This book is dedicated to Prabir Roy, Louis McAuley, Jonathan Seldin, Anil Nerode, and Maurice Boffa, my teachers, and to W. V. O. Quine and R. B. Jensen, the founders of this style of set theory.

Cahiers du Centre de logique
Volume 10

Contents

Cahiers du Centre de logique
Volume 10

Chapter 1

Introduction: Why Save the Universe?

This book is intended for one of two uses. It could be used as an introduction to set theory. It is roughly parallel in structure to Halmos's classic *Naive Set Theory*, though more topics have been added. The book contains exercises in most chapters, in line with its superficial character of being an elementary set theory text, but no representation as to pedagogical soundness is made. Some of the exercises are very hard (these are marked).

The other possible use of the book is to demonstrate the naturalness and effectiveness of an alternative set theory, Jensen's corrected version NFU of W. V. O. Quine's system "New Foundations" (NF), so-called because it was proposed in Quine's paper *New foundations for mathematical logic* in 1937. No introduction to set theory based on Quine's approach has appeared (to my knowledge) since J. B. Rosser's *Logic for Mathematicians*, which came out in 1953 (second edition 1978). As our title implies, NFU is a set theory in which there is a universal set.

Quine's "New Foundations" has a bad reputation. Its consistency relative to ZFC remains an open question. In 1953 (immediately after Rosser published his book *Logic for Mathematicians*, which was based on "New Foundations"), E. Specker proved that "New Foundations" proves the negation of the Axiom of Choice! No one has been able to derive a contradiction from NF, but the failure of AC makes it an unfriendly theory to work in.

None of this should be allowed to reflect on NFU. NFU is known to be consistent since the work of Jensen in 1969. It can be extended with the Axiom of Infinity (this is implicit in our Axiom of Projections). It is consistent with the Axiom of Choice. It can be extended with more powerful axioms of infinity in essentially the same ways that ZFC can be extended; we introduce an Axiom of Small Ordinals which results in a theory at least as strong as ZFC (actually considerably stronger, as has been shown recently by Robert Solovay). We believe that if Jensen's result had been given before Specker's, the subsequent history of interest in this kind of set theory might have been quite different.

These considerations are not enough to justify the use of NFU instead of ZFC. What positive advantages do we claim for this approach? The reason that we believe that NFU is a good vehicle for learning set theory is that it allows most of the natural constructions of genuinely "naive" set theory, the set theory of Frege with unlimited comprehension (it can be claimed that Cantor's set theory always incorporated "limitation of size" in some form, even before it was formalized). The universe of sets is actually a Boolean algebra (there is a universe; sets have complements). Finite and infinite cardinal numbers can be defined as equivalence classes under equipotence, following the original ideas of Cantor and Frege. Ordinal numbers can be defined as equivalence classes of well-orderings under similarity. The objects which cause trouble in the paradoxes of Cantor and Burali-Forti (the cardinality of the universe and the order type of the ordinals) actually exist in NFU, but do not have quite the expected properties. Many interesting large classes are actually sets: the set of all groups, the set of all topological spaces, etc. (most categories of interest, actually). The reason that the paradoxes are avoided is a restriction on the axiom of comprehension. These paradoxes, and the paradox of Russell, are discussed in the text.

The restriction on the axiom of comprehension which NFU shares with "New Foundations" is not very easy to justify to a naive audience, unless one starts with a presentation of the Theory of Types. We do not introduce NFU in this way; we use a finite axiomatization (Hailperin first showed that NF is finitely axiomatizable) to introduce NFU in a way quite analogous to the way that ZFC is usually introduced, as allowing the construction of sets and relations using certain basic operations. The basic operations used are quite intuitive. Set-builder notation is introduced early, and the impossibility of unrestricted comprehension is pointed out in the usual way using Russell's paradox. After the arsenal of basic constructions is assembled, the schema of stratified comprehension is proved as a theorem.

Another point for those familiar with the usual treatment of NF and NFU is that we use notation somewhat more similar to the notation usually used in set theory, avoiding some of the unusual notation going back to Rosser or Principia Mathematica which has become traditional in NF and related theories. Our notation does have some eccentricities, which are discussed in the section titled "Parentheses, Braces and Brackets" (p. 76).

The usual set theory of Zermelo and Fraenkel is not entirely neglected; there is an introduction to the usual set theory as an alternative, motivated in the context of NFU by a study of the isomorphism types of well-founded extensional relations, in section 19.2.

There is a study of somewhat more advanced topics in set theory at the end, including the proof of Robert Solovay's theorem that the existence of inaccessible cardinals follows from our Axiom of Small Ordinals. There is also a discussion of the analogous system of "stratified λ-calculus" in which the notion of *function* rather than the notion of *set* is taken as primitive.

All references are deferred to the section of Notes at the end (Chapter 24, p. 217).

Exercise

Find and read Quine's original paper *New foundations for mathematical logic*.

Cahiers du Centre de logique
Volume 10

Chapter 2

The Set Concept. Extensionality. Atoms

If the reader glances at the Introduction, he may expect an unusual treatment of set theory and its use as a foundation for mathematics in these pages. If our premise is correct, this will not be the case. The basic fact that mathematics is here founded on the undefined concepts of *set* and *membership* is unchanged. The notion of *ordered pair* is also treated as primitive, but we will indicate how it could be defined using the set concept. The techniques which are used to achieve this end differ only in technical detail from the techniques used in a more familiar treatment. We believe that the constructions given here are if anything more natural than the traditional constructions; that is why this book was written.

What is a set? This is not a question that we will answer directly. "Set" is an undefined notion. The reader will have to count on her intuitive notion of what a "set" or "collection" is at the start. She may find that some of the properties of the set concept that we develop here will be somewhat unfamiliar.

A set or collection has members. The basic relationship between objects in our theory, written $a \in b$, can be translated "a is an element of b" or "a is a member of b" or "a belongs to b". One way in which it should *not* be translated is "a is a part of b". The subset or inclusion relation, to be introduced later, is a better translation for the intuitive relation of part to whole. This is the major intuitive pitfall with set theory; a set does not have its elements as parts. An important difference between the membership relation and the relation of part to whole is that the latter is transitive but the former is not: if A is a part of B and B is a part of C, then A is a part of C; but it is not the case that given $A \in B$ and $B \in C$, $A \in C$ follows. We will see counterexamples to this shortly.

A set is exactly determined by its members. This can be summarized in our first axiom:

Axiom of Extensionality. *If A and B are sets, and for each x, x is an element of A if and only if x is an element of B, then $A = B$.*

The Axiom of Extensionality can be paraphrased in more colloquial English: "Sets with the same elements are the same".

Not all objects in our universe are sets. Objects which are not sets are called "atoms". You can think of ordinary physical objects, for instance, as being atoms. We certainly do not think of them as being sets! Atoms have no elements, since they are not sets:

Axiom of Atoms. *If x is an atom, then for all y, $y \notin x$ (read "y is not an element of x").*

An advantage of the presence of atoms is that we can suppose that the objects of any theory (or the objects of the usual physical universe) are available for discussion, even if we do not know how to describe them as sets or do not believe that they are sets. It turns out that our axioms will allow us to prove the existence of atoms, which is a rather surprising result!

Observe that the converse is not necessarily true: if an object has no elements, we cannot conclude that it is an atom. It is possible for there to be sets with no elements. What we can prove using the Axiom of Extensionality is that there is no more than one set with no elements:

Theorem. *If A and B are sets, and for all x, x is not an element of A and x is not an element of B, then $A = B$.*

PROOF. — For all x, if x is an element of A, it is an element of B and vice versa (a false statement implies anything...). By Extensionality, A and B must be equal. $\Box$

We will state axioms shortly which will guarantee the existence of a set with no elements. This unique set is called *the empty set* and denoted by the symbol $\{\ \}$.

A natural way to specify a set is to take the collection of all objects with some property. For example, we could consider the collection of all prime numbers greater than 17. This is a technique of specifying sets which has pitfalls (we will see some of these in later chapters), and we will not use it as our technique of choice for building sets at first, but it does inspire a very useful notation for sets. The set of all prime numbers greater than 17 can be written $\{x \mid x$ is a prime number and x is greater than $17\}$; in general, the collection of all objects with property ϕ can be written $\{x \mid x$ has property $\phi\}$, which is read "The set (or class, or collection) of all x such that x has property ϕ". The choice of the letter representing elements of the proposed set is indifferent: the expressions $\{x \mid x$ has property $\phi\}$, $\{y \mid y$ has property $\phi\}$ and $\{A \mid A$ has property $\phi\}$ are exactly the same (as long as appropriate substitutions of y or A for x are made in the description of the property ϕ: $\{x \mid x$ is a prime number and x is greater than $17\}$ is equivalent to $\{A \mid A$ is a prime number and A is greater than $17\}$).

It will turn out that not every set of the form $\{x \mid x$ has property $\phi\}$ actually exists; there are properties which *cannot* define sets. We will introduce such a property eventually. As a consequence, we will not use the general technique of collecting all objects with a given property to build sets; we will use a number of basic (and natural) constructions which experience indicates are safe to build sets. We will eventually use properties of these basic constructions to prove a theorem showing that a large class of properties (the "stratified" properties) do in fact define sets, and after the proof of this theorem we will use sets of the form $\{x \mid x$ has property $\phi\}$ much more freely. Some of the basic constructions which we use are incorrectly considered to be "dangerous" because they lead to problems in the context of the usual set theory; for instance, this set theory has a universal set. Illustrating the fact that such constructions are not dangerous is one of the aims of this work.

If the reader feels that arbitrary collections of objects of our theory *must* exist in some sense, he can understand the sentence “the set $\{x \mid x$ has property $\phi\}$ does not exist” as meaning not that there is no collection of all objects x of our theory such that ϕ, but that this collection cannot be regarded as a set in our theory; such collections which are not sets will sometimes be discussed (we call them “proper classes”).

Cahiers du Centre de logique
Volume 10

Chapter 3

Boolean Operations on Sets

We will now introduce the first of the basic set constructions promised in the last chapter. Two particular sets which we might expect to find are $\{\ \}$, the empty set mentioned in the previous chapter, and V, the universe, the set which contains everything. Notation for these sets might be $\{x \mid x = x\}$ or $\{x \mid \text{True}\}$ for V, and $\{x \mid x \neq x\}$ or $\{x \mid \text{False}\}$ for $\{\ \}$. We state an axiom:

Axiom of the Universal Set. $\{x \mid x = x\}$, *also called* V, *exists.*

We will not need a special axiom for $\{\ \}$, for reasons which will become evident shortly. Be warned: this is our first departure from the usual set theory ZFC; in ZFC, the Axiom of the Universal Set is *false*; the universe does not exist. The reasons for this will be made clear in a later chapter.

Given a set A, an obvious set which might leap to mind would be the set of all things not in A. If A were the set of beautiful things, the complementary set which would come to mind is the set of things which are not beautiful. We assert an axiom providing the existence of such sets (the symbol $\notin$ is read "is not an element of"):

Axiom of Complements. *For each set* A, *the set* $A^c = \{x \mid x \notin A\}$, *called the complement of* A, *exists.*

Two things to note here: the Axiom of the Universal Set and the Axiom of Complements together imply that the empty set $\{\,\}$ exists: the empty set can be constructed as the complement of the universe, i.e., $\{\,\} = V^c$; the Axiom of Complements is uniformly false in ZFC, where *no* set has a complement. We believe that the presence of the universe and of complements together actually give the system of set theory presented here a more intuitive flavour than the usual set theory.

When we have two sets A and B, a natural set to consider is the set which contains all the elements of both A and B. If A is the set of green objects and B is the set of red objects, the set we are interested is the set of all objects which are either green or red. In this case, the two sets do not overlap; if A were the set of college professors and B were the set of absent-minded people, the set we are interested in would be the set of people who are either college professors or absent-minded or both. We assert an axiom to provide for this kind of set:

Axiom of (Boolean) Unions. *If A and B are sets, the set*

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or both}\},$$

called the (Boolean) union *of A and B, exists.*

These are the only new primitive constructions and axioms required for most of this chapter. There are several derived set operations and relations among sets which we will need to define.

Given two sets A and B, another set which we might want to consider is the set of objects in the "overlap" between A and B; if A were the set of college professors and B were the set of absent-minded persons, the set of interest would be the set of absent-minded college professors. We can *prove* that such sets exist for every A and B:

Theorem. *For each set A and B, the set*

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\},$$

called the (Boolean) intersection *of A and B, exists.*

PROOF. — $A \cap B = (A^c \cup B^c)^c$. An object is an element of $(A^c \cup B^c)^c$ exactly if it is *not* an element of $A^c \cup B^c$. An object is *not* an element of

$A^c \cup B^c$ exactly if it is not the case either that it is an element of A^c or that it is an element of B^c; equivalently, if it is not the case either that it is not an element of A or that it is not an element of B. But this is true exactly if it is an element of A and an element of B. □

Note the possible confusion caused by two different uses of the word "and" in English: an object belongs to $A \cap B$ if and only if it belongs to A *and* belongs to B, but the set made up of the elements of A *and* the elements of B is $A \cup B$. We simply have to be careful: these two different uses of "and" are logically totally different from one another.

In some contexts, the "working universe" is not the whole of V. This causes problems when complements are to be taken. For instance, if we are working in arithmetic we think of the complement of the set of even numbers as being the set of odd numbers, rather than the set of all odd numbers and non-numbers. It is convenient to define a new concept, introduced in the following:

Theorem. *For each pair of sets A, B, the set*

$$B - A = \{x \mid x \in B \text{ and } x \notin A\},$$

called the relative complement of A with respect to B, *exists.*

PROOF. — $B - A = B \cap A^c$. □

A final operation on sets (only occasionally used):

Definition. *For A, B sets, we define the* symmetric difference $A \mathbin{\Delta} B$ *as* $(B - A) \cup (A - B)$.

A very important relation between sets is the "subset" relation, or "inclusion". We say that "A is a subset of B" or that "A is included in B" if every element of A is also an element of B. We give a

Definition. *$A \subseteq B$ (A is a* subset *of B, A is* included *in B) exactly if A and B are sets and for every x it is the case that if x is an element of A, then x is an element of B.*

$A \subset B$ (A is a proper subset *of B, A is properly included in B) exactly if A is a subset of B and A is not equal to B.*

$A \supseteq B$ and $A \supset B$ are the converse relations, read "A is a superset of B or A contains B", "A is a proper superset of B or A properly contains B".

Theorem. *$\{\,\} \subseteq A$ for every set A.*

PROOF. — If x is an element of $\{\,\}$... (anything, including "x is an element of A", follows). □

We can see that inclusion is not equivalent to membership; $\{\,\} \subseteq \{\,\}$ by the Theorem, but it is not the case that $\{\,\}$ is an element of $\{\,\}$! The relation between inclusion and the operations above is expressed in the following:

Theorem. *For all sets A, B, $A \subseteq B$ exactly if $A \cup B = B$.*

A formal proof is left to the reader. It should be obvious that adding all the elements of a subset of B to B will not change B!

We define another important relationship between sets:

Definition. *Sets A and B are said to be* disjoint *exactly if for every x, either x is not an element of A or x is not an element of B or both (i.e., no x belongs to both A and B).*

Sets are disjoint if they do not overlap; this insight is equivalent to the following result, which we state without proof:

Theorem. *Sets A, B are disjoint exactly if $A \cap B = \{\,\}$.*

There is no conventional symbol for the relationship of disjointness, so the result of this Theorem is used to represent disjointness symbolically. It is not correct to say that disjoint sets A and B have "no intersection"; they do have an intersection, namely the empty set, but this intersection has no elements. The notion of disjointness extends to more than two sets: for example, sets A, B, and C are said to be disjoint if there is no object which belongs to more than one of them. The condition $A \cap B \cap C = \{\,\}$ is weaker; this merely asserts that there is no object which belongs to all three sets, and allows for the possibility that an object may belong to two of them. For this reason, collections of disjoint sets with more than two elements are often referred to as *pairwise* disjoint sets; the real condition in the case above is that $A \cap B = A \cap C = B \cap C = \{\,\}$.

The operations and relations defined in this chapter comprise an interpretation of the field known as "Boolean algebra"; we will distinguish the operations defined here as "Boolean operations". We list some useful facts about them which are also axioms or theorems of Boolean algebra, expressed in our notation:

commutative laws: $A \cap B = B \cap A$
$A \cup B = B \cup A$

associative laws: $(A \cap B) \cap C = A \cap (B \cap C)$
$(A \cup B) \cup C = A \cup (B \cup C)$

distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

identity laws: $A \cup \{\,\} = A$
$A \cap V = A$

idempotence laws: $A \cup A = A$
$A \cap A = A$

cancellation laws: $A \cup V = V$
$A \cap \{\,\} = \{\,\}$

De Morgan's laws: $(A \cap B)^c = A^c \cup B^c$
$(A \cup B)^c = A^c \cap B^c$

double complement law: $(A^c)^c = A$

other complement laws: $A^c \cap A = \{\,\}$
$A^c \cup A = V$
$V^c = \{\,\}$
$\{\,\}^c = V$

inclusion principles: $A \subseteq B$ exactly if $A = A \cap B$
also exactly if $B = B \cup A$.

These laws should remind one to some extent of the ordinary rules of algebra for addition and multiplication; but notice the exact symmetry between the two operations, and that nothing in the usual algebra corresponds to the complement operation (in particular, not the additive inverse!)

The operations of union and intersection as we have defined them allow us to combine finitely many sets, or to find the common part of finitely many sets, by repeated application of the binary Boolean operations. We will sometimes want to define unions and intersections of collections of sets which are not necessarily finite; this is provided by the following

Axiom of Set Union. *If A is a set all of whose elements are sets, the set $\bigcup[A] = \{x \mid \text{for some } B, x \in B \text{ and } B \in A\}$, called the (set)* union *of A, exists.*

The construction of set intersections will require the assistance of axioms not yet provided. Since we chose Boolean union as a primitive and used it to define Boolean intersection, we will follow precedent and patiently wait until we have the required additional machinery needed to construct set intersections.

Exercises

(a) The smallest Boolean algebra consists of the sets $\{\ \}$ and V. Develop an interpretation of this Boolean algebra in mod 2 arithmetic, giving definitions of the operations of Boolean algebra in terms of the operations of the arithmetic. What Boolean algebra operations correspond to your arithmetic operations? Explain why there are two different nontrivial interpretations.

(b) Verify that the symmetric difference operation is commutative and associative, and that intersection distributes over it.

(c) Look up "Venn diagrams" in another source and verify some of the axioms given at the end of the chapter or the results of the previous exercise using Venn diagrams.

Cahiers du Centre de logique
Volume 10

Chapter 4

Building Finite Structures

So far, we have no assurance that there are any sets other than V and $\{\ \}$. All of the Boolean axioms are satisfied if these are the only sets. We provide a new axiom which will enable us to build a wide range of finite structures:

Axiom of Singletons. *For every object x, the set $\{x\} = \{y \mid y = x\}$ exists, and is called the* singleton *of x.*

Definition. *If x, y are objects, we define $\{x, y\}$, called the* unordered pair *of x and y as $\{x\} \cup \{y\}$; we define unordered triples $\{x, y, z\}$ as $\{x\} \cup \{y\} \cup \{z\}$ and all unordered n-tuples $\{x_1, ..., x_n\}$ in the same way (as $\{x_1\} \cup \{x_2, \ldots, x_n\}$, an inductive definition depending on the previous definition of the $(n-1)$-tuple).*

Observe that the expression $\{x, x\}$ represents $\{x\}$, and, similarly, a symbol like $\{x, x, y\}$ represents $\{x, y\}$; duplications in this notation for finite sets can be ignored. Also, $\{x, y\} = \{y, x\}$; order of items in the name of a finite set makes no difference to its meaning.

We can build more complicated structures in this way: for example, we can construct the objects $\{\ \}$, $\{\{\ \}\}$, $\{\{\{\ \}\}\}$, etc, or such things as $\{\{\{\ \}\}, \{\{V\}, \{\{V\}\}\}\}$. In the usual set theory, the numbers are constructed in this way; '0' is defined as $\{\ \}$, '1' is defined as $\{\text{'0'}\}$, '2' is defined as $\{‘0’,‘1’\}$, ‘3’ is defined as $\{‘0’,‘1’,‘2’\}$, and so forth. We use single quotes to distinguish these “von Neumann numerals” from the rather different objects which we will identify as the natural numbers. The individual von Neumann numerals exist in our system, but the set of all von Neumann numerals does not necessarily exist (although it is possible for it to exist).

One general kind of structure that can be constructed deserves its own

Definition. *We define $\langle x, y\rangle$, the* Kuratowski ordered pair of x and y *as* $\{\{x\}, \{x, y\}\}$.

It is straightforward to prove the following Theorem, whose proof is omitted:

Theorem. *$\langle a, b\rangle = \langle c, d\rangle$ exactly if $a = c$ and $b = d$. In particular, $\langle a, b\rangle = \langle b, a\rangle$ exactly if $a = b$ (note that $\langle a, a\rangle = \{\{a\}\}$).*

It is possible to develop the theory of pairs, and thus of functions and relations, using the Kuratowski pair, but it turns out to be inconvenient to do so in this set theory. In the usual set theory, $\langle a, b\rangle$ is used as the ordered pair; in a later chapter, we will discuss the problems with use of the Kuratowski pair in our set theory.

We will introduce the ordered pair as an independent primitive notion, since use of the Kuratowski pair introduces technicalities which we prefer to avoid. We start this by means of an

Axiom of Ordered Pairs. *For each a, b, the* ordered pair of a and b, (a, b), *exists; $(a, b) = (c, d)$ exactly if $a = c$ and $b = d$.*

Notice that we say nothing about whether (a, b) is a set or an atom (and thus nothing about what might be an element of an ordered pair). We will be able to prove some results about this later (it will be a consequence of axioms not yet introduced that “most” pairs must be atoms, a corollary of the fact, which we will prove, that “most” objects are atoms; this makes an actual definition of our pair in terms of set constructions impossible). Pairing, like the construction of singletons, allows us to build new finite structures:

Definition. *The* n-tuple $(x_1, x_2, \ldots, x_n)$ *is defined as* $(x_1, (x_2, \ldots, x_n))$*;*

this is an inductive definition, depending on the prior definition of the $(n-1)$*-tuple. For example,* $(x_1, x_2, x_3) = (x_1, (x_2, x_3))$*. Note that any* n*-tuple* $(x_1, \ldots, x_{n-1}, x_n)$ *is also an* $(n-1)$*-tuple,* $(x_1, \ldots, (x_{n-1}, x_n))$.

We close this short Chapter by providing for the construction of certain collections of ordered pairs :

Axiom of Cartesian Products. *For any sets* A, B*, the set*

$$A \times B = \{x \mid \text{for some } a \text{ and } b,\ a \in A, b \in B, \text{ and } x = (a, b)\},$$

briefly written $\{(a, b) \mid a \in A \text{ and } b \in B\}$*, called the* Cartesian product *of* A *and* B*, exists.*

Definition. *For any set* A*, we define* A^2 *as* $A \times A$ *and* A^{n+1} *as* $A \times (A^n)$ *for each natural number* $n > 1$*. Grouping is important here because the Cartesian product is not associative.*

We will hereafter use without comment the kind of extension of our set builder notation which uses finite structures in place of a simple variable to represent the typical element of a set, exemplified in the notation of this axiom. The uses of the specific sets introduced by the axiom will become apparent in the next chapter.

Exercises

(a) Prove the theorem about Kuratowski ordered pairs.

(b) Look up the alternative definition of the ordered pair in the Notes and prove that it satisfies the basic property of an ordered pair.

(c) Verify our observation that the Cartesian product is not associative. Can it ever be true that $(A \times B) \times C = A \times (B \times C)$?

(d) What interpretation might one place on the Cartesian powers A^1 and A^0?

Cahiers du Centre de logique
Volume 10

Chapter 5

The Theory of Relations

Just as we associate sets with (some) properties of objects, so we can associate sets with relations between two or more objects, with the assistance of the ordered pair construction. For instance, we interpret the relation "is less than" among numbers as the set of ordered pairs (m, n) of natural numbers such that $m < n$. The use of the pair is that it allows us to reinterpret a relation pertaining to a "configuration" of two or more objects as a property of a single "abstract" object associated with that configuration.

Definition. *A* relation *is a set of ordered pairs. An n-*ary relation *is a set of ordered n-tuples (note that any n-ary relation is also an $(n-1)$-ary relation, because all n-tuples are also $(n-1)$-tuples).*

We will use a distinctive notation for relations: if the name of a relation is R, we will write "$x\ R\ y$" for "x stands in the relation R to y". The exactly equivalent statement "(x, y) belongs to R" will be written "$(x, y) \in R$".

We will want to use certain infix operators which represent relations (for example, "$=$" and "$\subseteq$") as names of sets as well. To avoid confusion, we enclose an infix operator symbol in brackets (as for example "$[=]$" or "$[\subseteq]$") when it is being used as the name of a set rather than as an infix.

We provide some specific relations of interest:

Axiom of the Diagonal. *The set* $[=] = \{(x, x) \mid x \in V\}$ *exists (this is the* equality relation*).*

Axiom of Projections. *The sets* $\pi_1 = \{((x, y), x) \mid x, y \in V\}$ *and* $\pi_2 = \{((x, y), y) \mid x, y \in V\}$ *exist (these are the relations which an ordered pair has to its first and second terms, which are technically referred to as its* projections*). These names may be used as relation symbols.*

The names of the last two axioms reflect geometrical concepts; the graph of $y = x$ on the plane (the Cartesian product of the real line with itself) is a diagonal, and the coordinates x and y of a point (x, y) in the plane are its projections onto the coordinate axes. The Axiom of Projections has a special role which is not immediately obvious; it allows us to prove that there are infinitely many objects, as we will see below.

The Boolean operations give us operations on relations "ready-made". Observe that $\{\ \}$ is a relation (it has no elements which are not ordered pairs!), and clearly the "smallest" possible (no object stands in this relation to any other). The "largest" possible relation is $V \times V$, the Cartesian product of the universe and itself, i. e., the set of all ordered pairs (each object stands in this relation to each other). The relation symbol we will use for $V \times V$ will be V^2. (It is possible that $V^2 = V$, and it would be technically convenient if we postulated this, but we prefer not to add axioms for reasons of pure technical convenience.) The natural notion of complement for a relation is relative to V^2, so if R is a relation symbol, we will use R^c as the relation symbol corresponding to the set $V^2 - R$; we can write this as an equation: $R^c = V^2 - R$. Note that the relation symbol for the relation that never holds is $(V^2)^c$. The definitions of relation symbols $R \cap S$ and $R \cup S$ do not require comment. We summarize:

Definition. $V^2 = V \times V; R^c = V^2 - R.$

We now introduce some additional operations on relations. Both are natural constructions. Suppose that L stands for the relation "loves"; "$x\ L\ y$" stands for "x loves y". In English, we can also say "y is loved by x". The construction in the theory of relations which corresponds to the passive voice in English is called the "converse": the converse of L is written L^{-1}, and we write "$y\ L^{-1}\ x$" to mean "y is loved by x". The converse operation can be applied both to relation symbols and to set names: for instance, $(A \times B)^{-1} = (B \times A)$. We state the following

Axiom of Converses. *For each relation R, the set $R^{-1} = \{(x, y) \mid y\ R\ x\}$ exists; observe that $x\ R^{-1}\ y$ exactly if $y\ R\ x$.*

Consider the relation "is the paternal grandmother of". The fact that x is the paternal grandmother of y is mediated by a third party; x is the paternal grandmother of y exactly if there is a z such that x is the mother of z and z is the father of y. This is the model for the second primitive operation on relations that we introduce. If we write "$x\ M\ y$" for "x is the mother of y" and "$x\ F\ y$" for "x is the father of y", the operation we will introduce will enable us to write "$x\ M|F\ y$" for "x is the paternal grandmother of y". The operation defined here is called "relative product", and relative products are provided by an

Axiom of Relative Products. *If R, S are relations, the set*

$$(R|S) = \{(x, y) \mid \textit{for some } z,\ x\ R\ z \textit{ and } z\ S\ y\},$$

called the relative product *of R and S, exists.*

A useful notion extending this is the notion of a "power" of a relation:

Definition. *We define R^0 as $[=]$ and R^1 as R for each relation R; for each positive natural number n, we define R^n as $R|(R^{n-1})$.*

There is no problem with grouping in this definition, since the relative product is associative. There is the potential difficulty of ascertaining whether a Cartesian or relation power is intended, which one hopes will always be resolvable from the context.

Next, we consider how to get information about relations back down to the level of sets. If it is true that "$x\ L\ y$" (x loves y), we would like to say that x is a lover; we would like to be able to define the set of objects which have the relation L to something, without reference to what that something might be. The set of things which stand in the relation L to something is called the *domain* of L and represented by $\text{dom}(L)$; this is summarized in the

Axiom of Domains. *If R is a relation, the set*

$$\text{dom}(R) = \{x \mid \textit{for some } y,\ x\ R\ y\},$$

called the domain *of R, exists.*

If $\operatorname{dom}(L)$ stands for the set of lovers, we may complain that we have not provided for the set of beloveds, but this is simply the domain of L^{-1}. We call the domain of the converse of a relation the *range* of the relation:

Theorem. *If R is a relation, the set* $\operatorname{rng}(R) = \{x \mid$ *for some* $y, y\ R\ x\}$, *called the* range *of x, exists.*

PROOF. — $\operatorname{rng}(R) = \operatorname{dom}(R^{-1})$. □

We introduce a related notion:

Definition. *If R is a relation, we define the* field *or* full domain *of R, written* $\operatorname{Dom}(R)$, *as* $\operatorname{dom}(R) \cup \operatorname{rng}(R)$.

The final construction we provide here is the construction of "singleton images" of relations. We would certainly like to believe that for every R there is a relation $\mathrm{SI}\{R\}$ such that $\{x\}\ \mathrm{SI}\{R\}\ \{y\}$ exactly if $x\ R\ y$; it turns out that such relations are important.

Axiom of Singleton Images. *For any relation R, the set*

$$\mathrm{SI}\{R\} = \{(\{x\}, \{y\}) \mid x\ R\ y\},$$

called the singleton image *of R, exists.*

The existence of "singleton images" of sets is an easy theorem:

Theorem. *For each set A, the set $\mathcal{P}_1\{A\} = \{\{x\} \mid x \in A\}$ exists.*

PROOF. — $\mathcal{P}_1\{A\} = \operatorname{dom}(\mathrm{SI}\{A \times A\})$. □

The reader might want to prove this

Theorem. $\bigcup[\mathcal{P}_1\{A\}] = A$

With these axioms for sets and relations, we have arrived at a certain "critical mass" of primitive concepts and propositions. We will spend the next chapters elaborating the consequences of these axioms; it turns out that they have a great deal of logical power. We will not introduce more axioms until we have "digested" some of the consequences of the axioms we already have.

We close with a list of axioms for the "algebra of relations" taken from a book by Tarski and Givant; the "algebra of relations" is generally a less familiar subject than Boolean algebra. Understanding and verifying these axioms could be a useful exercise.

$$\begin{aligned}
&A \cup B = B \cup A;\\
&(A \cup B) \cup C = A \cup (B \cup C);\\
&(A^c \cup B)^c \cup (A^c \cup B^c)^c = A;\\
&A|(B|C) = (A|B)|C;\\
&(A \cup B)|C = (A|C) \cup (B|C);\\
&A|[=] = A;\\
&(A^{-1})^{-1} = A;\\
&(A \cup B)^{-1} = A^{-1} \cup B^{-1};\\
&(A|B)^{-1} = (B^{-1}|A^{-1});\\
&(A^{-1}|((A|B)^c)) \cup B^c = B^c.
\end{aligned}$$

Exercises

(a) Verify some or all of the axioms for relation algebra given at the end of the chapter. In particular, prove that the relative product is associative.

(b) (open ended) What difficulties would you encounter in trying to develop a system for reasoning about relations using diagrams (analogous to the method of Venn diagrams which you were asked to read about in an earlier exercise)? Do you think that such a system could be developed?

Cahiers du Centre de logique
Volume 10

Chapter 6

Sentences and Sets

We now pause to analyze the structure of the language we are using. We are going to define the notion "sentence (of first-order logic)"; we will not introduce any special symbolism, but rather a careful technical definition of the meaning of certain English expressions.

6.1. The Definition of "Sentence"

The first notion we need to define is "atomic sentence". Informally, an atomic sentence asserts that an object x has a property P (we can write this Px) or that objects x, y stand in a relation R to one another (where "relation" does not mean "set of ordered pairs") (we can write this Rxy), or that a list of objects $x_1 \ldots x_n$ satisfy an "n-ary relation" G (we write this $G[x_1, \ldots, x_n]$). Informal examples: Gx, meaning "x is green"; Lxy, meaning "x loves y"; $Txyz$, meaning x takes y from z. An atomic sentence can be thought of as expressing a simple fact about the world (or, at least, one that we have not analyzed logically).

We now give a semi-formal

Definition. *An* atomic sentence *is a sentence of the form $P[x_1, \ldots, x_n]$, where P is a predicate (property or relation) applicable to a list of*

n objects. The brackets and commas in the sentence can be omitted where this is convenient.

The first kind of complex structure we consider is the building of complex sentences using words like "not", "and", "or", "implies", and "if and only if" (usually abbreviated "iff"; we often say "exactly if" or "exactly when"). These words are called "propositional connectives". We regard negation and conjunction as primitive, and we define other English propositional connectives in terms of these. Their meanings in common English usage are less precise.

disjunction: (ϕ or ψ) is defined as not((not ϕ) and (not ψ)). Notice that this is the "non-exclusive or", written "and/or" in legal documents.

implication: (ϕ implies ψ) or, equivalently, (if ϕ then ψ) is defined as ((not ϕ) or ψ), an odd definition, but one which grows on you.

if and only if: (ϕ iff ψ) is defined as ((ϕ implies ψ) and (ψ implies ψ)).

Notice that we allow the non-English expedient of parentheses to clarify the structure of sentences built with propositional connectives.

The remaining construction of sentences is the use of "quantifiers" such as "for all x", "for some x", "for exactly one x". We give examples of sentences using these: "All men are mortal" translates to "For all x, if x is a man, then x is mortal". A mathematical sentence using two quantifiers is "Every number has prime divisors", which translates to "For all x, if x is a number then there is a y such that y divides x and y is prime".

We take "for some x" as primitive. "there is an x such that..." and "there exists x such that ..." and similar constructions are regarded as synonymous. We now give the complete semi-formal definition of *sentence*:

Definition (sentence). *The following clauses define our technical meaning of* sentence. *(It may be useful to be aware that logicians normally use the term "formula" for what we call a "sentence" and reserve the term "sentence" for formulas containing no free variables.)*

atomic: *An atomic sentence is a sentence.*

negation: *If ϕ is a sentence, "not ϕ" is a sentence.*

conjunction: *If ϕ and ψ are sentences, "ϕ and ψ" is a sentence.*

existential quantifier: *If ϕ is a sentence and x is a variable, "for some x, ϕ" is a sentence.*

completeness of definition: *Every sentence is constructed using these clauses alone.*

We give the definition of the universal quantifier:

Definition (universal quantifier). *"for all x, ϕ" is defined as "not(for some x, (not ϕ))". "for each x" and "for any x" are regarded as synonymous alternatives to this.*

We define some basic notions useful in discussing quantifiers:

Definition. *An occurrence of a variable x is said to be* bound *in a sentence when it is part of a sentence "for some x, ϕ". (and so when it is part of a sentence "for all x, ϕ" as well). An occurrence of a variable which is not bound is said to be* free.

Definition. *When ϕ is a sentence and y is a variable, we define $\phi[y/x]$ as the result of substituting y for x throughout ϕ, but only in case there are no bound occurrences of x or y in ϕ. (We note for later, when we allow the construction of complex names a which might contain bound variables, that $\phi[a/x]$ is only defined if no bound variable of a occurs in ϕ (free or bound) and* vice versa*).*

It is possible to give a more general definition of substitution, but this one is simple. The reason that it is adequate is that it is always possible to rename a bound variable without changing the meaning of a sentence. A substitution blocked by the restriction on the definition of substitution can be carried out by first changing the names of bound variables as indicated in the following

Fact. The meanings of "for some x, ϕ" and "for all x, ϕ" are unaffected by substitutions yielding "for some y, $\phi[y/x]$" and "for all y, $\phi[y/x]$", when the substitutions are meaningful.

For example, "All men are mortal" is equally well translated by "for all x, if x is a man then x is mortal" and "for all y, if y is a man then y is mortal". This should remind you of similar considerations regarding "set-builder" notation $\{x \mid \phi\}$; the x in this notation is also a bound variable, as we will see below.

We can now give another

Definition (uniqueness quantifier). *"for exactly one x, ϕ" is defined as "(for some x, ϕ) and (for all x, for all y, (ϕ and $\phi[y/x]$) implies $x = y$)". It is required that $\phi[y/x]$ be defined, which may require renaming of bound variables in ϕ.*

We now show how to introduce names into our language. At the moment, there are no nouns in our language, only pronouns (variables). The basic construction for nouns is (the x such that ϕ), where ϕ is a sentence. The meaning of this name will be "the unique object x such that ϕ, if there is indeed exactly one such object, otherwise the empty set." A special sentence construction which does not fall under this definition is "(the x such that ϕ) exists", which is taken to mean "there is exactly one x such that ϕ".

These names cover not only the case of unique objects as in "$0 =$ (the natural number n such that for all natural numbers m, $n+m = m+n = m$)" but also the case of definitions of *operations*: the union $A \cup B$ of two sets can be defined as (the C such that for all x, $x \in C$ iff ($x \in A$ or $x \in B$)) and the set builder notation $\{x \mid \phi\}$ can be defined as (the A such that for all x, $x \in A$ iff ϕ). Notice that x is a bound variable in this "complex name"; there is an additional requirement that there be no occurrences of A in ϕ. Variables free in a complex name (as in the definition of Boolean union) are called *parameters* of the name.

