The One and Only True Monty Hall Paradox

Richard D. Gill*

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Abstract

Short rigorous solutions to three mathematizations of the famous Monty Hall problem are given: asking for an unconditional probability, a conditional probabiliity, or for a game theoretic strategy. It is concluded which mathematicization ought to be considered as the Only True Solution of the True Monty Hall Problem: the little known Game Theoretical version.

Introduction

The famous Monty Hall Problem as quoted by Marilyn vos Savant in her "Ask Marilyn" column in *Parade* magazine (p. 16, 9 September 1990), as a problem posed to her by a correspondent Mr. Craig Whitaker, was the following:

> Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

The literature on this problem is enormous, and at the end of this paper I simply cite the references which for me have been most valuable: the 2008 paper by Jeff Rosenthal and the 2009 book by Jason Rosenhouse. The latter has a huge reference list and discusses the pre-history and the post-history of vos Savant's problem, as well as many variants. Aside from these two references, the wikipedia discussion pages (on English and on Dutch wikipedia)

*Mathematical Institute, Leiden University; http://www.math.leidenuniv.nl/∼gill

are a fabulous though every-changing resource, and everything that I write here was learnt from those pages.

Despite making homage here to both cited authors for their wonderful work, I emphasize that I strongly disagree with both Rosenhouse ("the canonical problem") and Rosenthal ("the original problem") on what the essential Monty Hall problem is. I am more angry with certain other authors, who will remain nameless but for the sake of argument I'll just call Morgan et al, for unilaterally declaring their Monty Hall problem to be the only possible sensible problem, for arrogantly calling everyone who solved different problems stupid, and for getting an incorrect theorem published in the peer reviewed literature.

But: deciding unilaterally (Rosenhouse, 2009) that a certain formulation is *canonical* is asking for a schism and an excommunication. And calling a version *original* (Rosenthal, 2008) without historical research is asking for a contradiction (in view of the pre-history of the problem, the notion is not well defined). Monty Hall is part of folk-culture, culture is alive, the Monty Hall problem is not owned by a particular kind of mathematician who looks at such a problem from a particular point of view, adding for them "natural" extra assumptions which merely have the role of allowing their solution to work.Thus any canonical Monty Hall problem is actually an example of *solution driven science* — you have learnt a clever trick and want to show that it solves lots of problems.

I will distinguish three different Monty Hall problems. One of them is simply to answer the question literally posed by Marilyn vos Savant, "would you switch?". The other two are popular mathematizations, particularly among experts or teachers of elementary probability theory: one asks for the unconditional probability that switching gets the car, the other asks for a conditional probability, given the choices made so far:

Q-0: Marilyn vos Savant's question "should you switch?"

Q-1: A mathematician's question "what is the unconditional probability that switching gives the car?"

Q-2: A mathematician's question "what is the conditional probability that switching gives the car?"

The free internet (and freely editable) encyclopedia Wikipedia is the scene of a furious debate as to which mathematization Q-1 or Q-2 is the right starting point for answering the original verbal question Q-0; and after that, what supplementary conditions are obviously implicitly being made (the assumptions which ensure that the question has a nice answer). My own humble opinion is "neither, though the unconditional approach comes closer; and no supplementary conditions should be made".

My analysis of the mathematical problems yields the good answer "2/3" for both unconditional and conditional probabilities, under minimal assumptions, and almost without computation or algebraic manipulation. I use Boris Tsirelson's proposal on the wikipedia talk pages to use symmetry to deduce the conditional probability from the unconditional one. (Boris graciously gave me permission to cite him here, but this should not be interpreted to mean that anything written here also has his approval). I finally use a game-theoretic point of view to answer the the original question posed by Marilyn vos Savant: if you were the player, should you switch doors? I make one supplementary assumption: that you know game theory, and therefore in advance have randomized your choice of door.

Let the three doors be numbered in advance 1, 2, and 3. Introduce four random variables taking values in the set of door-numbers $\{1, 2, 3\}$:

C: the quiz-team hides the Car behind door C,

P: the Player chooses door P,

Q: the Quizmaster opens door Q,

S: he asks the player if she'ld like to Switch to door S.

