On Efficient "Time Travel" in Gödel Spacetime*

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Abstract

Here we show that there exist closed timelike curves in Gödel spacetime with total acceleration less than $2\pi\sqrt{9+6\sqrt{3}}$. This settles a question posed by Malament (1985, 1986, 1987).

1 Introduction

Gödel spacetime [7], although not a reasonable model of our own universe, has provided wonderful insights concerning the possibilities of large-scale spacetime structure. The model is an exact solution of Einstein's equation in which the matter content is a perfect fluid [8, 11]. It contains five gobal independent Killing fields and is completely homogenous [21, 22]. Further, the model exhibits uniform, rigid rotaion [10, 17].

The causal structure of the Gödel universe is of particular interest: there exist closed timelike curves through each spacetime point [24, 19]. In addition, the model contains no spacelike hypersufaces without boundary. The exotic features of the Gödel model have been studied under a variety of pertubations [1, 23]. And recently Gödel-type models have been shown to be exact solutions of minimal supergravity in five dimensions [6, 25]. This has sparked a flurry of activity on a variety of topics (see, for example, [2, 9] and the citations there).

The literature on Gödel spacetime is vast but a number of recent historical and conceptual surveys have appeared which collect together a large subset of it [5, 15, 16, 18, 20]. Here, we will focus on a small handful of classical

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questions concerning total acceleration efficiency along closed timelike curves [3, 12, 13, 14].

The first question of interest is this: (Q1) Are there any closed timelike geodesics? In other words, can one "time travel" without accelerating? It has been known for some time that the answer is negative [4, 24]. Next, one wonders whether a would be "time traveler" can get by with arbitrarily small amounts of acceleration. Let us make this precise.

Let γ be a closed timelike curve[1] with tangent field ξ^b. Let the acceleration vector field be $\alpha^b = \xi^a \nabla_a \xi^b$ and the magnitude of acceleration be $a = (-\alpha^b \alpha_b)^{1/2}$. The *total acceleration* of γ is given by

$$TA(\gamma) = \int_\gamma a \; ds$$

where s is elapsed proper time along γ.

So the second question, posed by Chakrabarti, Geroch, and Liang [3], is this: (Q2) Is there some number $k > 0$ such that, for all closed timelike curves γ in Gödel spacetime, $TA(\gamma) \geq k$? Malament [12] showed there is indeed such a number: $\ln(2+\sqrt{5})$ will do.[2] Now, let GLB be the largest k such that, for all closed timelike curves γ in Gödel spacetime, $TA(\gamma) \geq k$. Our next question, which was posed by Malament [12, 13, 14], is the following: (Q3) What is GLB? This has yet to be settled. The smallest known value of total acceleration for a closed timelike curve in Gödel spacetime is $2\pi\sqrt{9+6\sqrt{3}}$. This means that $\ln(2+\sqrt{5}) \leq \text{GLB} \leq 2\pi\sqrt{9+6\sqrt{3}}$. But although we know that GLB falls within this range, pinning it down seems to be a somewhat difficult task.

Malament [12, 13, 14] also asked a related question: (Q4) Are there any closed timelike curves in Gödel spacetime with total acceleration less than $2\pi\sqrt{9+6\sqrt{3}}$? Malament believed there were not. To him, it seemed "overwhelmingly likely" that GLB$=2\pi\sqrt{9+6\sqrt{3}}$ but he was unable to prove the claim [14, p. 2430]. In this paper, we show that Malament's conjecture is false. Our result turns on the fact that closed timelike curves are not required to be smooth everywhere: at the initial (=terminal) point a "kink" is

[1] Timelike curves are those that are smooth everywhere unless they are closed, in which case smoothness will be allowed to fail at initial (=terminal) points [12, 14].