We give a semi-formal definition of complex names (this is a variation on Bertrand Russell's Theory of Descriptions):

Definition. *A sentence $\psi[(\text{the } y \text{ such that } \phi)/x]$ is defined as "((there is exactly one y such that ϕ) implies (for all y, ϕ implies $\psi[y/x]$)) and ((not(there is exactly one y such that ϕ)) implies (for all x, (x is the empty set) implies ψ))". Renaming of bound variables may be needed.*

Definition. *A sentence "(the x such that ϕ) exists" is defined as "there* *is exactly one x such that ϕ". "exists" is not to be understood as a predicate.*

A useful logical construction is the restriction of bound variables to sets. We give definitions.

Definition. *"for all $x \in A$, ϕ" means "for all x, if $x \in A$, then ϕ".*

"for some $x \in A$, ϕ" means "for some x, $x \in A$ and ϕ".

"for exactly one $x \in A$, ϕ" means "for exactly one x, $x \in A$ and ϕ".

"the $x \in A$ such that ϕ" means "the x such that $x \in A$ and ϕ" (this may serve to explain the idiom "the natural number n such that..." used above without comment).

6.2. The Representation Theorem

Our interest in first-order logic is to explain conditions under which sets $\{x \mid \phi\}$ exist. What we will do is provide set-theoretical constructions which parallel the sentence constructions above, using the primitive set and relation constructions we have given in the last few chapters.

The "predicates" (properties and relations) in atomic sentences correspond (sometimes) to sets and relations (in the set theory sense "sets of ordered pairs or n-tuples"). We note this in the context of a

Definition. *A property G is said to be* realized *by the set $\{x \mid Gx\}$ if this set exists. We call the set $G!$ when confusion with the predicate needs to be avoided.*

A relation $R[x_1, ..., x_n]$ among n objects is said to be realized by the set $\{(x_1, \ldots, x_n) \mid R[x_1, \ldots, x_n]\}$ when this set exists. We will call this set $R!$ when confusion with the predicate needs to be avoided.

The constructions of sentences not(P) and $(P$ and $Q)$ have obvious parallels in our arsenal of basic constructions: if $\{x \mid \phi\}$ and $\{x \mid \psi\}$ exist, $\{x \mid \text{not}(\phi)\} = \{x \mid \phi\}^c$ and $\{x \mid (\phi \text{ and } \psi)\} = \{x \mid \phi\} \cap \{x \mid \psi\}$. Similarly, $\{x \mid \phi \text{ or } \psi\} = \{x \mid \phi\} \cup \{x \mid \psi\}$.

The construction of sentences with quantifiers involves the use of variables; we need to develop set-theoretical tools analogous to the use of variables. What we will do is restrict our attention to sentences ϕ which involve only n variables $x_1, \ldots, x_n$; clearly, we can analyze any sentence if we take n large enough. We then represent any sentence ϕ by the collection of tuples $(a_1, \ldots, a_n)$ such that ϕ would be true if $x_1 = a_1, \ldots, x_i = a_i, \ldots, x_n = a_n$. This set can conveniently be written $\{(x_1, ..., x_n) \mid \phi\}$.

We go back to atomic sentences, where we have a difficulty: not all atomic sentences will involve all the variables, or involve them in the given order. We need to represent atomic sentences like $P[x_2]$, $Q[x_2, x_1]$, or $R[x_1, x_1, x_3, x_2]$ as well as sentences $S[x_1, \ldots, x_n]$ whose representation is obvious. The trick here is to provide relations among m-tuples (for varying m) which manage the shuffling and reduplication of variables. Once we deal with this, the operation analogous to quantification will turn out to be very easy to represent.

We consider a sentence $P[x_i]$ first. This will be represented by

$$\{(x_1, \ldots, x_i, \ldots, x_n) \mid P[x_i]\}.$$

We could define this as the domain of an intersection of relations $X_i \cap (V \times P!)$, where X_i is the relation between any $(x_1, \ldots, x_i, \ldots, x_n)$ and its "projection" x_i and $V \times P!$ is the relation between anything and an object of which P is true. The difficulty is now to construct the "projection" relation X_i. Recall that $(x_1, \ldots, x_n) = (x_1, (x_2, (x_3, \ldots (x_{n-1}, x_n))))$. Clearly $X_1 = \pi_1$. Analysis reveals that $X_2 = \pi_2|\pi_1, X_i = (\pi_2^{i-1})|\pi_1$ for $2 \leqslant i < n$, while $X_n = \pi_2^{n-1}$. The definition of X_n is the only one which depends on the particular value of n. So $P[x_i]$ is represented by $\mathrm{dom}(X_i \cap (V \times P!))$, which exists by our axioms if $P!$ exists.

We now consider more complex atomic sentences. The trick is to find the correct relations to represent *lists* of variables, as the X_i's represent individual variables. In order to represent $R[x_{k_1}, \ldots, x_{k_m}]$, where for each j, $1 \leqslant k_j \leqslant n$, we will need the relation which holds between any n-tuple $(x_1, \ldots, x_n)$ and the m-tuple $(x_{k_1}, \ldots, x_{k_m})$. For this, we use the following

Definition. *Where R, S are relations, $R \otimes S$ is defined as $(R|\pi_1^{-1}) \cap (S|\pi_2^{-1})$. Analysis reveals that $R \otimes S = \{(x, (y, z)) \mid x\ R\ y \text{ and } x\ S\ z\}$. Parentheses in expressions $(R_1 \otimes R_2 \otimes \cdots \otimes R_n)$ are supplied from the right;*

for example, $(R_1 \otimes R_2 \otimes R_3) = (R_1 \otimes (R_2 \otimes R_3))$*; this is done to parallel the grouping of ordered pairs in the definition of the* n*-tuple.*

The desired relation is then $(X_{k_1} \otimes \cdots \otimes X_{k_m})$. We can then show that the set $\{(x_1, \ldots, x_n) \mid P[x_{k_1}, \ldots, x_{k_m}]\}$ is represented by $\mathrm{dom}((X_{k_1} \otimes \cdots \otimes X_{k_m}) \cap (V \times P!))$. Thus any atomic sentence can be represented in our current scheme if the relation (in the sense of first-order logic) in it is represented by a (set) relation in the natural way.

The apparently simple problem of representing atomic sentences turned out to be rather involved. The problem of representing negation and conjunction is trivial: use complement and intersection as indicated above. The representation of quantification turns out to be fairly simple. The idea is that if a sentence ϕ is represented by a set $\phi! = \{(x_1, \ldots, x_n) \mid \phi\}$, the set of n-tuples such that "(for some x_i, ϕ)" is true for the listed values of x_j's is the set of $(x_1, \ldots, x_i', \ldots, x_n)$'s obtained by substituting any x_i' at all for the specific value of x_i in an n-tuple $(x_1, \ldots, x_i, \ldots, x_n)$ which belongs to $\phi!$. The point is that the statement (for some x_i, ϕ) says nothing about the value of x_i; it says that *some* value of x_i works; whereas ϕ might require some specific value of x_i to be true, (for some x_i, ϕ) allows x_i to "float free". Now this set is very easy to represent: it is $\mathrm{dom}((X_1 \otimes \cdots \otimes X_{i-1} \otimes V^2 \otimes X_{i+1} \otimes \cdots \otimes X_n) \cap (V \times \phi!))$.

These results taken together show that $\{(x_1, \ldots, x_n) \mid \phi\}$ is definable for any sentence ϕ involving only the variables $x_1, \ldots, x_n$ in which every relation or property mentioned in an atomic sentence is realized. Suppose that we want to define the set $\{x_1 \mid \phi\}$, with fixed values a_i for each variable x_i with $i > 1$; this can be constructed as the domain of the intersection of $\{(x_1, \ldots, x_n) \mid \phi\}$ and the sets interpreting the conditions "$x_i = a_i$" (the condition "$x = a$" on a variable x is realized by $\{a\}$). Thus, we can define $\{x \mid \phi\}$ for any variable x and sentence ϕ of first-order logic, as long as all properties and relations mentioned in atomic sentences are "realized". The operations we have defined on n-ary relations are basically the operations of the "cylindrical algebra" of Tarski, which is to first-order logic as Boolean algebra is to propositional logic (the logic of "and", "not" and their relatives without quantifiers).

We state the result of our discussion as the

Representation Theorem. *For any sentence ϕ in which all predicates mentioned in atomic sentences are realized, the set $\{x \mid \phi\}$ exists.*

A technical point which ought to be made (but which should not be belabored) is that statements like the Representation Theorem which begin with phrases like "for any sentence ϕ" should not be regarded as involving quantification over sentences. When we say "for all x, ϕ", we are thinking of ϕ as a sentence about objects in our domain of discourse, any of which can be substituted for the variable x to give a true statement. We do not think of "$\{x \mid \phi\}$ exists" as a sentence about the sentence ϕ; we have not considered the question as to whether we regard sentences themselves as inhabitants of our universe of discourse. The "sentence" (for all ϕ, $\{x \mid \phi\}$ exists) is not found in our language. This remark will apply to other theorems or axioms stated later as well, such as the Stratified Comprehension Theorem and the Axiom of Small Ordinals. One should think of an axiom or theorem of this kind as an infinite list of sentences, one for each specific sentence ϕ, rather than as a single sentence of our language at all. Axioms and theorems of this kind are properly called "axiom schemes" and "meta-theorems" respectively.

Exercises

(a) Express the sentence "All swans are white" in our formalized language.

(b) Express the sentences "Everybody is loved by somebody" and "Somebody loves everybody" in our formalized language. Are these sentences equivalent in meaning? Explain.

(c) Define the set of women with at least two children as in the proof of the Representation Theorem. You may assume that appropriate basic predicates are realized by sets.

(d) Express the sentence "The set A has exactly two elements" in our formalized language, using the membership and equality relations as the only predicates. Can you express it in terms of membership only?

Cahiers du Centre de logique
Volume 10

Chapter 7

Russell's Paradox and the Theorem of Stratified Comprehension

Every set that we have defined, we have defined in the format $\{x \mid \phi\}$, where ϕ is a sentence of first-order logic. The reader should be wondering by now why we bother with all these special cases instead of adopting the following

$*$**Axiom.** *For each sentence* ϕ, $\{x \mid \phi\}$ *exists.*

This axiom is called the (naive) Axiom of Comprehension. It is false. Consider the relation symbol $\in$ for membership. Notice that we did not include "the set $[\in] = \{(x, y) \mid x \in y\}$ exists" as an axiom in the natural place at the end of the last chapter. We have a good reason for this:

Theorem. $[\in]$ *does not exist.*

Proof (Russell). — Suppose that $[\in]$ existed. We construct the set $\mathrm{dom}([=] - [\in])$. $[=] - [\in]$ would be the set of ordered pairs (x, y) such that $x = y$ and x is not an element of y, i.e., the set of ordered pairs (x, x) such that x is not an element of x, and $\mathrm{dom}([=] - [\in])$ would be the set of all x such that x is not an element of itself, conveniently abbreviated $\{x \mid \mathrm{not}(x \in x)\}$. If this set is an element of itself, it is not an element of itself; if it is not an element of itself, it is an element of itself. The contradiction indicates that our original assumption that $[\in]$ existed must have been in error. $\square$

The core of this argument, called Russell's Paradox, is a *reductio ad absurdum* of the proposed Axiom of Comprehension: the set $\{x \mid \text{not}(x \in x)\}$ cannot exist, and $\text{not}(x \in x)$ is certainly a sentence of first-order logic. The specific theorem proved is an artifact of the set theory we are working in.

Since we cannot have $[\in]$, we settle for something less which proves adequate. We append a last axiom to our list of axioms asserting the existence of specific relations:

Axiom of Inclusion. *The set* $[\subseteq] = \{(x, y) \mid x \subseteq y\}$ *exists.*

Using the results of the previous chapter, we see that we can define $\{x \mid \phi\}$ if ϕ mentions no primitive relation other than $=$, π_1, π_2, or $\subseteq$. It turns out that a wide class of sentences ϕ which mention the relation $\in$ of membership, which is *not* realized, as we saw above, are nonetheless realized by sets; there is a technique of translation which converts some sentences with $\in$ to sentences without $\in$.

We now define the class of "stratified" sentences of first-order logic.

Definition. *A sentence ϕ of first-order logic involving no relation other than $\in$, π_1, π_2, or $=$ is said to be* stratified *if it is possible to assign a non-negative integer to each variable x in ϕ, called the* type *of x, in such a way that:*

(a) *Each variable has the same "type" wherever it appears.*

(b) *In each atomic sentence $x = y$, $x \; \pi_1 \; y$, $x \; \pi_2 \; y$ in ϕ, the "types" of the variables x and y are the same.*

(c) *In each atomic sentence $x \in y$ in ϕ, the "type" of y is one higher than the "type" of x.*

Stratified Comprehension Theorem. *For each stratified sentence ϕ, the set $\{x \mid \phi\}$ exists.*

PROOF. — We would like to apply the results of the previous chapter, but $\in$ is not realized. We observe that "$y \in z$" is precisely equivalent to "$\{y\} \subseteq z$". We choose a natural number N larger than the type assigned to any variable in ϕ. Our strategy is to replace all reference to each variable y of type n with reference to the $(N - n)$-fold iterated singleton of y.

All occurrences of the relations π_1, π_2 and $=$ are thus replaced by occurrences of iterated singleton images of these relations, which are realized; all occurrences of $\in$ are thus replaced by occurrences of iterated singleton images of the relation $\subseteq$ (since the singleton operation will be applied once more on the left than on the right, by the definition of stratification of atomic membership sentences). Since each variable occurs with only one type, references to the $(N-n)$-fold iterated singleton of the variable y of type n (for example) can be replaced with references to a variable y "restricted" to the collection of $(N-n)$-fold singletons, a collection whose existence follows from our axioms (it is the result of the singleton image operation on sets applied $N-n$ times to V). The restriction is achieved by replacing each "(for some y, ϕ)" with "(for some y "y is an $(N-n)$-fold singleton" and ϕ')" (n being the type of y, and ϕ' being the result of replacing references to the $(N-n)$-fold singleton of y in ϕ with references to y) and similarly replacing references to the $(N-m)$-fold singleton of x (where m is the type of x) with references to x and imposing the condition that the new x is an $(N-m)$-fold singleton. Since the set of $(N-n)$-fold singletons exists, the property of being an $(N-n)$-fold singleton is "realized" for each n. Recall that the type of the variable x in the set definition $\{x \mid \phi\}$ has been taken to be m; by applying the result of the previous chapter at this point (all properties and relations appearing in the modified ϕ are now "realized"), we succeed in defining the set of $(N-m)$-fold singletons of objects x for which ϕ is true; we apply the axiom of set union $N-m$ times to obtain the desired set $\{x \mid \phi\}$. The proof of the Stratified Comprehension Theorem is complete. □

One question which might remain is that we have not explicitly referred to the presence of atoms in our theory in this discussion. It is sufficient to note that atoms are exactly those objects which have no subsets; atomhood is expressible in terms of $\subseteq$.

It is worth mentioning that the Axiom of Singleton Images and the Axiom of Set Union, though they were introduced in earlier chapters, were not needed for the proof of the Representation Theorem and have found their first major application in the proof of the Stratified Comprehension Theorem.

Exercises

(a) Construct the set $\mathcal{P}\{A\}$ of all subsets of a given set using the primitive operations for constructing sets now available. (The solution appears later in the book).

(b) Verify that the intersection $\bigcap[A]$ of all elements of a set A (the set of all x which belong to each element B of A) has a stratified definition, and so exists by the theorem of Stratified Comprehension. Then construct it using the primitive set constructions.

Cahiers du Centre de logique
Volume 10

Chapter 8

Philosophical Interlude: What is a Set, Anyway? Motivating Stratification

In contrast with the rather natural basic constructions given in the axioms, the Stratified Comprehension Theorem is not at all intuitive. The usual way in which the set theory we are developing is initially presented is to take as axioms Extensionality, the axiom of Atoms, and Stratified Comprehension. Our axiom of Ordered Pairs would also be needed if our precise development were to be followed. The reason we have not done this is that Stratified Comprehension is not particularly easy to motivate. However, Stratified Comprehension is a very powerful tool, which we will want to be able to use. In this section, we will try to provide motivation for stratified comprehension as a criterion for set existence.

8.1. What is a Set, Anyway?

We begin by returning to the vexed question of set existence on a more fundamental level; what is a set supposed to be, anyway? We outline an informal answer to this question.

We don't intend to give any kind of formal definition of what a set is; we stated at the outset that set is an undefined notion of our theory. Nonetheless, we should be now be enough acquainted with a set to say something about this on an informal level.

How do sets come into human experience? An obvious example is that of herds of animals or sets of tools. Collections of discrete physical objects are our usual first model for the notion of "set".

But we need to be careful. We think of a herd of animals as a large physical object of which the individual animals are parts. We have learned that we cannot view the relation of element to set as the relation of part to whole. For example, a is an element of $\{a\}$, and $\{a\}$ is an element of $\{\{a\}\}$, but a is not an element of $\{\{a\}\}$; this makes no sense if "element" is to be interpreted as "part", because the relation of part to whole is transitive. The occurrence of singleton sets in our example is telling; notice that we do not distinguish between a herd of one animal and that single animal (though we do regard the former as a very peculiar notion!).

We give an example where we can tell in a mostly pre-mathematical context that the relation of element to set must be distinguished from the relation of whole to part. Suppose that John, Sam, and Tim are members of a committee. We ask how many subcommittee rosters of two members there can be; we comfortably answer "three". We might be asked for the set of all possible commmittee rosters; we would probably list seven elements (the empty committee can't get much done). But observe that, while we might be reasonably comfortable viewing each committee as an object formed by conglomerating its members, if we try to view sets of committee rosters as conglomerations of their elements, we cannot draw distinctions that we recognize as real; the set $\{\{$John, Sam$\}, \{$John, Tim$\}\}$ of two subcommittees could not be distinguished from the set $\{\{$John, Sam$\}, \{$John, Tim$\}, \{$Sam, Tim$\}\}$ of three subcommittees.

Even collections of disjoint physical objects present a difficulty. Consider the set of all human beings. This is a set we all "know", whose elements we can count, more or less. Now consider the set of all human cells! This is a set we all "know", can count, more or less, and which has (more or less) the same physical extent as the previous set – but it is not the same set because it does not have the same number of elements by many orders of magnitude! The "sets" of our experience, whose elements are discrete physical objects, need for their exact identification not only their physical extent but the kind of individual object into which they should be partitioned (they can be thought of as "typed" collections). Not all parts of a "set" thus understood are its "elements"; only those distinguished parts of the understood "type".

We noted above that the correct analogue of the relation of part to whole in the world of sets is the subset relation rather than the membership relation. Now use the fact (which we used in the proof of the Stratified Comprehension Theorem) that the membership relation can be reduced to the inclusion relation and the singleton operator: $x \in y$ iff $\{x\} \subseteq y$. We have explained inclusion as the intuitively familiar relation of part to whole. It remains to explain the singleton set construction.

The singletons are a collection of pairwise disjoint sets; if we understand inclusion as the relation of part to whole, we can put this informally by saying that the singletons, whatever they are, are a collection of discrete, non-overlapping objects. Singletons are disjoint objects like the physical objects which are "elements" (actually distinguished parts) of the "sets" of our everyday experience.

The singleton construction allows us to associate with each object in the universe of our set theory an object taken from a domain of discrete, non-overlapping objects.

The construction of a "conglomerate" object from a collection of arbitrary sets corresponds to taking the union of the collection (the minimal collection which has every element of the original collection as a *subset*). Quite different collections of arbitrary sets can have the same union. But distinct collections of singletons do have distinct unions.

We interpret the singleton construction as choosing a "token" or "counter" to correspond to each object in our universe of discourse (the universe of things of which we want to build sets). The counters are disjoint objects, which makes them satisfactory for building sets by aggregation. The singleton considered as a counter "stands for" its element in the construction of sets. So we can construe the relation between singleton and element as a special case of the familiar relation of symbol and referent; a singleton is a kind of *name* for its element. If we focus on the extension, we can think of the singleton as a kind of label affixed to it to make it possible for the extension to be itself an element of further sets.

In terms of this metaphor, a set will be understood to be an object which has as its parts names (or tokens, counters, labels) for its elements. (In everyday terms, it can be thought of as a *list* or *catalogue*.) The relation "x is an element of y" will mean "the name of x is part of y, and y is an aggregate of names". Objects which are not aggregates of names will be "atoms" in the sense of our set theory (though they will not be "atomic" objects in terms of this metaphor).

This does not complete the informal definition of "set", though. It seems intuitively reasonable that every aggregate of names, however defined, should exist as an object of some sort, and we already know that certain of these cannot be sets. Consider the aggregate R made up of all sets x such that $x \notin x$. This would be Russell's impossible "set", if it were a set! This aggregate has as parts all "names" $\{x\}$ such that $x \notin x$, that is, such that $\{x\}$ is not part of x. It seems reasonable to suppose that this object would exist, though it might not be in our universe of discourse. We maintain informally that it *does* exist. Is $R \in R$? $R \in R$ is defined as holding if and only if $\{R\}$ is part of R (by the informal definition of $\in$), but also as holding if and only if $\{R\}$ is not a part of R (by the definition of R itself)! But notice that we have not made any provision for an object $\{R\}$ (a "name" of R)! We can't: for if $\{R\}$ existed, we would be forced to conclude that $R \in R$ iff $R \notin R$.

We have not arrived at a contradiction; we have arrived at the conclusion that R cannot have been assigned a singleton (a "name", in terms of our metaphor), which means that it is not in our universe of discourse. This shows us how to qualify the definition of "set". This is the missing element of the definition of "set": a set is to be understood as an aggregation of names *which itself has a name*; our conclusion about R is that it exists (as an aggregate object) but is not a set. Other aggregations of names may be understood as real objects (our metaphor certainly suggests that they exist) but they are not sets. We may informally define "class" to mean "aggregation of names, not necessarily having a name itself". Classes which are not sets may be termed "proper classes". Proper classes have elements but are not eligible to be elements.

Because we accept informally that classes which are not sets have some sort of existence, we will not be *too* wary in the sequel about using set-builder notation $\{x \mid \phi\}$ (and "function-builder notation" $(x \mapsto T)$ when this is introduced) to refer to classes which might not be sets. But we will always attempt to warn the reader when this may be happening; we will not formally admit proper classes as objects of our theory.

Our discussion so far is a philosophical analysis of the term "set" (and related terms) which should not prejudice us in our choice of a formal system of set theory. The Russell paradox may lose some of its "paradoxical" flavor as a result of this informal analysis (we hope so). We think of the paradox as not representing any kind of difficulty with the nature of reality or fundamental limitation on human understanding (as some seem to think), but as representing a mistake in reasoning: the failure to appreciate the difference between "arbitrary collection of labels of objects" and "arbitrary collection of labels of objects which itself has a label". It is quite reasonable that any condition at all should define an arbitrary collection of labels of objects; but it is also quite reasonable that a collection defined in terms of one's scheme for labelling collections may prove impossible to label!

8.2. Motivating Stratified Comprehension

The choice of a particular set theory is determined by what conditions one thinks *ought* to define sets. We turn to a motivation of our particular choice (stratified comprehension).

The motivation we propose here is related to the discipline of abstract data types in computer science. Properties of the concrete implementation of an abstract object which do not reflect properties of the object being implemented should not be used by the programmer when manipulating the implemented abstraction. The abstraction that we are trying to implement with sets is the notion of a collection, a pure extension. This coincides with the notion of *class* in our intuitive picture. A set implements a class, but it has an additional feature, the name or label associated with it. A "type-safe" property of sets or operation on sets should not depend on details of which objects "label" which collections of objects. Any relationship between the "label" attached to an extension and the elements of the extension is an "implementation-dependent" feature of a set, and not a feature of the class it implements. Self-membership of a set, for example, is the property of identity between the label and one of the parts of the extension, which is clearly not a property of the extension of the set; if the same extension were labelled differently, it would cease to be an element of itself.

This gives us an *a priori* reason to reject $\{x \mid x \in x\}$, for example, as a set definition, prior to any consideration of paradox (in this case, there is no paradox to consider unless we go on to allow other set constructions, such as complement). The same reasoning allows us to reject the definition of the Russell class as a reasonable specification for a set *before* discovering that it is impossible to implement.

One possible analysis of the reason for rejecting $\{x \mid x \in x\}$ (or its paradoxical complement) is that the same object x appears in the sentence $x \in x$, further analyzed as $\{x\} \subseteq x$, in two different roles, as a label (in which role we are concerned with it only as an unanalyzed bare object) and as representing an extension. Further analysis reveals that there is a hierarchy of further roles. An object can be considered as a bare object (as a label), or as the associated extension, a class. But, further, we may consider an object via the associated extensions of the members of its associated extension, viewing it as a class of classes of bare objects. We may consider classes of classes of classes of objects, and so forth. These roles can be indexed by the natural numbers: an object being considered by itself is taking role 0, while an object being considered as a collection of objects themselves taking role n is taking role $n + 1$. The relationship between any two distinct roles of one and the same object can be perturbed by permuting the underlying "labelling" scheme; we should reject any definition of a set in which any object appears in more than one role as "implementation-dependent". This is exactly the criterion of stratification: in any sentence "$x \in y$", the role taken by y should be expected to have index one higher than that of the role taken by x if a coherent assignment of roles to objects is possible; the relations of equality and left and right projection (π_1 and π_2) are understood as relations between bare objects, and so make sense only for pairs of objects playing the same role.

Historically, the motivation of stratification was as a simplification of Russell's type theory, in which the different "roles" (object, class of objects, class of classes of objects, etc.) are played by objects of different sorts or "types". Our motivation here is different from the historical motivation in assuming from the outset that there is only one sort of object. We will use the word "type" to refer to the different "roles", sometimes qualifying it as "relative type" to remind ourselves that we do not assume the existence of different kinds of object, as in Russell's original theory, but of different levels of permitted access to one kind of object.

There is some formal mathematical support for our motivation; it has been shown that the stratified conditions are exactly those which are invariant under all redefinitions of the membership relation by permutation which preserve the totality of represented sets in a certain technical sense.

There is nothing in our metaphor to suggest that every object that has a label (and so is capable of being an element of our universe) is an aggregate of labels (a set). In fact, the Axiom of Choice will allow us to prove that there must be elements which are not sets.

8.3. Technical Remarks

It needs to be noted that just because a set is defined by an unstratified condition does not mean that it *cannot* exist; it simply means that it *might not* exist; we are free to reject such a set if it turns out that its existence would lead to problems. An example of a set which is impossible to define in a stratified manner, but which can (but need not) exist in our set theory is the collection having as members exactly the von Neumann numerals '0', '1', '2',... defined earlier.

The Stratified Comprehension Theorem is stated for sentences ϕ in which the only names of objects used are variables. But recall that we can use the Theory of Descriptions to introduce names (the x such that ϕ), where ϕ is a sentence.

It is straightforward to determine by examining the way in which names (the x such that ϕ) are eliminated that a sentence containing one or more such descriptions will be stratified after the term is eliminated under the following refinement of the usual conditions: relative types should be assignable to all terms (descriptions as well as variables) with the same rules for atomic sentences, and, in addition, the same type must be assigned to the variable x as to any name (the x such that ϕ) in which it is bound (since instances of (the x such that ϕ) will actually be replaced with instances of x when the description is eliminated). Notice that a name (the x such that ϕ) cannot appear in a stratified sentence unless it is itself stratified (since the sentence ϕ will still appear as part of the larger sentence when the name is eliminated).

Operations in our set theory are theoretically implemented by complex names with free variable parameters. If a name has variables in it, it is important to be aware of the difference in type ("role") between the name and the free variables which appear in it. Examples: (x, y) means "The z such that $z\ \pi_1\ x$ and $z\ \pi_2\ y$"; here z, and so the whole name, has the same type as x or y; on the other hand, $\{x\}$ means "the y such that $x \in y$ and for all z, if $z \in y$ then $z = x$"; here y, and so $\{x\}$ must have type one higher than that of x. Both of these examples are easily handled using the intuitive concept of "roles"; the projections of a pair play the same "role" as the pair itself (we regard pairing as a construction acting on bare objects), while if x is playing the role of a bare object, it should be clear that $\{x\}$ is playing the role of a collection of bare objects! This, in turn, would enable us to conclude that the relation which every x has to its singleton $\{x\}$ is not legitimate. We are not planning to make explicit use of the Theory of Descriptions; we noted its existence to justify the use of names in general and their role in relation to the notion of stratification.

Another fact about stratification restrictions should be noted: a variable free in ϕ in a set definition $\{x \mid \phi\}$ may appear with more than one type without preventing the set from existing, as long as this set definition is not itself embedded in a further set definition. The reason for this is that such a set definition can be made stratified by distinguishing all free variables: the resulting definition is supposed to work for *all* assignments of values to those free variables, including those in which some of the free variables (even ones of different type) are identified with one another. For example, the set $\{x, \{y\}\}$ has a stratified definition; the set $\{x, \{x\}\}$ has an unstratified definition, but the existence of $\{x, \{y\}\}$ for all values of x and y ensures the existence of $\{x, \{x\}\}$ for any x. But a term $\{x, \{x\}\}$ *cannot* appear in the definition of a further set in which x is bound.

The Stratified Comprehension Theorem can be used as an axiom (technically, an axiom scheme) of this set theory. If it is used as an axiom, the only other axioms which are needed (of those introduced so far) are those of Extensionality, Atoms, and Ordered Pairs. Look at the definitions given in set-builder notation for the objects stated to exist by each other axiom, and you will see that each of the other axioms is actually a case of Stratified Comprehension.

The use of a primitive ordered pair is not strictly necessary for the development up to this point, although it is far more convenient than the alternative. The Kuratowski pair $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ could have been used instead, but the forms of certain axioms would have been very hard to motivate without the notion of stratification having already been introduced. The main point is that the relations π_1 and π_2 do not exist for the Kuratowski pair, because if x and y are assigned type n, the Kuratowski pair $\langle x, y \rangle$ is assigned type $n+2$. The relations κ_1 and κ_2 which $\{\{x\}\} = \langle x, x \rangle$ and $\{\{y\}\} = \langle y, y \rangle$, respectively, have to $\langle x, y \rangle$ have to be used instead. The other axioms are true for the Kuratowski pair. The results on representation of sentences of first-order logic are also true for the Kuratowski pair, but the Axiom of Set Union has to be used to collapse sets of objects $\langle x, x \rangle$ (for example) down to the corresponding sets of objects x at certain points in the proof, and the Axiom of Singleton Images has to be used to reverse this process. The use of a primitive ordered pair satisfying our form of the Axiom of Projections does have some additional strength; it allows one to prove that the universe is infinite, which we could not do with the other axioms we have up to this point. We will not consider the use of the Kuratowski pair further.

Cahiers du Centre de logique
Volume 10

Chapter 9

Special Kinds of Relations: Equivalence and Order

We now consider some interesting properties which may belong to particular relations. A relation R is said to be *reflexive* if $x\ R\ x$ for every x in $\mathrm{Dom}(R)$ (recall that this denotes the full domain of R, the union of its domain and range). This can be readily expressed as

$$([=] \cap (\mathrm{Dom}(R) \times \mathrm{Dom}(R))) \subseteq R.$$

A relation R is said to be *symmetric* if $x\ R\ y$ implies $y\ R\ x$. This can be expressed very briefly as $R^{-1} = R$. A relation is said to be *transitive* if ($x\ R\ y$ and $y\ R\ z$) implies $x\ R\ z$. This can be abbreviated $R|R \subseteq R$.

9.1. Equivalence Relations and Partitions

A relation is called an *equivalence relation* if it is reflexive, symmetric, and transitive. The "smallest" equivalence relations are subsets of $[=]$; the "largest" equivalence relations are of the form $X \times X$. Each equivalence relation R is related to a "partition" of $\mathrm{dom}(R)$ into disjoint sets:

Definition. *A set P of non-empty subsets of a set X is called* a partition *of X if for any sets A, B in P, A and B are disjoint (P is a pairwise disjoint collection of non-empty sets) and $\bigcup[P] = X$ (the collection P "covers" all of X).*

Definition. *Let X be a set. An equivalence relation R is called an* equivalence relation on X *exactly if* $\text{dom}(R) = X$.

Theorem. *Let X be a set, and let R be an equivalence relation on X. Let $[x] = \{y \mid y\ R\ x\} = \text{dom}(R \cap (V \times \{x\})$ for each x in X. Then the collection X/R of all sets $[x]$ exists and is a partition of X. Moreover, for any partition P of X, the relation $[\sim_P] = \{(x, y) \mid$ for some $A \in P$, $x \in A$ and $y \in A\}$ exists, is an equivalence relation on X, and $X/[\sim_P] = P$. Each equivalence relation uniquely determines the associated partition and vice versa.*

PROOF. — The collection $X/R = \{A \mid$ for some x, for all y, $y \in A$ iff $y\ R\ x\}$ exists by the Stratified Comprehension Theorem. It is a partition of X: each x belongs to $[x]$, since $x\ R\ x$; any y in $[x]$ must belong to X, since $y\ R\ x$ and $X = \text{dom}(R)$; thus the union of all the $[x]$'s covers all of X and no more than X; if $[x]$ and $[y]$ meet, they are equal, because if z is in both, we have $z\ R\ x$ and $z\ R\ y$, thus, by symmetry and transitivity, $x\ R\ y$ and $y\ R\ x$, thus any $w\ R\ x$ or $w\ R\ y$ implies $w\ R\ y$ or $w\ R\ x$, respectively, and $[x]$ and $[y]$ are the same set; thus two sets $[x]$ and $[y]$ are disjoint if they are different. The relation $[\sim_P]$ obviously exists, by Stratified Comprehension, and is easily seen to be an equivalence relation. The one-to-one correspondence between equivalence relations and partitions is easy to verify. □

The classes $[x]$ are called "equivalence classes"; we suggest the notation $[x]_R$ for them if the dependence on R must be made explicit. An equivalence relation on a set X is often used to indicate that certain objects in X are to be identified for some purpose; if $x\ R\ y$, the corresponding objects $[x]$ and $[y]$ in X/R are actually equal, so a "version" of X can be constructed in which the desired identifications are actually realized.

A caution: in a stratified set definition, $[x]$ is not at the same "type" as x. As a result, the relation between x and $[x]$ does not necessarily exist. However, the relation between $\{x\}$ and $[x]$ is definable in a stratified way. The use of equivalence classes is slightly more difficult technically for this reason than in the usual set theory; the compensation for this is that many concepts can be modelled in a natural way using equivalence classes which cannot be modelled in this way in the usual set theory!

A solution to this difficulty is to choose an element from each equivalence class $[x]$ as a "representative" of that equivalence class. The advantage of this is that the relation between x and the representative element of $[x]$ is stratified; the representative element of $[x]$ is playing the same "role" as x in this construction. Although the construction of a set by such choices seems intuitively reasonable, it is not enabled by any of our axioms so far, so we introduce the dreaded

Axiom of Choice. *For each set P of pairwise disjoint non-empty sets, there is a set C, called a* choice set *from P, which contains exactly one element of each element of P.*

On this axiom, there will be a great deal more later. For the moment, we observe that the Axiom of Choice allows us to "collapse" equivalence classes to single elements without introducing a stratification problem.

9.2. Order

A class of important kinds of relation are the various kinds of "order". A relation R is called *antisymmetric* if for all x, y, $(x\ R\ y$ and $y\ R\ x)$ implies $x = y$; equivalently, R is antisymmetric iff $R \cap R^{-1} \subseteq [=]$.

Definition. *A relation R is called a* (partial) order *if it is reflexive, antisymmetric and transitive.*

It is called a partial order on X *if* $\mathrm{dom}(R) = X$.

The most natural example of an equivalence relation is equality itself; the most natural example of a partial order is inclusion. Another interesting partial order is the extension relation on functions (introduced below; it is the partial order on functions induced by inclusion).

Where R is a partial order and $X = \mathrm{dom}(R)$ (and thus $X = \mathrm{rng}(R)$ as well), one often sees mention of R suppressed and X termed a "partially ordered set" with R understood. We will attempt to avoid this; we will refer to X as the domain of the partial order R, which itself is sufficient to determine X. Similarly, we consider the mathematical structure of the natural numbers (re Peano arithmetic) to be the ordered pair (addition, multiplication); all other elements usually taken to be parts of this structure (such as the set $\mathcal{N}$ itself) can be derived from these by set-theoretical operations.

If one wonders why general orders are called "partial", one needs only to attend to the following

Definition. *A* linear *or* total *order is an order R such that for each x, y in* $\text{dom}(R)$*, either $x\ R\ y$ or $y\ R\ x$.*

The most natural example of a linear order is the usual order relation "less than or equal to" on the real numbers. By analogy with this example, orders in many contexts (including partial orders) are written $\leqslant$. When $\leqslant$ is a general order, we say $x < y$ when $x \leqslant y$ and $x \neq y$. If $\leqslant$ is an order, its converse is also an order, which we write $\geqslant$, and $>$ is defined similarly. The relations $<$ and $>$ are called "strict orders". Words like "smaller", "larger", "less", "greater", will be used in conformity with the analogy with the usual order from arithmetic. An element s of the domain of $\leqslant$ is said to be "least" if $s \leqslant x$ for all x in the domain; s is said to be "minimal" if there is no x such that $x < s$. These notions coincide for linear but not for partial orders. The notions of "greatest" and "maximal" are defined similarly.

Suppose that A is a subset of the domain of $\leqslant$. Then any b such that $b \leqslant a$ for all a in A is called a *lower bound* of A; if there is a greatest among the lower bounds of A, it is called the greatest lower bound or glb of A relative to $\leqslant$ and written $\inf_{\leqslant}[A]$ ($\leqslant$ may be omitted where understood). The notions of *upper bound* and *least upper bound* (lub; written $\sup_{\leqslant}[A]$) are defined analogously. A partial order relative to which all subsets of the domain have glb's and lub's is called *complete*.

