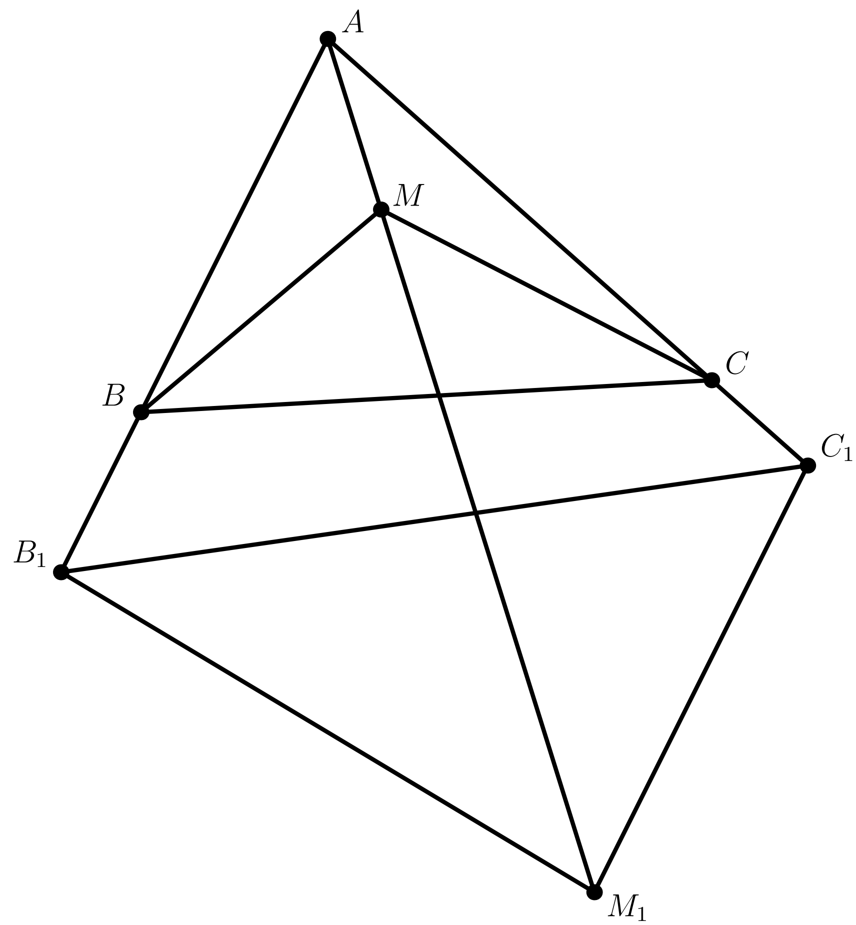

Figure 3. Inversion of ΔBCM

$\triangle AM_1B_1$, $\triangle ACM \sim \triangle AM_1C_1$, and $\triangle ABC \sim \triangle AC_1B_1$. The corresponding ratios of similarity are

$$\frac{BM}{B_1M_1} = AM \cdot AB, \frac{CM}{C_1M_1} = AM \cdot AC, \text{ and } \frac{BC}{B_1C_1} = AB \cdot AC.$$

These immediately imply that

$$\frac{B_1M_1}{BM \cdot AC} = \frac{C_1M_1}{CM \cdot AB} = \frac{B_1C_1}{AM \cdot BC} = \frac{1}{AM \cdot AB \cdot AC}.$$

This proves that $\triangle B_1M_1C_1$ is similar to a triangle having sides $BM \cdot AC$, $CM \cdot AB$, and $AM \cdot BC$. $\square$

Figure 3 assumes that the lengths AB and AC are less than 1 (the radius of inversion). All other configurations can be dealt with in a similar manner. Alternatively, one can work with a general inversion of radius r that yields the configuration of Figure 3 and adapt the argument accordingly. This remark applies also to the proof of Theorem 2 below; see also the comment immediately following the proof of this theorem.

3. Pompeiu's Area Formula

A further investigation led us to a beautifully symmetric formula that computes the area of the triangle proved to exist in Theorem 1.