[2] Malament [12, p. 776] calculated that a rocket ship traversing a curve with total acceleration $\ln(2+\sqrt{5})$ must have at least 76% of its initial mass as fuel.

permitted.[3]

Physically, an observer traveling along a kinked closed timelike curve $\gamma : [s_i, s_f] \to M$ in a spacetime (M, g_{ab}) will have different initial and final velocity vectors ξ_i^a and ξ_f^a at the point $\gamma(s_i) = \gamma(s_f)$. Of course, since the time traveler cannot, at this kink point, instantaneously switch from ξ_f^a back to ξ_i^a (that would imply an infinite acceleration) this means that the trip cannot be immediately repeated. This contrasts with the smooth case where the trip may be repeated any number of times.

The set up is certainly an idealization. But if consistency worries arise regarding the kink point, the proposition below can be understood simply as the following claim: There exists some $k < 2\pi\sqrt{9 + 6\sqrt{3}}$ such that for all sufficiently small open sets O in Gödel spacetime, there exists a (smooth, non-closed) timelike curve γ which leaves O and then returns to it with $TA(\gamma) < k$. In other words, one may return arbitrarily closely to a previously visited spacetime point with less total acceleration than was known.

2 Preliminaries

Here we review some basic facts concerning Gödel spacetime (M, g_{ab}). Here the manifold M is just $\mathbb{R}^4$. The metric g_{ab} is such that for any point $p \in M$, there is a global adapted (cylindrical) coordinate system t, r, φ, y in which $t(p) = r(p) = y(p) = 0$ and

$$\begin{aligned} g_{ab} \;\; = \;\; & (\nabla_a t)(\nabla_b t) - (\nabla_a r)(\nabla_b r) - (\nabla_a y)(\nabla_b y) \\ & + j(r)(\nabla_a \varphi)(\nabla_b \varphi) + 2k(r)(\nabla_{(a} \varphi)(\nabla_{b)} t) \end{aligned}$$

where $j(r) = \sinh^4 r - \sinh^2 r$ and $k(r) = \sqrt{2} \sinh^2 r$. Here $-\infty < t < \infty$, $-\infty < y < \infty$, $0 \leq r < \infty$, and $0 \leq \varphi \leq 2\pi$ with $\varphi = 0$ identified with $\varphi = 2\pi$.

The vector field $(\frac{\partial}{\partial \varphi})^a$ is a rotational Killing field with squared norm $j(r)$. The closed integral curves of $(\frac{\partial}{\partial \varphi})^a$ (curves with constant t, r, and y values) will be called *Gödel circles*. Let r_c be such that $\sinh r_c = 1$ (so $j(r_c) = 0$). Gödel circles with radius less than r_c are closed spacelike curves. If the radius

[3] We know that Malament took the possibility of kinked closed timelike curves very seriously. At one point, he devotes a paragraph to explaining that only the possibility of kinked closed timelike curves kept him from doubling his minimal acceleration requirement [12, p. 776]. Also see [14, p. 2430]

is larger than r_c, the Gödel circles are closed timelike curves. Gödel circles with radius r_c are closed null curves. Because of the simple nature of these curves, it is fairly straightforward to calculate the total acceleration of Gödel circles as a function of r. Because these curves play a central role in our argument, we carry out the calculation here.

Lemma 1. A Gödel circle γ with radius $r > r_c$ has total acceleration $\pi \sinh 2r(2\sinh^2 r - 1)j(r)^{-1/2}$.

Proof. The unit timelike vector field for a Gödel circle or radius r is $\xi^a = j(r)^{-1/2}(\frac{\partial}{\partial\varphi})^a$. We know that $\xi^a \nabla_a j(r)^{-1/2} = 0$. So the acceleration vector $\alpha_b = \xi^a \nabla_a \xi_b$ is $j(r)^{-1}(\frac{\partial}{\partial\varphi})^a \nabla_a (\frac{\partial}{\partial\varphi})_b$. But because $(\frac{\partial}{\partial\varphi})^a$ is a Killing field, this is just $-j(r)^{-1}(\frac{\partial}{\partial\varphi})^a \nabla_b (\frac{\partial}{\partial\varphi})_a = -\frac{1}{2}j(r)^{-1}\nabla_b j(r)$. Differentiating, we have $\alpha_b = -\frac{1}{2}j(r)^{-1}\sinh 2r(2\sinh^2 r - 1)\nabla_b r$. Thus, $a(r) = (-\alpha^b\alpha_b)^{1/2} = \frac{1}{2}j(r)^{-1}\sinh 2r(2\sinh^2 r - 1)$. Next we compute $\frac{d\varphi}{ds} = \xi^a \nabla_a \varphi = j(r)^{-1/2}$. So, integrating, we have