If $\leqslant$ is a partial order, X is its domain and x is an element of X, the segment determined by x, written $\text{seg}_{\leqslant}\{x\}$, or $\text{seg}\{x\}$ if $\leqslant$ is understood, is the set $\{y \in X \mid y < x\}$. Note that the strict order is used in this definition, and that the type of $\text{seg}\{x\}$ is one higher than the type of x. If the non-strict order is used instead, we get the "weak segment"; we use $\text{seg}^{+}\{x\}$ to denote the weak segment for x. The partial order $\leqslant$ on X is precisely parallel to the inclusion relation on the set of weak segments relative to $\leqslant$, for any partial order $\leqslant$, with a type differential; this reduces all partial orders to the canonical one of inclusion, in some sense.

Observe that there is a set containing all partial orders in this theory; this will be true of many natural categories of mathematical structures which would be "too large" to collect in orthodox set theory.

Exercises

(a) Prove or give a counterexample: a transitive, symmetric relation must be reflexive. (The outcome here is affected by a difference between my definition of reflexivity and the usual one).

(b) Give an example of a collection of sets $\mathcal{S}$ such that the restriction of inclusion to $\mathcal{S}$ is a linear order.

(c) Verify our assertion that the definition of the set X/R is stratified.

(d) Let $\leqslant$ be a partial order. Does the Stratified Comprehension Theorem allow us to assert the existence of a set $\{(x, \mathrm{seg}^+_{\leqslant}\{x\}) \mid x \in \mathrm{dom}(\leqslant)\}$ witnessing the parallelism in structure between a partial order and the inclusion relation on its weak segments? If not, how could this set definition be modified so that it would work?

Cahiers du Centre de logique
Volume 10

Chapter 10

Introducing Functions

Definition. *A* function *is a relation F with the property that for each x in* dom(F)*, there is exactly one y such that $x\ F\ y$. We introduce $F(x)$ as a name for this unique y; we define $F(x)$ as (the y such that (F is a function and $x\ F\ y$)). Notice that this definition causes values of non-functions and values of functions outside their domains to become* $\{\,\}$.

Observe that the type of F is one higher than the type of x when $F(x)$ appears in a stratified sentence. The word "map" is used as a synonym for "function"; F is said to "map" x to $F(x)$.

We will generally write $F(x) = y$ instead of $(x, y) \in F$. y is called the value of F at x; we also say that F sends x to y; x is called the *argument* of F in the expression $F(x)$ The symbol $f : X \to Y$ may be used to mean "f is a function from X to Y", or, more explicitly, "f is a function, dom(F) $= X$ and rng(F) $\subseteq Y$".

Definition. *The set $[X \to Y] = \{f \mid f \text{ is a function from } X \text{ to } Y\}$ exists by the Stratified Comprehension Theorem. Another notation frequently used for this set is Y^X. Observe that the type of $[X \to Y]$ is one higher than the type of X or Y in a stratified sentence.*

Since functions are relations, we have already defined the notions of *domain* and *range* for functions. We define some related notions.

Definitions. *A function in* $[X \to Y]$ *with the property that* $\mathrm{rng}(f) = Y$ *is said to be* onto Y *and is said to be a* surjection *(relative to* Y*).*

A function f *with the property that* f^{-1} *is also a function is said to be* one-to-one *or an* injection *(not only does* $x\ F\ y$ *uniquely determine* y*, given* x*, but it also uniquely determines* x*, given* y*;* F *associates each* x *in* $\mathrm{dom}(F)$ *with one and only one* y*).*

A map in $[X \to Y]$ *which is one-to-one and onto* Y *is called a* bijection *from* X *onto* Y *(or* between X *and* Y*). (Any injection from* X *to* Y *might be referred to as a bijection from* X into Y*).*

We pause to breathe before introducing more

Definitions. *Suppose that* $\mathrm{dom}(F) = X$ *and* $A \subseteq X$*. We define the* restriction *of* F *to* A*,* $F\lceil A = F \cap (A \times V)$*.* F *itself is said to be an* extension *of* $F\lceil A$*.*

The image *of* A *under* F*,* $F[A]$*, is defined as* $\mathrm{rng}(F\lceil A)$*; this is the set of all values of* F *at elements of* A*. The* inverse image $f^{-1}[A]$ *is defined as* $\mathrm{dom}(F \cap (V \times A))$*; although this may seem a less natural concept than the image, it turns out to have much nicer properties (notice that* f^{-1} *is not always a function).*

The notions of image and inverse image make sense for general relations, not just for functions.

A related notion is the preimage *of an element* x *of the range of a relation* R *(not necessarily a function) under that relation: it is defined as the inverse image of* $\{x\}$ *under* R*.*

If $X \subseteq Y$, there is an obvious function in $[X \to Y]$ which takes each element of X to itself; this is called the *inclusion map* of X (and actually does not depend on Y at all; it is simply $[=] \cap (X \times X)$). The inclusion map of X into X (the same object) is called the *identity map* on X. Note that $[=]$ is the identity map on the universe. The identity map on Y, restricted to $X \subseteq Y$, yields the inclusion map from X into Y.

Observe that $[=]$ is not alone among our primitive relations in being a function; the projection relations π_1 and π_2 are also functions, with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all x, y. Note that we write $f(x, y)$ instead of $f((x, y))$. We define *projections* of subsets of Cartesian products as their images under the projection functions; of course, there is a conflict here, since we also call the values under the projection functions "projections". It should always be possible to resolve this issue in context. The words "domain" and "range" are also available for these images, of course.

Any function F from X to Y can be used to define an equivalence relation (and a partition) on X: the equivalence relation is $R = \{(x, y) \mid x, y \in X$ and $F(x) = F(y)\}$, and the corresponding partition is the collection of sets $F^{-1}[\{y\}]$, the inverse images under F of singletons of elements y of Y. The map which sends $F^{-1}[\{y\}]$ to $\{y\}$ for each y is a one-to-one map from X/R to $\mathcal{P}_1\{Y\}$; we can use the Axiom of Choice to define a one-to-one map from representatives of the equivalence classes in X/R onto Y.

We can associate a function with each set in a natural way. We introduce the definitions of the numbers 0 and 1: 0 is defined as $\{\{\ \}\}$ (the set of all sets with zero elements) and 1 is defined as $\mathcal{P}_1\{V\}$, the set of all sets with one element. The *characteristic function* $\text{Char}(A)$ of a set A is the function $\{(x, y) \mid \text{if } x \in A \text{ then } y = 1 \text{ and if not}(x \in A) \text{ then } y = 0\}$; $\text{Char}(A)(x) = 1$ if x is an element of A and 0 otherwise. The operation Char is a function, since the type of A is the same as the type of $\text{Char}(A)$ in a stratified sentence.

The collection $[X \to \{0, 1\}]$ of restrictions of characteristic functions to X, which is more usually called the collection of characteristic functions on X, encodes all the subsets of X. We introduce the set of all subsets of X:

Definition. *$\mathcal{P}\{X\}$ is defined as $\text{dom}([\subseteq] \cap (V \times \{X\}))$, the collection of all subsets of X. We call $\mathcal{P}\{X\}$ the* power set *of X. Notice that the type of $\mathcal{P}\{X\}$ in a stratified sentence would be one higher than the type of X.*

Theorem. *There is a natural bijection between $\mathcal{P}\{X\}$ and $[X \to \{0, 1\}]$.*

PROOF. — The map which takes each subset A of X to $\text{Char}(A)\lceil X$ can be shown to exist using Stratified Comprehension. □

We introduce "function-builder" notation analogous to the "set builder" notation $\{x \mid \phi\}$:

Definition. *If T is an expression and x is a variable, we define $(x \mapsto T)$ as (the function F such that for all x, $F(x) = T$). There must be no occurrences of F in T.*

An example is the typical function $f(x) = 2x + 1$ from algebra, which would be written $(x \mapsto 2x + 1)$ in this notation. $(x \mapsto T)$ will be defined if the expressions x and T would be assigned the same type in a stratified sentence (which includes the implicit condition that T *can* appear in such a sentence; some expressions, like $x \cap \{x\}$ or $x(x)$, cannot appear in such a sentence at all, so certainly cannot appear with the same type as x). Another notation for $(x \mapsto T)$ is the notation $(\lambda x.T)$ of the "lambda-calculus". The construction here is called "function abstraction" from the name or "term" T, and is formally analogous to set comprehension $\{x \mid \phi\}$ from sentences ϕ.

The notation $(x \mapsto T)$ will also sometimes be used informally for "functions" which actually do not exist as sets in our theory. We will for example refer from time to time to the "map" $(x \mapsto \{x\})$, which, as we will see below, is not a set. You should be able to see now that its definition is not stratified. We also occasionally refer to restrictions of proper class functions and images under proper class functions (which may be proper classes) in the sequel. All instances of such notations should be possible to eliminate in principle, as we do not formally admit that there are such objects as proper classes or proper class functions.

Exercises

(a) Verify our assertions about the stratification and relative type of $[X \to Y]$.

(b) Develop a definition of the number 2 by analogy with the definitions of 0 and 1 given in the chapter. Give the definition in set-builder notation, then give it in terms of the primitive operations for constructing sets.

(c) What is the type of $(x \mapsto T)$ relative to the term T? To take a specific example, is the definition of the function $(a \mapsto (x \mapsto ax))$ which takes a number a to the function "multiply by a" a stratified definition (on reasonable assumptions about multiplication)?

(d) Verify our assertion that $(x \mapsto \{x\})$ cannot be shown to exist using Stratified Comprehension, by expanding its definition into set-builder notation and verifying that it is not stratified.

Cahiers du Centre de logique
Volume 10

Chapter 11

Operations on Functions and Families of Sets

11.1. Operations on Functions

The basic operations on functions are analogous to those on relations.

Definition. *If f and g are functions, we define the* composition *of f and g, denoted by $(f \circ g)$ or fg, as $g|f$, the relative product of g and f.*

The change in order is motivated by the fact that $(f \circ g)(x) = f(g(x))$ (when $\mathrm{rng}(g) \subseteq \mathrm{dom}(f)$). There are mathematicians who feel that it is more natural to put the function name after the argument, writing $(x)f$ instead of $f(x)$; these will write $g|f$ as gf as well. In some sense, the order of the relative product is more natural than the order of composition, because our feeling about $f(g(x))$ is that we use g to transform x to $g(x)$, *then* use f to transform $g(x)$ to $f(g(x))$. There is another minority opinion among mathematicians that we ought to define a function f as $\{(y, x) \mid y = f(x)\}$; holders of this view have no trouble with the notation for composition.

The converse f^{-1} of a function is not usually a function. If f^{-1} is a function (i.e., if f is one-to-one), then it is called the *inverse* of f. The relation of inverse functions to composition deserves to be noted:

$$(f \circ g)^{-1} = (g^{-1} \circ f^{-1}).$$

We observe that the operation of taking inverse images "commutes with Boolean operations":

$$f^{-1}[A^c] = \mathrm{dom}(f) - f^{-1}[A] \qquad \text{(complement relative to dom}(f));$$
$$f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B].$$

One might think that the "image" under f, a more natural notion on the surface, would have nicer behaviour, but this is not the case. We give some typical results:

$$f[A \cup B] = f[A] \cup f[B];$$
$$f[A \cap B] \subseteq f[A] \cap f[B] \qquad \text{(equality does not hold);}$$
$$\mathrm{rng}(f) - f(A) \subseteq f[A^c];$$

and indicate the relation between image and inverse image (they are not inverse operations!):

$$f[f^{-1}[A]] = A;$$
$$f^{-1}[f[A]] \supseteq A.$$

11.2. Operations on Families of Sets

We now turn our attention to some operations in families of sets. One of these, the operation of set union, we have already introduced. We will use the set union $\bigcup[A]$ to illustrate the connection with functions.

There is a tendency to consider a family of sets A via a function F from an index set I to the set A; such a function F is called an "indexed collection" representing the family $A = \mathrm{rng}(F)$ via the "index set" $I = \mathrm{dom}(F)$. $F(i)$ is often written a_i, leaving the function F anonymous; operations on indexed collections, like the Cartesian product $\prod[F]$ which we define below, would be written $\prod_{i \in I}[a_i]$ in this style, or simply $\prod[a_i]$. In certain cases, the presence of the index set I and the anonymous function from I to A is simply redundant; one context in which it does make sense is when the operations on families of sets are being considered as infinite analogues of binary operations; the index formalism makes it possible to formalize infinitary analogues of the commutative, associative, distributive, and other algebraic laws, because they give the "infinite algebraic expressions" more structure. Another case is where duplications of objects in the family A have an effect on the result of the operation. We will suppress the use of indexed collections where we can; for example, we do not define set union or intersection for indexed collections.

It is worth noting a relationship between the power set operation and the set union operation:

$$\bigcup[\mathcal{P}\{A\}] = A; \qquad \mathcal{P}\{\bigcup[A]\} \supseteq A.$$

We now formally introduce the operation of set intersection.

Theorem. *$\bigcap[A] = \{x \mid$ for all B, if $B \in A$, then $x \in B\}$, called the* intersection of the set of sets A*, exists.*

Proof. — Use the Stratified Comprehension Theorem. □

$\bigcap[A]$ is the complement of the union of the set of complements of elements of A, a fact analogous to the relationship between binary intersection and union.

An observation worth making is that the operation of taking inverse images under a function "commutes with" set intersection and set union just as it does with the Boolean operations.

Like set intersection, the next operation we introduce extends a binary operation we have already defined for sets.

Theorem. *Let F be a function whose range is A. $\prod[F] = \{f \mid f$ is a function from $\mathrm{dom}(F)$ to $\mathcal{P}_1\{\bigcup[A]\}$ (the set of singletons of elements of elements of A) such that for each $i \in \mathrm{dom}(F)$, $f(i) \subseteq F(i)\}$, called the* Cartesian product *of F, exists. We may use $\prod[A]$ to abbreviate $\prod[[=]\lceil A]$; we signal this by saying that we use A to index itself.*

Proof. — Use the Stratified Comprehension Theorem (the application of stratified comprehension in this theorem is fully analyzed in the next section). The use of functions from elements of A to singletons of elements of elements of A, rather than to the elements of the elements of A themselves, is dictated by stratification requirements. Note that the functions which are elements of $\prod[A]$ each encode the choice of one element from each element of A. $\square$

The analogy between the Cartesian product of a family of sets and the Cartesian product of a pair of sets is not as close as the analogies in the cases of set vs. Boolean intersection and union. Since the presence of duplicate sets in a Cartesian product makes a difference, we must use the function F, whose domain is an "index set" (the use of an index set here can be used to see how an index set would be used in the cases of set union and intersection). The elements of the Cartesian product of a set are not n-tuples (this would not be feasible if the size of A were not any finite n!), but functions with domain $\text{dom}(F)$. There is a type-differential between the elements of a set Cartesian product and the elements of the sets in the product, which is not the case in the binary Cartesian product. The Cartesian product of a two-element set is formally related to the Cartesian product of the two sets, but it is not the same object ($\prod[\{A, B\}]$ has the function which sends A to $\{a\}$ and B to $\{b\}$ in place of each (a, b) in $A \times B$).

The use of any of these operations involves attention to type differentials: the types of $\bigcup[A]$ and $\bigcap[A]$ are one lower than the type of A, while the type of $\prod[F]$ is one higher than the type of F (similarly for $\prod[A]$). This means that $(A \mapsto \prod[A])$, for example, is not a function; this is also the case for the other two operations. A function which does exist is the operation $(A \mapsto \mathcal{P}\{\bigcup[A]\})$ alluded to in passing above; the type-raising effect of power-set and the type-lowering effect of set union cancel out.

Finally, we observe that the apparently obvious assertion that the Cartesian product of any collection of non-empty sets is non-empty is precisely equivalent to the Axiom of Choice. The proof will motivate the introduction of a new binary operation on sets and the corresponding operation on families of sets.

The Axiom of Choice allows us to select an element from each compartment of a partition of any set. This is precisely what is needed to construct an element of $\prod[A]$ if A is a pairwise disjoint collection of sets. The trick needed for the proof of the general result is a device for constructing a disjoint family of sets from a family of sets which is not disjoint. This is the set-theoretical operation of forming "disjoint sums".

Definition. *If A and B are sets, the* disjoint sum *of A and B, written $A \oplus B$, is the set $\{(x, y) \mid (x \in A$ and $y = 0)$ or $(x \in B$ and $y = 1)\}$.*

This is a binary operation, so not immediately useful to us. Note that the types of A, B, and $A \oplus B$ in a stratified sentence will all be the same. The corresponding operation on families of sets is less straightforward (it requires "indexing").

Definition. *If F is a function, $\sum[F]$, the* disjoint sum *of F, is the set $\{(\{b\}, i) \mid b \in F(i)$ and $i \in \mathrm{dom}(F)\}$. We may use $\sum[A]$ to abbreviate $\sum[[=]\lceil A]$, the sum of the identity function on A; we say then that we use A to index itself.*

Notice that the elements of $\sum[F]$ are not "labelled" elements of elements of $\mathrm{rng}(F)$, but "labelled" singletons of elements of elements of $\mathrm{rng}(F)$; this is needed to satisfy stratification requirements. $\sum[F]$ is actually of the same type as F, unusually. $\sum[F]$ is naturally partitioned into the pairwise disjoint sets $\{(\{b\}, i) \mid b \in F(i)\}$ for each i in $\mathrm{dom}(F)$. As with the Cartesian product, the disjoint sum of a one-to-one function whose range has two elements is not the same as the disjoint sum of the two elements of that range, although they are closely related.

We are now ready to prove the promised

Theorem. *The Axiom of Choice is equivalent to the assertion that all Cartesian products of indexed collections of nonempty sets are nonempty.*

Proof. — We indicate how to use the Axiom of Choice to construct an element of the Cartesian product of any "indexed collection" F of nonempty sets. Observe that the disjoint sum $\sum[F]$ can be partitioned in a natural way into the collections $\mathcal{P}_1\{F(i)\} \times \{i\}$ for each i in $\mathrm{dom}(F)$. We use the Axiom of Choice to construct a choice set for this partition; this will be a set with a single element $(\{b\}, i)$ such that $b \in F(i)$ for each i in $\mathrm{dom}(F)$. This set is a relation, whose converse will be a function taking each element of $\mathrm{dom}(F)$ to the singleton of an element of $F(i)$, which will be an element of $\prod[F]$ by the definition of that set.

Given a collection P of pairwise disjoint nonempty sets, take $\prod[[=]\lceil P]$ (the self-indexed product of the collection P, which we might write $\prod[P]$

if we were being slightly less formal); given that $\prod[[=]\lceil P]$ is nonempty, we can construct a choice set for P as the union of the range of an element of the self-indexed Cartesian product of P. This completes the proof. □

11.3. A Case Study in Stratification

In this section, we will analyze one of the constructions of the last section with an eye to justifying assertions about relative type and stratification, and with an eye to giving a reader a model of how to go about supplying such justifications himself for later assertions of this sort which are not given explicit support.

Our example will be the definition of the Cartesian product of a collection of sets "indexed" by a function F. We reproduce the relevant theorem from above:

Theorem. *Let F be a function whose range is A. $\prod[F] = \{f \mid f$ is a function from $\mathrm{dom}(F)$ to $\mathcal{P}_1\{\bigcup[A]\}$ (the set of singletons of elements of elements of A) such that for each $i \in \mathrm{dom}(F)$, $f(i) \subseteq F(i)\}$, called the* Cartesian product *of F, exists. We may use $\prod[A]$ to abbreviate $\prod[[=]\lceil A]$; we signal this by saying that we use A to index itself.*

We want $\prod[F]$ to contain elements representing each way to choose an element from each element of the range of the function F. We need to assign "roles" or relative types to all objects involved in this construction.

Observe that each element of the range A of the function F is supposed to be a set, since we need to choose an element from each of them. These elements appear to be the objects of lowest type in the construction, so we assign these relative type 0 (we think of these as "bare objects"). This implies that each element B of the range A of F should be assigned relative type 1 (we think of these as "classes of bare objects"). F itself is a collection of ordered pairs of the form (i, B), where i is an element of some index set I (the domain of F), and B is a set belonging to A, the range of F. The set B has already been assigned relative type 1. An ordered pair will have the same relative type as each of its elements, so we see that we must assign (i, B) and the index i relative type 1 as well. This means that F (as a class containing (i, B)), the index set I (as a class containing i) and the range A of F (as a class containing B) must be assigned relative type 2 (we think of I and A as "classes of classes of bare objects"; F is a "class of pairs of classes of bare objects", and we assign the same role to pairs of classes that we assign to classes). We have completed the assignment of roles or relative types to objects that we are given before the construction starts.

An element of the product we are constructing is supposed to associate each element i of the index set I with an element x of the set $A = F(i)$. Naively, we would like to make this element a function f with elements of the form (i, x). But this is forbidden by considerations of relative type: we have already seen that we must assign relative type 1 to each index $i \in I$ and relative type 0 to each element x of an element B of the range A of F. The standard solution to this difficulty is to recall that we can associate the singleton $\{x\}$ of relative type 1 with the object x of relative type 0. We revise our intentions accordingly, using the pair $(i, \{x\})$ to code the choice of the element x from the set $F(i)$. A function f belonging to the product to be constructed will be a set of pairs $(i, \{x\})$ for $i \in I$ and $x \in F(i)$. Note that such functions f will be assigned relative type 2 in this definition, since it is a collection of pairs $(i, \{x\})$ which must be assigned the same relative type 1 as their projections i and $\{x\}$. Moreover, such functions will have domain $I = \mathrm{dom}(F)$ as stated in the Theorem. Each element x belongs to the union of the range A of F, so each singleton $\{x\}$ belongs to $\mathcal{P}_1\{\bigcup[A]\}$, as stated in the theorem: the need to use the $\mathcal{P}_1$ construction reflects the underlying need to use the singleton construction. This justifies the assertion in the theorem that each function f belonging to the product will be a function from $\mathrm{dom}(F)$ to $\mathcal{P}_1\{\bigcup[A]\}$; we further select as elements of $\prod[F]$ only those pairs $(i, \{x\}) = (i, f(i))$ such that $x \in F(i)$, or, equivalently, $\{x\} = f(i) \subseteq F(i)$. We have now motivated the form of the definition of $\prod[F]$ in terms of considerations of stratification. Since we can see that the definition of $\prod[F]$ is stratified, we see that $\prod[F]$ exists (the need to verify this is the reason that the "definition" of $\prod[F]$ is actually a theorem).

Further, we see that $\prod[F]$ must be assigned relative type 3, since its elements f are assigned relative type 2. The parameter F in the definition of $\prod[F]$ was assigned relative type 2 in our analysis, so we see that the type of $\prod[F]$ in any construction will be $3 - 2 = 1$ type higher than the type of its parameter. It is important to recall that our initial assignment of type

0 to the elements of the elements of the range of F was arbitrary; in the context of a larger construction, we might have assigned a different relative type to these elements, with the effect of displacing all other types in the construction by a uniform amount. This would leave invariant the fact that the relative type of $\prod[F]$ must be one higher than that of F, which is the only fact we need to recall for the use of the product construction in further constructions of sets.

11.4. Parentheses, Braces, and Brackets

This is a natural point at which to explain certain conventions that we have followed so far. We have the notation $f(x)$ for the result of applying a function f to an argument x. The observant reader may have noticed that some of the operations we have defined above, such as the domain $\mathrm{dom}(x)$ have had the same format, while others, such as the power set $\mathcal{P}\{x\}$ or the set union $\bigcup[X]$ use different delimiters for their arguments. We do attempt to follow a consistent convention: functions and those operators which can be realized as functions (the domain operation, for example, is actually a function) use parentheses. Operations which raise relative type (usually, but not always, by one) usually use braces, as in $\mathcal{P}\{x\}$. The fact that the braces resemble the braces in the singleton $\{x\}$ is not accidental; an operation on x which raises type by one *can* be understood as the application of a function to $\{x\}$! Other operations (those which lower type or those which abuse it in worse ways) use brackets as delimiters. An exception is that negative "powers" of type-raising operations, though they in fact lower type, seem naturally to call for braces. The use of brackets in the image operation should be distinguishable from the last use in principle: the difference between $f[X]$ (the image of the set X under the function f) and $\bigcup[X]$ (the set union of X) is that the symbol on the left is the name of a set in one case and a special operator in the other. Brackets may also be used for images of sets under type-raising operations. A uniform exception is that all of the "operations on families of sets" introduced in this chapter take brackets, although some *could* take parentheses or braces. None of these uses should conflict with the special use of brackets to convert infix operators to set names as in $[=]$, with the notation $[X \to Y]$, or with the notation for equivalence classes.

Exercises

(a) Explain the precondition $\mathrm{rng}(g) \subseteq \mathrm{dom}(f)$ on the identity $(f \circ g)(x) = f(g(x))$.

(b) Prove the correctness of our assertions about the relationships between the inverse image and image operations and the Boolean operations.

(c) Verify our assertions about the stratification and relative type of the disjoint sum of an indexed family of sets.

Cahiers du Centre de logique
Volume 10

Chapter 12

The Natural Numbers

12.1. Definition of the Natural Numbers. Induction and Infinity

We noted earlier that the numbers are defined in a more usual treatment of set theory as follows: '0' = $\{\ \}$, '1' = $\{$'0'$\}$, '2' = $\{$'0', '1'$\}$, and so forth. These are called the von Neumann numerals, and the existence of each von Neumann numeral is a consequence of our axioms. Observe that the successor 'n'$^+$ of each von Neumann numeral 'n' is 'n' $\cup \{$'n'$\}$, and you will see why these numerals are unsatisfactory from our standpoint. The expression $x^+ = x \cup \{x\}$ cannot appear in a stratified sentence, since it contains x with two different types! If we cannot define the successor operation for these numerals, we cannot use them effectively.

Fortunately, there is a much more natural definition of the natural numbers (actually the original set-theoretic definition of number) which succeeds here (and fails in the more usual treatment of set theory).

We have already defined 0 as $\{\{\ \}\}$ and 1 as $\mathcal{P}_1\{V\}$. 0 is defined as the set of all sets with 0 elements, and 1 is defined as the set of all sets with 1 element (but without the circularity suggested by the English phrases). This seems quite natural. The number 3, for example, can be thought of as a property shared by all sets with three elements; properties are represented by sets, so 3 might be expected to be the set of all sets with three elements.

Now we need to address the question of whether the number 3 as defined above actually exists. What we will do is define the successor operation:

Definition. *For any set of sets A, we define $A + 1$ as the collection*

$$\{a \cup \{x\} \mid a \in A \textit{ and } \textit{not}(x \in a)\}$$

of unions of elements a of A and singletons of objects x not in a.

If the set of all sets with n elements exists (call it "n" for the moment), then it should be clear that "n" + 1 will be the collection of all sets with $n + 1$ elements, for any concrete natural number n. Since we have 0 and 1, we can construct $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$, and so forth.

We are now in a better position than with the von Neumann numerals in two ways: the numerals we are using are more natural objects, and (as can easily be checked) the function $\mathrm{Inc} = (A \mapsto A + 1)$ which realizes the successor operation actually exists. We can go further: we can define the set of natural numbers (or something which looks as if it *ought* to be the set of natural numbers!): it would appear that any set which contains 0 and is closed under the function Inc will contain all natural numbers (by the principle of mathematical induction) and it seems unreasonable that any other objects would belong to all such sets. We make some definitions:

Definitions. $\mathrm{Inc} = (A \mapsto A + 1)$.

Ind, *the collection of* inductive sets, *is defined as* $\{A \mid 0 \in A \textit{ and } \mathrm{Inc}[A] \subseteq A\}$*; an inductive set is one which contains 0 and contains the "successor" of each of its elements.*

$\mathcal{N}$*, the collection of* natural numbers*, is defined as* $\bigcap[\mathit{Ind}]$*, the collection of objects which belong to all inductive sets.*

Fin*, the collection of* finite sets*, is defined as* $\bigcup[\mathcal{N}]$.

A set which is an element of a natural number n may be referred to as a set of size n *or* a set with n elements*; the natural number to which a finite set A belongs may be referred to as the* size of A.

It is straightforward to check that the existence of each of the sets defined above follows from Stratified Comprehension. The definition of Fin may require a little explanation: the union of $\mathcal{N}$ consists of all the elements of the elements of $\mathcal{N}$; the elements of a natural number n are the sets with n elements, so $\bigcup[\mathcal{N}]$ consists of the sets A such that for some natural number n (depending on A), A has n elements, which seems a reasonable definition for the collection of finite sets.

An immediate corollary of the definition of the set $\mathcal{N}$ is the familiar

Principle of Mathematical Induction. *Suppose that S is a set of natural numbers. If $0 \in S$ and for all n, $n+1 \in S$ if $n \in S$, it follows that $S = \mathcal{N}$; all natural numbers are in S.*

A formulation in terms of properties of numbers instead of sets of numbers will work as well, as long as the properties are required to be stratified, so that they define sets. A later axiom will have as a consequence that induction holds for unstratified conditions as well, but induction on unstratified conditions is not needed for arithmetic (it has very strong consequences in set theory).

We prove a perfectly obvious

Theorem. *$\mathcal{N}$ is a partition of* Fin*.*

using the following

Definition. *Sets A and B are said to be* equivalent *if there is a bijection f between A and B.*

and

Lemma. *Equivalence is an equivalence relation on sets.*

PROOF OF THE LEMMA. — Equivalence exists as a relation by Stratified Comprehension. To prove reflexivity, observe that identity functions are bijections. To prove symmetry, observe that inverses of bijections exist and are bijections. To prove transitivity, observe that compositions of bijections are bijections. □

PROOF OF THE THEOREM. — We prove that for any $A \in n$, $B \in n$ iff A and B are equivalent. We prove this by induction: consider the set of all n

such that for any $A \in n$, $B \in n$ iff there exists a bijection f between A and B. Certainly 0 belongs to this set: it has only one element, $\{\ \}$, and there is a bijection between a set and $\{\ \}$ only if the set is itself empty. Suppose that n belongs to this set: then any $A \in n+1$ equals $A' \cup \{x\}$, where $A' \in n$ and x is not in A'. Similarly, any $B \in n+1$ equals $B' \cup \{y\}$ with parallel conditions. By inductive hypothesis, there is a bijection between A' and B', which we extend by mapping x to y. Now suppose that there is a bijection f between an element A of $n+1$ and a set B: $A = A' \cup \{x\}$, and $B = f[A'] \cup \{f(x)\}$. $f[A']$ belongs to n by hypothesis, so B belongs to $n+1$. It follows that if two natural numbers share an element, they are equal (since equivalence is an equivalence relation), so the collection $\mathcal{N}$ of natural numbers is pairwise disjoint as a collection of sets, so partitions its set union Fin. □

There is one possible problem. Suppose that the universe actually contained only n objects for some natural number n. We would then find that the numeral for $n+1$ and each subsequent numeral would be $\{\ \}$, the empty set! Fortunately, we can prove that this is not the case. Usually, a set theory has an Axiom of Infinity to take care of this case; ours is no exception, but we have already smuggled in the Axiom of Infinity in the guise of the Axiom of Projections (this axiom ensures that the relative types of the projections of an ordered pair are the same as the relative type of the pair itself). We can see from above that if the universe were finite, $\{\ \}$ would be a natural number. We prove the

Theorem of Infinity. $\{\ \}$ *is not a natural number.*

PROOF. — We show that for any set A belonging to a natural number n, $A \times \{0\}$ belongs to the same natural number. We do this by induction on the property of n "for all $n \in \mathcal{N}$, $A \in n$ implies $A \times \{0\} \in n$". This is clearly true of 0, since $\{\ \}$ (the sole element of 0) is equal to $\{\ \} \times \{0\}$. Suppose that it is true of n; then the elements of $n+1$ are exactly the objects $A \cup \{x\}$ for $A \in n$ and x not in A; but $A \times \{0\}$ is in n for each such A by inductive hypothesis, and so $(A \cup \{x\}) \times \{0\} = (A \times \{0\}) \cup \{(x, 0)\}$ belongs to $n+1$ (since $(x, 0)$ cannot belong to $A \times \{0\}$, x not being in A). Now our condition on n defines a set, which we have just shown to be inductive, so all elements of $\mathcal{N}$ belong to it. Now we show that the collection of nonempty elements of $\mathcal{N}$ is inductive; certainly 0 belongs to it; if n is nonempty, take an element A of n, and form $(A \times \{0\}) \cup \{(0, 1)\}$, which must belong to $n+1$. □

Corollary. *Every natural number n has an element which is not V.*

Proof. — This is a corollary of the construction: if X belongs to n, $X \times \{0\}$ belongs to n and is not equal to V. This means that whenever we choose an element X of n, we can do it in such a way that there is an x which is not in X. □

The following theorem gives us the capability of inductive *definition* of functions, without any relaxation of stratification restrictions:

Recursion Theorem. *If X is a set, x is an element of X, and f is a function in $[X \to X]$, there is a unique function g in $[\mathcal{N} \to X]$ such that $g(0) = x$ and $g(n+1) = f(g(n))$ for each natural number n.*

Proof. — The function g is defined very much as $\mathcal{N}$ is: it is the intersection of all sets which contain $(0, x)$ and are closed under the function $((n, y) \mapsto (n+1, f(y)))$. □

Definition. *For f a function, x any object, and n a natural number, we define $f^n(x)$ as $g(n)$, where g is the function with domain $\mathcal{N}$ such that $g(0) = x$ and $g(n+1) = f(g(n))$ for each natural number n.*

12.2. Peano Arithmetic

We could give inductive definitions of the operations of addition and multiplication, but we prefer to give "natural" ones. After defining the operations, we will prove the axioms of Peano arithmetic.

Definition. *For $m, n \in \mathcal{N}$, $A, B \in m, n$ respectively, we define $m + n$ as the natural number which contains $A \oplus B$, if there is one.*

Theorem. *The two definitions of $m + 1$ agree.*

Proof. — The typical element of $m+1$ as originally defined is $M \cup \{x\}$, for $M \in m$ and x not in M; the typical element of $m+1$ as defined here is $M \times \{0\} \cup (\{x\} \times \{1\}) = M \times \{0\} \cup \{(x, 1)\}$; but $M \times \{0\} \in m$ and $(x, 1)$ is certainly not in $M \times \{0\}$. □

Definition. *For $m, n \in \mathcal{N}$, $A, B \in m, n$ respectively, we define mn as the natural number which contains $A \times B$, if there is one.*

The numbers $m+n$ and mn are necessarily unique by the result that $\mathcal{N}$ is pairwise disjoint. We will prove below that they actually exist in all cases.

Theorem. $0 \in N$.

PROOF. — True by definition of $\mathcal{N}$. □

Theorem. *If $n \in \mathcal{N}$, $n + 1 \in \mathcal{N}$.*

PROOF. — True by definition of $\mathcal{N}$. □

Theorem. *If $n \in \mathcal{N}$, $n + 1 \neq 0$.*

PROOF. — $\{\ \}$, the sole element of 0, is not of the form $A' \cup \{x\}$ for an $A' \in n$ and x not in A'; all elements of $n + 1$ must be of this form. □

Theorem. *If $n \in \mathcal{N}$ and $n \neq 0$, then $n = m + 1$ for some m.*

PROOF. — Trivial induction. □

Theorem. *If $n, m \in \mathcal{N}$ and $n + 1 = m + 1$, then $n = m$.*

PROOF. — We proceed by induction on the property of n "for all m, if $n + 1 = m + 1$, then $n = m$". If $0 + 1 = m + 1$, then every singleton $\{x\}$ (element of $0 + 1$) is of the form $A \cup \{y\}$, for $A \in m$ and y not in A; but y clearly must be x, and so A must be empty, i.e., $m = 0$. This shows that 0 has the property. Suppose that for all m, if $n + 1 = m + 1$, then $n = m$. Now suppose that $n + 1 + 1 = m + 1$. This means that any set of the form $A \cup \{x\} \cup \{y\}$, where $A \in n$ and x, y are distinct and both not in A, is also a set of the form $B \cup \{z\}$, where $B \in m$ and z is not in B (there are such sets because all numbers are nonempty). We see that B is nonempty (it must contain x or y or both) so $m = q + 1$ for some q. If z is x or y, we see that $q + 1 = n + 1$, since then $B = A \cup \{$the other of x and $y\}$, so $q = n$ and $m = n + 1$ by inductive hypothesis. If z is not x or y, we consider $B' = B - \{x\} \cup \{z\}$; this belongs to m, because there is a bijection between it and B. Now $B \cup \{z\} = B' \cup \{x\} = A \cup \{x\} \cup \{y\}$, and, by deleting x from both sets, we see that $q + 1 = m = n + 1$, as before, so

$q = n$ by inductive hypothesis, and $m + 1 = n + 1$. We have shown that $n+1$ has the property if n does, so the proof is complete by induction (on a stratified condition which defines a set). □

Theorem. *For each m, $m + 0 = m$.*

Proof. — If $A \in m$, $B \in 0$, $B = \{\,\}$ and $A \oplus \{\,\} = A \times \{0\} \in m$. □

Theorem. *For each m, n for which $m+n$ exists, $m+(n+1) = (m+n)+1$.*

Proof. — Let M, N belong to m, n respectively. An element of $m+(n+1)$ is $M \times \{0\} \cup ((N \cup \{x\}) \times \{1\})$ for some x not in N, or $M \times \{0\} \cup N \times \{1\} \cup \{(x, 1)\}$. But $M \times \{0\} \cup N \times \{1\}$ belongs to $m+n$, and $(x, 1)$ belongs to neither $M \times \{0\}$ nor $N \times \{1\}$, so this set also belongs to $(m + n) + 1$. □

Corollary. *$m + n$ exists for all m, n.*

Proof. — By induction on the condition on n, "$m+n$ exists for all m". This condition, the closure property of addition, is important for the next section. □

Theorem. *For all $m \in N$, $m0 = 0$.*

Proof. — Let M be an element of m. $M \times \{\,\} = \{\,\}$. □

Theorem. *For all $m, n \in \mathcal{N}$, $m(n + 1) = mn + m$.*

Proof. — Let M, N belong to m, n and let x not belong to N. $M \times (N \cup \{x\})$ belongs to $m(n+1)$, and is equal to $(M \times N) \cup (M \times \{x\})$. The function which sends (y, z) to $((y, z), 0)$ for $y \in M$, $z \in N$ and sends (y, x) to $(y, 1)$ for $y \in M$ is well-defined and a bijection between this set and $(M \times N) \oplus M$, an element of $mn + m$, so these two numbers are equal. □

Corollary. *mn exists for each pair of natural numbers m, n.*

Proof. — By induction on the condition on n, "mn exists for all m". □

All conditions expressed in the language of first-order logic with equality, addition and multiplication, with all variables restricted to $\mathcal{N}$, are stratified, so the definition of $\mathcal{N}$ provides us with the Axiom of Induction for

Peano arithmetic (we do not need the stronger theorem provable for our full set theory). We have proven all the axioms of Peano arithmetic as interpreted in set theory. Of course, with our "natural" definitions of operations, the usual algebraic properties of the system of natural numbers are fairly easy to prove.