Because of the standard story of the Monty Hall show, we certainly have:

$Q \neq P$, the quizmaster opens a door different to the player's first choice,

$Q \neq C$, opening that door reveals a goat,

$S \neq P$, the player is invited to switch to another door,

$S \neq Q$, no player wants to go home with a goat.

It does not matter for mathematical analysis whether probabilities are subjective (Bayesian) or objective (frequentist); nor does it matter whose probabilities they are supposed to be, at what stage of the game. Some writers think of the player's initial choice as fixed. For them, P is degenerate. I simply write down some assumptions and deduce consequences of them.

1 Solution to problem 1: unconditional chance that switching wins

By the rules of the game and the definition of S, if $P \neq C$ then $S = C$, and vice-versa. A "switcher" would win the car if and only if a "stayer" would lose it. Therefore:

> *If* $\text{Prob}(P = C) = 1/3$ *then* $\text{Prob}(S = C) = 2/3$*, since the two events are complementary.*

2 Solution to problem 2: conditional chance that switching wins

Take the door chosen by the player as fixed, $P \equiv x$, say. We are to compute $\text{Prob}(S = C|Q = y)$ for a further specific value $y \neq x$. Let y' denote the remaining door-number, besides x and y. *Assume that all doors are equally likely to hide the car and assume that the quizmaster chooses completely at random when he has a choice*:C is uniform; and the distribution of Q given C is uniform on the available possibilities. From the first assumption

$$\text{Prob}(P = C) \;=\; \text{Prob}(C = x) = 1/3$$

and therefore by the solution to Problem 1, $\text{Prob}(S = C) = 2/3$. This unconditional probability is the weighted average of the two conditional probabilities $\text{Prob}(S = C|Q = y)$ and $\text{Prob}(S = C|Q = y')$, weighted by the unconditional probabilities $\text{Prob}(Q = y)$ and $\text{Prob}(Q = y')$. The distribution of C is unchanged on exchanging y and y'. Regarding the set of conditional laws of Q given each value of C,

$$\text{Prob}(Q = y|C = x) \;=\; \text{Prob}(Q = y'|C = x) \;=\; 1/2,$$

$$\text{Prob}(Q = y'|C = y) \;=\; \text{Prob}(Q = y|C = y') \;=\; 1.$$

This set of conditional laws is symmetric on interchange of y and y'. Finally, regarding the set of conditional laws of S given C and Q, S is simply defined as the unique door different from Q and x. This definition too is symmetric under exchange of y and y'.

> *If P is fixed, C is uniform, and Q is symmetric, then that switching gives the car is independent of the quizmaster's choice:*

$$\text{Prob}(S = C | Q = y) \ = \ \text{Prob}(S = C | Q = y') \ = \ \text{Prob}(S = C) \ = \ 2/3.$$

Instead of starting with fixed P, one can *without loss of generality pretend* that is random and uniform, and add the uniformity of C given P and that of Q given C and P. The joint law of C, P, Q, S is now invariant under renumberings of the three doors. Hence $\text{Prob}(S = C | P = x, Q = y)$ is the same for all $x \neq y$ and hence equal to the unconditional $\text{Prob}(S = C) = 2/3$.

3 Answer to Marylin vos Savant: would I switch doors?

Yes. Recall, *I only know that Monty Hall always opens a door revealing a goat.* I didn't know what strategy the quiz-team and quizmaster were going to use for their choices of the distribution of C and the distribution of Q given P and C, so naturally I had picked my door uniformly at random. My strategy of choosing C uniformly at random guarantees that $\text{Prob}(C = P) = 1/3$ and hence that $\text{Prob}(S = C) = 2/3$.

In fact, I know elementary game theory as well as elementary probability theory, so it was easy for me to find out that this combined strategy, which I'll call "symmetrize and switch", is my minimax strategy.

On the one hand, "symmetrize and switch" guarantees me a 2/3 (unconditional) chance of winning the car, whatever strategy used by the quizmaster and his team.

On the other hand, if the quizmaster and his team use their "symmetric" strategy "hide the car uniformly at random and toss a fair coin to open a door if there is choice", then I cannot win the car with a *better* probability than 2/3.