$$TA(\gamma) = \int_\gamma a(r)\ ds = \int_0^{2\pi} a(r)j(r)^{1/2}\ d\varphi = 2\pi a(r)j(r)^{1/2}$$

So the total acceleration is $\pi \sinh 2r(2\sinh^2 r - 1)j(r)^{-1/2}$ as claimed. $\square$

Note that the total acceleration of a Gödel circle approaches infinity as $r \to r_c$ *and* as $r \to \infty$. The total acceleration is minimized when r is such that $\sinh^2 r = (1+\sqrt{3})/2$ (call this optimal radius r_o). The total acceleration of this optimal Gödel circle is $2\pi(9+6\sqrt{3})^{1/2}$.

Next, for ease of presentation, we give a list of identities that are true in Gödel spacetime.

Lemma 2. Let (M, g_{ab}) be Gödel spacetime. The following are true:

(i) $(\frac{\partial}{\partial r})^a \nabla_a (\frac{\partial}{\partial r})_b = 0$

(ii) $(\frac{\partial}{\partial\varphi})^a \nabla_a (\frac{\partial}{\partial r})_b = (\frac{\partial}{\partial r})^a \nabla_a (\frac{\partial}{\partial\varphi})_b$

(iii) $(\frac{\partial}{\partial\varphi})^a \nabla_a (\frac{\partial}{\partial\varphi})_b = -\frac{1}{2}(\frac{\partial j}{\partial r})\nabla_b r$

(iv) $(\frac{\partial}{\partial r})^a \nabla_a (\frac{\partial}{\partial\varphi})_b = (\frac{dj}{dr})\nabla_b \varphi + (\frac{dk}{dr})\nabla_b t$

Proof. We know (i) is true because $(\frac{\partial}{\partial r})^a \nabla_a (\frac{\partial}{\partial r})_b = -(\frac{\partial}{\partial r})^a \nabla_a \nabla_b r$. But because r is a scalar field, this is just $-(\frac{\partial}{\partial r})^a \nabla_b \nabla_a r$. This becomes $(\frac{\partial}{\partial r})^a \nabla_b (\frac{\partial}{\partial r})_a$ which is the zero vector.

To see why (ii) holds, note that $(\frac{\partial}{\partial r})^a \nabla_a (\frac{\partial}{\partial \varphi})_b = -(\frac{\partial}{\partial r})^a \nabla_b (\frac{\partial}{\partial \varphi})_a$ because $(\frac{\partial}{\partial \varphi})_b$ is a Killing field. But this is just $(\frac{\partial}{\partial \varphi})^a \nabla_b (\frac{\partial}{\partial r})_a$. We rewrite this as $-(\frac{\partial}{\partial \varphi})^a \nabla_b \nabla_a r$, switch the differential operators because r is a scalar field, and wind up with $-(\frac{\partial}{\partial \varphi})^a \nabla_a \nabla_b r$ which is just $(\frac{\partial}{\partial \varphi})^a \nabla_a (\frac{\partial}{\partial r})_b$ as claimed.

Because $(\frac{\partial}{\partial \varphi})^a$ is a Killing field, $(\frac{\partial}{\partial \varphi})^a \nabla_a (\frac{\partial}{\partial \varphi})_b = -(\frac{\partial}{\partial \varphi})^a \nabla_b (\frac{\partial}{\partial \varphi})_a$. But this is just $-\frac{1}{2}\nabla_b j(r) = -\frac{1}{2}(\frac{\partial j}{\partial r})\nabla_b r$ as claimed. So (iii) is true.