12.3. Finite Sets. The Axiom of Counting

Theorem. *Any subset of a finite set is finite.*

PROOF. — We restate the theorem in the form "for each natural number n, any subset of an element of n is finite". We prove this theorem by induction. It is clearly true in the case $n = 0$. Now let k be a natural number such that all subsets of any set of size k are finite. Let A be a set of size $k + 1$ and let x be any element of A. Let B be any subset of A. If $x \notin B$, then $B \subseteq A - \{x\}$, a set of size k, and B is finite by inductive hypothesis. If $x \in B$, then $B - \{x\}$ is finite by inductive hypothesis, belonging to some natural number m, and B itself is then seen to belong to $m+1$, and so is seen to be finite. The proof of the theorem is complete. □

Theorem. *If A and B are finite sets, $A \cup B$ is finite.*

PROOF. — There is a bijection between $A \cup B$ and $A \oplus (B - A)$. A is finite by hypothesis, $B - A$ is finite by the preceding theorem (it is a subset of B), and disjoint sums of finite sets are finite by the closure property of addition, proved in the previous subsection. □

Theorem. *The union of any finite set of finite sets is finite.*

PROOF. — We restate this as "for each natural number n, the union of any collection of n finite sets is finite". We prove this by induction on n. The case $n = 0$ is trivial; the union of the empty set is empty. Let k be a natural number such that any union of k finite sets is finite. Consider a set $\mathcal{A}$ of $k + 1$ finite sets, and let $A \in \mathcal{A}$. The union of $\mathcal{A} - \{A\}$ is finite by hypothesis, and A is finite by hypothesis. The union of $\mathcal{A}$ is then the Boolean union of two finite sets, which we know to be finite by the preceding theorem. The proof of the theorem is complete. □

Theorem. *The power set of any finite set is finite.*

Proof. — We restate this as "for each natural number n, the power set of any element of n is a finite set". We prove this by induction on n. The case $n = 0$ is obvious; the power set of the element of 0 has one element. Let k be a natural number such that the power set of any element of k is finite. Let A be an element of $k+1$ and let x be an element of A. $\mathcal{P}\{A - \{x\}\}$ is finite by hypothesis, and so belongs to a natural number m. There is an obvious bijection between $\mathcal{P}\{A\}$ and $\mathcal{P}\{A - \{x\}\} \oplus \mathcal{P}\{A - \{x\}\}$; send each subset which does not contain x to its pair with 0 and each subset B which does contain x to $(B - \{x\}, 1)$. Thus, we see that $\mathcal{P}\{A\} \in m + m$, and so is finite. The proof of the theorem is complete. □

Theorem. *The "singleton image" $\mathcal{P}_1\{A\}$ of a finite set A is finite.*

Proof. — The power set of A is finite by the preceding theorem, and the "singleton image" of A is a subset of this finite set, so finite. □

Theorem. *Every natural number n contains a set of the form $\mathcal{P}_1\{A\}$.*

Proof. — By induction. This is certainly true for $n = 0$ (the empty set is its own "singleton image"). Suppose that it is true for k; we are given a set $\mathcal{P}_1\{A\} \in k$, and we consider $\mathcal{P}_1\{A\} \cup \{\{y\}\} = \mathcal{P}_1\{A \cup \{y\}\}$ for $y \notin A$; this is clearly an element of $k+1$ of the desired form. The proof of the theorem is complete. □

What we *cannot* prove is the following obvious

$*$**Theorem** *If $A \in n$, $\mathcal{P}_1\{A\} \in n$.*

We can prove this for each concrete natural number, but the obvious proof by induction does not work, because the condition "$A \in n$ iff $\mathcal{P}_1\{A\} \in n$" is not stratified. Methods beyond the scope of this book show that the purported theorem *cannot* be proven from our axioms given so far.

It is straightforward to show that A and B are equivalent iff $\mathcal{P}_1\{A\}$ and $\mathcal{P}_1\{B\}$ are equivalent. This justifies the following

Definitions. *$T\{n\}$ is defined as the natural number containing $\mathcal{P}_1\{A\}$ for each $A \in n$. Notice that the relative type of $T\{n\}$ is one higher than that of n.*

$T^{-1}\{n\}$ is defined as the natural number m such that $T\{m\} = n$. Notice that the relative type of $T^{-1}\{n\}$ is one lower than that of n.

Theorem. *$T^{-1}\{n\}$ is well-defined for every n.*

PROOF. — By a theorem above, there is a set $\mathcal{P}_1\{A\} \in n$. A is finite, because it is the union of this finite collection of finite (one-element) sets; it clearly belongs to $T^{-1}\{n\}$. □

We now introduce a new axiom (temporarily; it will turn out to be a consequence of the Axiom of Small Ordinals introduced later):

Axiom of Counting. *For all natural numbers n, $T\{n\} = n$.*

Corollary. *For all natural numbers n, $T^{-1}\{n\} = n$.*

The reader may wonder why we went to the trouble of defining the T operation and its inverse. The answer lies in the following result:

Restricted Subversion Theorem. *If a variable x in a sentence ϕ is restricted to $\mathcal{N}$, then its type can be freely raised and lowered; i.e., such a variable can safely be ignored in making type assignments for stratification.*

PROOF. — Any occurrence of a variable n restricted to $\mathcal{N}$ can be replaced by $T\{n\}$ (which lowers the type of n by one) or by $T^{-1}\{n\}$ (which raises the type of n by one). This can be repeated as needed to give any occurrence of a variable restricted to $\mathcal{N}$ any desired type. □

A more general theorem of the same kind appears in a later chapter.

We prove some obviously true theorems which require the Axiom of Counting:

Theorem. $\{1, \ldots, n\} \in n$.

PROOF. — It is straightforward to show that $\{1, \ldots, n\} \in T^2\{n\}$ for each natural number n, by induction. The double application of T is needed to stratify the sentence so that the induction can be carried out; the Axiom of Counting allows us to eliminate it. This obviously true statement was the original form in which the Axiom of Counting was proposed by Rosser. □

Theorem. *A union of m disjoint sets each belonging to n belongs to mn (this is shorthand for "if $A \in m \in \mathcal{N}$ and $A \subseteq n \in \mathcal{N}$ then $\bigcup[A] \in mn$").*

Proof. — It is straightforward to show by induction that the union of m disjoint sets each belonging to n has $T^{-1}\{m\}n$ elements. Also notice that for $A \in m \in \mathcal{N}$, $B \in n \in \mathcal{N}$, it is easy to see that $A \times B$ is the union of $T\{m\}$ sets, each of size n; the $T\{m\}$ sets are, of course, the "rows" $\{(a, b) \mid b \in B\}$ for each a: there is a stratified definition of a bijection between the set of "rows" and the set of singletons $\{a\}$ of elements of A. The application of T or T^{-1} are needed for stratification, and are eliminable by the Axiom of Counting. □

Exercises

(a) Prove that $2 + 2 = 4$.

(b) Give a set-theoretical proof of the distributive law of multiplication over addition.

(c) Prove that the Cartesian product of two finite sets is a finite set.

(d) Define the function $(n \in \mathcal{N} \mapsto 2^n)$ recursively (give a definition of the function as a set of ordered pairs). Try to prove the theorem "for all $n \in \mathcal{N}$, $A \in n$ implies $\mathcal{P}\{A\} \in 2^n$". If you do this carefully, you should find that there is an obstruction to the natural proof involving stratification. Complete the proof using the Axiom of Counting; what is the form of the theorem you could prove without using the Axiom of Counting?

Cahiers du Centre de logique
Volume 10

Chapter 13

The Real Numbers

In this section, we outline the construction of the continuum (the system $\mathcal{R}$ of real numbers).

13.1. Fractions and Rationals

We start by restricting ourselves to the set $\mathcal{N}^+$ of positive natural numbers. This will have the effect of simplifying our constructions.

We will first construct the system $\mathcal{F}$ of *fractions* (a preliminary implementation of the positive rational numbers).

The basic idea of the construction is that a fraction $\frac{m}{n}$, where m and n are positive natural numbers, is a finite structure which we might just as well identify as (m, n). The difficulty which arises immediately is that there need to be identifications among fractions. We expect that $\frac{m}{n} = \frac{r}{s}$ will hold if $ms = nr$. We make the following

Observation. The relation $\sim$ defined by "$(m, n) \sim (r, s)$ iff $ms = nr$" is an equivalence relation on pairs of positive natural numbers. (This use of $\sim$ is a nonce notation).

A typical approach now would be to define the positive rational numbers as equivalence classes under this relation. We prefer not to do this, because of the difference in relative type between equivalence classes and their representatives. An example of the difficulty is that the natural map taking each positive natural number n to the fraction $\frac{n}{1}$, which would be the equivalence class $[(n,1)]_\sim$ under this scheme, would have an unstratified definition and so could not be relied upon to be defined (the Axiom of Counting introduced in the previous section *does* imply that it is well-defined, though).

The solution to this problem in general, as we noted above, is the use of the Axiom of Choice to choose a representative from each equivalence class. This is not necessary in this case, as there is a familiar way to choose a canonical fractional representation for a positive rational number.

Definition. *We define the function*

$$\text{simplify}(m,n) = \Big(\frac{m}{\gcd(m,n)}, \frac{n}{\gcd(m,n)}\Big)$$

for $m, n \in \mathcal{N}^+$. Observe that this function sends each element of each equivalence class under $\sim$ to a unique representative element of that class. Division of positive natural numbers (and greatest common divisors) are understood to be defined in the usual way.

Definition. *We define the set $\mathcal{F}$ of* fractions *as $\{\text{simplify}(m,n) \mid m,n \in \mathcal{N}^+\}$. We define $\frac{m}{n}$ as the fraction simplify(m,n) for each pair of positive natural numbers m, n.*

We define basic operations and relations on fractions.

Definition. *Let $\dfrac{m}{n}$ and $\dfrac{r}{s}$ be fractions. We define $\dfrac{m}{n} + \dfrac{r}{s}$ as $\dfrac{ms+nr}{ns}$ and $(\dfrac{m}{n})(\dfrac{r}{s})$ as $\dfrac{mr}{ns}$. Notice that the definition of fraction notation ensures that simplifications are understood to be carried out.*

Definition. *Let $\dfrac{m}{n}$ and $\dfrac{r}{s}$ be fractions. We define a relation $\dfrac{m}{n} < \dfrac{r}{s}$ (and $\dfrac{r}{s} > \dfrac{m}{n}$)as holding when $ms < nr$. Observe that this relation implements the usual linear order on positive rational numbers.*

Definition. *Let* $\dfrac{m}{n}$ *and* $\dfrac{r}{s}$ *be fractions, with* $\dfrac{m}{n} > \dfrac{r}{s}$*. We define* $\dfrac{m}{n} - \dfrac{r}{s}$ *as* $\dfrac{ms - nr}{ns}$*; the order condition on the fractions is exactly what is needed for the subtraction of natural numbers in the numerator to be well-defined, and it is straightforward to verify that this subtraction operation has the correct relationship to addition of fractions.*

We now construct the system $\mathcal{Q}$ of rational numbers. The idea, similar to that underlying the construction of the system $\mathcal{F}$ of fractions, is to let ordered pairs (p, q) of fractions stand for differences $p - q$. Of course, we wish to make certain identifications among these ordered pairs:

Observation. The relation $\sim$ on pairs of fractions defined by "$(p, q) \sim (r, s)$ iff $p + s = q + r$" is an equivalence relation.

Once again, we want to avoid using equivalence relations on pairs for reasons of stratification.

Definition. *We define a function*

$$\mathrm{simp}(p, q) = \Big(\frac{1}{1} + p - \min(p, q), \frac{1}{1} + q - \min(p, q)\Big).$$

Observe that the function simp collapses each equivalence class under $\sim$ *to the unique member of that class one of whose projections is* $\frac{1}{1}$*.*

Definition. *We define the set* $\mathcal{Q}$ *of* rational numbers *as* $\{\mathrm{simp}(p, q) \mid p, q \in \mathcal{F}\}$*. Notice that the rational numbers are of three kinds: there is a unique element* $(\frac{1}{1}, \frac{1}{1})$ *which we denote by 0 (an admitted abuse of notation), elements* $(\frac{m}{n} + \frac{1}{1}, \frac{1}{1})$ *which we might denote by* $+\frac{m}{n}$ *and elements* $(\frac{1}{1}, \frac{m}{n} + \frac{1}{1})$ *which we might denote by* $-\frac{m}{n}$*.*

We do not intend to be so careful; we will use the usual notation for rational numbers, with the attending confusion of positive rationals with fractions and postive integral fractions and rational numbers with the corresponding positive natural numbers. We introduce the notation $p - q$, where p and q are fractions, for $\mathrm{simp}(p, q)$, to simplify the notation of the next definitions; the notion of subtraction of fractions defined above, for which we have the same notation, will not be used below.

We define basic operations and relations on the rational numbers:

Definition. *Let $p - q$ and $r - s$ be rational numbers (where p, q, r, s are fractions). We define $(p - q) + (r - s)$ as $(p + r) - (q + s)$. We define $(p - q)(r - s)$ as $(pr + qs) - (ps + qr)$. The notions of addition and multiplication on the right of each equation are those defined on fractions, so these are definitions.*

Definition. *Let $p - q$ and $r - s$ be rational numbers (where p, q, r, s are fractions). We define $(p - q) < (r - s)$ as holding when $p + s < q + r$ (this is a definition because the order relation on the right is that defined on fractions).*

Definition. *The set $\mathcal{Z}$ of* integers *is defined as the collection of all rational numbers of the forms $\frac{n}{1} - \frac{1}{1}$ or $\frac{1}{1} - \frac{n}{1}$, for $n \in \mathcal{N}^+$. Note that this usually distinguished set played no special part in our construction.*

This development is certainly not the only one possible. All developments of number systems in set theory are "implementations" of the familiar systems and must have some essentially arbitrary features. Nor is this development rigorous; it would be possible to verify all familiar properties of the rational numbers starting with the axioms for the natural numbers proved in the previous chapter, but we have not done this. We have presumed familiarity with the properties of the number systems being implemented.

An advantage of this implementation is that we need never worry about zero denominators. The price we pay is the special role of $\frac{1}{1}$ in the "simplest form" for differences; zero is not available to us! Another advantage of this development is that it is more historically accurate; positive rational numbers were understood before negative numbers, at least by the Greeks.

13.2. Magnitudes, Real Numbers, and Dedekind Cuts

Our official implementation of the real numbers will be eccentric in not being based on the rationals!

We return to the system $\mathcal{F}$ of fractions. We observe that the positive real numbers correspond exactly to the open initial segments of the set of positive rational numbers. We define a set $\mathcal{M}$ of *magnitudes*, a preliminary implementation of the positive real numbers.

Definition. *$\mathcal{M}$ is defined as $\{A \mid A \subset \mathcal{F}, A \neq \{\ \}$ and for all $p \in \mathcal{F}, p \in A$ iff for some $q \in A, q > p\}$. An element of $\mathcal{M}$ is called* a magnitude. *A magnitude is a proper initial segment of $\mathcal{F}$ with no greatest element. For some purposes, $\mathcal{F}$ might be regarded as a special "magnitude" ∞.*

Definitions of operations and relations on $\mathcal{M}$ are extremely simple:

Definition. *If $A, B \in \mathcal{M}$, $A + B = \{a + b \mid a \in A, b \in B\}$ and $AB = \{ab \mid a \in A, b \in B\}$. $A < B$ iff $A \subset B$.*

The construction of the system $\mathcal{R}$ of *real numbers* from the system $\mathcal{M}$ is precisely analogous to the construction of the system $\mathcal{Q}$ of rational numbers from the system $\mathcal{F}$ of fractions; for the details see the previous subsection. Constructing a system of "extended reals" including $\pm\infty$ would involve some technicalities.

For this nonstandard approach, we could claim once again the authority of the ancient Greeks, who understood positive reals (or at least some of them) without ever acknowledging negative numbers or zero. We also think that there is something very Greek about defining arbitrary magnitudes as *segments*.

We indicate the approach of Dedekind, who based his construction of $\mathcal{R}$ on the full system $\mathcal{Q}$.

Definition. *A* Dedekind cut *is a pair (L, R) of subsets of $\mathcal{Q}$ such that*

(*a*) *L and R are nonempty.*

(*b*) *Each element of L is less than each element of R.*

(*c*) *L has no greatest element and R has no least element.*

(*d*) *$\mathcal{Q} - (L \cup R)$ has at most one element.*

It is generally understood (a draft said "intuitively clear" here, which is certainly not true!) that Dedekind cuts correspond exactly to individual real numbers. The definitions of operations on Dedekind cuts are somewhat more complicated than the very simple definitions of operations on $\mathcal{M}$.

A difficulty with type differentials which we avoided in the constructions of fractions and rational numbers by using representatives instead of equivalence classes is unavoidable in our construction of magnitudes (and so of reals). The identification of a fraction $\frac{p}{q}$ with the corresponding magnitude $\{\frac{r}{s} \mid \frac{r}{s} < \frac{p}{q}\}$ cannot be shown to be witnessed by a function from $\mathcal{F}$ to $\mathcal{M}$ without an appeal to the Axiom of Counting; its definition is unstratified. To show that such a function exists using Counting, check that the map taking $\frac{T\{p\}}{T\{q\}}$ to $\{\frac{r}{s} \mid \frac{r}{s} < \frac{p}{q}\}$ does have a stratified definition, and then observe that the Axiom of Counting allows us to ignore the T operation.

Magnitudes and real numbers, unlike fractions and rationals, have an essentially set theoretical definition; their definition cannot be reduced to finite structures. This is not surprising, since the additional property which $\mathcal{R}$ has, over and above the axioms of an ordered field which are also satisfied by $\mathcal{Q}$, is the Least Upper Bound Property, which is an essentially set theoretical property.

We state and prove the Least Upper Bound Property for $\mathcal{M}$; the development of the property for $\mathcal{R}$ involves merely technical complications.

Definition. *Let A be a set of magnitudes and let m be a magnitude. We call m an* upper bound *for A iff for all $n \in A$, $m \geqslant n$.*

Least Upper Bound Property. *Let A be a nonempty set of magnitudes with an upper bound. Then A has a least upper bound; i.e., there is a magnitude m which is an upper bound of A such that for all upper bounds n of A, $m \leqslant n$.*

PROOF. — Recall throughout this proof that the order on $\mathcal{M}$ coincides with inclusion: subsets are less than supersets.

Let $B = \{m \in \mathcal{M} \mid m \text{ is an upper bound for } A\}$. We claim that the least upper bound of A is $\bigcap[B]$. An intersection of open proper initial segments of $\mathcal{M}$ will either be an open proper initial segment of $\mathcal{F}$ (and so an element of $\mathcal{M}$), a closed proper initial segment of $\mathcal{F}$, or the empty set. $\bigcap[B]$ will not be empty because every element of B is a superset of any given element of A. Suppose that $\bigcap[B]$ is the closed interval determined by an element p of $\mathcal{F}$. $\{q \in \mathcal{F} \mid q < p\}$, the element of $\mathcal{M}$ corresponding to p, would clearly be an upper bound of A, so a superset of $\bigcap[B]$, so $p \notin \bigcap[B]$, contrary to the purported choice of p. Thus $\bigcap[B] \in \mathcal{M}$. $\bigcap[B]$ will be an upper bound of A because it is a superset of each element of A; it is less than or equal to each upper bound of A because it is the intersection of all upper bounds of A. □

Definition. *If A is a nonempty subset of $\mathcal{M}$ or $\mathcal{R}$ with an upper bound, we define* $\sup A$ *as the least upper bound of A. If A is a nonempty set with a lower bound, we define* $\inf A$ *as the greatest lower bound of A (the existence of greatest lower bounds is an easy corollary of the existence of least upper bounds).*

We can define infinite sums of magnitudes (and of real numbers with more work); it is very convenient to allow the "magnitude" ∞ in this context.

Definition. *A function with domain a nonempty initial segment (proper or otherwise) of $\mathcal{N}$ will be called* a sequence. *A sequence whose range is included in a set A will be called* a sequence of elements of A. *A sequence whose domain is a proper subset of $\mathcal{N}$ will be called a* finite sequence*; a sequence which is not finite will be called an* infinite sequence. *The alternative notation s_n is available for $s(n)$ when s is a sequence and $n \in \mathcal{N}$.*

Definition. *The sum $\sum[s]$ of a finite sequence of fractions (the form of the definition will be the same for other number systems) is defined inductively as follows: $\sum[s\lceil\{0\}] = s(0)$ (this defines the notion of sum for one-element sequences); $\sum[s\lceil\{0, \ldots, n+1\}] = \sum[s\lceil\{0, \ldots, n\}] + s(n+1)$.*

Definition. *The sum $\sum[s]$ of an infinite sequence s of magnitudes is defined as $\{\sum[f] \mid f$ is a finite sequence such that for each $n \in \mathrm{dom}(f), f(n) \in s(n)\}$. This is unstratified ($n$ cannot be consistently typed); the Axiom of Counting assures us that this definition succeeds (the stratified form of the definition is $\{\sum[f] \mid f$ is a finite sequence such that for each $n \in \mathrm{dom}(f), f(n) \in s(T\{n\})\}$.*

The problem can be avoided completely by the following alternative definition: first define finite sums of magnitudes in the same way as *finite sums of fractions, then define (for s an infinite sequence of magnitudes)* $\sum[s] = \bigcup\{\sum[f] \mid$ *for some* $n \in \mathcal{N}, f = s\lceil\{0, 1, \ldots, n\}\}$.

Observation. Any sum of magnitudes is itself either a magnitude (in which case the sum is said to *converge*) or ∞.

Exercises

(a) Attempt proofs of some or all of the basic rules of algebra in the systems presented in this chapter.

(b) Prove that for each positive natural number n and magnitude $x \in \mathcal{M}$, there is a unique magnitude $\sqrt[n]{x} \in \mathcal{M}$ such that $(\sqrt[n]{x})^n = x$.

(c) (hard) How would you define addition of two real numbers defined as Dedekind cuts in $\mathcal{Q}$? How would you define multiplication of real numbers defined in this way?

(d) (hard) Present a set theoretical definition of the base e of natural logarithms.

Cahiers du Centre de logique
Volume 10

Chapter 14

Equivalents of the Axiom of Choice

We now turn to an investigation of various forms of the Axiom of Choice and their consequences. We do not consider the Axiom of Choice to be a problematic mathematical assumption. It is rather like the axiom of parallels in Euclidean geometry; we have the "choice" of systems with and without the Axiom of Choice, which can be shown to be equally "safe". The one curious thing about the Axiom of Choice in the context of this particular kind of set theory with Stratified Comprehension is that (as we will see in a later chapter) we can use the Axiom of Choice to prove the existence of atoms; we do not see any way to prove the existence of atoms without the Axiom of Choice, but it remains an open problem whether this can be done.

We restate the Axiom and an equivalent of the Axiom which we have already established above:

Axiom of Choice. *For each set P of pairwise disjoint non-empty sets, there is a set C of elements of elements of P which contains exactly one element of each element of P.*

Theorem. *The Cartesian product of any family of non-empty sets is non-empty.*

If A is a family of sets, we call an element of $\prod[A]$ (a function which takes each element a of A to a singleton $\{c\} \subseteq a$) a *choice function* for A. We cannot in general have a function taking each a to the corresponding c due to stratification restrictions. We consider the special case of the nonempty subsets of some set X; an element of $\prod[\mathcal{P}\{X\} - \{\{\ \}\}]$, the Cartesian product of the collection of nonempty subsets of X, is called a choice function for the set X – the choice function sends each nonempty subset of X to the singleton of one of its elements.

Our discussion of equivalents for the Axiom of Choice involves the introduction of a new kind of order:

Definition. *A linear order $\leqslant$ is a* well-ordering *if every nonempty subset of* $\mathrm{dom}(\leqslant)$ *has a least element relative to $\leqslant$. If $X = \mathrm{dom}(\leqslant)$, $\leqslant$ is said to be a* well-ordering of X. *If $\leqslant$ is a linear order which is a well-ordering, the corresponding strict linear order $<$ is said to be a* strict well-ordering.

Note that the usual order relation on the natural numbers is a well-ordering, but that the usual order on the integers is not a well-ordering (the whole set of integers does not have a least element) so the usual order of the reals, for example, is not a well-ordering.

We now introduce and prove another theorem equivalent to the Axiom of Choice, with a preliminary definition:

Definition. *If $\leqslant$ is a partial order, a* chain *in $\leqslant$ is the domain of a subset of $\leqslant$ which is a linear order.*

Zorn's Lemma. *If $\leqslant$ is a partial order such that each chain in $\leqslant$ has an upper bound relative to $\leqslant$, then the domain of $\leqslant$ has a maximal element relative to $\leqslant$.*

PROOF. — We first motivate the proof. The Axiom of Choice allows us to make many simultaneous, independent choices. What is needed in the proof of Zorn's Lemma (or, for example, for a direct proof of the result that the universe is well-ordered given below), is a technique for making a long sequence of choices each of which depends on choices made previously. The way we do this is to consider the set of all situations in which we might find ourselves in the course of such a sequence of choices, and all possible "next choices" in each such situation. We make independent choices of "next choices" in all possible situations, and from this we are able to construct a sequence of choices depending on choices made earlier.

For each chain c in $\leqslant$, we define a set X_c as follows: if there is any upper bound x of c in $\leqslant$ such that $x \notin c$, then X_c is the collection of sets $c \cup \{x\}$ for all such upper bounds x. If there is no such upper bound, then $X_c = \{c\}$. Note that in either case, each element of X_c is a chain, and, in the latter case, any upper bound for c (there is only one) belongs to c and is a maximal element for $\leqslant$.

The collection of sets $\{c\} \times X_c$ for c a chain in $\leqslant$ is a set of pairwise disjoint sets. A choice set for this collection will be a function taking each chain c to a selected element of X_c. Choose such a function and call it G.

We call a chain c in $\leqslant$ *authentic* if it is well-ordered by $\leqslant$ and has the property that for each $x \in c$, $G(\{y \in c \mid y < x\}) = \{y \in c \mid y \leqslant x\}$. (We define $x < y$ as "$x \leqslant y$ and $x \neq y$" in accordance with our stated convention). So $x = \max G(\{x \in c \mid y < x\})$ (we use the notation $\max A$ here for the largest element of a set A with respect to an understood order).

The following should be obvious: any initial segment $\{y \in c \mid y < x\}$ or $\{y \in c \mid y \leqslant x\}$ of an authentic chain is an authentic chain; the empty chain is authentic; if c is an authentic chain, $G(c)$ is an authentic chain.

We now need to prove a technical property. Consider two authentic chains c and c' and let a be an element of c such that $\{x \in c \mid x < a\} = \{x \in c' \mid x < a\}$. Then $a \in c'$ or $c' = \{x \in c \mid x < a\}$. Indeed, suppose that $a \notin c'$ and that $c' \neq \{x \in c \mid x < a\}$. So $c' \neq \{x \in c' \mid x < a\}$ and there exists some $a' \in c'$ which is greater or equal to a. As c' is authentic, it is well-ordered and we can assume that a' is the minimum of $\{x \in c' \mid x \geqslant a\}$. This implies that $\{x \in c' \mid x < a\} = \{x \in c' \mid x < a'\}$ and also that $\{x \in c \mid x < a\} = \{x \in c' \mid x < a'\}$. But then,

$$a = \max G(\{x \in c \mid x < a\}) = \max G(\{x \in c' \mid x < a'\}) = a' \in c'.$$

So $a \in c'$, a contradiction.

Now, let c_1 and c_2 be distinct authentic chains. We want to prove that one of these two chains is an initial segment of the other one. Without loss of generality, suppose that there is an element of c_1 which is not in

c_2; let a be the minimal element of c_1 not in c_2. So $\{x \in c_1 \mid x < a\} \subseteq \{x \in c_2 \mid x < a\}$.

If $\{x \in c_1 \mid x < a\} \neq \{x \in c_2 \mid x < a\}$, then we can find some element which is in c_2 but not in c_1 and which is smaller than a. Let b be the minimum of $\{x \in c_2 \mid x \notin c_1 \text{ and } x < a\}$. As b is minimal and $b < a$ and $\{x \in c_1 \mid x < a\} \subseteq \{x \in c_2 \mid x < a\}$, it is easy to prove that $\{x \in c_2 \mid x < b\} = \{x \in c_1 \mid x < b\}$. By the technical property, $b \in c_1$ or $c_1 = \{x \in c_2 \mid x < b\}$. As, by definition, $b \notin c_1$, we conclude that c_1 is an initial segment of c_2. But this contradicts the assumption that $a \in c_1 - c_2$.

So $\{x \in c_1 \mid x < a\} = \{x \in c_2 \mid x < a\}$. Then, by the technical property, $a \in c_2$ or $c_2 = \{x \in c_1 \mid x < a\}$. As, by definition, $a \notin c_2$, we conclude that c_2 is an initial segment of c_1.

Thus, more generally, we have proved that authentic chains are linearly ordered by inclusion.

It is straightforward to show that the union of a collection of chains which is linearly ordered by inclusion will be a chain. It is then easy to see that the union of any collection of authentic chains will be an authentic chain. In particular, the union of all authentic chains in $\leqslant$ is an authentic chain; call it C. $G(C)$ must also be an authentic chain, but this implies that $G(C) = C$ (since each can be seen to be a subset of the other), which further implies that C has no upper bound not in C (by the definition of the function G); C has an upper bound by the assumed properties of $\leqslant$, which must be a maximal element for $\leqslant$. The proof of Zorn's Lemma is complete. □

Theorem. *Zorn's Lemma implies the Axiom of Choice.*

PROOF. — Let P be a pairwise disjoint collection of nonempty sets. Let Q be the set of choice sets for subsets of P. It is easy to see that inclusion on Q satisfies the conditions of Zorn's Lemma (take the union of a chain of such sets with respect to inclusion to get an upper bound), and that a maximal element with respect to this order must be a choice set for P. □

Zorn's Lemma allows us to prove this attractive

Theorem. *There is a well-ordering of V.*

Proof. — Consider the collection of all well-orderings (this exists by Stratified Comprehension). We define the following order on well-orderings: we say $R \leqslant S$ if $R \subseteq S$ and for each r in $\mathrm{dom}(R)$ and s in $\mathrm{dom}(S) - \mathrm{dom}(R)$, rSs; the well-ordering S extends R by adding elements at the end only; we say that S "continues" R.

Now let C be a chain of well-orderings with respect to continuation; we claim that the union of C is also a well-ordering, and a continuation of each of the elements of C. Consider a nonempty subset A of the domain of the union of C. Since it is nonempty, it has an element x, which belongs to the domain of some $R \in C$. Since R is a well-ordering, $A \cap \mathrm{dom}(R)$ has a minimal element a. Any element of the domain of the union of C less than a would belong to $\mathrm{dom}(R)$, and so would not belong to A; thus a is the least element of A with respect to the union of C, and the union of C is a well-ordering. That it is a continuation of each of the elements of C is easy to see.

We have shown that the collection of well-orderings ordered by continuation satisfies the conditions of Zorn's Lemma. A maximal well-ordering must have domain V, or we could extend it by making some single object greater than all the previously provided objects. □

Corollary. *For any set X, there is a well-ordering on X.*

Theorem. *The existence of a well-ordering on V implies the Axiom of Choice.*

Proof. — Let P be a pairwise disjoint collection of nonempty sets. Let $\leqslant$ be a well-ordering of V. Let C be the set of least elements with respect to $\leqslant$ of elements of P; this C is clearly a choice set for P. Note that we can *define* choice sets for any partition given a well-ordering of the universe. □

We have given four equivalent forms for the Axiom of Choice, the original statement, the statement that Cartesian products of collections of nonempty sets are nonempty, Zorn's Lemma, and the statement that there is a well-ordering of the universe.

Exercises

(a) Prove that every finite collection of disjoint sets has a choice function without using the Axiom of Choice.

(b) Use Zorn's Lemma to show that any infinite set contains the range of some bijection with domain $\mathcal{N}$.

(c) Use Zorn's Lemma to prove that every vector space has a basis.

(d) Bertrand Russell used the following example in a philosophical discussion of the Axiom of Choice. Suppose we have an infinite collection of pairs of shoes; it is easy to see that there is a choice function for this collection of pairs (explain). But the case of an infinite collection of pairs of socks is more difficult (explain). Use Zorn's Lemma to demonstrate that we can choose one sock from each of the pairs in the infinite collection.

(e) Look up the results of Banach and Tarski on the decomposition of 3-spheres using the Axiom of Choice in the library. What do you think of our assertion that the Axiom of Choice is not a problematic assumption?

Cahiers du Centre de logique
Volume 10

Chapter 15

Ordinal Numbers

15.1. Well-Orderings and Ordinals

Let W be the domain of a well-ordering $\leqslant$; then we can justify a method of proving statements about elements of W somewhat analogous to mathematical induction:

Theorem of Transfinite Induction. *Let W be the domain of a well-ordering $\leqslant$. Let A be a set with the property that for each element a of W, $a \in A$ if $\operatorname{seg}_{\leqslant}\{a\} \subseteq A$; i.e, A contains a if it contains each element of W less than a. Then $W \subseteq A$; all elements of W belong to A.*

Proof. — If the set $W - A$ is nonempty, then it has a least element a. But if a is the least element of $W - A$, then $\operatorname{seg}\{a\} \subseteq A$, and a must be in A. It follows that $W - A$ must be empty, and $W \subseteq A$. □

The difference between conventional mathematical induction and this variety is that in conventional mathematical induction we consider the immediate predecessor of an element to show that it is in the set, while in transfinite induction we consider the set of all predecessors. This kind of induction can be used on the natural numbers as well; the greater generality is necessary because not all elements of domains of general well-orderings have immediate predecessors. The basis step of the usual mathematical induction handles the problem that 0 does not have an immediate predecessor in the natural numbers; the "basis step" in a transfinite induction is trivial, since the set of predecessors of the first element is the empty set, all elements of which have any property one would care to propose.

We can also prove a recursion theorem:

Transfinite Recursion Theorem. *Let W be the domain of a well-ordering $\leqslant$. Let F be a function from the collection of functions with domains segments of $\leqslant$ to the collection of singletons. Then there is a unique function f such that $\{f(a)\} = F(f\lceil\text{seg}\{a\})$ for each $a \in W$.*

PROOF. — By transfinite induction. Let A be the set of elements a of W such that there is a uniquely determined function g on $\text{seg}^+\{b\}$ with the indicated property for each $b < a$. We want to deduce that there is a uniquely determined function f on $\text{seg}^+\{a\}$ with the indicated property. The desired function is the union of the union g of all the functions on the $\text{seg}^+\{b\}$'s (which must agree with one another by uniqueness) and $\{a\} \times F(g)$ (remember that $F(g)$ is the singleton of the desired value). The function on all of W is the union of the functions on the weak segments. □

The function F gives instructions on how to compute the value of f at a point in W given the values already computed on the values before that point.

We now use transfinite induction to define an equivalence relation on the class of well-orderings.

Definition. *Let R and S be well-orderings. We say that R is* similar *to S if there is a bijection s between $\text{dom}(R)$ and $\text{dom}(S)$ such that $s(a)\, S\, s(b)$ iff $a\, R\, b$; we call the map s a* similarity *between R and S.*

Theorem. *Similarity is an equivalence relation. Moreover, if R is similar to S, the similarity between R and S is uniquely determined; if R is not similar to S, then either S is a continuation of an order similar to R or R is a continuation of an order similar to S (not both).*

PROOF. — It is easy to check that similarity is an equivalence relation; check that the identity map on the domain of a well-ordering is a similarity, that inverses of similarities are similarities, and that compositions of similarities are similarities. The rest follows from the Transfinite Recursion Theorem: define $s(a)$ as the least element of $\mathrm{dom}(S) - s[\mathrm{seg}_R\{a\}]$ (if any) for each a in $\mathrm{dom}(R)$ – this defines a unique function. Induction shows that $s(a)$ is the only possible candidate for the value at a of the similarity between R and S, if any. If s is defined on all of $\mathrm{dom}(R)$, then S is a continuation of an order similar to R; if s is only defined on a segment of $\mathrm{dom}(R)$ (because $\mathrm{dom}(S)$ is exhausted) then R is a continuation of an order similar to S; the fact that s is uniquely defined ensures that only one of the three possible situations can occur. □

Since similarity is an equivalence relation, it induces a partition of the set of all well-orderings into equivalence classes. These equivalence classes are objects of some interest to us.