The fact that my "symmetrize and switch" strategy gives me "at least" 2/3, while the quizmaster's "symmetry" strategy prevents me from doing better, proves that these are our respective minimax strategies, and 2/3 is the game-theoretic value of this two-party zero-sum game.

There is not much point for me in worrying about my conditional probability of winning given my specific initial choice and the specific door opened by the quizmaster, say doors 1 and 3 respectively. I don't know this conditional probability anyway, since I don't know the strategy used by quiz-team and the quizmaster. (Even though I know probability theory and game theory, they maybe don't). However, it is maybe comforting to learn, by easy calculation, that if the car is hidden uniformly at random, then the conditional probability cannot be *smaller* than 1/2. So in that case at least, it certainly never *hurts* to switch door.

4 Discussion

Above I tried to give short clear mathematical solutions to three mathematical problems. Two of them were problems of elementary probability theory, the third is a problem of elementary game theory. As such, it involves not much more than elementary probability theory and the beautiful minimax theorem of John von Neumann (1929). That a finite two-party zero-sum game has a saddle-point, or in other words, that the two parties in such a game have matching minimax strategies (if randomization is allowed), is in my opinion not obvious, but its proof is not especially difficult. It seems to me that probabilists ought to know more about game theory, since every ordinary non-mathematician who hears about the problem starts to wonder whether the quiz-master is trying to cheat the player, leading to an infinite regress: if I know that he knows that I know that....

It am told that the literature of mathematical economics and of game theory is full of Monty Hall examples, but no-one can give me a nice reference to a nice game-theoretic solution of the problem. Probably game-theorists like to keep their clever ideas to themselves, so as to make money from playing the game. Only losers write books explaining how the reader could make money from game theory.

Then there is a sociological or historical question: who "owns" the Monty Hall problem? I think the answer is obvious: no-one. A beautiful mathematical paradox, once launched into the real world, lives it own life, it evolves, it is re-evaluated by generation after generation. This point of view actually makes me believe that Question 0: *would you switch* is the right question, and no further information should be given beyond the fact that you know that the quizmaster knows where the car is hidden, and always opens a door exhibiting a goat. Question 0 is a question you can ask a non-mathematician at a party, and if they have not heard of the problem before, they'll give the wrong answer (or rather, one of the two wrong answers: *no* because nothing is changed, or *it doesn't matter* because it's now 50–50). My mother, who was one of Turing's computers at Bletchley Park during the war, but who had had almost no schooling and in particular never learnt any mathematics, is the only person I know who immediately said: *switch*, by immediate intuitive consideration of the 100-door variant of the problem. The problem is a *paradox* since you can next immediately convince anyone (except lawyers, as was shown by an experiment in Nijmegen), that their initial answer is wrong.

The mathematizations Questions 1 and 2 are not (in my humble opinion!) *the* Monty Hall problem; they are questions which probabilists might ask, anxious to show off Bayes' theorem or whatever. Some people intuitively try to answer Question 0 via Questions 1 and 2; that is natural, I do admit. And sometimes people become very confused when they realize that the answer to Question 2 can only be given its pretty answer "2/3" under further conditions. It is interesting how in the pedagogical mathematical literature, the further conditions are as it were held under your nose, e.g. by saying "three *identical* doors", or replacing Marilyn's "say, door 1" by the more emphatic "door 1".

It seems to me that adding into the question explicitly the remarks that the three doors are equally likely to hide the car, and that when the quizmaster has a choice he secretly tosses a fair coin to decide, convert this beautiful paradox into a dull probability puzzle, which any right-minded amateur wouldn't *want* to solve.

It also converts the problem into one version of the three prisoner's paradox. The three prisoners problem is isomorphic to the conditional probabilistic three doors problem. I always thought of it as a silly problem, and my probability students always found it silly too.

Marilyn vos Savant's question is actually ambigious. Are the mentioned door numbers the actual door numbers painted on front of the doors in advance, or are we just for convenience *naming* the doors by the choices of the actors in our game. It turned out that Marilyn was actually thinking of the latter; but she was not entirely self-consistent in her solutions to her problem.

This little paper containing nothing new, and only almost trivial mathematics, is a plea for future generations to preserve the life of The True Monty Hall paradox, and not let themselves be misled by probability purists who say "you *must* compute a conditional probability".

References

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