To see why (iv) holds, consider the following. $\nabla_a (\frac{\partial}{\partial \varphi})_b = \nabla_{[a} (\frac{\partial}{\partial \varphi})_{b]}$ because $(\frac{\partial}{\partial \varphi})^a$ is a Killing field. So we can rewrite this with the exterior derivative operator as $d_a (\frac{\partial}{\partial \varphi})_b$. This is the same as $d_a (j \nabla_b \varphi + k \nabla_b t)$. But this is just $\nabla_a j \nabla_b \varphi + \nabla_a k \nabla_b t$. Differentiating, we have $(\frac{dj}{dr})\nabla_a r \nabla_b \varphi + (\frac{dk}{dr})\nabla_a r \nabla_b t$. So $(\frac{\partial}{\partial r})^a \nabla_a (\frac{\partial}{\partial \varphi})_b = (\frac{dj}{dr})\nabla_b \varphi + (\frac{dk}{dr})\nabla_b t$ as claimed. □

Let S be any submanifold of M on which $t = \text{const}$ and $y = \text{const}$. In this paper, we will be concerned only with closed timelike curves which are contained entirely within S. We now find an expression for the magnitude of acceleration of this limited class of curves.

Lemma 3. Let $\xi^a = f(r,\varphi)(\frac{\partial}{\partial \varphi})^a + g(r,\varphi)(\frac{\partial}{\partial r})^a$ be the unit tangent to some curve $\gamma : I \to S$. Then the acceleration $a(r,\varphi)$ at a point on $ran[\gamma]$ is

$$[-f^2(\frac{\partial f}{\partial \varphi})^2 j - 2f(\frac{\partial f}{\partial \varphi})g(\frac{\partial f}{\partial r})j - 4f^2(\frac{\partial f}{\partial \varphi})g(\frac{dj}{dr}) + \frac{1}{4}f^4(\frac{dj}{dr})^2 + f^3(\frac{\partial g}{\partial \varphi})(\frac{dj}{dr})$$
$$+f^2 g(\frac{\partial g}{\partial r})(\frac{dj}{dr}) + f^2(\frac{\partial g}{\partial \varphi})^2 + 2fg(\frac{\partial g}{\partial r})(\frac{\partial g}{\partial \varphi}) - g^2(\frac{\partial f}{\partial r})^2 j - 4g^2(\frac{\partial f}{\partial r})(\frac{dj}{dr})f$$
$$+g^2(\frac{\partial g}{\partial r})^2 + 4(\frac{dj}{dr})^2 g^2 f^2 m - 8(\frac{dj}{dr})g^2 f^2(\frac{dk}{dr})km + 4(\frac{dk}{dr})^2 g^2 f^2 jm]^{1/2}$$

where $m(r) = 1/(\sinh^4 r + \sinh^2 r)$.

Proof. Let ξ^a be as above. Consider the acceleration vector $\alpha_b = \xi^a \nabla_a \xi_b$:

$$\alpha_b = [f(\frac{\partial}{\partial \varphi})^a + g(\frac{\partial}{\partial r})^a][(\nabla_a f)(\frac{\partial}{\partial \varphi})_b + f\nabla_a(\frac{\partial}{\partial \varphi})_b$$
$$+(\nabla_a g)(\frac{\partial}{\partial r})_b + g\nabla_a(\frac{\partial}{\partial r})_b].$$

By (i) and (ii) of Lemma 2 and direct computation, we know that α_b becomes

$$f\frac{\partial f}{\partial\varphi}(\frac{\partial}{\partial\varphi})_b + f^2(\frac{\partial}{\partial\varphi})^a\nabla_a(\frac{\partial}{\partial\varphi})_b + f\frac{dg}{d\varphi}(\frac{\partial}{\partial r})_b + g\frac{\partial f}{\partial r}(\frac{\partial}{\partial\varphi})_b$$
$$+2fg(\frac{\partial}{\partial r})^a\nabla_a(\frac{\partial}{\partial\varphi})_b + g(\frac{\partial g}{\partial r})(\frac{\partial}{\partial r})_b \qquad .$$