Definition. *An* ordinal number *is an equivalence class under similarity.* Ord *is the set of all ordinal numbers.*

Definition. *The* order type *of a well-ordering is the ordinal number of which it is an element.*

The first ordinal number is the similarity class of the empty relation, which is certainly a well-ordering; this is the ordinal number 0. The ordinal numbers of well-orderings of finite sets are the finite ordinal numbers $0, 1, 2, \ldots$ (not to be confused with the (cardinal) natural numbers defined above). An example of an infinite ordinal number is the equivalence class "ω" of the usual order on the natural numbers.

We feel the need of some structure:

Definition. *If α and β are ordinal numbers, and $R \in \alpha$, $S \in \beta$ are well-orderings, we say that $\alpha \leqslant \beta$ iff S is a continuation (possibly non-strict) of a well-ordering similar to R.*

Theorem. *The relation $\leqslant$ defined on ordinal numbers is a well-ordering.*

Proof. — It is easy to show that $\leqslant$ is well-defined (does not depend on the choice of the well-orderings R and S representing α and β in thee definition) and that it is a linear order (using the previous theorem). If A is a set of ordinal numbers, consider a well-ordering R with domain W belonging to an element of A; consider the set of elements a of W such that R restricted to $\text{seg}_R\{a\}$ belongs to an ordinal in A. If it is empty, then every ordinal less than the ordinal to which R belongs is not in A, and this ordinal is itself the smallest ordinal in A; if it is nonempty, it has a smallest element a, and the ordinal of $\text{seg}_R\{a\}$ must be the smallest ordinal in A. $\square$

We define some operations on ordinal numbers:

Definition. *If α and β are ordinal numbers, and R and S are well-orderings, $\alpha + \beta$ is the ordinal which contains the order on $\text{dom}(R) \oplus \text{dom}(S)$ produced by using the order on R to order $\text{dom}(R) \times \{0\}$, the order on S to order $\text{dom}(S) \times \{1\}$, and letting all elements of $\text{dom}(R) \times \{0\}$ be "less than" all elements of $\text{dom}(S) \times \{1\}$.*

Definition. *If α and β are ordinal numbers, and R and S are well-orderings, $\alpha\beta$ is the ordinal which contains the order on $\text{dom}(R) \times \text{dom}(S)$ induced by comparing the second projection using R, then comparing the first projection using S if the first projections are the same (reverse lexicographic order).*

These operations coincide with the usual arithmetic operations on finite ordinals, but they do not have desirable algebraic properties. For instance, $\omega + 1$ is the ordinal containing the order $0 < 1 < 2 < \ldots < x$, where a new object x is added following all the numbers; $1 + \omega$ is the ordinal containing $x < 0 < 1 < 2 < \ldots$, which is simply ω itself. An exponentiation operation can also be defined using transfinite recursion. Ordinal addition and multiplication can be shown to be functions using Stratified Comprehension.

We give some examples of well-orderings. The usual order on the integers is not a well-ordering, because there is no least element of the whole domain. But if we redefined the order so that negative numbers were all less than the positive numbers and zero (as before) but now ordered among themselves with respect to absolute value, we would have the order

$$-1 < -2 < -3 < \ldots < 0 < 1 < 2 < \ldots$$

of order type $\omega + \omega = \omega 2$. A natural order on linear functions with natural number coefficients would be

$$0 < 1 < 2 < \ldots x < x + 1 < x + 2 < \ldots < 2x < 2x + 1 < 2x + 2 < \ldots,$$

an order of type $\omega\omega = \omega^2$. The natural order on all polynomials of finite degree with natural number coefficients which extends this order is an example of order type ω^ω (which we have not as yet defined).

15.2. The "Paradox" of Burali-Forti meets T

Observe that every ordinal α has a successor $\alpha+1$, which is strictly greater than α. The ordinals have a natural well-ordering $\leqslant$, indicated above. For any ordinal α, the order $\leqslant$ restricted to $\text{seg}\{\alpha\}$ belongs to a uniquely determined ordinal $T^2\{\alpha\}$; observe that the type of $T^2\{\alpha\}$ is two types higher than the type of α in a stratified sentence; this is the reason for the superscript 2.

The following "inductive" argument might convince us that $T^2\{\alpha\} = \alpha$; consider the smallest ordinal β such that $T^2\{\beta\}$ is not equal to β. Now for each $\alpha < \beta$ it is easy to see that $T^2\{\alpha\} < T^2\{\beta\}$. By examining the order on ordinals, we can see that every ordinal less than $T^2\{\beta\}$ must be $T^2\{\gamma\}$ for some $\gamma < \beta$. But it follows that $T^2\{\beta\}$ is the smallest ordinal greater than all $T^2\{\alpha\}$'s for $\alpha < \beta$, that is, β itself — so there can be no such β.

Now consider the well-ordering $\leqslant$ itself. The ordinal number containing $\leqslant$ is called Ω. By the "result" of the previous paragraph, we find that $T^2\{\Omega\} = \Omega$; the ordinal number of $\leqslant$ restricted to $\text{seg}\{\Omega\}$ is the same as the ordinal number of $\leqslant$; but this is clearly impossible, since the latter well-ordering is a strict continuation of the former, including such additional ordinals as $\Omega + 1$.

This is the Burali-Forti paradox, the paradox of the largest ordinal number. If you were alert, you saw that it does not really work in our set theory; the point is that "$T^2\{\alpha\} = \alpha$" is not a stratified condition, so does not define a set, and we can only do transfinite induction on conditions which define sets. Instead of the paradox, we obtain the following (perhaps startling)

Theorem. $T^2\{\Omega\} < \Omega$; *i.e,* $\leqslant$ *on* $\text{seg}_{\leqslant}\{\Omega\}$ *is not similar to* $\leqslant$ *on all ordinals.*

A genuinely surprising result is the following:

Theorem. *The singleton image of $\leqslant$ is not similar to $\leqslant$.*

PROOF. — The relation between x and $\operatorname{seg}\{x\}$ is not stratified (for x in $W = \operatorname{dom}(R)$, $R \in \alpha$), but the relation between $\{x\}$ and $\operatorname{seg}\{x\}$ is stratified. Thus, we can define a similarity between $\leqslant$ on $\operatorname{seg}\{\Omega\}$ and the double singleton image $\mathrm{SI}^2\{\leqslant\}$ (the relation which obtains between $\{\{\alpha\}\}$ and $\{\{\beta\}\}$ when α and β are ordinals and $\alpha \leqslant \beta$). Since $\mathrm{SI}\{\mathrm{SI}\{\leqslant\}\}$ is not similar to $\leqslant$, clearly $\mathrm{SI}\{\leqslant\}$ is not similar to $\leqslant$. □

Definition. *We can now define $T\{\alpha\}$ as the order type of the singleton image $\mathrm{SI}\{R\}$ of an element R of α. $T^2\{\alpha\}$ will have the intended meaning.*

Theorem. *The function $(x \mapsto \{x\})$ does not exist.*

PROOF. — If it did, it would be easy to define a similarity between $\leqslant$ and its singleton image. □

Theorem. *The inclusion relation restricted to the proper initial segments of a well-ordering of type α has order type $T\{\alpha\}$.*

PROOF. — The similarity is supplied by the obvious bijection between proper initial segments of the domain of the well-ordering and singletons of elements of the domain of the well-ordering. □

T is traditionally used in this kind of set theory for a class of similar operations on classes of sets or relations involving the singleton; it should always be possible to figure out which one is intended from context. All such operations are "singleton image" operations in one way or another.

We define another important operation on the ordinals:

Definition. *If A is a set of ordinals, $\lim A$ (or $\sup A$) is the smallest ordinal greater than all elements of A.*

For example, if F is the set of finite ordinals, $\omega = \lim F$. Of course, some "limits", like $\lim \mathrm{Ord}$, do not exist. If we wanted a function lim, $\lim(A)$ would have to be the singleton of $\lim A$.

In general, ordinals fall into three classes: 0, which has no predecessors, ordinals $\alpha + 1$ (successor ordinals), which have immediate predecessors, and ordinals $\lim A$ (limit ordinals), where A is nonempty with no largest element. An example of a limit ordinal is ω (the smallest one).

If W is a well-ordering, we may interpret W as a "sequence" indexed by the ordinals as follows:

Definition. *Let W be a well-ordering. For each ordinal α, W_α is defined as the element w of $\mathrm{dom}(W)$ such that $\mathrm{seg}_W\{w\} \in \alpha$ (if there is such an element).*

Notice that the ordinal index α is two types higher than the term W_α. By the results above, we cannot hope in general to get a *function* mapping an initial segment of the ordinals onto $\mathrm{dom}(W)$; some well-orderings are longer than the natural well-ordering of the ordinals.

The argument for the Burali-Forti paradox above showed that the set of ordinals such that $T\{\alpha\} = \alpha$ cannot exist (otherwise the induction could be carried out). Nonetheless, we are interested in this "property" of ordinals, and provide a

Definition. *A* Cantorian *ordinal is an ordinal α such that $T\{\alpha\} = \alpha$.*

Theorem. *The set of Cantorian ordinals does not exist.*

Proof. — From the Burali-Forti argument. □

The "class" of Cantorian ordinals can be thought of as the "small" ordinals of our theory. We could already prove with no difficulty that each of the concrete finite ordinals is Cantorian and without undue difficulty that the infinite ordinal ω and many of its relatives are Cantorian. The ordinals which we can show not to be Cantorian are the very large ones like Ω. The reason for the use of the term "Cantorian" will be given in a later chapter.

Using Ω, we can build another example of a "class" which, like the "class" of Cantorian ordinals, is not a set. We first state a theorem about the T operation on ordinals:

Theorem. *For α, β ordinals, $\alpha \leqslant \beta$ iff $T\{\alpha\} \leqslant T\{\beta\}$; $T\{\alpha+\beta\} = T\{\alpha\} + T\{\beta\}$; and $T\{\alpha\beta\} = T\{\alpha\}T\{\beta\}$.*

Proof. — The first fact (the one we will use here) is obvious; we omit the proofs (easy) of the other facts. □

Theorem. *The "class" consisting of Ω, $T\{\Omega\}$, $T^2\{\Omega\}$, $T^3\{\Omega\}, \ldots$, constructed by iterating the T operation on Ω, does not exist.*

Proof. — We have proven above that $T\{\Omega\} < \Omega$. Repeated application of the previous Theorem yields the result that $T^{n+1}\{\Omega\} < T^n\{\Omega\}$ for each natural number n. If the "class" described above were actually a set, it would have no least element relative to the natural ordering on the ordinals, which is impossible. □

15.3. The Axiom of Small Ordinals

The last theorem is somewhat disquieting; it means that in some "external" sense the ordinals of our set theory are *not* well-ordered. We will introduce an axiom which will help allay this disquiet for the Cantorian or "small" ordinals, first verifying that ω is Cantorian (and so intuitively "small"), as claimed above.

Definition. *A "small" ordinal is an ordinal which is less than some Cantorian ordinal.*

Theorem. *ω is Cantorian.*

Proof. — One element of ω is the order $\leqslant$ on the natural numbers; one element of $T\{\omega\}$ is the singleton image of $\leqslant$, an order on the singletons of natural numbers. A similarity between these orders is defined by $s(n) = \{T^{-1}\{n\}\}$ (the T operation on natural numbers was defined in the section on finite sets and the Axiom of Counting; results in that section show that this map is a bijection). Note that we cannot conclude that $s(n) = \{n\}$ without appealing to the Axiom of Counting, but we don't need to do this to establish the result. □

Corollary. *Each finite ordinal is small.*

Proof. — Each finite ordinal is less than the Cantorian ordinal ω. □

Axiom of Small Ordinals. *For any sentence ϕ in the language of set theory, there is a set A such that for all x, x is a small ordinal such that ϕ iff ($x \in A$ and x is a small ordinal).*

An informal way of stating this (in terms of proper classes) is that for any sentence ϕ there is a set A such that the (possibly proper) class $\{x \mid \phi\}$ and the set A have the same intersection with the (proper) class of small ordinals.

We briefly discuss the intuition behind the Axiom of Small Ordinals. It is certainly true that our well-orderings up to any finite ordinal are "genuine"; there is no "external" collection which has no least element. Up to some point, our ordinals are "standard" in this sense. A property of "standard" ordinals α is that they contain $\text{seg}\{\alpha\}$ relative to the natural order on ordinals; but this is an "illegitimate" (unstratified) property of ordinals in our intuitive model of set theory, and thus so is the stronger property of being a "standard" ordinal (with its reference to *all* subcollections of the domain of a well-ordering, rather than to those with "labels" — i.e., those realized by sets). The idea underlying the Axiom of Small Ordinals is that we let the "illegitimate" notion of "standard" (which justifies defining sets in terms of any sentence ϕ, not just stratified sentences) coincide with the "illegitimate" notion "small" (which says that the "illegitimate" relation between ordinals and segments has not yet totally slipped its leash). We assert that all subsets of the "standard" ordinals are realized by sets, although large subsets are realized only by extensions (the class of all Cantorian ordinals is realized by Ord, for instance).

When we apply the Axiom of Small Ordinals, we are likely to consider classes which may or may not exist as sets (and some which *definitely* do not exist as sets). We warn the reader that we will sometimes use the notations $\{x \mid \phi\}$ and $(x \mapsto T)$ as notations of convenience for "classes" and "functions" which do not exist as sets in our theory.

Mathematical Induction Theorem. *For each condition ϕ and variable x, if $\phi[0/x]$ holds and "for all $n \in \mathcal{N}$, if $\phi[n/x]$ then $\phi[n+1/x]$" holds, then "for all $n \in \mathcal{N}$, $\phi[n/x]$" holds.*

PROOF. — It is easy to define the bijection between the finite ordinals and the corresponding natural numbers. By the Axiom of Small Ordinals, there is a set A whose small ordinal members are precisely the small ordinals x such that x is finite and $\phi[n/x]$ holds of the natural number n corresponding to x. Let Fin be the set of finite ordinals. Then the image of $A \cap \text{Fin}$ in $\mathcal{N}$ consists of exactly the elements n of $\mathcal{N}$ such that $\phi[n/x]$ and is inductive, so includes all of $\mathcal{N}$. □

Theorem of Counting. *For each natural number n, $T\{n\} = n$.*

PROOF. — Easy induction on the unstratified condition. □

We see that the Axiom of Counting is redundant once the Axiom of Small Ordinals is introduced.

The Axiom of Small Ordinals allows induction on unstratified conditions on general Cantorian well-orderings as well as on the natural numbers:

Cantorian Transfinite Induction Theorem. *Let $\leqslant$ be a well-ordering belonging to a Cantorian ordinal α. Let ϕ be any sentence. Suppose that for all $x \in \text{dom}(\leqslant)$, (if (for all $y \leqslant x$, $\phi[y/x]$) then ϕ); ϕ is true of x if it is true of all its predecessors in the well-ordering. It follows that for all $x \in \text{dom}(\leqslant)$, ϕ.*

PROOF. — Recall that the notation $[\leqslant]_\beta$ represents the element (if any) of $\text{dom}(\leqslant)$ determining a segment of order type β, for each ordinal β. The class of all small ordinals β such that not $\phi[[\leqslant]_\beta/x]$ would be the intersection of some set B with the small ordinals. All ordinals less than α would be small by definition of "small", so $B \cap \text{seg}\{\alpha\}$ would be a set. If this set were nonempty, it would have a smallest element γ. For all $x \leqslant [\leqslant]_\gamma$ we would have ϕ, but not $\phi[[\leqslant]_\gamma/x]$. This is impossible. □

The concept of a "small" ordinal is actually a nonce concept, as the following reveals:

Theorem. *An ordinal is small iff it is Cantorian.*

PROOF. — Let β be a small ordinal. By the definition of "small", there is an ordinal α greater than β which is Cantorian. α is small because $\alpha+1$ is Cantorian. 0 is Cantorian; successors of Cantorian ordinals are Cantorian; limits of Cantorian ordinals are Cantorian (because the supremum of a collection of ordinals $T\{\beta_\alpha\}$ will be the image under T of the supremum of the collection of β_α's; thus the limit of a collection of fixed points of T will itself be a fixed point of T). The order type of ordinals up to α is

$T^2\{\alpha\} = \alpha$, so we can prove by induction (on an unstratified condition) that all ordinals $< \alpha$ (including β) are Cantorian. All small ordinals are Cantorian, so "small" simply means "Cantorian". $\square$

15.4. Von Neumann Ordinals

We note that the difference between the representation of natural numbers in the usual set theory and our representation of natural numbers carries over into the infinite: just as each von Neumann natural number is the collection of all smaller natural numbers, so each ordinal in the usual set theory is the collection of all smaller ordinals (with the numerals as the finite ordinals). Each of the von Neumann numerals is a set in our set theory, but this is not necessarily the case for the larger von Neumann ordinals; as we have observed already, even the von Neumann "ω", which would be the collection of all the finite von Neumann ordinals, is not necessarily a set in our set theory, though it is possible for it to exist. We give a definition:

Definition. *A* von Neumann ordinal *is a set A such that $\bigcup[A] \subseteq A$ and the restriction of $\in$ to A is a strict well-ordering of A.*

Definition. *For each von Neumann ordinal A, we define A^+, the* successor *of A, as $A \cup \{A\}$.*

Lemma. *For each von Neumann ordinal A, A^+ is a von Neumann ordinal.*

Proof. — $\bigcup[A \cup \{A\}] = \bigcup[A] \cup A = A$ (because $\bigcup[A] \subseteq A$) $\subseteq A \cup \{A\}$. The restriction of $\in$ to A is a strict well-ordering; the restriction of $\in$ to A^+ extends this relation by adding a single object "greater" than the elements of A, which will still be a strict well-ordering. $\square$

Note that this is an unstratified condition! We would like to say that a von Neumann ordinal A corresponds naturally to the order type of the restriction of $\in$ to A, but there is a problem with the von Neumann ordinals '0' $= \{\ \}$ and '1' $= \{\{\ \}\}$, on which $\in$ determines the same empty relation. The correct formulation is that each von Neumann ordinal corresponds to the unique ordinal α such that the order type of the restriction of $\in$ to A^+ is $\alpha + 1$. The same technical point arises in our treatment of ordinals in relation to well-founded extensional relations in a later chapter.

Lemma. *An element of a von Neumann ordinal A is a "proper initial segment" of A in the sense of the order induced by $\in$ and is itself a von Neumann ordinal.*

PROOF. — The fact that $\in$ is a strict linear order of A and that each element of an element of A is an element of A (because $\bigcup[A] \subseteq A$) ensures that an element of A is a proper initial segment of A in the sense of the order induced by $\in$. ($A \in A$ is impossible because the ordering of A by $\in$ must be strict; this is why the initial segments are proper).

Let $B \in A$. Let $D \in C \in B$ be an arbitrary element of $\bigcup[B]$; we will show $D \in B$. $C \in A$ because $\bigcup[A] \subseteq A$ (elements of elements of A are in A); a second application of the same fact gives $D \in A$. Since C and D are in A and $\in$ is a linear order on A, $D \in C \in B$ implies $D \in B$. Thus $\bigcup[B] \subseteq B$. B is strictly well-ordered by $\in$ because it is a subset of A, which is strictly well-ordered by the same relation. Thus any element B of A is a von Neumann ordinal. □

It is useful to note that this result is correct in the case of the peculiar ordinal '1'; seg{'0'} defined with respect to the (empty) order on '1' does come out correctly as '0'.

Lemma. *The intersection of two von Neumann ordinals is a von Neumann ordinal.*

PROOF. — Let C be the intersection of von Neumann ordinals A and B. $\bigcup[C] \subseteq (\bigcup[A] \cap \bigcup[B]) \subseteq (A \cap B) = C$. $\in$ clearly strictly well-orders C, since it strictly well-orders its supersets A and B. □

Lemma. *Each proper "initial segment" of a von Neumann ordinal A (in the sense of $\in$ restricted to A) is an element of A.*

PROOF. — There is an $\in$-least element B of A which is not an element of the proper initial segment; each of its elements must be in the proper initial segment by the fact that it is $\in$-least; if B did not contain some element C of the initial segment, then C would contain B (because $\in$ linearly orders A) and B would belong to the initial segment, contradicting the choice of B. Thus B must contain exactly the elements of (and so be equal to) the proper initial segment. □

Lemma. *For any two von Neumann ordinals A and B, we have either $A \in B$, $B \in A$, or $A = B$.*

Proof. — The intersection of two von Neumann ordinals is a von Neumann ordinal. It is also easy to see that the intersection will be an initial segment of each of the two originally chosen ordinals. If it is a proper initial segment of each of them, it will be an element of each of them and so of itself, which is impossible (this would violate strict well-ordering by membership). Thus, the intersection of two distinct von Neumann ordinals must be equal to one of them, which will be an element of the other by the preceding Lemma. $\square$

Lemma. *Each von Neumann ordinal A is the set of all von Neumann ordinals B such that the strict well-ordering of B^+ by $\in$ has smaller order type than the strict well-ordering of A^+ by $\in$.*

Proof. — By the Lemmas above, each von Neumann ordinal A has as elements exactly those von Neumann ordinals B which are proper initial segments of A with respect to the order determined by $\in$. B is a proper initial segment of the order on A iff the order type of membership on B^+ is smaller than the order type of membership on A^+; we have to consider the successors to avoid the strange case of '0' and '1'. $\square$

Theorem. *The set of von Neumann ordinals does not exist.*

Proof. — Suppose that there was a set N whose members were exactly the von Neumann ordinals. Any element of a von Neumann ordinal is a von Neumann ordinal, so $\bigcup[N] \subseteq N$ would hold. It follows from the lemmas above that $\in$ coincides with inclusion on von Neumann ordinals, so the restriction of $\in$ to N would be a set. $\in$ would be a strict linear order on N by a preceding Lemma. Every initial segment of N is a von Neumann ordinal, on which $\in$ is a strict well-ordering; it is easy to see that a strict linear order every initial segment of which is a strict well-ordering is a strict well-ordering itself. Thus N would itself be a von Neumann ordinal, from which it would follow that $N \in N$, but no von Neumann ordinal can be an element of itself! $\square$

Theorem. *The order type of $\in$ restricted to a von Neumann ordinal is Cantorian.*

PROOF. — Let α be the order type of $\in$ restricted to a von Neumann ordinal A. The inclusion order on the set of proper initial segments of A with respect to the order $\in$ has the order type $T\{\alpha\}$. But this is the same set with the same order as A ordered by $\in$, so $\alpha = T\{\alpha\}$ is Cantorian. □

It follows that "large" ordinals such as Ω cannot have corresponding von Neumann ordinals. It is not possible to prove in our set theory that there are any infinite von Neumann ordinals; it turns out to be consistent with our axioms that every Cantorian ordinal has a corresponding von Neumann ordinal.

Exercises

(a) Construct examples of well-orderings and ordinal numbers. Take sums and products of some of these order types and describe well-orderings belonging to the sums and products.

(b) Present definitions of addition and multiplication of ordinal numbers using transfinite recursion. Prove that they are equivalent to the definitions given. Propose a definition of ordinal exponentiation using transfinite recursion.

(c) Construct a well-ordering $\leqslant$ with the property that for any sequence s of elements of $\mathrm{dom}(\leqslant)$ which is increasing in the sense of $\leqslant$, there is an element x of $\mathrm{dom}(\leqslant)$ with the property that $x > s_n$ for each $n \in \mathcal{N}$.

Hint: consider the class of order types of well-orderings of $\mathcal{N}$.

(d) (hard) Finite sets can be modelled in the natural numbers using the relation $m \in^{\mathcal{N}} n$ defined by "the mth binary digit of n is 1", where the mth binary digit is the coefficient of 2^m in the binary expansion: the digit in the "ones place" is the 0th binary digit. Show that the relation $\in^{\mathcal{N}}$ is represented by a set in our theory. A permutation π is defined as interchanging each natural number n with $\{m \mid m \in^{\mathcal{N}} n\}$. Show that the permutation π exists (you need the Axiom of Counting for this). Now define the relation $x \in^{\pi} y$ as $x \in \pi(y)$. The relation $\in^{\pi}$ is not a set (the relative type of y is one type higher than the relative type of x). Prove that if $\in^{\pi}$ is used as the membership relation instead of $\in$, all of our axioms will still hold (you can neglect Small Ordinals, though it works as well) and the set of all finite von Neumann ordinals will exist.

Cahiers du Centre de logique
Volume 10

Chapter 16

Cardinal Numbers

We defined the relationship of "equivalence" above as holding between sets A and B whenever there is a bijection f between A and B. We proved that "equivalence" is an equivalence relation. We showed that two finite sets belong to the same natural number n (i.e., are "the same size") exactly if they are equivalent. What we will do in this chapter is generalize the application of "equivalence" as meaning "the same size" to all sets, and use the notion of "equivalence" to define the notion of cardinal number for general sets. The intuitive idea that two sets are the same size if we can set up a one-to-one correspondence between them, taken from our experience with finite sets, extends to infinite sets fairly well.

16.1. Size of Sets and Cardinals

One counterintuitive consequence of adopting this notion of "size" of sets is that a proper part of a set need not be strictly smaller than the set: for instance, to use a classical example due to Galileo Galilei, there is a bijection between the set of natural numbers $\mathcal{N}$ and its proper subset, the set of perfect squares. In fact, in the presence of the Axiom of Choice, this failure of our intuition for an infinite set defines the notion of "infinite set" exactly:

Theorem (Dedekind). *A set is infinite exactly if it is equivalent to one of its proper subsets.*

PROOF. — Each set belonging to 0 is not equivalent to any of its proper subsets. That is, { } is not equivalent to any of its (nonexistent) proper subsets. Suppose that no element of n is equivalent to any of its proper subsets. An element A of $n+1$ is of the form $A' \cup \{x\}$, where $A' \in n$ and not$(x \in A')$. Suppose that A were equivalent to one of its proper subsets. Remove x from A and the corresponding element from the proper subset; we may suppose without loss of generality that x does not belong to the proper subset. A one-to-one correspondence between A' and a proper subset of A' is constructed, contrary to hypothesis. Thus no finite set is equivalent to a proper subset of itself. Now suppose that X is infinite. It is sufficient to prove that X contains a subset equivalent to $\mathcal{N}$, since we can then use a bijection between $\mathcal{N}$ and one of its proper subsets to construct a bijection between X and one of its proper subsets. Consider the partial order of inclusion on one-to-one maps from subsets of $\mathcal{N}$ into X. This order satisfies the conditions of Zorn's Lemma (since X is not finite); a maximal element in this order is a one-to-one map from $\mathcal{N}$ into X. □

We would like to say that a set X is less than or equal to a set Y in size if there is a one-to-one map from the set X into the set Y. This requires a little work; it is easy to see that X is less than or equal to X in size (identity map) and that if X is less than or equal to Y in size and Y is less than or equal to Z in size, X is less than or equal to Z in size (use a composition). We also expect, however, that if X is less than or equal to Y in size and Y is less than or equal to X in size, that X and Y will be the same size, that is, equivalent. This is the not entirely trivial

Schröder–Bernstein Theorem. *If X and Y are sets, and f is a one-to-one map from X into Y and g is a one-to-one map from Y into X, then X and Y are equivalent.*

PROOF. — We need to construct a bijection h between X and Y. It is sufficient to show that X is equivalent to $g[Y] = g[f[X]] \subseteq X$. The map $g \circ f$ is a one-to-one map from X into $g[Y]$. Let the set Pushset be defined as the smallest set containing $X - g[Y]$ and closed under $g \circ f$. Define Push as the union of the restriction of $(g \circ f)$ to Pushset and the restriction of the identity map to $X - \text{PushSet}$. It is easy to see that Push is a bijection from X onto $g[Y]$. $g^{-1} \circ \text{Push}$ is the desired h. □

The Axiom of Choice enables us to show that this is a total order on size:

Theorem. *For any sets X, Y, either X is less than or equal to Y in size or Y is less than or equal to X in size.*

Proof. — Consider well-orderings R, S on X, Y respectively. If R and S are similar, X and Y are of the same size; if S is the continuation of a relation similar to R, X is less than or equal to Y in size; if R is the continuation of a relation similar to S, Y is less than or equal to X in size; there are no other alternatives. □

We also need the Axiom of Choice to show the following:

Theorem. *If f is a map from X onto Y, then Y is less than or equal to X in size.*

Proof. — Build a choice set C for the collection of inverse images of singletons $\{y\}$ of elements y of Y; map each y to the unique element x of C such that $f(x) = y$. □

This removes some possible doubts about the following

Definition. *Let A be a set. We define the* cardinal number *of A, written $|A|$, as $\{B \mid B$ is equivalent to $A\}$, which exists by the Stratified Comprehension Theorem. Note that the type of $|A|$ is one higher than the type of A.*

Observation. The natural numbers are cardinal numbers. We call them "finite cardinal numbers".

Definition. *We define $|A| \leqslant |B|$ iff A is equivalent to a subset of B.*

We have already shown that $\leqslant$ is a linear order on cardinal numbers, but we can do better.

Theorem. *The order $\leqslant$ on cardinal numbers is a well-ordering.*

Proof. — Take any set A of cardinals, and choose a cardinal κ from the set. Choose a set $W \in \kappa$ and choose a well-ordering R of W. We look at the cardinal numbers of initial segments of R; consider the set of all a in W such that $|\operatorname{seg}_R\{a\}| \geqslant \lambda$ for some λ in A. If this set is empty, κ is the smallest cardinal in A; if it is nonempty, it has a least element m, and $|\operatorname{seg}_R\{m\}|$ is the smallest cardinal in A. □

16.2. Simple Arithmetic of Infinite Cardinals

We define the arithmetic operations of addition and multiplication just as we defined them for the natural numbers.

Definition. *If A and B are sets, $|A|+|B| = |A \oplus B|$ and $|A|.|B| = |A \times B|$.*

It is easy to show that the results of these operations do not depend on the choice of A and B. We will now introduce our first infinite cardinal number.

Definition. $\aleph_0 = |\mathcal{N}|$. *A set of cardinality $\aleph_0$ is said to be "countable".*

We prove some theorems about $\aleph_0$ (read "aleph-null" — $\aleph$ is the first letter of the Hebrew alphabet) and general cardinal numbers which will enable us to show that a great many sets are countable; in fact, we will begin to wonder if any infinite sets are *not* countable.

Theorem. *If κ is an infinite cardinal number and n is a finite cardinal number (a natural number), then $\kappa + n = \kappa$.*

PROOF. — Let $\kappa = |A|$. A has a countable subset (a subset equivalent to $\mathcal{N}$) by the theorem of Dedekind given above (p. 122). Let f be a bijection from $\mathcal{N}$ into A. $\kappa + n$ is the cardinality of $A \times \{0\} \cup \{1, \ldots, n\} \times \{1\}$, applying Rosser's Counting Theorem to get a set with n elements (the use of the Counting Theorem is easily avoided). Map each element $(a, 0)$ of $A \times \{0\}$ to $f(f^{-1}(a)+n)$ if a is in the range of f and to a itself otherwise, then map $(i, 1)$ to $f(i-1)$ for $1 \leqslant i \leqslant n$, obtaining a bijection onto A. □

Theorem. $\aleph_0 + \aleph_0 = \aleph_0$.

PROOF. — Map $\mathcal{N} \times \{0\}$ to the even numbers and $\mathcal{N} \times \{1\}$ to the odd numbers in the natural way. □

Theorem. $\kappa + \kappa = \kappa$, *if* κ *is an infinite cardinal.*

Proof. — Let $\kappa = |A|$. Apply Zorn's Lemma to the relation of inclusion restricted to the set of bijections between $X \oplus X$ and X for $X \subseteq A$. There are such maps because A has countable subsets. These are easily seen to satisfy the conditions of the Lemma. A maximal element will be a bijection from $Y \oplus Y$ to Y for some $Y \subseteq A$. We prove that the set $A - Y$ is finite. If it were not, then $A - Y$ would have a countable subset M; it would then be easy to define a bijection from $(Y \oplus M) \oplus (Y \oplus M)$ to $(Y \oplus M)$ extending the "maximal" bijection by exploiting the bijection from $(M \oplus M)$ to M. Now we have shown that $|Y| + |Y| = |Y|$, but, since $A - Y$ is finite, $|Y| = |A|$ by an earlier theorem. □

Theorem. $\kappa + \lambda = \lambda + \kappa = \kappa$, *if* λ *is an infinite cardinal and* $\kappa \geqslant \lambda$.

Proof. — Commutativity of addition is obvious. If $\kappa = |A|$ and $\lambda = |B|$, choose a subset C of A equivalent to B, then define a bijection between $C \oplus B$ and B and use it in the obvious way to build a bijection from $A \oplus B$ to A. □

Theorem. $\aleph_0 \aleph_0 = \aleph_0$.

Proof. — First list the ordered pairs of natural numbers which add up to 0, then the ordered pairs which add up to 1 (increasing the first projection and decreasing the second), then the ordered pairs which add up to 2, and so forth. This sets up a bijection between $\mathcal{N} \times \mathcal{N}$ and $\mathcal{N}$. □

Theorem. $\kappa\kappa = \kappa$, *for* κ *an infinite cardinal.*

Proof. — Let $\kappa = |A|$. Apply Zorn's Lemma to the inclusion order restricted to bijections between $X \times X$ and X, where $X \subseteq A$. Since A has a countable subset, there are such maps. A maximal element of this order is a bijection from $Y \times Y$ to Y for some $Y \subseteq A$. Now suppose that $|Y| < |A|$. It would follow that $|A - Y| = |A|$, since $|A - Y| + |Y| = |A|$ and the sum of two infinite cardinals is their maximum. It follows that $A - Y$ has a subset Z equivalent to Y. Now consider the set $(Y \oplus Z) \times (Y \oplus Z)$; this is the union of four Cartesian products (take one term of each disjoint sum in each possible way), each having cardinality $|Y| \times |Y| = |Y| = |Z|$. The union of the three Cartesian products not in the domain of our "maximal" function must have cardinality $|Y| = |Z|$ by our theorems on addition, and so there is a bijection from this set to Z, which can be used then to extend the "maximal" bijection to a bijection from $(Y \oplus Z) \times (Y \oplus Z)$ onto $(Y \oplus Z)$, contradicting maximality. Thus $|Y| = |A|$, and we are done. □

Theorem. *$\kappa\lambda = \lambda\kappa = \kappa$ if λ is an infinite cardinal and $\kappa \geqslant \lambda$.*

PROOF. — $\kappa \leqslant \kappa\lambda \leqslant \kappa\kappa = \kappa$; apply the Schröder–Bernstein Theorem. □

Corollary. *For any infinite cardinals κ, λ, $\kappa + \lambda = \kappa\lambda = \max\{\kappa, \lambda\}$.*

This is all very well, and remarkably simple, but it does not give us any way of getting any infinite cardinal larger than $\aleph_0$! The results of this chapter can be used to show that the set $\mathcal{Z}$ of integers is countable ($(\mathcal{N} \oplus \{0\}) \oplus \mathcal{N}$ is clearly of the same cardinality as the set of integers), that the set of rational numbers is countable (!) (the set $\mathcal{Q}$ of rational numbers has a subset equivalent to $\mathcal{Z}$ and is equivalent in an obvious way to a subset of $\mathcal{Z} \times \mathcal{Z}$); the same results apply to intuitively "bigger" sets such as the set of all points with rational coordinates in three-dimensional space. Are all infinities the same?

Exercises

(a) Give an algebraic formula for a bijection $p \in [(\mathcal{N} \times \mathcal{N}) \to \mathcal{N}]$ (an explicit witness for $\aleph_0\aleph_0 = \aleph_0$).

(b) Construct a bijection between the natural numbers and the integers.

(c) Construct a bijection between the natural numbers and the rational numbers (your description should enable me to compute the rational number corresponding to each natural number).

(d) Prove that the set of all points with rational coordinates in three-dimensional space is countable.

Cahiers du Centre de logique
Volume 10

Chapter 17

Exponentiation and Three Major Theorems

17.1. Exponentiation meets T. Infinite Sums and Products

The arithmetic operation which we have avoided, even in the realm of finite sets, is exponentiation. The natural definition of exponentiation is

$*$**Definition** $|A|^{|B|} = |[B \to A]|$.
Special case: $2^{|A|} = |[A \to \{0,1\}]| = |\mathcal{P}\{A\}|$.

The idea is that we must make $|A|$ independent choices from a set of size $|B|$. The Axiom of Choice protects us from certain possible embarrassments. This definition does not work (in the sense that it does not define a function) because it is not stratified; the types of A and B are one higher on the right than on the left.

The solution to the problem is to introduce another T operation (extending the T operation already defined on natural numbers):

Definition. $T\{|A|\} = |\mathcal{P}_1\{A\}|$.

It is easy to show that this is a valid definition (it does not depend on the choice of A). It is less easy to see that it is nontrivial, but we need to remember that we have already shown that the natural bijection $(x \mapsto \{x\})\lceil A$ between A and $\mathcal{P}_1\{A\}$ (the set of one-element subsets of A) does not exist in at least the case $A = V$. Note that $T\{\kappa\}$ has type one higher than that of κ. In the definitions below we will use the inverse T^{-1} of this operation, which proves to be a well-defined partial operation on cardinals (defined only on cardinals less than or equal to $|\mathcal{P}_1\{V\}|$) and has the effect of lowering relative type rather than raising it.

We can then give a genuine (partial)

Definition. $|A|^{|B|} = T^{-1}\{|[A \to B]|\}$; $2^{|A|} = T^{-1}\{|\mathcal{P}\{A\}|\}$.

Corollary. $T\{|A|\}^{T\{|B|\}} = |[A \to B]|$; $2^{T\{|A|\}} = |\mathcal{P}\{A\}|$.

We give the function $(\kappa \mapsto 2^\kappa)$ a name:

Definition. $\exp = (\kappa \mapsto 2^\kappa)$.