Theorem 2. *Let $\triangle ABC$ be acute with side-lengths a, b, and c and area S_0, and let M be a point in its interior. Denote by S the area of a triangle having side-lengths $MA \cdot BC$, $MB \cdot CA$, and $MC \cdot AB$ and by S_1, S_2, S_3 the areas of $\triangle BCM, \triangle CAM$, and $\triangle ABM$. Then*

$$SS_0 = a^2 S_2 S_3 + b^2 S_3 S_1 + c^2 S_1 S_2.$$

Proof. We let $\sigma(XYZ)$ denote the area of $\triangle XYZ$. With the notation in the second proof of Theorem 1 (see Figure 3 again), we have

$$\sigma(AB_1M_1) = S_3 \cdot \frac{1}{AM^2 \cdot AB^2}, \sigma(AC_1M_1) = S_2 \cdot \frac{1}{AM^2 \cdot AC^2}, \text{ and}$$

$$\sigma(AB_1C_1) = S_0 \cdot \frac{1}{AB^2 \cdot AC^2}. \tag{3}$$

A triangle with sides $MA \cdot BC$, $MB \cdot CA$, and $MC \cdot AB$ is similar to $\triangle C_1B_1M_1$ with similarity ratio

$$\frac{1}{AM \cdot AB \cdot AC}.$$

Thus

$$\sigma(C_1B_1M_1) = S \cdot \frac{1}{AM^2 \cdot AB^2 \cdot AC^2}. \tag{4}$$

We show that M_1 cannot be in the interior of $\triangle AB_1C_1$. Suppose it were. Then $\angle AM_1B_1$ or $\angle AM_1C_1$ would be obtuse. Hence, by similarity, so would $\angle ABM$ or $\angle ACM$ be. This contradicts the hypothesis that $\triangle ABC$ is acute. Therefore, M_1 must fall on the exterior of $\triangle AB_1C_1$, hence $\square AB_1M_1C_1$ must be a convex quadrilateral.

Using (3) and (4) in the obvious equality

$$\sigma(AB_1M_1) + \sigma(AC_1M_1) = \sigma(AB_1C_1) + \sigma(C_1B_1M_1),$$

we obtain

$$S_3 \cdot AC^2 + S_2 \cdot AB^2 = S_0 \cdot MA^2 + S$$

or, if $u = MA \cdot BC$,

$$u^2 = \frac{a^2c^2S_3 + a^2b^2S_2 - a^2S}{S_0}.$$

Note that we can repeat the argument above by performing an inversion centered at B respectively C. Therefore, similar equalities hold for $v = MB \cdot CA$ and $w = MC \cdot AB$.

Substituting these values into Heron's formula

$$16S^2 = 2(u^2v^2 + v^2w^2 + w^2u^2) - (u^4 + v^4 + w^4),$$

and simplifying, we get the desired formula. The proof is complete. □

We remark that the hypothesis that M is in the interior of an acute triangle in Theorem 2 can be replaced with the assumption that M is in the interior of an arbitrary triangle such that $\angle AMB, \angle BMC$ and $\angle CMA$ are all obtuse. Indeed, if we perform an inversion centered at A with r greater than AB and AC, the point M_1 must fall again on the outside of $\triangle AB_1C_1$. From this point on, the proof is the same.

Clearly, Pompeiu's theorem is an immediate consequence of Theorem 1. And yet Pompeiu's name is attached to this corollary and it has become to be known as such because of his fresh approach to this old plane geometry problem. We name the following corollary of Theorem 2 after him as well.

Pompeiu's Area Formula. *Let $\triangle ABC$ be an equilateral triangle with area S_0, and let M be a point in its interior. Let S^* denote the area of a triangle having side-lengths MA, MB, and MC, and let S_1, S_2, S_3 denote the areas of $\triangle BCM, \triangle CAM$, and $\triangle ABM$. Then*

$$S^* S_0 = S_1 S_2 + S_2 S_3 + S_3 S_1.$$

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