Let $m(r) = 1/(\sinh^4 r + \sinh^2 r)$. Now we compute $a = (-\alpha_b\alpha^b)^{1/2}$. By (iii) and (iv) of Lemma 2 and direct computation, we have our result.[4] $\square$

3 Result

In this section we present our result. It will be useful to have a general idea of how we will go about proving our claim. Eventually, we seek to answer (Q3) by showing there exists a curve in Gödel spacetime with total acceleration less than $2\pi\sqrt{9+6\sqrt{3}}$. We will do this by considering the behavior of a particular curve $\gamma : I \to S$ contained entirely in the submanifold S.

We can think of γ as three separate curves joined together. From $0 \leq \varphi \leq \epsilon$ for some ϵ the curve γ makes its way from the point $(r_o, 0)$ to (r_ϵ, ϵ) where $r_c < r_\epsilon < r_o$. We will call this portion of the curve γ_1. From $\epsilon \leq \varphi \leq 2\epsilon$, γ makes its way from the point (r_ϵ, ϵ) to $(r_o, 2\epsilon)$. This portion of the curve we will call γ_2. From $2\epsilon < \varphi < 2\pi$, γ is simply the optimal Gödel circle of radius r_o. We call this portion of the curve γ_3. We are careful to make the three portions of γ join together smoothly except at the point (r_ϵ, ϵ). Thus, at this point, there will be a "kink" and so we stipulate that this will be the initial (and therefore the final) point of the closed timelike curve (see Figure 1).

The basic structure of our proof is simple. We show that along γ, (a) the acceleration of γ_1 is always decreasing (from the constant acceleration of the optimal Gödel circle) and (b) the acceleration of γ_2 is always increasing (up to the constant acceleration of the optimal Gödel circle). With this information we can integrate along γ to show that the total acceleration from $0 \leq \varphi \leq 2\epsilon$ is less than the total acceleration of the optimal Gödel circle from $0 \leq \varphi \leq 2\epsilon$. Because the total acceleration of γ_3 just is that of the optimal Gödel circle from $\varphi = 2\epsilon$ to $\varphi = 2\pi$, we have our result.

[4] It is helpful during the calculation to have the inverse to g_{ab}. It is given by $g^{ab} = -j(r)m(r)(\frac{\partial}{\partial t})^a(\frac{\partial}{\partial t})^b - (\frac{\partial}{\partial r})^a(\frac{\partial}{\partial r})^b - (\frac{\partial}{\partial y})^a(\frac{\partial}{\partial y})^b - m(r)(\frac{\partial}{\partial\varphi})^a(\frac{\partial}{\partial\varphi})^b + 2k(r)m(r)(\frac{\partial}{\partial\varphi})^{(a}(\frac{\partial}{\partial t})^{b)}$. See [12, p. 777].

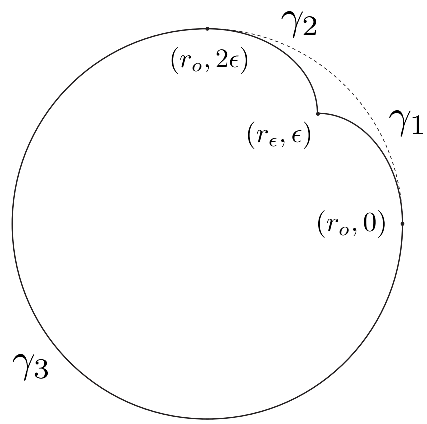

Figure 1: *The three portions of the kinked curve* γ *which has total acceleration less than* $2\pi\sqrt{9+6\sqrt{3}}$.

Proposition. There exists a closed timelike curve in Gödel spacetime with total acceleration less than $2\pi\sqrt{9+6\sqrt{3}}$.