Exponentiation is a special case of the more general problem of sums and products of infinite sets of cardinals; we give relevant definitions:

Definition. *Let G be an indexed family of sets. Let F be the associated infinite family of cardinals, defined by $F(\{i\}) = |G(i)|$ for each i in $\mathrm{dom}(G)$.*

We define $\sum[F]$, the sum of F, as $T^{-1}\{|\sum[G]|\}$, the result of applying T^{-1} to the cardinality of the disjoint sum of G.

We define $\prod[F]$, the product of F, as $T^{-2}\{|\prod[G]|\}$, the result of applying T^{-2} to the cardinality of the Cartesian product of G.

Self-indexing may be used to define sums and products of sets. Context cues will of course be required to distinguish these notations from the identical notations for Cartesian products and disjoint sums of sets!

The criterion for determining the applications of T in these definitions is that the type of individual cardinals in the infinite sums and products should be the same as the type of the resulting cardinals.

For the sake of completeness, we develop a definition of exponentiation of ordinals. Infinite sums and products of ordinals can also be defined, but we will not do this. A natural operation on ordinals should, of course, correspond to a natural operation on well-orderings. The product of ordinals corresponded to the order type of the "reverse lexicographic" ordering of pairs of elements taken from well-orderings taken from the factors. For example, ω^2 is the order type of the lexicographic ordering on pairs of elements taken from an ordering of type ω (natural numbers, for example).

This suggests that ω^ω, for example, should be the reverse lexicographic ordering on strings of numbers of length ω. Unfortunately, one does not obtain a well-ordering (or even a total order) in this way. Experiment reveals that one does obtain a well-ordering of items taken from an order of type β ordered with type α if one observes the restriction that all but finitely many of the α-ordered items are the smallest element of the order of type β. Here is the formal definition:

Definition. *Let α, β be ordinal numbers. Choose well-orderings R, S, with least elements r, s respectively, from α, β, respectively. We define a well-ordering W on the set of all elements of $[\mathrm{dom}(R) \to \mathrm{dom}(S)]$ with the property that all but finitely many values of the function are s, the least element of $\mathrm{dom}(S)$. For f, g in this set of functions, we say that $f\ W\ g$ if $f(x)\ S\ g(x)$, where x is the R-maximal element at which the values of f and g differ (there must be such a maximal element because of the restriction on the functions used). The order type of W will be $T\{\beta^\alpha\}$; if the order type of W is not in the range of the T operation, β^α is not defined (this actually never happens).*

Ordinal exponentiation can also be defined by transfinite recursion in a fairly natural way. The need to assign higher priority to positions in sequences which are later in the α-order motivates the use of reverse lexicographic order here (where it is necessary) and in the definition of ordinal multiplication (where it seems a bit unnatural; I would rather write 2ω than $\omega 2$, but $2\omega = \omega$).

Ordinal exponentiation does not parallel cardinal exponentiation in its effects on size. Note, for example, that ω^ω is the order type of a well-ordering of a countable set. In fact, the cardinality of the full domain of a well-ordering of type β^α is no larger than the larger of the sizes of full domains of representatives of α and β. A consequence of this is that β^α is always defined.

It should be evident that the T operation "commutes" with all operations of cardinal (and ordinal) arithmetic. This observation is recorded in the following:

Lemma. *If κ and λ are cardinal numbers,*

$T\{\kappa\} \leqslant T\{\lambda\}$ *iff* $\kappa \leqslant \lambda$ *(thus* $T\{\kappa\} = T\{\lambda\}$ *iff* $\kappa = \lambda$*);*

$T\{\kappa + \lambda\} = T\{\kappa\} + T\{\lambda\}$*;*

$T\{\kappa\lambda\} = T\{\kappa\}T\{\lambda\}$*;*

$T\{\kappa\}^{T\{\lambda\}} = T\{\kappa^\lambda\}$ *— if the latter exists*
(thus $\exp(T\{\kappa\}) = T\{\exp(\kappa)\}$ *— if* $\exp(\kappa)$ *exists).*

PROOF. — Omitted. Use the fact that any structure on a collection of objects corresponds to an exactly parallel structure on the collection of singletons of those objects. □

17.2. Cantor's Theorem. Cantor's "Paradox" meets $\mathcal{P}_1$

We now prove a powerful theorem, whose "naive" form is the last of the three classical paradoxes of set theory.

Theorem (Cantor). $T\{\kappa\} < \exp(T\{\kappa\})$.

PROOF. — Let $\kappa = |A|$. Clearly $T\{\kappa\} = |\mathcal{P}_1\{A\}| \leqslant |\mathcal{P}\{A\}| = \exp(T\{\kappa\})$. If we had $T\{\kappa\} = \exp(T\{\kappa\})$, we would have a one-to-one map f from $\mathcal{P}_1\{A\}$ onto $\mathcal{P}\{A\}$; it suffices to show that the existence of such a map is impossible. If such an f existed, we could define the set $R = \{x \in A \mid \text{not}(\{x\} \subseteq f(\{x\})\}$. Now $f^{-1}(R) = \{r\}$ for some $r \in A$; we see that $r \in R$ exactly if $\text{not}(\{r\} \subseteq f(\{r\}))$, i.e., exactly if $\text{not}(r \in R)$, which is impossible. □

Corollary. $\kappa < \exp(\kappa)$.

In completely naive set theory, where we do not doubt that $\mathcal{P}_1\{A\}$ is the same size as A, we get the following

$*$**Corollary (Cantor's Paradox).** $|V| = |\mathcal{P}_1\{V\}| < |V|$.

Of course, what we actually get is the still disturbing

Corollary. $|\mathcal{P}_1\{V\}| < |V|$ *(note that we can write this* $|1| < |V|$ *or* $T\{|V|\} < |V|$*, as well).*

which should not surprise us too much, as we have already shown that the natural "bijection" $(x \mapsto \{x\})$ does not exist.

Note that we have now proven that there are uncountable sets; $\aleph_0 = |\mathcal{N}| = T\{|\mathcal{N}|\} < \exp(T\{|\mathcal{N}|\}) = |\mathcal{P}\{\mathcal{N}\}|$; there are more sets of natural numbers than there are natural numbers. We point out that $\exp(\aleph_0)$ is also the cardinal number of $\mathcal{R}$, the set of reals: the union of the reals between 0 and 1 and a countable set (the set of decimal expansions ending in infinitely many 9's — countable because equivalent to the rationals between 0 and 1) is equivalent to the set of decimal expansions with zero to the left of the decimal point, which is equivalent to the set of characteristic functions of subsets of $\mathcal{N}$, which is equivalent to $\mathcal{P}\{\mathcal{N}\}$. For this reason, $\exp(\aleph_0)$ is called c, the cardinality of the continuum.

Since the natural order on cardinals is a well-ordering, there is a first uncountable cardinal, called $\aleph_1$. It is an open problem whether c is equal to $\aleph_1$; i.e., whether there are uncountable sets of real numbers which are not equivalent to $\mathcal{R}$ itself. It is known that this question cannot be decided by the usual axioms of set theory (or by ours!). This famous question is called the Continuum Hypothesis; it was raised by Cantor at the very beginning of the subject. An even more powerful assumption, also known to be undecidable by the axioms, is the Generalized Continuum Hypothesis (GCH), which asserts that $\exp(\kappa)$ is the smallest cardinal greater than κ for every cardinal κ for which $\exp(\kappa)$ is defined.

17.3. König's Theorem

We prove a theorem relating the sizes of infinite products and infinite sums, of which Cantor's Theorem is only a special case.

König's Theorem. *Let F and G be functions with the same nonempty domain I all of whose values are cardinal numbers. Suppose further* *that $F(x) < G(x)$ for all $x \in I$, where the order is the natural order on cardinals. It follows that $\sum[F] < \prod[G]$.*

PROOF. — Consider concrete sets A and B of cardinality $\sum[F]$ and $\prod[G]$, respectively.

We may suppose that A is the union of disjoint nonempty sets $\mathcal{A}(x)$ for x in a domain of size $T^{-1}\{|I|\}$; we assume henceforth that I is a set of singletons and that the domain of the function $\mathcal{A}$ is the union of I, which we will call J. No generality is lost by assuming that each set $\mathcal{A}(x)$ is nonempty, i.e., that each $F(\{x\})$ is nonzero.

We choose sets $\mathcal{B}(x)$ of cardinality $G(\{x\})$ for each $x \in J$ and a bijection h from $\mathcal{P}_1^2\{B\}$ onto the set of functions $f : J \to \mathcal{P}_1\{V\}$ such that $f(x) \subseteq \mathcal{B}(x)$. The functions f are the elements of the Cartesian product of $\mathcal{B}$; they are two types above the elements of B.

Consider any map $M : A \to B$. We construct an element p of B which cannot be an element of $\text{rng}(M)$. It should be clear that this will be sufficient to prove the theorem. $h(\{\{p\}\})$ needs to differ from each $h(\{\{M(q)\}\})$ at some element of J. We describe the procedure for construction of $h(\{\{p\}\})(x)$ for a fixed $x \in J$; once we have constructed a suitable function, the fact that h is a bijection allows us to obtain a suitable p. We want to choose one element from $\mathcal{B}(x)$ in such a way as to frustrate the identification of $h(\{\{p\}\})$ with any $h(\{\{M(q)\}\})$ for $q \in \mathcal{A}(x)$. This is possible for each x, because there are only $|\mathcal{A}(x)| = F(\{x\}) < G(\{x\}) = |\mathcal{B}(x)|$ values of the form "the element of $h(\{\{M(q)\}\})(x)$" to avoid, and $|\mathcal{B}(x)|$ values to choose from. It should be clear that this procedure prevents the identification of $h(\{\{p\}\})$ with any $h(\{\{M(q)\}\})$ for q in any $\mathcal{A}(x)$, i.e., for any $q \in A$. The proof of König's Theorem is complete. □

The proof of König's Theorem involves an essential appeal to the Axiom of Choice, as is not the case for Cantor's Theorem.

17.4. There are Atoms: Specker's Theorem

We now prepare to prove a further theorem, due to E. Specker. This is the surprising result that there are atoms! It is not surprising that there are atoms; it is surprising that we can prove that there must be atoms, since on the face of it we have not talked about anything that could not be a set (our ordered pairs are not necessarily sets, but we could have used the pair $\langle x, y\rangle$ of Kuratowski and done everything in pure set theory). The Axiom of Choice seems to be necessary for this proof. It is unknown whether the existence of atoms can be proven from the other axioms of our theory; this leaves open the possibility of a consistent "atomless" theory in which the Axiom of Choice is false [1].

Definition. *For each cardinal κ, we define the* Specker sequence *of κ as the intersection of all sets which contain $(0, \kappa)$ and contain $(n+1, \exp(\lambda))$ whenever they contain (n, λ) and $\exp(\lambda)$ is defined (the construction here is the construction from the proof of the Recursion Theorem modified for a partial function). The Specker sequence is a (possibly finite) sequence; we use the notation $\exp^i(\kappa)$ for its ith term. We call a cardinal κ a* Specker number *iff $\exp^n(\kappa) = |V|$ for some natural number n (a Specker number is usually defined as a cardinal with a finite Specker sequence, but our more restrictive definition is better for our version of the proof).*

Theorem (Specker). $|\mathcal{P}\{V\}| < |V|$*, and thus there are atoms.*

PROOF. — Consider the set of Specker numbers. It must have a smallest element μ, since the natural order on cardinals is a well-ordering and the Specker numbers make up a nonempty set. Let s be the Specker sequence of μ. Since μ is a Specker number, there is an N such that $s(N) = |V|$. Now define the function T$[s]$ as the Specker sequence of $T\{\mu\}$. An easy induction (on an unstratified condition) using the fact that exp commutes with T (where defined) proves that $T[s](n) = T\{s(n)\}$ for each natural number n, and so $T[s](N) = T\{|V|\}$. Now the assertion that there are no atoms implies the assertion that $\mathcal{P}\{V\} = V$ ($\mathcal{P}\{V\}$ is the collection of sets), which implies that $\exp(T\{|V|\}) = \exp(|\mathcal{P}_1\{V\}|) = |\mathcal{P}\{V\}| = |V|$. If there were no atoms, we would have $T[s](N+1) = \exp(T[s](N)) = \exp(T\{|V|\}) = |V|$, and $T\{\mu\}$ would be a Specker number. But this is absurd; we would then have $T\{\mu\} \geqslant \mu$, since μ is the smallest Specker number, and so, by an easy induction, we would have $T[s](n) \geqslant s(n)$ for each n (if $\kappa \leqslant \lambda$, $\exp(\kappa) \leqslant \exp(\lambda)$), and so we would have $T\{|V|\} = T[s](N) \geqslant s(N) = |V|$, which is impossible. □

1. For the non-naive reader, this is the question as to whether Quine's original theory "New Foundations" is consistent.

It is worth noting that the use of mathematical induction on an unstratified condition, which depends on the Axiom of Small Ordinals (the Axiom of Counting suffices), can be avoided; there is a rather more complicated proof which requires nothing but Extensionality, Stratified Comprehension, and the Axiom of Choice (without which we could not show that the natural order on the cardinal numbers is a well-ordering).

Since the proof actually shows that $|\mathcal{P}\{V\}| < |V|$, we have shown that most objects in V are atoms, which also shows that most ordered pairs (there are $|V|.|V| = |V|$ ordered pairs) are atoms; we cannot hope for a set theoretical definition of our pair (x, y).

17.5. Cantorian and Strongly Cantorian Sets

In the usual set theory, Cantor's Theorem is taken to prove that the power set of a set is larger than the set. This depends on the intuition that the singleton map $(x \mapsto \{x\})$ is a function, or at least that $|\mathcal{P}_1\{A\}| = |A|$, which is not reliable. As we have just seen, there is a set (the universe) which is *larger* than its power set!

However, the argument of Cantor will work exactly as intended if the set A to be compared with $\mathcal{P}\{A\}$ has certain properties:

Definition. *A set A is* Cantorian *if it is equivalent to $\mathcal{P}_1\{A\}$; A is* strongly Cantorian *if the singleton map restricted to A exists as a set (that is, if there is a function $(x \mapsto \{x\})\lceil A$ such that for each $x \in A$, $((x \mapsto \{x\})\lceil A)(x) = \{x\}$). If A is a Cantorian or strongly Cantorian set, $|A|$ is said to be a Cantorian or strongly Cantorian cardinal. The order type of a strongly Cantorian well-ordering is said to be a strongly Cantorian ordinal.*

Observations. If A is strongly Cantorian, it is Cantorian; if A is Cantorian, $|A| = |\mathcal{P}_1\{A\}| < |\mathcal{P}\{A\}|$. If A is the domain of a well-ordering R which in turn belongs to an ordinal α, then the ordinal α is (strongly) Cantorian exactly if A is (strongly) Cantorian.

The truly powerful property is that of being strongly Cantorian. Without introducing the Axiom of Small Ordinals, we can already "subvert" stratification restrictions if we have strongly Cantorian sets:

Subversion Theorem. *If a variable x in a sentence ϕ is restricted to a strongly Cantorian set A, then its type can be freely raised and lowered; i.e., such a variable can safely be ignored in making type assignments for stratification.*

Proof. — Let K be the singleton map restricted to A. Any occurrence of x can be replaced with an occurrence of $\bigcup[K(x)]$, raising the type of x by one, or with an occurrence of $K^{-1}(\{x\})$, lowering the type of x by one, without affecting the meaning of the sentence. $\square$

As a result, unstratified conditions can be used extensively where strongly Cantorian sets are involved, but variables not subject to stratification conditions must be "bounded" in a strongly Cantorian set.

Unfortunately, the only sets which can be shown to be strongly Cantorian with Extensionality, Atoms, Ordered Pairs, and Stratified Comprehension alone are "standard finite sets"; each finite set for which we can actually list the elements can be proven to be strongly Cantorian, but we cannot even prove that all finite sets are strongly Cantorian, much less that $\mathcal{N}$ is strongly Cantorian (although we can prove that $\mathcal{N}$ is Cantorian). Even adopting as an axiom our Axiom of Counting (which shows that $\mathcal{N}$ and all finite sets are strongly Cantorian; we have already enunciated the special case of the Subversion Theorem which applies the Axiom of Counting) strengthens the theory considerably. The power of the Axiom of Small Ordinals or a similar principle is that it enables us to evade stratification restrictions on a wider range of sets.

It is worth noting that the argument given above to show that all von Neumann ordinals are Cantorian actually demonstrates that they are strongly Cantorian.

Exercises

(a) Construct a bijection between $\mathcal{P}\{\mathcal{N}\}$ and $\mathcal{R}$. Construct a bijection between $\mathcal{R}$ and $\mathcal{R}^2$.

(b) We indicate an analytic proof that $\mathcal{R}$ is uncountable. Suppose otherwise. Then we would have a sequence r such that each real number

$x = r_i$ for some $i \in \mathcal{N}$. Start with a closed interval $[a_0, b_0]$ in the reals which has r_0 neither in its interior nor as an endpoint. We construct a sequence of intervals $[a_i, b_i]$. At step i, we divide the current interval in the sequence into three parts of equal length (why do we need three?) and let the next interval in the sequence be chosen to have r_{i+1} neither in its interior nor as an endpoint. Complete the argument that this allows us to construct a real number not equal to any r_i.

(c) Show that Cantor's Theorem is a special case of König's Theorem.

(d) Analyze the proof of König's Theorem from the standpoint of stratification; assign relative types to all the objects involved in the argument and check that the relative types are correct for the relationships postulated among the objects.

(e) Prove that there are infinitely many cardinals κ such that every set of size κ contains atoms.

(f) (hard) We outline the approach of Specker to proving that the Axiom of Choice implies that there are atoms without our appeal to Counting: assume that $\mathcal{P}\{V\} = V$, so that $\exp(T\{|V|\}) = |V|$. Define "Specker numbers" as cardinals κ such that $\exp^n(\kappa)$ exists for only finitely many n, and define μ as the smallest Specker number. Prove that $\mu = T\{\mu\}$ by proving that $T\{\mu\}$ and $T^{-1}\{\mu\}$ are both Specker numbers (one has to prove that the latter exists), then considering the definition of μ. Let n be the largest natural number such that $\exp^n(\mu)$ is defined. Considering the fact that $\mu = T\{\mu\}$, show that $n = T\{n\} + 1$ or $n = T\{n\} + 2$. Why is this impossible? Complete the details of the proof.

(g) Suppose that A and B are strongly Cantorian sets. Prove that $A \cap B$, $A \cup B$, $A \times B$, $\mathcal{P}\{A\}$, and $[A \to B]$ are strongly Cantorian sets, without appealing to the Subversion Theorem.

Cahiers du Centre de logique
Volume 10

Chapter 18

Sets of Real Numbers

In this section, we resume our treatment of the familiar systems of rational and real numbers.

18.1. Intervals, Density, and Completeness

We first consider a special property of the total order defined on the rational and real numbers.

Definition. *A linear order* $\leqslant$ *is said to be* dense *iff for every pair* $a < b$ *of elements of* $\mathrm{dom}(\leqslant)$*, there is an element* c *of* $\mathrm{dom}(\leqslant)$ *such that* $a < c < b$*.*

We also state a definition of sets important to consider in connection with linear orders:

Definition. *Let* $\leqslant$ *be a linear order. For any pair of points* $a < b$ *in* $\mathrm{dom}(\leqslant)$*, we define* $[a, b]$ *as* $\{c \mid a \leqslant c \leqslant b\}$*. We call this the* closed interval *determined by* a *and* b*. We define* (a, b)*, the corresponding* open interval*, as* $[a, b] - \{a\} - \{b\}$*. We define* $[a, b)$ *and* $(a, b]$ *as* $[a, b] - \{b\}$ *and* $[a, b] - \{a\}$*, respectively. We allow the use of the symbols* ∞ *and* $-\infty$ *as upper or lower limits:* $[a, \infty) = \{c \mid c \geqslant a\}$*, for example.*

The subsets of dom($\leqslant$) *described in this definition are called* intervals, *including the ones with open "infinite" upper and lower limits.*

Observation. For any $a < b$ in the domain of a dense linear order $\leqslant$, the interval (a, b) is an infinite set.

The linear order on the rationals (and on the reals) is easily seen to be dense. We see in the next theorem that the order on the rationals is very special (indeed, in a sense, unique):

Theorem. *Let $\leqslant$ be a dense linear order with a countable domain and without a greatest or least element. Then there is a bijection f from* dom($\leqslant$) *onto $\mathcal{Q}$ such that $f(x) \leqslant f(y)$ (the order here is the usual order on the rationals) iff $x \leqslant y$. In other words, all such orders have the same structure as the order on the rationals.*

PROOF. — Choose a sequence s which is a bijection from $\mathcal{N}$ onto dom($\leqslant$). Choose a sequence q which is a bijection from $\mathcal{N}$ onto $\mathcal{Q}$. We construct the map f in stages. At stage 0, we set $f(s_0) = q_0$. At each subsequent stage, we will define up to two values of f; we guarantee that at least $f(s_i)$ for each $i \leqslant n$ and $f^{-1}(q_i)$ for each $i \leqslant n$ will be defined at the end of stage n. A finite number of values of f will have been determined at each stage. At stage $n + 1$, we check whether $f(s_{n+1})$ has been defined; if it has not, we define it as q_j, where j is the smallest index such that $f^{-1}(q_j)$ has not already been determined and q_j has the same order relations to each $f(s_i)$ already defined that s_{n+1} has to the corresponding s_i (there will be such a q_j because the order is dense with no largest or smallest element and there are only finitely many s_i's to consider). Then check whether $f^{-1}(q_{n+1})$ has been defined; if it has not, define it as s_k, where k is the smallest index such that $f(s_k)$ has not already been determined and s_k has the same order relations to each $f^{-1}(q_i)$ already defined that q_{n+1} has to the corresponding q_i's (the reasons that this works have already been stated). Observe that the conditions which are to hold at each stage will hold at stage $n + 1$ if they held at stage n. Induction shows that each stage can be carried out, giving a complete definition of a function f with the desired properties. This kind of proof is called a "back-and-forth argument". $\square$

The form of this theorem inspires a

Definition. *Two linear orders $\leqslant$ and $\leqslant^*$ are said to be* isomorphic *iff there is a bijection f from* $\mathrm{dom}(\leqslant)$ *onto* $\mathrm{dom}(\leqslant^*)$ *such that $x \leqslant y$ iff $f(x) \leqslant^* f(y)$. It is easy to see that isomorphism is an equivalence relation on linear orders. The bijection f is said to be an* isomorphism.

Observation. The previous theorem can be restated as "All dense linear orders without endpoints and with countable domains are isomorphic".

The order on the real numbers is dense, and also complete in a stronger sense. The order on the rationals has gaps in it in a sense in which the order on the reals does not.

Definition. *A subset A of the domain of a linear order $\leqslant$ is said to be* convex *iff for each $a, b \in A$ such that $a < b$, $(a, b) \subset A$.*

Interval Classification Theorem. *Each nonempty subset of $\mathcal{R}$ convex with respect to the usual order is an interval.*

Proof. — If the set has no upper bound, the upper limit of the interval will be ∞; if the set has no lower bound, the lower limit of the interval will be $-\infty$. If the interval has an upper bound, take the upper limit of the interval to be the least upper bound of the set, taking the interval to be closed or open at the upper endpoint as appropriate. If the interval has a lower bound, take the lower limit of the interval to be the greatest lower bound of the set, taking the interval to be open or closed at the lower endpoint as appropriate. □

Definition. *A linear order with respect to which every convex set is an interval is said to be* complete*.*

This theorem can be understood as telling us that we cannot cut the real line in a way which misses any real number; for surely the sets above and below a cut would be convex! The subset $\{p \in \mathcal{Q} \mid p^2 < 2\}$ is an example of a convex subset of the rationals with respect to the usual order which is not an interval.

We now propose a further extension of our use of the word *dense*:

Definition. *A subset D of the domain of a linear order $\leqslant$ is said to be* dense *if for every pair $a < b$ in* $\mathrm{dom}(\leqslant)$*, the interval (a, b) contains an element of D.*

For example, $\mathcal{Q}$ is a dense subset of $\mathcal{R}$ with respect to the usual ordering on these sets (and with respect to the usual identification of $\mathcal{Q}$ with a certain subset of $\mathcal{R}$).

We now give a characterization of the structure of the usual order on $\mathcal{R}$:

Theorem. *Any complete dense linear order $\leqslant$ with no endpoints and a countable dense subset D is isomorphic to the usual linear order on $\mathcal{R}$.*

PROOF. — Choose an isomorphism f between the order $\leqslant$ restricted to D and the usual order on the rationals. Each element x of $\text{dom}(\leqslant) - D$ uniquely determines the partition of D into subsets $(-\infty, x) \cap D$ and $(x, \infty) \cap D$ (any $y \neq x$ has a different relation to elements of the nonempty set $(x, y) \cap D$ than x). This determines a partition of the rationals $\mathcal{Q}$ into two convex sets via the isomorphism f, which in turn determines a unique real number r between the two sets. We define $f(x) = r$. This process is reversible; we can determine $f^{-1}(r)$ for each $r \in \mathcal{R}$ in exactly the same way. □

18.2. Topology of $\mathcal{R}$

We introduce elementary concepts of topology of the real line.

Definition. *A subset of $\mathcal{R}$ is said to be* open *iff it is the union of some set of open intervals. Equivalently, a set $A \subseteq \mathcal{R}$ is open iff for each element x of A, there is an open interval $(y, z) \subseteq A$ of which x is an element. Such an interval (y, z) is said to be a* neighborhood *of x; any open set containing x may also be called a neighborhood of x.*

Definition. *A set $A \subseteq \mathcal{R}$ is said to be* closed *iff $\mathcal{R} - A$ is open.*

We prove some easy theorems.

Theorem. *For any set $\mathcal{O}$ of open sets, $\bigcup[\mathcal{O}]$ is an open set.*

PROOF. — The union of a collection of unions of open intervals is still the union of a collection of open intervals. □

Theorem. *For any finite set $\mathcal{A}$ of open sets, $\bigcap[\mathcal{A}]$ is an open set.*

PROOF. — For each element x of $\bigcap[\mathcal{A}]$, we need to find an open interval $(y, z) \subseteq \bigcap[\mathcal{A}]$ containing x. To do this, choose an open interval (y_i, z_i) containing x from each element A_i of $\mathcal{A}$; the intersection of this finite collection of open intervals containing x will be an open interval containing x and also a subset of $\bigcap[\mathcal{A}]$. □

Theorem. *For any set $\mathcal{C}$ of closed sets, $\bigcap[\mathcal{C}]$ is a closed set.*

PROOF. — We need to show that $\mathcal{R} - \bigcap[\mathcal{C}]$ is open. But this is immediate: another way of writing this set is $\bigcup[\{\mathcal{R} - C \mid C \in \mathcal{C}\}]$, which is the union of a collection of open sets, so open. □

Theorem. *The union of any finite collection $\mathcal{B}$ of closed sets is closed.*

PROOF. — This is proved in the same way as the previous theorem; the complement of the union of a finite collection of closed sets is the intersection of the finite collection of (open) complements of those closed sets. □

We define a new concept.

Definition. *A real number r is a* limit point *of a set A iff each open interval containing r meets A. The* closure *of a set A is the set of all limit points of A. A variation on this definition: a real number r is an* accumulation point *of a set A iff each open interval containing r meets $A - \{r\}$. A limit point of A which is not an accumulation point of A must be an element of A and is said to be an* isolated point *of A.*

Theorem. *A set is closed iff it contains all of its limit points. A set is closed iff it is equal to its closure. The closure of any set is closed.*

PROOF. — Suppose that the set C is closed. Each point in C is certainly a limit point of C. Each point y in $\mathcal{R} - C$ has an open interval containing it which lies entirely in $\mathcal{R} - C$, because the complement of C is open. This implies that no point not in C is a limit point of C. It follows that the set of limit points of C coincides with C.

Suppose that a set L contains all of its limit points. It follows that each element y of $\mathcal{R} - L$ belongs to an open interval not meeting L (because it is not a limit point of L). This is exactly what it means for $\mathcal{R} - L$ to be open, and for L to be closed!

Suppose that C is the closure of a set A. A limit point of C is a point any open interval about which will meet a point x of C. We know that any open interval containing $x \in C$ will meet A (because C is the closure of A). Thus any limit point of C is also an element of C (a limit point of A), and we have already seen that a set which contains all of its own limit points is closed.

The proof of the theorem is complete. □

We give a result depending on the existence of the countable dense subset $\mathcal{Q}$ of $\mathcal{R}$.

Theorem. *Any system $\mathcal{O}$ of pairwise disjoint open sets (or, more specifically, open intervals) is at most countable.*

PROOF. — Each element of $\mathcal{O}$ contains an open interval as a subset, which in turn contains a rational number as an element. Choose one rational number to associate with each element of $\mathcal{O}$ (this does not require the Axiom of Choice; one could choose the first rational number falling in $\mathcal{O}$ from a fixed sequence containing all the rationals). A different rational must be associated with each element of $\mathcal{O}$ because the elements of $\mathcal{O}$ are disjoint; $\mathcal{O}$ can be no larger than the collection of such rationals, and so no more than countable. (Stratification would require us to associate the singleton of a rational element with each element of $\mathcal{O}$, but the collection $\mathcal{P}_1\{\mathcal{Q}\}$ is countable as well). □

Here is an important theorem depending on the least upper bound property:

Definition. *A sequence A of sets is said to be* nested *if $A_{i+1} \subseteq A_i$ for each i.*

Theorem. *A nested sequence of closed intervals $[r_i, s_i]$ has nonempty intersection.*

PROOF. — Consider sup $\{r_i \mid i \in \mathcal{N}\}$. This is clearly greater than or equal to each r_i, and also less than or equal to each s_i, and so belongs to each interval. □

We introduce another family of concepts:

Theorem. *Any infinite subset A of a closed interval $[r, s]$ has an accumulation point.*

PROOF. — We construct a sequence of intervals $[r_i, s_i]$ as follows:

- $[r_0, s_0] = [r, s]$;
- $[r_{i+1}, s_{i+1}] = [r_i, \frac{r_i+s_i}{2}]$ if this interval contains infinitely many elements of A;
- otherwise $[\frac{r_i+s_i}{2}, s_i]$

(this is a definition by recursion). A simple induction shows that each $[r_i, s_i]$ contains infinitely many elements of A. The diameter of $[r_i, s_i]$ will be $\frac{s-r}{2^i}$. The intersection of all the r_i's is nonempty (by the previous theorem) and contains no more than one point (because the diameters of the $[r_i, s_i]$'s converge to zero). Call this point p. Any open interval containing p contains some entire interval $[r_i, s_i]$ (one can take i sufficiently large that the diameter of $[r_i, s_i]$ will be smaller than the distance from p to either endpoint of the open interval about p), so contains points of A distinct from p itself; thus p is an accumulation point of A in $[r, s]$. The proof of the theorem is complete. □

Definition. *If A is a subset of $\mathcal{R}$ and $\mathcal{O}$ is a set of open subsets of $\mathcal{R}$, $\mathcal{O}$ is said to be an* open cover *of A iff $A \subseteq \bigcup[\mathcal{O}]$.*

Definition. *A subset of the reals is said to be* compact *if it is a closed subset of a closed interval $[r, s]$ (i.e., it is closed and bounded above and below).*

Theorem. *A subset K of the reals is compact iff each open cover of K has a finite subcover (i.e., a finite subset which is also an open cover).*

PROOF. — If K is not bounded, the set of intervals $(r, r+1)$ will provide an open cover of K with no finite subcover. If K is not closed, choose a limit point p of K which is not an element of K, and the collection of all complements of closed intervals centered at p will provide an open cover of K with no finite subcover.

Now suppose that K is closed and bounded. Any open cover of K has a countable subcover; it suffices to choose one element of the cover containing each open interval with rational endpoints that is a subset of some element of the cover. Thus every open cover of K is the range of a sequence of open sets O. Suppose that O is a sequence of open sets whose range has no finite subcover; we can require further that no $O_i \cap K$ is covered by the sets $O_j \cap K$ for $j < i$ (we can eliminate redundant sets O_i). Choose a sequence of points $p_i \in K$ such that $p_i \in O_i$ for each i (note the failure of stratification, which is all right given the Axiom of Counting) and $p_i \notin O_j$ for any $j < i$. The sequence p must have an accumulation point q (because K is a subset of a closed interval), which must be an element of K (because K is closed). q cannot belong to any open set of the sequence O, because if it did, so would infinitely many p_i's, which is impossible by the construction of the sequence of p_i's. But the range of O purports to be a cover of K. This contradiction completes the proof of the theorem. $\square$

We return to the distinction between an accumulation point and a mere limit point.

Definition. *Let $A \subseteq \mathcal{R}$. We say that A is* perfect *iff A is closed, nonempty, and has no isolated points.*

Theorem. *The cardinality of a perfect set is $c = 2^{\aleph_0}$, the cardinality of the continuum.*

PROOF. — Let P be a perfect set. $|P| \leqslant c = |\mathcal{R}|$ is immediate; we need to establish $|P| \geqslant c$.

We do this by defining an injection from $[\mathcal{N} \to \{0,1\}]$, a set known to have cardinality c, to a subset of $\mathcal{P}$.

P contains at least one point x, because it is nonempty. It contains at least one other point y, because any open interval about x contains a point of P other than x. We associate with each of x, y an open interval in such a way that the corresponding closed intervals do not intersect. We now consider each of the points x and y and repeat this process: for example, y is in an interval (u, v) chosen at the previous step; since P is perfect, there is another real number y' in $(u, v) \cap P$; choose disjoint open intervals about y and y' in such a way that the corresponding closed intervals do not intersect.

In this way we obtain a countable set F with the following properties: it is the union of finite sets F_i for $i \in \mathcal{N}$ such that $F_0 = \{x\}$, $|F_i| = 2^i$, each set F_i is associated with a set O_i of open intervals such that the corresponding closed intervals are pairwise disjoint and each element of O_i contains exactly one element of F_i, $F_i \subset F_{i+1}$ and each element of O_i contains two elements of F_{i+1}. We may further require that the radius of each interval in $O_i < 2^{-i}$. We associate a finite sequence with each element of each F_i for $i > 0$ as follows: with the two elements of F_1 we associate the one-term sequence with 0 as its sole term (this goes with x) and the one-term sequence with 1 as its sole term (this goes with y). The map on F_{i+1} is determined by assigning to each element of F_i the sequence obtained by adding one more value, a 0, to the sequence associated with it at the previous step; the new element of each interval in O_i is assigned the extension of the finite sequence associated with the old element of the same interval with one new value, a 1. Assignment of relative types to the various occurrences of i in this discussion would require some use of the T operation, which the Axiom of Counting allows us to neglect.

The closure of the set F is a subset of P, because P is perfect and so closed. Each element of $[\mathcal{N} \to \{0,1\}]$ (infinite sequence of binary digits) is associated with a unique element of the closure of F and so of P by considering the intersection of the nested sequence of intervals from which points associated with finite initial segments of the sequence are chosen. There are c such sequences, so there are at least c elements in P. The proof of the theorem is complete. □

We continue the development with a

Definition. *For each set $A \subseteq \mathcal{R}$, we define the* derivative *A' of A as the set of accumulation points of A.*

Theorem. *The derivative of a closed set C is a closed subset of C.*

Proof. — The derivative is a set of limit points of C, so certainly a subset of C.

It is sufficient to show that for any closed set C, any limit point of accumulation points of C is itself an accumulation point of C. A limit point of accumulation points of C is a point c any open interval about which contains an accumulation point a of C. If $a = c$ we are done. If $a \neq c$, choose an open interval about a included in the interval already chosen about c and not containing c. This interval must contain a point of C, which cannot be c, so the original interval about c contained such a point, so c is an accumulation point of C. The proof of the theorem is complete. $\square$

Theorem. *The set of isolated points of any set of reals A is at most countable.*

PROOF. — For each isolated point, we can choose an open interval containing that point which contains no other point of A. We can further refine this by halving the radius of each chosen interval; the resulting family of intervals will be pairwise disjoint. By a theorem above, no set of pairwise disjoint open intervals can be more than countable. The proof of the theorem is complete. $\square$

We prove a special case of the Continuum Hypothesis:

Theorem. *Each closed subset of $\mathcal{R}$ is either finite, countable, or of cardinality c.*

The full Continuum Hypothesis would be obtained if we could drop the word "closed". The theorem follows immediately from the following

Lemma. *Each closed subset A of $\mathcal{R}$ is the union $C \cup P$ of an at most countable set C and a set P which is either perfect or empty.*

This is of course a "Lemma" only for our purposes; it is a substantial theorem in its own right!

PROOF OF THE LEMMA. — Let $\leqslant$ be any well-ordering of an uncountable set. We can define a function f from $\mathrm{dom}(\leqslant)$ to subsets of $\mathcal{R}$ by transfinite recursion such that $f(w) = A \cap \bigcap[\{(f(v))' \mid v < w\}]$. The first set in the resulting transfinite sequence of sets is A; the successor of any set in the sequence is its derivative; intersections are taken at limits. It is easy to see that each $f(w)$ is a closed subset of A.

If we choose the well-ordering $\leqslant$ with large enough domain, the process described by f must terminate: each set that we construct is a subset of the previous one (because all are closed), so the number of steps we can go through is bounded by the number of disjoint subsets we can take from $\mathcal{R}$. Assume that we choose $\leqslant$ so that the process terminates; thus, there is a u such that we have $f(u) = f(v)$ for all v such that $u \leqslant v$. We define $P = f(u)$ and $C = A - f(u)$. $P = P'$, so by the definition of "perfect", P is either perfect or empty.