Proof. The first step is to define our curve. Consider the vector field $\xi^a(r,\varphi) = f(r,\varphi)(\frac{\partial}{\partial\varphi})^a + g(\varphi)(\frac{\partial}{\partial r})^a$ defined for all values of $r > r_c$ and on some interval $[0,\epsilon]$ of φ. Let $f(r,\varphi) = j(r)^{-1/2}h(\varphi)$ where $h(\varphi) = (1+e^{-2/\varphi})^{1/2}$. Let $g(\varphi) = -e^{-1/\varphi}$ For continuity considerations later, let $h(0) = 1$ and $g(0) = 0$. Clearly, ξ^a is a unit timelike vector field. Now, for some interval $I \subseteq \mathbb{R}$, let $\gamma_1 : I \to S$ be such that its tangent vector at each point is ξ^a and $(r_o, 0) \in ran[\gamma_1]$ (i.e. γ_1 is an integral curve of ξ^a).

We have also chosen ξ^a to be such that at $\varphi = 0$, it joins smoothly with $j(r)^{-1/2}(\frac{\partial}{\partial\varphi})^a$ (the unit tangent field associated with Gödel circles). Finally, we note two important facts concerning our functions f and g. The first is a relationship between f and g and their derivatives with respect to φ. The second states that as φ approaches zero from above, g and $dg/g\varphi$ both go to zero more quickly than $d^2g/d\varphi^2$. These facts will play a crucial role in our argument. They are easily verifiable and so we present them here without any proof:

(1) $f\frac{\partial f}{\partial\varphi} = j^{-1}g\frac{dg}{d\varphi}$

(2) $\lim_{\varphi\to 0^+} g/\frac{d^2g}{d\varphi^2} = \lim_{\varphi\to 0^+} \frac{dg}{d\varphi}/\frac{d^2g}{d\varphi^2} = 0$

Let a_1 be the magnitude of acceleration at any point on $ran[\gamma_1]$. Next, consider the expression $\xi^b\nabla_b a_1 = f\frac{\partial a_1}{\partial\varphi} + g\frac{\partial a_1}{\partial r}$. This is the rate of change of the magnitude of the acceleration in the direction of ξ^b. The claim is that this quantity will be negative when evaluated at points $(r,\varphi) \in ran[\gamma_1]$ very close to $(r_o, 0)$. We can differentiate the expression for a given in Lemma 3 to find that $f\frac{\partial a_1}{\partial\varphi} + g\frac{\partial a_1}{\partial r}$ is a (very long) string of terms. Using the identity (1) we can rewrite the string of terms such that all the terms *except for one* contain, as a factor, either g or $\frac{dg}{d\varphi}$. We choose the one exception to be the term $\frac{1}{2}a_1^{-1}f^4(\frac{d^2g}{d\varphi^2})(\frac{dj}{dr})$ (call this term ω). The following can also be verified:

(3) *All* of the terms in $f\frac{\partial a_1}{\partial\varphi} + g\frac{\partial a_1}{\partial r}$ approach zero as the point $(r_o, 0)$ is approached.

(4) *All* of the various factors of the terms approach real numbers as the point $(r_o, 0)$ is approached (none of them "blow up").

(5) ω goes to zero as $\frac{d^2g}{d\varphi^2}$ does.

We know that (2)-(5) imply that, as the point $(r_o, 0)$ is approached, ω becomes the dominate term (it goes to zero slower than any term containing g or $\frac{dg}{d\varphi}$). To illustrate this, we can pick any term in $\xi^b\nabla_b a_1$ (other than ω) and show that it must go to zero faster than ω as the point $(r_o, 0)$ is approached. Take, for example, the term $-\frac{1}{2}a_1^{-1}f^2(\frac{\partial f}{\partial\varphi})^3 j$. This is one of the terms that results in taking the partial derivative (with respect to φ) of the first term in the expression for a_1 in Lemma 3 and multiplying by f. Using (1) we can rewrite this term as $-\frac{1}{2}a_1^{-1}f(\frac{\partial f}{\partial\varphi})^2 g(\frac{\partial g}{\partial\varphi})$. We know that as $(r_o, 0)$ is approached, a_1 goes to some positive real number (the acceleration of the optimal Gödel circle). Similarly, f approaches some positive real number (the number is $j(r_o)^{-1/2}$). The remaining three factors all go to zero as $(r_o, 0)$ is approached. So, the entire term approaches zero. How fast does it go? We know it must go at least as fast as any one of the factors. So, it must go at least as fast as g. But now consider ω. We know that it goes to zero as $\frac{d^2g}{d\varphi^2}$ does. We also know that ω must go to zero slower than the example term that we picked (it dominates the term as the point (r_o) is approached).