It only remains to show that C is countable. C is partitioned into disjoint sets $f(w)-f(w+1)$ for $w \in \mathrm{dom}(\leqslant)$. $f(w)-f(w+1)$ is the set of isolated points of $f(w)$ for each w. Let J_n be a sequence whose range is all of the open intervals with rational endpoints (this is a countable set). For each element c of C, define I_c as the first interval in the sequence J such that for some $w \in \mathrm{dom}(\leqslant)$, $I_c \cap f(w) = \{c\}$. Since there is a set $f(w)-f(w+1)$ which contains c, and c is an isolated point of this $f(w)$, there is such an I_c. It is straightforward to show that no two points $c, d \in C$ can have $I_c = I_d$, and so C is at most countable. It is worth noting that this implies that there can be no more than countably many distinct $f(w)$'s. The proof of the Lemma is complete. $\square$

Proof of Theorem. — A closed subset of $\mathcal{R}$ is seen by the Lemma to be the union of a set which is either finite or countable with a set which is either empty or has cardinality c; the union of two such sets is either finite, countable, or has cardinality c. $\square$

We reluctantly leave this fascinating area of application of set theory.

18.3. Ultrafilters. An Alternative Definition of $\mathcal{R}$. Nonstandard Analysis

We outline an elegant alternative definition of the real numbers, which gives us an occasion to introduce the notions of *filter* and *ultrafilter* (along with the dual notions of *ideal* and *prime ideal*). We go on to discuss the application of ultrafilters to construction of "nonstandard models" of sets, and sketch the application of this construction to analysis.

Definition. *Let $\leqslant$ be a partial order. A* filter on X *is a subset of* $\mathrm{dom}(\leqslant)$ *with the following properties:*

(*a*) *If* $p \in F$ *and* $p \leqslant q$, $q \in F$.

(*b*) *For any* $p, q \in F$, *there is* $r \in F$ *such that* $r \leqslant p$ *and* $r \leqslant q$.

Notice that $\mathrm{dom}(\leqslant)$ *can be a filter under this definition, but only in case every pair of elements of* $\mathrm{dom}(\leqslant)$ *has a lower bound. If there are pairs of elements of* $\mathrm{dom}(\leqslant)$ *which do not have a lower bound (as will be the case for the orders we will consider), then filters on* $\leqslant$ *must be proper subsets of* $\mathrm{dom}(\leqslant)$.

An ultrafilter *is a filter which has no proper superset which is a filter.*

An ideal (prime ideal) *on* $\leqslant$ *is a filter (ultrafilter) on* $\geqslant$.

Theorem. *For any filter* F *on* $\leqslant$, *there is an ultrafilter* U *on* $\leqslant$ *which extends* F, *i.e., such that* $F \subseteq U$.

PROOF. — Consider the order on filters on $\leqslant$ determined by inclusion. The union of any chain of filters in this order is a filter, so the conditions of Zorn's Lemma are satisfied and there is a maximal filter in this order. The proof of the theorem is complete. □

Our alternative definition of the real numbers follows:

Definition. *A* real number *is an ultrafilter on the inclusion order on the set of bounded closed intervals* $[p, q] = \{x \mid p \leqslant x \leqslant q\}$ *for* $p \leqslant q$ *elements of* $\mathcal{Q}$. *The real number corresponding to a rational* p *is simply the set of closed intervals that contain* p. $\mathcal{R}$ *is the set of all real numbers.*

Definition. *Let* r, s *be real numbers.* $r + s$ *is the set of intervals* I *such that there are intervals* $J \in r$ *and* $K \in s$ *such that* $I = \{p + q \mid p \in J$ *and* $q \in K\}$. *The product* rs *is defined analogously.* $r \leqslant s$ *is said to hold iff there are intervals* $J \in r$, $K \in s$ *such that for all* $p \in J$, $q \in K$, $p \leqslant q$.

Verification that this definition "works" is left to the reader. One way of understanding it is to think of the closed interval elements of the ultrafilter as "estimates" $p \pm q$ (p and q rational, $q \geqslant 0$); this estimate denotes the interval $[p - q, p + q]$. A real number is a maximal collection of rational estimates of this kind which are consistent with one another. A sufficient condition for a filter on the inclusion order on the closed intervals to be an ultrafilter is that it contain closed intervals of every positive diameter; the existence of ultrafilters extending any given filter on this order can be proved without an appeal to Zorn's Lemma!

We outline the verification of the least upper bound property. Consider a nonempty set $A \subset \mathcal{R}$ which is bounded above and below. Now consider the set $\bigcap[A]$ of closed intervals in $\mathcal{Q}$ which belong to all elements of A. The set $\bigcap[\bigcap[A]]$ is the set of rational numbers which are "between" elements of A in a suitable sense. Choose closed intervals of each positive diameter which contain an element of $\bigcap[\bigcap[A]]$ and an upper bound of $\bigcap[\bigcap[A]]$; the unique real number containing all of these intervals as elements will be the least upper bound of A. It is easy to deduce from the least upper bound property for nonempty sets bounded below that the least upper bound property holds for all nonempty sets.

It is usual to consider filters and ideals over the inclusion order on families of sets. There is a nice characterization of ultrafilters on the family of nonempty subsets of a set X:

Theorem. *Let U be a filter on the inclusion order on $\mathcal{P}\{X\} - \{\{\ \}\}$ for some set X. U is an ultrafilter iff for each set $A \subseteq X$, either $A \in U$ or $X - A \in U$.*

PROOF. — Suppose that for some set Y, $Y \notin U$ and $X - Y \notin U$. We show that U is not maximal. Consider the set

$$U' = U \cup \{(A \cap Y) \cup Z \mid A \in U \text{ and } Z \subseteq X\}.$$

It is straightforward to verify that U' is a filter on nonempty subsets of X properly extending U. On the other hand, if U does contain each set or its relative complement, it is easy to see that U is maximal; if $Y \notin U$, it follows that $X - Y \in U$, so we cannot adjoin Y to U and still satisfy the definition of a filter on the inclusion order on nonempty subsets of X; there is no nonempty subset of X below a set and its relative complement! □

Observation. For each element x of a set X, the set of subsets of X which contain x as an element is an ultrafilter on the inclusion order on nonempty subsets of X. Such an ultrafilter is called a *principal* ultrafilter on X.

We describe a way to make an infinite set X look like a larger object X^* with the same properties (in a suitable sense) by adjoining some "non-standard" objects using an ultrafilter.

Theorem. *Let A be an infinite set. There is a nonprincipal ultrafilter on the inclusion order on nonempty subsets of A.*

PROOF. — The set of sets X such that $X - A$ is finite (these are called *cofinite* subsets of A) makes up a filter on the inclusion order on nonempty subsets of A. This filter can be extended to an ultrafilter, and this ultrafilter cannot be principal: each element a of A fails to belong to the cofinite set $\{a\}^c$. $\square$

Let A be an infinite strongly Cantorian set (the Axiom of Counting allows us to assert that there are such sets), and let U be a nonprincipal ultrafilter on the inclusion order on nonempty subsets of A. Let X be an infinite set. We make some definitions.

Definition. *We define X^*_U (or X^* where U is understood) as a partition of $[A \to X]$. The equivalence relation $\sim_U$ which determines the partition is defined thus: $f \sim_U g$ iff $\{a \in A \mid f(a) = g(a)\} \in U$. It is straightforward to verify that $\sim_U$ actually is an equivalence relation. Each element x of X corresponds to an element x^* of X^*, the equivalence class of the constant function on A with value x. x^* is two types higher than x. The set X^*_U is called an* ultrapower.

Definition. *For any predicate $R[x_1, \ldots, x_n]$ of objects x_i in X, we define a predicate $R^*_U[[f_1], \ldots, [f_n]]$ (we may omit U where it is understood) of objects $[f_i]$ (equivalence classes of functions f_i) in X^*_U as holding iff $\{a \in A \mid R[f_1(a), \ldots, f_n(a)]\} \in U$. Here is where the strongly Cantorian character of A becomes significant; the variable a does not need to be assigned a consistent relative type, because it is restricted to a fixed strongly Cantorian set, so it is possible to define the set needed for the definition of R^* even when R is a predicate whose arguments have different relative types (such as $\in$). For any sentence ϕ, we can define a sentence ϕ^* by replacing all predicates R with predicates R^* and references to specific constants c with c^* (it is useful to suppose here that we have eliminated all complex names from ϕ using the theory of definite descriptions), and restricting all quantifiers to X^*.*

Łoś's Theorem. *Suppose either that the set X whose ultrapower we consider is strongly Cantorian or that we restrict all sentence variables to the stratified sentences. For any sentence ϕ with free variables $a_1, \ldots, a_n$, ϕ^* holds for specific values of its free variables a_i iff $\{a \in A \mid \phi'\} \in U$, where ϕ' is obtained from ϕ by replacing free variables a_i with terms $a_i'(a)$ with $a_i' \in X^*$ representing the value to be assigned to the free variable a_i. A special case of this is that ϕ holds iff ϕ^* holds in the special case where there are no free variables, or in the case where all free variables code standard elements of X (are equivalence classes of constant functions).*

Proof. — We proceed by induction on the structure of ϕ.

(a) If ϕ is atomic, this follows by the definition of predicates R^*.

(b) Suppose that the result holds for ψ and ξ. That it holds for "not ψ" follows from the fact that U contains a set iff it does not contain its complement relative to A. That it holds for "ψ and ξ" follows from the fact that U is closed under Boolean intersection.

(c) Suppose that the result holds for ψ. We want to determine whether it holds for "for some x, ψ". x is free in ψ. If "for some $x \in X^*$, ψ^*" holds, then ψ^* holds for a specific $x \in X^*$, so $\{a \in A \mid \psi'\} \in U$ holds by inductive hypothesis (using that specific value of $x \in X^*$ as x'), and $\{a \in A \mid (\text{for some } x,\, \psi)'\} \in U$ holds because a superset of an element of U is an element of U. If $\{a \in A \mid (\text{for some } x,\, \psi)'\} \in U$, we want to show that for some $x' \in X^*$, $\{a \in A \mid \psi'\} \in U$. A specific value of x' which works is any function which takes each a in $\{a \in A \mid (\text{for some } x,\, \psi)'\}$ to an x such that ψ and takes other objects to anything; we don't care what. For such an x', the sets $\{a \in A \mid (\text{for some } x,\, \psi)'\}$ and $\{a \in A \mid \psi'\}$ are the same set, which is known to be in U. Caution is required here: the function x' can be relied upon to exist only under restricted conditions: one that works is "X is strongly Cantorian"; another that works is "ϕ is stratified". □

We usually expect that there will be "new" objects in an ultrapower X^*: e.g., the identity function on A will represent an element of A^* different from all analogues of standard elements of A (it differs from each constant function on a cofinite subset of A).

We present applications to "nonstandard analysis", a way of defining basic concepts of the calculus using infinitesimals.

Our observations on "nonstandard analysis" will be brief. Carry out the construction above with $A = \mathcal{N}$. We can work with $X = V$ here because notions of analysis can be presented in stratified form. The objects in V^* will be equivalence classes of sequences of elements of V. Consider the equivalence class of a sequence of positive real numbers converging to 0. It will be an element of $(\mathcal{R}^+)^*$ less than all elements of $(\mathcal{R}^+)^*$ corresponding to standard real numbers and greater than 0^*; in other words, it will be a positive infinitesimal.

Let s be a sequence of real numbers. We assert (as usual) that $\lim s = L$ iff for each real number $\epsilon > 0$, there is a natural number N such that for each $i \geqslant N$, $|s_i - L| < \epsilon$.

We present an equivalent notion in terms of ultrapowers: let s^* be the sequence of real numbers corresponding to s in an ultrapower V_U^*, U an ultrafilter over the natural numbers. We claim that $\lim s = L$ iff for each nonstandard i in $\mathcal{N}^*$, we have $|s_i^* - L^*|$ infinitesimal (or zero). The proof is straightforward: if $\lim s = L$ is true, let $N = [t]$ be any nonstandard natural number; there will be a standard function M such that for any $\epsilon > 0$, and natural number k, $k > M(\epsilon)$ implies $|s_k - L| < \epsilon$; the set $\{i \mid t_i > M(\epsilon)\}$ is in U for each (standard) $\epsilon > 0$, consequently the set $\{i \mid |s_{t_i} - L| < \epsilon\}$ is in U too, showing that $|s_N^* - L^*|$ is infinitesimal. On the other hand, if $\lim s = L$ is not true, a strictly increasing standard sequence t can be defined such that $|s_{t_i} - L| > \epsilon$ for all i with a *fixed* positive real ϵ; the element of the ultrafilter corresponding to the sequence t will be a nonstandard integer N such that $|s_N^* - L^*| > \epsilon^*$ for the analogue ϵ^* of that fixed ϵ.

This kind of reasoning can be extended to allow the presentation of the basic notions of the calculus in terms of infinitesimals. For example, the derivative f' of a function f from the reals to the reals can be defined as the unique function f' (if any) such that the absolute value of the difference between $f'^*(x)$ and $\dfrac{f^*(x + dx) - f^*(x)}{dx}$ is infinitesimal for each nonzero infinitesimal dx in $\mathcal{R}^*$.

Exercises

(a) (hard: Baire Category Theorem) Prove that the intersection of a countable collection of open dense subsets of the reals is dense. We define a *nowhere dense* set as a set which is not dense in any interval. Prove that the union of a countable collection of nowhere dense subsets of the real line is not the entire real line. The union of a countable collection of nowhere dense subsets is called a *meager* set.

(b) (hard: Borel sets) The set $\mathcal{B}$ of *Borel sets* is defined as the smallest subset of $\mathcal{R}$ which contains all intervals and is closed under unions and intersections of its countable subcollections. Show that all open sets and all closed sets are Borel sets. Show that $|\mathcal{B}| = |\mathcal{R}|$ (so most sets of reals are not Borel). A hierarchy of sets indexed by the natural numbers can be obtained by starting with the open sets as stage 1, then adding all countable intersections of stage 1 sets to get stage 2 sets, all countable unions of stage 2 sets to get stage 3 sets, and so forth, taking countable intersections to get even stages and countable unions to get odd stages. Show that each of the stages contains new sets. Show that there are Borel sets which are not constructed at stage n of this process for any $n \in \mathcal{N}$.

(c) Find a book on nonstandard analysis and read the proofs of some familiar theorems of analysis.

Cahiers du Centre de logique
Volume 10

Chapter 19

Strongly Cantorian Sets and Conventional Set Theory

19.1. Consequences of the Axiom of Small Ordinals

We "take the gloves off" and start using the Axiom of Small Ordinals.

Theorem. *All Cantorian sets (thus cardinals, ordinals) are strongly Cantorian.*

PROOF. — Let A be a Cantorian set. Let R be a well-ordering of A; R then belongs to a Cantorian ordinal α, and R is similar to the natural order on $\text{seg}\{\alpha\}$. The similarity which witnesses this fact is a bijection between A and $\text{seg}\{\alpha\}$. It is sufficient to show that $\text{seg}\{\alpha\}$ is strongly Cantorian. Define a map by transfinite recursion: $f(0) = \{0\}$; $f(\beta+1) = \{\gamma+1\}$ if $f(\beta) = \{\gamma\}$; $f(\lim B) =$ the singleton of the limit of the collection of $\bigcup[f(\beta)]$'s for $\beta \in B$. It is straightforward to prove by (stratified) transfinite induction that $f(\alpha) = \{T^{-1}\{\alpha\}\}$ for each ordinal α which is an image under T; it follows that $f(\alpha) = \{\alpha\}$ for each Cantorian α. We have already established that each ordinal less than a Cantorian ordinal is Cantorian, so $f(\beta) = \{\beta\}$ for each $\beta < \alpha$, so f witnesses the fact that $\text{seg}\{\alpha\}$ is strongly Cantorian. $\square$

This shows that the concepts of Cantorian and strongly Cantorian sets, ordinals, or cardinals collapse together in the presence of the Axiom of Small Ordinals. We will use the term "Cantorian" hereafter for this concept.

Theorem. *Every set of Cantorian ordinals is Cantorian.*

PROOF. — The map f defined in the proof of the preceding theorem sends each Cantorian ordinal to its singleton; its restriction to any set A of Cantorian ordinals witnesses the Cantorian character of A. □

We show that the Cantorian sets are closed under certain constructions.

Theorem. *Any pair $\{a, b\}$ is Cantorian.*

PROOF. — Obvious. □

Theorem. *There is an infinite Cantorian set.*

PROOF. — $\mathcal{N}$. □

Theorem. *If A is Cantorian, $\{x \in A \mid \phi\}$ exists for any sentence ϕ.*

PROOF. — If A is Cantorian, there is a bijection f between A and $\mathrm{seg}\{\alpha\}$ for a Cantorian ordinal α. $f[\{x \in A \mid \phi\}]$ is a collection of small ordinals definable by a sentence, and so has exactly the small ordinal elements of a set B. The set $f^{-1}[B \cap \mathrm{seg}\{\alpha\}]$ is the desired set. □

This gives us comprehension for any property at all (that we can express in our language) as long as we restrict our attention to elements of a fixed Cantorian set.

Theorem. *If A is Cantorian, $\mathcal{P}\{A\}$ is Cantorian.*

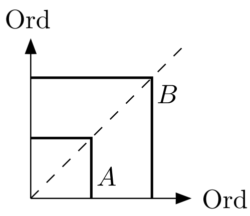
PROOF. — If K is the map which takes a to $\{a\}$ for each a in A, the map $(B \mapsto K[B])$ for B in $\mathcal{P}\{A\}$ has a stratified definition, and sends each B to $\mathcal{P}_1\{B\}$. The map $(\mathcal{P}_1\{B\} \mapsto \{B\})$ has a stratified definition; composition yields the singleton map on $\mathcal{P}\{A\}$. □

Since $|\mathcal{P}\{A\}| > |A|$ for A Cantorian, this enables us to build larger and larger Cantorian sets.

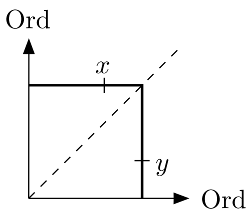
Definition. *We use transfinite recursion to code pairs of ordinals as ordinals:*

- *define* $\langle\langle 0, 0\rangle\rangle$ *as 0;*
- *define* $\langle\langle \beta, \gamma + 1\rangle\rangle$ *as* $\langle\langle \beta, \gamma\rangle\rangle + 2$, *where* $\gamma < \beta$*;*
- *define* $\langle\langle \alpha, \beta\rangle\rangle$ *as* $\langle\langle \beta, \alpha\rangle\rangle + 1$ *whenever* $\alpha < \beta$ *(notice that the first projection is greater than or equal to the second in each other case defined here);*
- *define* $\langle\langle \alpha, \lim B\rangle\rangle$ *as* $\lim\{\langle\langle \alpha, \beta\rangle\rangle \mid \beta \in B\}$ *when* B *has no greatest element and* α *is greater than all elements of* B*;*
- *define* $\langle\langle \beta + 1, 0\rangle\rangle$ *as* $\langle\langle \beta, \beta\rangle\rangle + 1$*;*
- *define* $\langle\langle \lim B, 0\rangle\rangle$ *as*
  lim $\{\langle\langle \beta, \gamma\rangle\rangle \mid \beta, \gamma \in B\}$, *where* B *has no greatest element.*

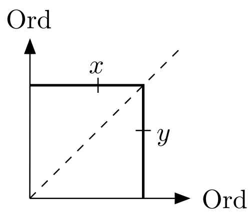
This definition can be pictured as follows:



Each point of B is greater than each point of A.



When $\pi_1(x) > \pi_2(y)$, $x > y$ because x is farther up the diagonal than y ($\pi_1(x) < \pi_2(y)$ would imply $y > x$)



If the projections of x and y onto the diagonal are equal ($\pi_1(x) = \pi_2(y)$) then $x > y$ because x is higher than y.

Notice that the projections of a coded pair are at the same relative type as the coded pair. Observe that the order on the codes of pairs of ordinals is determined by considering first the maximum of the two ordinals, then the smaller ordinal, then the position of the larger ordinal; technically, the order on codes $\langle\langle \alpha, \beta\rangle\rangle$ is induced by the lexicographic order on triples $(\max(\alpha, \beta), \min(\alpha, \beta), i)$, where $i = 0$ if $\alpha > \beta$ and 1 otherwise. It is straightforward to demonstrate that $\langle\langle\alpha,\beta\rangle\rangle$ exists for each α and β: let W be a well-ordering such that W_α and W_β exist; then consider the order R on $\mathrm{dom}(W)\times\mathrm{dom}(W)$ defined by considering first the maximum of the projections of a pair, then its smaller projection, then the position of the larger projection (as above): $(W_\alpha,W_\beta)=R_{\langle\langle\alpha,\beta\rangle\rangle}$ will hold. It is straightforward to show that $T\{\langle\langle\alpha,\beta\rangle\rangle\}=\langle\langle T\{\alpha\},T\{\beta\}\rangle\rangle$, for any ordinals α, β. It follows from this that pairs of Cantorian ordinals are coded exactly by the Cantorian ordinals.

Replacement Theorem. *If A is Cantorian and ϕ is a sentence such that "for each x in A, there is exactly one Cantorian ordinal y such that ϕ" holds, then $\{y \mid y$ is a Cantorian ordinal and for some $x\in A,\phi\}$ exists.*

PROOF. — Let f be a bijection from A to $\mathrm{seg}\{\alpha\}$ for some Cantorian ordinal α. Consider the collection of small ordinals $\{\langle\langle f(x),y\rangle\rangle \mid x\in A, y$ is a Cantorian ordinal, and $\phi\}$; there is a set B which contains exactly these small ordinals. Now consider the set $C=\{y \mid$ for some $x\in A, y$ is the smallest ordinal such that $\langle\langle f(x),y\rangle\rangle\in B\}$; this is the desired set of ordinals. Any unwanted elements of the "image" of B are disposed of by choosing the smallest y for each $f(x)$, which must be the unique Cantorian y. □

This theorem says that if we can "replace" each element of a "small" set with a Cantorian ordinal, the replacements themselves make up a set. We cannot have such a theorem about "replacement" with general objects: the numbers 0,1,2... can be "replaced" with $\Omega, T\{\Omega\}, T^2\{\Omega\},\ldots$ which do not make up a set. One consequence of the theorem is that no Cantorian set can contain all of these objects.

Theorem. *If A is a Cantorian set, and each element of A is a Cantorian set, $\bigcup[A]$ is Cantorian.*

PROOF. — Consider $\sum[A]$, the disjoint sum of A (indexed by A itself); it is partitioned into disjoint sets, on each of which we know how to construct the singleton map; the question is whether we can put all these maps together uniformly. It is larger than $\mathcal{P}_1\{\bigcup[A]\}$ (there is a natural map from $\sum[A]$ onto $\mathcal{P}_1\{\bigcup[A]\}$), so if it is Cantorian $\mathcal{P}_1\{\bigcup[A]\}$ and so $\bigcup[A]$ itself must be Cantorian. The trick is to use the coding of pairs of ordinals by ordinals again: choose a bijection f_B of B onto $\mathrm{seg}\{\beta\}$ for some ordinal β for each element B of A, and an embedding g of A onto $\text{seg}\{\alpha\}$ for some ordinal α, then replace each element $(\{x\}, B)$ of $\sum[A]$ with $\langle\langle T\{f_B(x)\}, g(B)\rangle\rangle$ (the T is actually redundant, since $f_B(x)$ is a Cantorian ordinal in each case) to get a set of ordinals of the same size; this is a set of strongly Cantorian ordinals, so strongly Cantorian; it is the same size as $\sum[A]$, so $\sum[A]$ is strongly Cantorian. The proof of the theorem is complete. $\square$

Theorem. *A "class" $\{x \mid \phi\}$ of Cantorian ordinals fails to be realized by a set of Cantorian ordinals exactly if there is a sentence which describes a "one-to-one correspondence" between $\{x \mid \phi\}$ and the proper class of Cantorian ordinals.*

Proof. — By the Axiom of Small Ordinals, there is a set B whose "intersection" with the small ordinals is $\{x \mid \phi\}$. Define a function f on Ord by transfinite recursion which takes each β to the smallest ordinal element of B greater than all $f(\gamma)$ for $\gamma < \beta$, or to $\{\ \}$ if there is no such element. If some Cantorian ordinal is sent to a non-Cantorian ordinal or to $\{\ \}$, there is a smallest such ordinal α (the Axiom is applied here), and $f[\text{seg}\{\alpha\}]$ is a Cantorian set of Cantorian ordinals, certainly not in one-to-one correspondence with all Cantorian ordinals (smaller than the Cantorian set $\mathcal{P}\{\text{seg}\{a\}\}$). If a set of Cantorian ordinals could be placed in a "one-to-one correspondence" defined by a sentence with all the Cantorian ordinals, the class of Cantorian ordinals would be a set by the Replacement Theorem, which is impossible. If each Cantorian ordinal is sent to a Cantorian ordinal, "$f\lceil$ Cantorian ordinals" is the desired "one-to-one correspondence" (observe that $f(\alpha) \geqslant \alpha$ for any ordinal α, so a non-Cantorian ordinal will not be mapped to a Cantorian ordinal); in this event, no set of Cantorian ordinals can realize $\{x \mid \phi\}$, because we could then define the set of all Cantorian ordinals as $f^{-1}[\{x \mid \phi\}]$. $\square$

19.2. Interpreting ZFC in the Cantorian Ordinals

The theorems we have been proving above will look familiar if one has experience with conventional set theory. We introduce the usual set theory ZFC (Zermelo–Fraenkel set theory) by showing how to interpret it using the Cantorian ordinals, and stating its axioms as theorems.

Construction

Each Cantorian ordinal to be interpreted as a set of Cantorian ordinals in such a way that every set of Cantorian ordinals is interpreted by some ordinal.

We construct a relation E on the ordinals by recursion. We define a sequence of ordinals ρ_β indexed by the ordinals, by recursion: $\rho_0 = 0$, $\rho_{\beta+1} =$ the first ordinal γ such that $|\operatorname{seg}\{\gamma\}| = \exp(|\operatorname{seg}\{\beta\}|)$, and $\rho_{\lim B} = \sup\{\rho_\beta \mid \beta \in B\}$, where B has no greatest element. When we have defined the part E_β of the relation E with domain $\operatorname{seg}\{\rho_\beta\}$, we want the following conditions: $\operatorname{rng}(E_\beta) = \operatorname{seg}\{\rho_{\beta+1}\} - \{0\}$; for each subset A of $\operatorname{seg}\{\rho_\beta\}$, there is exactly one element α of $\operatorname{seg}\{\rho_{\beta+1}\}$ such that $\beta E \alpha$ iff $\beta \in A$, for all β. We prove by transfinite induction that this can be achieved. It is certainly achievable for ρ_0. Suppose it has been achieved for ρ_β. We show how to extend E so that it is achieved on $\rho_{\beta+1}$. Let F be a bijection between $\mathcal{P}_1\{\operatorname{seg}\{\rho_{\beta+2}\}\} - \mathcal{P}_1\{\operatorname{seg}\{\rho_{\beta+1}\}\}$ and $\mathcal{P}\{\operatorname{seg}\{\rho_{\beta+1}\}\} - \mathcal{P}\{\operatorname{seg}\{\rho_\beta\}\}$ (it is straightforward to show that the cardinalities work out correctly); now define E as obtaining between $\alpha \in \operatorname{seg}\{\rho_{\beta+1}\}$ and $\beta \in \operatorname{seg}\{\rho_{\beta+2}\} - \operatorname{seg}\{\rho_{\beta+1}\}$ iff $\alpha \in F(\{\beta\})$. Suppose it has been achieved for all ρ_γ for γ in a set B with no greatest element; let $\beta = \lim B$; to achieve it for ρ_β, we need a bijection F between $\mathcal{P}_1\{\operatorname{seg}\{\rho_{\beta+1}\}\} - \mathcal{P}_1\{\operatorname{seg}\{\rho_\beta\}\}$ and $\mathcal{P}\{\operatorname{seg}\{\rho_\beta\}\} - \bigcup[\{\mathcal{P}\{\operatorname{seg}\{\rho_\gamma\}\} \mid \gamma \in B\}]$, which we use just as in the last case.

The relation E is certainly defined for all Cantorian ordinals, and determines a "one-to-one correspondence" between Cantorian ordinals and sets of Cantorian ordinals: any Cantorian ordinal is associated with a set of smaller ordinals, which must thus be Cantorian; any set of Cantorian ordinals is bounded by a Cantorian ordinal α, and its "code" is found below the first ordinal β such that $|\operatorname{seg}\{\beta\}| = \exp(|\operatorname{seg}\{\alpha\}|)$, which is certainly Cantorian.

The following axioms are satisfied if E is interpreted as membership and the class of Cantorian ordinals is interpreted as the universe of objects:

Extensionality: Sets with the same elements are the same.

PROOF. — Obvious. □

Pairing: If a, b are sets, $\{a, b\}$ is a set.

PROOF. — a and b are actually Cantorian ordinals; the actual set $\{a, b\}$ is coded by an ordinal. □

Union: If A is a set, the set $\bigcup[A] = \{x \mid \text{for some } B, x \in B \text{ and } B \in A\}$ exists.

PROOF. — If $\in$ is replaced by E in the condition, we have a condition which defines a set of Cantorian ordinals, which will be coded by an ordinal. □

Power Set: If A is a set, the power set $\mathcal{P}\{A\} = \{x \mid x \subseteq A\}$ exists.

PROOF. — Same as for Union. □

Infinity: There is a set which contains $\{\ \}$ and which contains $x \cup \{x\}$ whenever it contains x.

PROOF. — Same as for Union. □

These axioms allow us to build some specific sets.

Choice: Pairwise disjoint collections of nonempty sets have choice sets.

PROOF. — An easy consequence of Choice in our theory. □

Foundation: For each property ϕ, (for all A, ((for all $x \in A, \phi$) implies $\phi[A/x]$)) implies for all x, ϕ. (i.e., if ϕ holding of all elements of a set implies that it hold of the set as well then ϕ holds for all sets).

PROOF. — All elements of the set coded by an ordinal are smaller ordinals; this follows by transfinite induction. □

Choice and Foundation are "structural" axioms.

Now we get to the meat of the difference between conventional set theory and the theory we have presented.

Separation: For any set A and sentence ϕ of the language of set theory, $\{x \in A \mid \phi\}$ exists.

PROOF. — The ordinals interpreting elements of $\{x \in A \mid \phi\}$ are all less than the ordinal interpreting A. The ordinals less than the ordinal interpreting A make up a Cantorian set, and any condition we can express on elements of a Cantorian set defines a set by a theorem in the previous subsection. □

This is the particular restriction on the inconsistent axiom of unrestricted comprehension which motivates this style of set theory. The intuitive idea is "limitation of size"; the "illegitimate" classes , like Russell's class, are very large. The previous axioms give us the material for Separation to work on (the particular sets A).

Note that an arbitrary sentence ϕ about our interpreted ZFC is usually unstratified, even though E is a set relation, because it is likely to involve quantification over all sets of ZFC, which is interpreted as quantification over all Cantorian ordinals: the definition of "Cantorian ordinal" is unstratified. Thus, we cannot prove Separation in the simple way we proved Power Set, Union and Infinity above. A sentence ϕ in which every quantifier is restricted to a set does translate to a stratified sentence. The same comments apply to the Axiom of Replacement discussed below.

Theorem. *There is no universe.*

PROOF. — If the universe V existed, we could find $\{x \in V \mid \text{not}(x \in x)\}$ using the Axiom of Separation. □

The set theory with the axioms given so far is the set theory of Zermelo, which is slightly stronger than our set theory without the Axiom of Small Ordinals (or Counting). The full conventional set theory ZFC is completed with the

Axiom of Replacement: If ϕ is a sentence and A is a set, and we have "for each $x \in A$, for exactly one y, ϕ", then $\{y \mid \text{for some } x \in A, \phi\}$ exists.

PROOF. — Use our Replacement Theorem above (p. 158). □

We have shown that our full set theory is at least as strong as the usual set theory ZFC; in fact, it is much stronger. Preliminary results along these lines will be established below; the full results are stated and their proofs sketched, but they are really beyond the scope of this book. The "universe" of ZFC is built in stages, like our model of it in the Cantorian ordinals: one starts with the empty set, then builds successive power sets indexed by the ordinals (the von Neumann ordinals are used; each ordinal is identified with the set of smaller ordinals) to get larger and larger sets as needed. Definitions of mathematical objects using equivalence classes on "large" relations, such as we have used for various kinds of numbers here, do not work in ZFC because the "large" sets do not exist. This is sometimes evaded in ZFC by allowing "large" collections as "classes" which have elements but are not elements. We believe that the set theory we present here is better in its ability to work with intuitively appealing "large" collections. The additional discipline of attention to stratification (usually involving references to singletons, singleton images or T operators) seems not to be that hard (in fact, it has been claimed that stratification is the rule in mathematical practice, mod systematic confusion of objects with their singletons) and to present the advantage of permitting access to a larger mathematical world. A suitable redefinition of the membership relation by permutation will actually cause the membership relation E of our model to coincide with the "real" membership $\in$ in a corner of our universe which can then be taken to be the universe of ZFC.

Exercises

(a) Verify the existence of the bijections whose existence is required for the Construction in the chapter.

(b) (hard) The map F taking each ordinal α in $\mathrm{Dom}(E) \cap T[\mathrm{Ord}]$ (where E is the simulated membership relation in the ordinals) to

$$\{\beta \mid \beta \mathrel{E} T^{-1}\{\alpha\}\}$$

has a stratified definition. The use of T^{-1} is necessary to preserve stratification; note that T^{-1} fixes all Cantorian ordinals, and that all Cantorian ordinals are in the domain of F. Define a permutation π as interchanging α and $F(\alpha)$ for all $\alpha \in \mathrm{Dom}(E) \cap T[\mathrm{Ord}]$ and fixing all other objects. Define a new "membership relation" $x \in^\pi y$ as $x \in \pi(y)$. Demonstrate that all axioms of our theory hold if $\in^\pi$ replaces $\in$, and that the axioms of ZFC hold on the original domain of Cantorian ordinals with the new membership relation.

Cahiers du Centre de logique
Volume 10

Chapter 20

Well-Founded Extensional Relations and Conventional Set Theory

20.1. Connected Well-Founded Extensional Relations

In this chapter we will develop the theory of "connected well-founded extensional relations", and use this to develop an alternative interpretation of the usual set theory ZFC. Some of the following definitions are repeated from above.

Definition. *The* full domain *of a relation R, denoted by* $\mathrm{Dom}(R)$ *is defined as* $\mathrm{dom}(R) \cup \mathrm{rng}(R)$, *the union of the domain and range of R.*

Definition. *Let R be a relation and let S be a subset of* $\mathrm{Dom}(R)$. *An element x of S is said to be a* minimal element *of S with respect to R exactly when there is no y such that $y \; R \; x$.*

Definition. *A relation R is said to be* well-founded *if each nonempty subset S of* $\mathrm{Dom}(R)$ *has a minimal element with respect to R.*

Definition. *A relation R is said to be* extensional *exactly when for all $x, y \in \mathrm{Dom}(R)$, $x = y$ if and only if for all z, $z \; R \; x$ iff $z \; R \; y$, i.e., $x = y$ if and only if $R^{-1}[\{x\}] = R^{-1}[\{y\}]$.*

Definition. *A well-founded relation R is said to have t as its* top *exactly when for each element x of* $\mathrm{Dom}(R)$*, there is a finite sequence $x = x_0, \ldots, x_n = t$ with $x_i \; R \; x_{i+1}$ for each appropriate value of i. A well-founded relation with a top is called a* connected *well-founded relation.*

Note that the empty relation is a connected well-founded extensional relation with any object t whatsoever as its top (vacuously).

Lemma. *Let R be a nonempty connected well-founded relation. Then R has a unique top.*

Proof. — Observe that there can be no cycles in R, i.e., there can be no finite sequence $x = x_0, \ldots, x_n = x$ with $n \geqslant 1$ and $x_i \; R \; x_{i+1}$ for each appropriate value of i. For the set C of elements of $\mathrm{dom}(R) \cup \mathrm{rng}(R)$ belonging to cycles would clearly have no minimal element. By the definition of "top", the existence of two distinct tops of R would allow us to construct a cycle in R. □

The motivation for these definitions lies in the fact that the membership relation of the usual set theory is a well-founded extensional relation (a proper class, of course), and that the membership relation of the usual set theory restricted to a set A, its elements, the elements of its elements, and so forth, is a connected well-founded extensional relation. Sets in the usual set theory are uniquely determined by the structure of the associated connected well-founded extensional relation, which motivates our study of connected well-founded extensional relations as representing the objects of ZFC in our working set theory.

Definition. *Relations R and S are said to be* isomorphic *exactly when there is a bijection f from* $\mathrm{Dom}(R)$ *onto* $\mathrm{Dom}(S)$ *such that $x \; R \; y$ iff $f(x) \; S \; f(y)$.*

The relation of isomorphism is easily seen to be an equivalence relation. It should also be clear that a relation isomorphic to a relation having the properties of being (connected) well-founded or extensional is likewise (connected) well-founded or extensional.

Definition. *We define Z as the set of equivalence classes of connected well-founded extensional relations under isomorphism. The existence of Z*

follows from stratified comprehension. The element of Z to which a connected well-founded extensional relation belongs is called its isomorphism type.

Definition. *Let R be a well-founded relation and let x be an element of* $\mathrm{Dom}(R)$. *We define the* component *of R determined by x as the maximal subrelation of R which is a connected well-founded relation with x as top, and denote it by* $\mathrm{comp}_R\{x\}$. *We call a component of R an* immediate component *if its top stands in the relation R to the top of R. Note that the component associated with an element x of* $\mathrm{Dom}(R)$ *is at the relative type of R, one type higher than the relative type of x.*

Notice that the empty relation will be the component associated with a minimal element of $\mathrm{Dom}(R)$ relative to R. The fact that any object is a top of the empty relation makes this work out correctly.