The claim is that if we repeated this process and compared all the terms in $\xi^b \nabla_b a_1$, ω would dominate them all.

What is the behavior of ω near $(r_o, 0)$? It is negative. So, there exists an ϵ_1 such that for all $\varphi \in (0, \epsilon_1]$, $\xi^b \nabla_b a_1 < 0$ (moving along γ_1 away from $(r_o, 0)$ the value of acceleration decreases).

Now we define another curve γ_2. Pick any point (r_o, δ) in the optimal Gödel circle. Let $f'(r, \varphi) = j(r)^{-1/2} h'(\varphi)$ where $h'(\varphi) = (1+e^{-2/(\delta-\varphi)})^{1/2}$ and $g'(\varphi) = e^{-1/(\delta-\varphi)}$ (for continuity considerations, let $h'(\delta) = 1$ and $g'(\delta) = 0$). Let $\eta^a = f'(\frac{\partial}{\partial\varphi})^a + g'(\frac{\partial}{\partial r})^a$ and let $\gamma_2 : I' \to M$ be such that its tangent vector at each point is η^a and $(r_o, \delta) \in ran[\gamma_2]$. Note that for all points (r, φ) where $0 \geq \varphi \geq \delta$ we have $f'(r, \delta - \varphi) = f(r, \varphi)$ and $g'(\delta - \varphi) = -g(\varphi)$. Thus, under that same interval of φ, it is the case that $ran[\gamma_1]$ is the mirror image of $ran[\gamma_2]$ across the line of symmetry $\varphi = \delta/2$.

Let a_2 be the magnitude of acceleration for any point on γ_2. By an argument very similar to the one made above for γ_1 we can establish that there exists some ϵ_2 such that for all $\varphi \in [\epsilon_2, \delta)$, $\eta^b \nabla_b a_2 > 0$ (moving along γ_2 toward (r_o, δ) the value of acceleration increases). Let $\epsilon = min\{\epsilon_1, \delta - \epsilon_2\}$. Because δ was arbitrarily chosen and because $\epsilon \leq \delta - \epsilon_2$, we know (if we let $\delta = 2\epsilon$) that for all $\varphi \in [\epsilon, 2\epsilon)$, $\eta^b \nabla_b a_2 > 0$. Of course, because $\epsilon \leq \epsilon_1$ we know that for all $\varphi \in (0, \epsilon]$, $\xi^b \nabla_b a_1 < 0$.

Let $\gamma_3 : I'' \to S$ be that portion of the optimal Gödel circle from $\varphi = 2\epsilon$ to $\varphi = 2\pi$. Let γ be such that $ran[\gamma] = ran[\gamma_1] \cup ran[\gamma_2] \cup ran[\gamma_3]$.

Now we integrate. We reparametrize a_1 along γ_1 so that it is only a function of φ. Next, note that $\frac{d\varphi}{ds}$ for γ_3 is $j(r)^{-1/2}$ while $\frac{d\varphi}{ds}$ for γ_1 is $j(r)^{-1/2} h(\varphi)$. We also reparametrize $j(r)$ along γ_1 so that it is a function of φ. Since along γ_1, $j(\varphi)^{1/2} \leq j(0)^{1/2}$ and $h(\varphi) \geq 1$ we may conclude that

$$\int_0^\epsilon a_1(\varphi) j(\varphi)^{1/2} h(\varphi)^{-1} d\varphi \leq j(0)^{1/2} \int_0^\epsilon a_1(\varphi) d\varphi.$$

Let $a_3(\varphi)$ be the acceleration at any point in the optimal Gödel circle. Because $\xi^b \nabla_b a_1 < 0$ along γ_1, for all $0 < \varphi \leq \epsilon$, we know that $a_3(\varphi) > a_1(\varphi)$ over that same interval and, of course, $a_1(0) = a_3(0)$. From Lemma 1 we know that the total acceleration of the optimal Gödel circle over this interval is $\epsilon(9 + 6\sqrt{3})^{1/2}$. So, we have

$$j(0)^{1/2} \int_0^\epsilon a_1(\varphi) d\varphi < j(0)^{1/2} \int_0^\epsilon a_3(\varphi) d\varphi = \epsilon(9 + 6\sqrt{3})^{1/2}.$$

So, we have

$$TA(\gamma_1) = \int_0^{\epsilon} a_1(\varphi) j(\varphi)^{1/2} h(\varphi)^{-1} d\varphi < \epsilon(9 + 6\sqrt{3})^{1/2}.$$

A similar argument establishes that for γ_2, we have

$$TA(\gamma_2) = \int_{\epsilon}^{2\epsilon} a_2(\varphi) j(\varphi)^{1/2} h(\varphi)^{-1} d\varphi < \epsilon(9 + 6\sqrt{3})^{1/2}.$$

Finally, for γ_3 we have

$$TA(\gamma_3) = \int_{2\epsilon}^{2\pi} a_3(0) j(0)^{1/2} d\varphi = (2\pi - 2\epsilon)(9 + 6\sqrt{3})^{1/2}.$$

So we may conclude that

$$TA(\gamma) = TA(\gamma_1) + TA(\gamma_2) + TA(\gamma_3) < 2\pi(9 + 6\sqrt{3})^{1/2}.$$

Thus, there exists a closed timelike curve in Gödel spacetime with total acceleration less that $2\pi(9 + 6\sqrt{3})^{1/2}$. $\square$

4 Conclusion

So, we have answered (Q4) concerning closed timelike curves in Gödel spacetime. We have shown there exists a curve (and therefore a family of curves) with total acceleration less than $2\pi\sqrt{9 + 6\sqrt{3}}$. It is uncertain if a curve of the type we have proposed will actually approach GLB. As previously mentioned, (Q3) remains open.

In addition to finding the answer to (Q3), other work remains. We wonder if the result presented here applies to other spacetimes, such as Kerr-Newman, which also contain closed timelike curves but no closed timelike geodesics [26]. The more general question is this: (Q5) In any spacetime, if there exists a smooth non-geodesic closed timelike curve γ, does there also exist a kinked closed timelike curve γ' such that $TA(\gamma') < TA(\gamma)$? The following is an argument sketch in support of an affirmative answer.

Consider any spacetime (M, g) with a smooth non-geodesic closed timelike curve $\gamma : [s, s'] \rightarrow M$. We know there will be a point p in $ran[\gamma]$ such that the scalar acceleration of γ at p is non-zero. Smoothness conditions near p

guarantee that there will be some convex normal neighborhood O of p such that the scalar acceleration of γ restricted to O is everywhere non-zero. But, within any convex normal neighborhood, any two points may be connected by a unique geodesic contained in O. So let r and r' be such that $s < r < r' < s'$ and $\gamma(r)$ and $\gamma(r')$ are in O. Now let γ' be the closed timelike curve whose image is exactly the same as the image of γ except for the portion running from $\gamma(r)$ to $\gamma(r')$. Let the the image of γ' between those points be identical to the unique geodesic connecting them. The resulting (unphysical) curve has *two* kinks but is clearly such that $TA(\gamma') < TA(\gamma)$. However, it might be possible to "smooth out" one of the kinks while maintaing the result.

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