The immediate components of the connected well-founded extensional relation associated with each set of the usual set theory are the connected well-founded extensional relations associated with its elements; thus this relation will be used to define the analogue of membership in the interpretation we are constructing:

Definition. *We define a relation $E \subseteq Z \times Z$: $x \; E \; y$ exactly when $x \in Z$, $y \in Z$, and x is the isomorphism type of an immediate component of an element of y (thus of an immediate component of each element of y). The existence of E as a set follows from stratified comprehension.*

Induction Lemma. *E is a well-founded relation.*

PROOF. — It is sufficient to show that each subset S of Z has a minimal element with respect to E. Choose any element s of S, and choose a connected well-founded extensional relation R from s. Consider the intersection of S with the set of equivalence classes under isomorphism of components of R. The set of elements of $\mathrm{Dom}(R)$ determining these components has an R-minimal element, the isomorphism class of whose component must be an E-minimal element of S. The proof is complete. $\square$

We demonstrate that the set Z with the relation E has properties we would expect of a sort of set theory with E as membership.

Extensionality Lemma. *For all $x, y \in Z$, $x = y$ if and only if for all z, $z\ E\ x$ iff $z\ E\ y$.*

PROOF. — If the lemma is false, there is $R \in x \in Z$ and $S \in y \in Z$ with $y \neq x$ such that the immediate components of S represent the same elements of Z as the immediate components of R. If each isomorphism class has at most one representative among the immediate components of each of R and S, then it would be easy to construct an isomorphism between R and S, contradicting our hypothesis. Now choose an component of R with R-minimal top t_1 such that there is an isomorphic component of R with a different top t_2; each of the elements of the preimage under R of t_1 must have a corresponding element in the preimage under R of t_2 with an isomorphic component, which by minimality of t_1 must be the same object, implying equality of t_1 and t_2 by extensionality of R. The same argument applies to S. We have a contradiction, establishing the truth of the lemma. □

Corollary. *E is a well-founded extensional relation.*

The following Lemma allows us to associate extensional well-founded relations with arbitrary well-founded relations:

Collapsing Lemma. *For each well-founded relation R, there is an extensional well-founded relation S and a map g from $\mathrm{Dom}(R)$ onto $\mathrm{Dom}(S)$ such that for each element x of $\mathrm{Dom}(R)$, and element z of $\mathrm{Dom}(S)$, $z\ S\ g(x)$ iff $z = g(y)$ for some y such that $y\ R\ x$ (i.e., $\{z \mid z\ S\ g(x)\} = \{g(y) \mid y\ R\ x\}$). Moreover, the relation S is determined up to isomorphism by R.*

PROOF. — For any equivalence relation $\sim$ on $\mathrm{Dom}(R)$, we define an equivalence relation $\sim^+$ by "$x \sim^+ y$ iff for each $z\ R\ x$ there is a $w\ R\ y$ such that $w \sim z$ and for each $z\ R\ y$ there is a $w\ R\ x$ such that $w \sim z$". The effect of $\sim^+$ is to identify elements of $\mathrm{Dom}(R)$ which have the same "extension" relative to R up to the equivalence relation $\sim$.

We make the following additional observations: whenever $[\sim_1] \subseteq [\sim_2]$, we will have $[\sim_1^+] \subseteq [\sim_2^+]$. We call an equivalence relation $\sim$ *nice* iff

$[\sim] \subseteq [\sim^+]$; it follows from the previous observation that for any nice $\sim$, $\sim^+$ is also nice. It is straightforward to establish that the union of any nonempty collection of equivalence relations on $\mathrm{Dom}(R)$ is also an equivalence relation on $\mathrm{Dom}(R)$; it follows from the previous observations that the union of a nonempty set of nice equivalence relations will be a nice equivalence relation.

We define a set Eq as the intersection of all sets A of equivalence relations on $\mathrm{Dom}(R)$ such that $[=]\lceil\mathrm{Dom}(R) \in A$, for each $[\sim] \in A$, $[\sim^+] \in A$, and, for each nonempty $B \subseteq A$, $\bigcup[B] \in A$. We have already observed that the union of a nonempty set of equivalence relations on $\mathrm{Dom}(R)$ is an equivalence relation on $\mathrm{Dom}(R)$. It is clear from the observations above and the fact that $[=]\lceil\mathrm{Dom}(R)$ is a nice equivalence relation that all relations in Eq are nice.

Now consider the relation $\bigcup[\mathrm{Eq}]$. It must be an equivalence relation; we denote it by $\sim_0$. We must further have $[\sim_0^+] = [\sim_0]$; $[\sim_0]$ is in Eq because it is a union of relations in Eq (all of them); from this it follows that $[\sim_0^+]$ is also an element of Eq, and so a subset of $[\sim_0]$; $[\sim_0]$ is a subset of $[\sim_0^+]$ because it is nice, so we have the desired equation.

Select a choice set for the partition of $\mathrm{Dom}(R)$ determined by $\sim_0$. Let g be the map sending each element of $\mathrm{Dom}(R)$ to the element of the choice set belonging to its equivalence class. Let S be the relation $\{(g(x), g(y)) \mid x\ R\ y\}$.

We verify that S is a well-founded relation. Let C be a nonempty subset of $\mathrm{Dom}(S)$; the set $g^{-1}[C]$ will be nonempty and have an R-minimal element x. We claim that $g(x)$ is an R-minimal element of C. Suppose that $g(x)$ had a preimage z under S in C; this means that $z = g(a)$ with $a\ R\ b$ and $b \sim_0 x$ for some a and b. We also have $b \sim_0^+ x$, so each preimage under R of b (including a) stands in the relation $\sim_0$ to some preimage under R of x. If we had $c \sim_0 a$ and $c\ R\ x$, we would have a contradiction, because $g(c) = g(a) = z \in C$, and x was chosen R-minimal in $g^{-1}[C]$. This establishes that S is well-founded.

Finally, if we have $z\,S\,g(x)$, we have $z = g(a)$ for some a such that $a\ R\ b$ and $b \sim_0 x$ for some b. Since we have $b \sim_0 x$, we also have $b \sim_0^+ x$; each preimage of b under R (and so a) stands in the relation $\sim_0$ to some preimage under R of x. So we have $z = g(a) = g(y)$ for some $y\ R\ x$, which is what we need to establish the desired property of g.

It should be clear that S is determined up to isomorphism by R. The proof of the Collapsing Lemma is complete. $\square$

It is worth noting that if the well-founded relation R happens to be extensional, the relation S obtained from this construction will be isomorphic to R.

A type-raising operation, denoted as usual by T, can be defined on Z. Observe that $\mathrm{SI}\{R\}$ is a connected well-founded extensional relation if and only if R is one.

Definition. *If $R \in x \in Z$, we define $T\{x\}$ as the uniquely determined element of Z such that $\mathrm{SI}\{R\} \in T\{x\} \in Z$.*

Isomorphism Lemma. *For all $x, y \in Z$, $x \mathrel{E} y$ iff $T\{x\} \mathrel{E} T\{y\}$.*

The truth of the Isomorphism Lemma is obvious, following from the parallelism of structure between relations R and $\mathrm{SI}\{R\}$.

Comprehension Lemma. *For each subset S of Z, there is an element s of Z such that for all x, $T\{x\} \mathrel{E} s$ iff $x \in S$.*

PROOF. — Choose any element R of an element x of S. Define a new relation

$$R' = \Big\{(\{a\}, R), (\{b\}, R) \mid a \mathrel{R} b\Big\}.$$

It is easy to see that R' exists by stratified comprehension and is isomorphic to $\mathrm{SI}\{R\}$, so belongs to $T\{x\}$. For each pair of elements R_1, R_2 of the set S, R_1' and R_2' are disjoint. Choose one representative R of each element of S, take the union of the corresponding relations R' and add a new object to the resulting relation which has as its preimage exactly the set of tops of the relations R'. The resulting relation will be well-founded and have immediate components representing exactly the elements of $T[S] = \{T\{x\} \mid x \in S\}$. Apply the Collapsing Lemma to obtain an well-founded extensional relation with the same properties (the isomorphism types of its proper components will not be affected by collapse, because these components are all extensional), whose equivalence class in Z will be the desired s. $\square$

The obstruction to a stronger Comprehension Lemma allowing us to construct s with S as its preimage under E is that, while we can choose a representative of each $x \in S$, we may not be able to choose representatives of each $x \in S$ which are compatible with one another in the sense that taking the union of the representatives will not introduce unintended relations between elements of their full domains, as we were able to do with the $T\{x\}$'s by taking pairwise disjoint representatives and applying the Collapsing Lemma. It is fortunate that there is such an obstruction, since we would otherwise be able to find an r such that $x \; E \; r$ iff not$(x \; E \; x)$, reproducing Russell's paradox!

Applying the Comprehension Lemma, we see that there is an element v of Z such that $x \in v$ if and only if $x = T\{u\}$ for some u; v has immediate components covering all and exactly the isomorphism types of singleton images of connected well-founded extensional relations. For example, $T\{v\} \; E \; v$. But then, by repeated applications of the Isomorphism Lemma, $T^{n+1}\{v\} \; E \; T^n\{v\}$, and there is an apparent "descending chain" in the relation E. Of course, the collection of $T^n\{v\}$'s is not a set (its definition is clearly unstratified) and so is not obligated to have an E-minimal element. This is reminiscent of the way that the Burali-Forti paradox is avoided in the ordinals.

20.2. A Hierarchy in Z

There is a close relationship between the structure of Z and the structure of the ordinals. Each strict well-ordering with a maximum element is a connected well-founded extensional relation; this sets up a natural correspondence associating each ordinal α with the equivalence class of the strict well-orderings obtained by rendering elements of $\alpha + 1$ irreflexive. We use the latter objects as ordinals in the rest of this chapter. With the following definitions, we introduce a hierarchical structure on Z which can be indexed by the ordinals:

Definition. *If S is a subset of Z, we define $P(S)$ as the collection of all elements of Z which have preimages under E which are subsets of S. Note that $P(S)$ and S have the same relative type; P is a function coded by a set. P codes a kind of "power set" operation on subsets of Z.*

Definition. *We define the set $H \subseteq \mathcal{P}\{Z\}$ as the intersection of all sets $\mathcal{H} \subseteq \mathcal{P}\{Z\}$ such that for each $A \in \mathcal{H}$, $P(A) \in \mathcal{H}$ and for each $\mathcal{A} \subseteq \mathcal{H}$, $\bigcup[\mathcal{A}] \in \mathcal{H}$. In other words, H is the smallest collection of subsets of Z which is closed under P and under the operation of taking set unions of subcollections.*

Definition. *A subset A of Z is said to be* transitive *if $A \subseteq P(A)$, i.e., if the preimages under E of elements of A are subsets of A.*

Lemma. *Each element of H is transitive.*

PROOF. — The result of applying P to a transitive subset of Z is transitive, and so is the set union of a collection of transitive subsets of Z. By the definition of H, H is a subset of the set of transitive subsets of Z. □

Lemma. *The relation of set inclusion restricted to H is a well-ordering, and each element of A is immediately succeeded by $P(A)$ in this order.*

PROOF. — Consider the collection of elements A of H such that the elements $B \subseteq A$ of H are well-ordered by inclusion, and for each such B either $B = A$ or $P(B) \subseteq A$. Observe that for any such A, the collection of elements of H which are either subsets of A or supersets of A will be closed under P and under set unions of its subcollections, and so will include all elements of H. Thus, the collection of such A's is well-ordered by inclusion. It is easy to see from the fact that it is ordered that the collection of such A's is closed under set unions of subcollections, and it should be clear that $P(A)$ belongs to this collection if A does, so this collection includes all of H. From this it follows that H is well-ordered by inclusion and that $P(A)$ is the successor of A in this well-ordering for each $A \in H$. □

The elements of H are subsets of Z constructed by a process of repeatedly taking "power sets" and taking unions at limit stages. We are interested in the question of when these "power sets" are genuine:

Definition. *A rank A (an element of H) is called a* complete rank *when it is the case that for each subset B of A, there is an element b of $P(A)$ such that the preimage under E of b is precisely B. A complete rank is one whose full power set is represented in Z in the obvious sense.*

Lemma. *There is an incomplete rank (a rank which is not complete).*

PROOF. — Since H is closed under unions, $\bigcup[H]$ is an element of H. This implies that $P(\bigcup[H]) = \bigcup[H]$, since H is closed under P and $\bigcup[H]$ must be its maximal element in the inclusion order. This implies further that $\bigcup[H] = Z$. Suppose otherwise and consider an E-minimal element of Z not in $\bigcup[H]$; clearly this would be an element of $P(\bigcup[H]) = \bigcup[H]$, which is absurd. So Z itself is a rank, and an incomplete one, since Z itself is clearly not represented by any element of $P(Z) = Z$ (cycles cannot occur in well-founded relations). $\square$

Definition. *We define Z_0 as the first incomplete rank in the order determined by inclusion.*

Lemma. *For any rank X at or before Z_0, $T[X]$ is also a rank.*

PROOF. — We define $T[S]$, for any set S of subsets of Z, as the set of all $T[X]$ for X in S.

Now consider $\mathcal{H}$ and $T[\mathcal{H}]$, where $\mathcal{H}$ is the set of ranks. If there is no X in $\mathcal{H}$ such that $T[X]$ is not in $\mathcal{H}$ then we have the desired result. Suppose on the contrary that X is the first rank such that $T[X]$ is not a rank, but each $T[Y]$ for Y a rank below X is a rank. X is either a successor rank, so $X = \mathcal{P}\{Y\}$, but $T[X]$ is not $T[\mathcal{P}\{Y\}]$ (this can only be true if Y is not a complete rank, which puts X above Z_0 as desired) or a limit rank, in which case X is the union of the Y's. This case is impossible, because $T[X]$ would then be the union of the $T[Y]$'s and itself a rank.

Since there *is* an incomplete rank, there is in fact an X in $\mathcal{H}$ (the successor $\mathcal{P}\{Z_0\}$ of the first incomplete rank Z_0) such that $T[X]$ is not in $\mathcal{H}$ ($T[\mathcal{P}\{Z_0\}]$ is a proper subset of $\mathcal{P}\{T[Z_0]\}$). $\square$

Remark. $Z_0 \neq T[Z_0]$. For every rank of the form $T[A]$ is complete by the Comprehension Lemma. This implies that the sequence of iterated images of Z_0 under T forms an external descending chain in the inclusion order on ranks, which cannot, of course, be a set.

Lemma. *Each complete rank contains an ordinal; each complete successor rank contains one "new" ordinal. (We remind the reader that by "ordinal" we mean here "isomorphism class of* strict *well-orderings with a largest element").*

PROOF. — Each ordinal has as its preimage under E the set of smaller ordinals. The proof proceeds by induction: consider the minimal complete rank for which this is not the case, and consider the element of this minimal rank which has as its preimage under E exactly the set of ordinals occurring in previous ranks (this exists because the rank is complete); this is clearly a new ordinal! The contradiction establishes that each complete rank contains an ordinal. It should also be clear from the construction that one new ordinal is added at each complete successor rank. □

Definition. *To each complete rank in H, we can assign the ordinal which first appears in its successor rank as an index (the first ordinal which does not belong to the rank). If we want the index to have the same relative type as the rank, we need to apply the T operation. Incomplete ranks can also be assigned indices in this way.*

The set Z_0 with the relation E interpreted as membership looks like a set theory of the same general kind as ZFC. We will prove later that there are ranks which are models of ZFC.

20.3. Interpreting ZFC in the Cantorian part of Z_0

Our aim here is to suggest that the study of the usual set theory can be understood in the context of our set theory as the study of isomorphism classes of strongly Cantorian connected well-founded extensional relations. The hierarchy coded in the set H looks like the hierarchy usually seen in ZFC, except for the fact that there is an external "isomorphism" T from Z_0 into itself sending very high levels of the hierarchy down to lower (but still very high) levels. The universe of ZFC is usually understood as being "constructed" in a sequence of stages indexed by the ordinals, with power sets being taken at each successor ordinal and unions being taken at each limit ordinal, just as the sequence H is constructed here.

We summarize the status of axioms of ZFC in an interpretation in the strongly Cantorian elements of Z_0:

Extensionality: The Extensionality Lemma establishes this.

Pairing: Use the Comprehension Lemma on the condition defining a pair of Cantorian elements of Z_0. The two elements are fixed under the T operation, so one gets the right extension.

Union: Use the Comprehension Lemma on the condition of being the isomorphism type of an immediate component of an immediate component of an element of a fixed Cantorian element of Z_0. Components of Cantorian elements of Z_0 are Cantorian, thus fixed by the T operation, so one gets the right extension.

Power Set: Use the Comprehension Lemma on the condition of being the isomorphism type of a relation having each immediate component isomorphic to an immediate component of an element of a fixed Cantorian element of Z_0. A relation isomorphic to a subrelation of a Cantorian relation will be Cantorian, so elements will be fixed under the T operation.

Infinity: Obviously holds; use the Cantorian ordinal ω.

Separation: A full proof of Separation would require an appeal to the Axiom of Small Ordinals. The difficulty is that quantifications over all sets would translate to quantifications over the domain of all Cantorian elements of Z_0, which are unstratified. It is certainly possible to use the Axiom of Small Ordinals to prove that Separation holds, but we will not do it here. We will note that any condition in which each quantifier is restricted to a set will translate to a stratified condition defining a set, and we would then be able to apply the Comprehension Lemma.

The subtheory of ZFC which omits the Axiom of Replacement and restricts Separation to sentences in which each quantifier is restricted to a set is called "bounded Zermelo set theory" or "Mac Lane set theory"; it is adequate for most mathematical purposes. The restriction of quantifiers to sets harmonizes with the restriction of the bound variables in set definitions in the Axiom of Separation. The discussion here should show that the Theorem of Counting is the only consequence of the Axiom of Small Ordinals needed to show that the Cantorian part of Z_0 provides an interpretation of Mac Lane set theory (the Theorem of Counting is needed for Infinity). The mathematical strength of our theory without the Axiom of Small Ordinals or the Axiom of Counting turns out to be the same as that of Mac Lane set theory.

Replacement: Verification of this would require the Axiom of Small Ordinals. For the verification of both Separation and Replacement, observe that the Cantorian elements of Z_0 can be coded into the Cantorian ordinals, producing essentially the same structure as the Construction of the previous chapter.

Foundation and Choice: These axioms are easily seen to hold; Foundation is immediate, while the construction of a choice set would require the Comprehension Lemma.

20.4. Interpreting our own set theory in Z_0. The Axiom of Endomorphism

It is further possible to interpret our own set theory (rather than ZFC) inside the structure of isomorphism classes, by defining membership in a slightly different way. The "membership" E of Z or Z_0 is a relation realized as a set, so there are no stratification restrictions on "comprehension" (instead, the restriction is a "size" restriction; collections of elements of Z are not represented by elements of Z if they are too large).

Lemma. *$P(T[Z_0])$ is a complete rank.*

PROOF. — Since $T[Z_0]$ is a complete rank, we would otherwise have $P(T[Z_0]) = Z_0$, the first incomplete rank. Now consider the set of elements x of H such that $P^n(x) = Z_0$ for some natural number n; this would coincide with the set of iterated images of Z_0 under T, which cannot be a set. This can be argued directly using mathematical induction on an unstratified condition, but this appeal to the Axiom of Small Ordinals can be avoided at the price of making the argument more technical: roughly, the set of ranks x such that $P^n(x) = Z_0$ for some n must be finite, since it is a descending sequence of ranks; it has either an odd or an even number of elements. We then observe that the image under T of this sequence clearly has the same parity (T preserves that kind of structure) and also opposite parity (since it is obtained from the original sequence by deleting just the largest element). This is impossible. □

Definition. *We define another pseudo-membership "relation" on Z_0: $x \epsilon y$ iff $T\{x\}$ E y and y E $P(T[Z_0])$ (i.e., each element of the preimage under E of y is of the form $T\{x\}$ for some x.)*

Theorem. *The theory with domain Z_0 and membership "relation" ϵ satisfies the axioms of our set theory.*

Proof. — Elements of $P(T[Z_0])$ are regarded as sets; other objects are regarded as atoms. The axioms of extensionality and atoms are easily seen to be satisfied. Stratified comprehension holds because the type differential of the "relation" ϵ is the same as that for membership, so sets of elements of Z_0 defined by stratified sentences in ϵ instead of $\in$ exist; the element of Z_0 coding such a set can be obtained by taking the image under T of the set and using the fact that $P(T[Z_0])$ is a complete rank. The Axiom of Small Ordinals holds because the sets of ordinals which it requires are all codable in the complete rank $T[Z_0]$. Verification of Ordered Pairs is technical; one could use the Kuratowski pair in the set theory with E as membership if Z_0 is known not to be a successor type; the existence of a suitable pair is established in any case by the fact that Z_0 is infinite and the theorem $\kappa.\kappa = \kappa$ (for infinite κ) of cardinal arithmetic. □

The way in which our set theory is coded into the "set theory" of Z is exactly analogous to a way that it can be interpreted in ZFC; it is sufficient to show that a nonstandard model of part of the hierarchy of ZFC can be constructed with an external automorphism which moves a rank downward. We find it very appealing that this set theory's interpretation of ZFC reveals how this set theory can in turn be interpreted inside ZFC.

The interpretation of our set theory inside Z satisfies the following additional axiom:

Axiom of Endomorphism. *There is a one-to-one map* Endo *from $\mathcal{P}_1\{V\}$ (the set of singletons) into $\mathcal{P}\{V\}$ (the set of sets) such that for any set B, $\mathrm{Endo}(\{B\}) = \{\mathrm{Endo}(\{A\}) \mid A \in B\}$.*

Theorem. *The Axiom of Endomorphism is satisfied in the interpretation of set theory in Z_0 outlined above.*

Indication of proof. — Let Endo be interpreted as the restriction to $P(T[Z_0])$ of the map which takes each element of Z_0 whose preimage under

E has exactly one element to that one element (the inverse of the singleton map in the set theory with membership E). Verification that the Axiom holds is straightforward. $\square$

The Axiom of Endomorphism has the appealing feature that it allows us to assign set-theoretical structure to atoms (an atom a is associated with a set $\text{Endo}(\{a\})$). It also allows us to define a type-level "membership" relation $x\,E\,y$ by $\text{Endo}(\{x\}) \in y$. A curious fact is that, while membership in our set theory is certainly not well-founded, and the type-level membership E externally parallels the usual membership relation $\in$, it is possible (e.g., in the interpretation of our set theory in Z_0) for the type level relation E to be well-founded, an analogue for our set theory of the Axiom of Foundation in ZFC.

Exercises

(a) We call a set A *well-founded and transitive* if every element of A is a subset of A and every subset of A is disjoint from one of its elements (i.e., every subset of A has an $\in$-minimal element). We call a set *well-founded* if it is an element of some well-founded transitive set. Show that the class of all well-founded sets cannot be a set. You need only Boolean operations on sets for the argument (if that much). This is called Mirimanoff's paradox (mod the exact definition of "well-founded"). Why is this not a problem for our set theory?

(b) Verify that the following definition of an ordered pair on Z_0 satisfies the basic properties of an ordered pair and can be used to witness the Axiom of Ordered Pairs in the interpretation of our set theory in Z_0. For purposes of this exercise, let numerals stand for the isomorphism types of strict well-orderings of finite sets. For each isomorphism type x, define x' as the isomorphism type resulting if each immediate component of an immediate component of an element of x which belongs to a numeral n is replaced by an element of the numeral $n+1$, while x'' results from the same operation with the additional step of adding an element of the numeral 0 as an immediate component of each immediate component of the element of x. The ordered pair $\langle x, y\rangle$ is defined (locally to this exercise) as the isomorphism type "$x' \cup y''$" an element of which has as immediate components the immediate components of an element of x' and an element of y''. This definition of the ordered pair (cast in terms of membership rather than immediate componenthood, of course) is due to Quine. Notice that the rank of $\langle x, y \rangle$ is no higher than the maximum of the ranks of x and y (if these ranks are infinite); this is the technical advantage of this pair.

Cahiers du Centre de logique
Volume 10

Chapter 21

Generalized Dedekind Cuts and Surreal Numbers

We develop the *surreal numbers* of Conway. These numbers are defined by Conway using induction on the membership relation, which is impossible in our set theory, but the (von Neumann) ordinals are also defined in this way in Zermelo–Fraenkel set theory, and we succeeded in defining the ordinals using the theory of well-orderings, which is a more natural way to define the ordinals in any case. The surreal numbers will be defined using concepts from the theory of linear orderings in a way analogous to the way in which the ordinals were defined.

This chapter is really only a sketch, not a full development of the surreal numbers. For the reader who wants to get the full picture, we recommend a reading of the zeroth part of John Conway's *On Numbers and Games*, followed by a re-reading of this chapter.

The intuitive idea behind the surreal numbers is a generalization of the concept of "Dedekind cut" originally used to define the real numbers.

Definition. *Let $\leqslant$ be a linear order. Then a* cut (L, R) *in $\leqslant$ is a partition of* $\mathrm{dom}(\leqslant)$ *into two sets L and R such that for each $x \in L$ and $y \in R$, $x < y$. Let $\leqslant^*$ be a linear order which extends $\leqslant$; an element c of* $\mathrm{dom}(\leqslant^*)$ *is said to* realize *a cut C of $\leqslant$ if $\{x \in \mathrm{dom}(\leqslant) \mid x <^* c\} = L$ and $\{x \in \mathrm{dom}(\leqslant) \mid x >^* c\} = R$. Note that c cannot be in* $\mathrm{dom}(\leqslant)$ *if it realizes a cut in $\leqslant$.*

The process of filling in linear orders by introducing objects which realize "cuts" can be iterated. We formalize this idea in the following definition:

Definition. *An* iterated cut system *is defined by a linear order* $\leqslant$ *and a transitive relation* S, *($x\ S\ y$ is read "x is simpler than y", or "x comes before y"), with the following properties:*

(a) *In each nonempty subset* A *of* $\mathrm{dom}(\leqslant) = \mathrm{dom}(S)$, *there is an element* a *such that for no* $b \in A$ *is* $b\ S\ A$ *true (we call this a* simplest element *of* A*).*

(b) *Let* x *be an element of* $\mathrm{dom}(\leqslant)$*; let* $\mathrm{ancestors}\{x\}$ *be the set* $\{y \in \mathrm{dom}(\leqslant) \mid ySx\}$ *(the set of objects simpler than* x*).* x *obviously realizes a cut in the restriction of* $\leqslant$ *to* $\mathrm{ancestors}\{x\}$*; we require further that each cut in the restriction of* $\leqslant$ *to* $\mathrm{ancestors}\{x\}$ *is realized.*

Definitions. *We define* $\mathrm{generation}\{x\}$ *as the collection of objects* c *such that* c *is the simplest object realizing some cut* C *in* $\mathrm{ancestors}\{x\}$*. It is obvious that* x *itself belongs to* $\mathrm{generation}\{x\}$.

We define $\mathrm{birthday}(x)$ *as the largest object in* $\mathrm{generation}\{x\}$*, the simplest object which realizes the cut* $(\mathrm{ancestors}\{x\},\{\ \})$.

We define On *as the set of* birthdays.

Obervations. Observe that for each x, y in On, $x\ S\ y$ exactly if $x < y$, so a simplest object in a subset of On is the least object in the subset; On is strictly well-ordered by $\leqslant$.

Observe further that for any object x and element a of On which is simpler than x, there is a natural "approximation" x_a in $\mathrm{generation}\{a\}$, namely, the simplest object in $\mathrm{generation}\{a\}$ realizing the cut in $\mathrm{ancestors}\{a\}$ between objects less than x and objects greater than x.

Let's look at the structure of an iterated cut system. There is at least one simplest element in $\mathrm{dom}(\leqslant)$ (if it is nonempty); call it 0. 0 must be defined by a cut in $\mathrm{ancestors}\{0\}$, which is the empty set; there is only one such cut, $(\{\ \},\{\ \})$. The next generation (made up of the simplest objects different from 0) comprises two elements, which may be called

-1 and 1, determined by the two possible cuts ({ },{0}) and ({0},{ }) of generation{0}.

We expand the development of the previous paragraph into a detailed example of an iterated cut system: the elements of its domain are the "dyadic rationals" $\pm\frac{a}{2^b}$, for natural numbers a and b. The generations are indexed by the natural numbers. Each generation is finite and is constructed from the finite union of all the preceding generations in the following way: the cut at the top end is realized by adding one to the largest element of the previous generation; the cut at the lower end is realized by subtracting one from the smallest element of the previous generation. Each cut between two successive elements of the union of all the preceding generations is realized by their arithmetic mean.

We exhibit the first few stages in this process:

0: 0

1: $-1, (0), 1$

2: $-2, (-1), -\frac{1}{2}, (0), \frac{1}{2}, (1), 2$

3: $-3, (-2), -\frac{3}{2}, (-1), -\frac{3}{4}, (-\frac{1}{2}), -\frac{1}{4}, (0), \frac{1}{4}, (\frac{1}{2}), -\frac{3}{4}, (1), \frac{3}{2}, (2), 3$

In each list, elements of previous generations are enclosed in parentheses.

The order $\leqslant$ on this system is the usual order relation. The simplicity relation is easily defined if we know how to compute the generation in which a particular dyadic rational appears: the dyadic rational whose simplest form is $\pm\frac{a}{2^b}$ appears in the same generation as the natural number $\left\lceil \frac{a}{2^b} \right\rceil + b$.

If we add one more generation to this iterated cut system, we add all the real numbers (in the obvious way preserving the correspondence between our relation $\leqslant$ and the relation of the same name on the reals) but we also need to add new objects $\pm\infty$ as the largest and smallest elements of the final generation and objects r^+ and r^- realizing the cuts immediately to the left and right of each dyadic rational r.

The structure of iterated cut systems is rigidly determined by the number of generations:

Theorem. *For any two iterated cut systems* $(\leqslant, S)$ *and* $(\leqslant^*, S^*)$*, the order type of whose generations is the same, there is a uniquely determined bijection* f *from* $\mathrm{dom}(\leqslant)$ *to* $\mathrm{dom}(\leqslant^*)$ *such that* $f(x) \leqslant^* f(y)$ *iff* $x \leqslant y$ *and* $f(x)\ S^*\ f(y)$ *iff* $x\ S\ y$*. It follows from this that the structure of iterated cut systems is entirely determined by the order type of* $\leqslant$ *restricted to* On.

PROOF. — Let $(\leqslant, S)$ and $(\leqslant^*, S^*)$ be two iterated cut systems as specified in the conditions of the theorem. We define a map f by transfinite recursion on the well-ordering of generations, as follows: when f from $\mathrm{dom}(\leqslant)$ to $\mathrm{dom}(\leqslant^*)$ is defined for members of each previous generation, we match each element x of the current generation over $\leqslant$ with the element of the corresponding generation over $\leqslant^*$ lying between (in the sense of $\leqslant^*$) the image under f of the set of elements of ancestors$\{x\}$ less than x and the image under f of the set of elements of ancestors$\{x\}$ greater than x. This definition succeeds in defining a unique $f(x)$ for each x in $\mathrm{dom}(\leqslant)$ such that the corresponding generation exists for $\leqslant^*$. $\square$

Note that iterated cut systems with different order types on their generations can still be compared using this result; the system with more generations has a subsystem consisting of the union of the initial segment of its generations similar to the generations of the other system which is seen by this result to be isomorphic with the system with fewer generations.

This motivates the following

Definition. *A* surreal number *is an equivalence class of triples* $(\leqslant, S, C)$*, where* $(\leqslant, S)$ *is an iterated cut system and* C *is a cut on* $\leqslant$*, under the equivalence relation of corresponding to one another via the map defined above between the domains of iterated cut systems. The set of surreal numbers is called* No.

Definition. *For surreal numbers* r *and* s*, we define* $r\ S\ s$ *and* $r \leqslant s$ *as follows: let* $(\leqslant', S', C')$ *be an element of* r *and let* $(\leqslant'', S'', C'')$ *be an element of* s*. Without loss of generality, we may suppose that each of the two iterated cut systems is included in the same iterated cut* *system, which we will call $(\leqslant^*, S^*)$. We assert that r S s if there are more generations in each element of s than there are in each element of r in the obvious sense. Suppose that one of r and s is simpler than the other; without loss of generality, let it be r. Then C' is realized by a simplest element with respect to $\leqslant^*$, which we will call r^*. We have $r^* \in \pi_1(C'')$, in which case we say $r < s$, or $r^* \in \pi_2(C'')$, in which case we say $s < r$. If neither r nor s is simpler than the other, then $r < s$ iff the intersection of $\pi_2(C')$ and $\pi_1(C'')$ is nonempty, and similarly for $s < r$. The only remaining case is that in which $r = s$. We have now defined relations S and $\leqslant$ ($r \leqslant s$ iff $r < s$ in the sense defined above or $r = s$) on the surreal numbers.*

Theorem. *$(\leqslant, S)$ is an iterated cut system with domain the set* No *of surreal numbers.*

PROOF. — A surreal number, which is an equivalence class of relations on elements of domains of iterated cut systems, is two types higher than the latter elements; there is a natural map, respecting $\leqslant$ and S, from No into the domain of the generations indexed by the elements of $\mathrm{T}^2[Ord]$ in a large enough iterated cut system. All objects of No have corresponding objects in an iterated cut system of "rank" $\lim[\mathrm{T}^2[Ord] = \Omega$ (to verify the evaluation of this limit, see the next chapter), and the local relations of order and simplicity will parallel those of the surreal numbers exactly.

Observation. Conway constructed the surreal numbers in the usual set theory using a recursion on membership; (L, R) being the simplest surreal number between L and R whenever L and R were sets of surreal numbers. This clearly will not work in our theory, since it is an unstratified construction. It also has the consequence that the "class" No of surreal numbers is not a set; otherwise, consider the "surreal number" (No,$\{\ \}$), seen to be greater than all surreal numbers! In our set theory, we *can* define an analogue to Conway's basic operation: Let (L, R) be taken to be the surreal number containing $(\leqslant, S, (L, R))$, where $\leqslant$ and S are the relations on the iterated cut system on No itself. But (L, R) is not the simplest surreal number between the sets L and R; it is the simplest surreal number between the images of L and R under the T^2 operation on surreal numbers (the type-raising operation induced by taking double singleton images of iterated cut systems). Just as Ω does not sit "past the end" of the ordinals, but is smaller than the largest ordinals, so (No, $\{\ \}$) is indeed a surreal number — it is the upper limit of the set of surreal numbers which are of the form $T^2\{r\}$. Larger or more complex surreal numbers cannot be expressed in the (L, R) form, but for any surreal r, we *can* express $T^2\{r\}$ in this form.

We close by defining arithmetic operations on No.

Definition. *If r is a surreal number, we let $L[r]$ be the set of surreal numbers less than r and simpler than r, and $R[r]$ be the set of surreal numbers greater than r and simpler than r.*

We define $r + s$ as the simplest number greater than all elements of $\{r\} + L[s]$ and $L[r] + \{s\}$ and less than all elements of $\{r\} + R[s]$ and $R[r] + \{s\}$.

We define $-r$ as the simplest number between $-R[r]$ below and $-L[r]$ above (and $r - s$ as $r + (-s)$).

We define rs as the simplest number between the sets $L[r]\{s\} + \{r\}L[s] - L[r]L[s]$ and $R[r]\{s\} + \{r\}R[s] - R[r]R[s]$ below and the sets $L[r]\{s\} + \{r\}R[s] - L[r]R[s]$ and $R[r]\{s\} + \{r\}L[s] - R[r]L[s]$ above.

Observations. Sums and products of sets are interpreted as sets of all possible sums or products of a term or factor from one and a term or factor from the other. These are definitions by transfinite recursion; observe that degrees of simplicity are indexed by ordinals, or develop a recursion theorem for definitions of functions on the surreals. Conway shows in his book *On Numbers and Games* that No makes up a field with these operations; a suitable subset of the surreals with birthday on or before ω can be regarded as the real numbers (it turns out that those with finite birthdays are the rational numbers with denominator a power of two, as in our example). No has a far more interesting elementary arithmetic than the other systems of transfinite numbers. As Conway points out, it is very interesting that we can get the real numbers "all at once" without going through the usual intermediate systems (a restriction on cuts used in iterated cut systems which gives the reals exactly: "in any cut (L, R), L is nonempty with no greatest element exactly if R is nonempty with no least element"). The definitions of arithmetic operations can be shown to be based on the principle of "choosing the simplest answer compatible with the axioms of an ordered field".

Exercise

Read the zeroth part of Conway's book *On Numbers and Games*. (There really is a zeroth part!)

Cahiers du Centre de logique
Volume 10

Chapter 22

The Structure of the Transfinite

22.1. Cardinals, Ordinals, and $\aleph$

We begin with some notions and results relating cardinal and ordinal numbers.

It is an obvious observation that two similar well-orderings are well-orderings of a set of the same cardinality, since a similarity between well-orderings induces a bijection on the underlying sets. Thus, each ordinal is associated with a uniquely determined cardinal, and each cardinal is associated with a class of ordinals which are order types of well-orderings of sets of that cardinality. For any infinite cardinal, this class of ordinals has many members. This discussion motivates the following

Definition. *For each ordinal α, we use the notation* $\mathrm{card}(\alpha)$ *to represent the cardinal of sets well-orderable with order type α, and for each cardinal κ, we use the notation* $\mathrm{init}(\kappa)$ *to denote the smallest ordinal α such that* $\mathrm{card}(\alpha) = \kappa$*. Ordinals* $\mathrm{init}(\kappa)$ *are called* initial ordinals.

There is a one-to-one correspondence between cardinals and initial ordinals. In the usual set theory, initial (von Neumann) ordinals and cardinals are identified. The notations introduced in the definition are not standard, but something like them is necessary in the absence of the usual identification of cardinals with initial ordinals.

We have already encountered one (necessarily partial) function (the exponential map) which sends each cardinal in its domain to a strictly larger cardinal. We now introduce another natural such function.

Definition. *For each cardinal κ, we define $\kappa+$ as the smallest cardinal greater than κ, i.e., the successor of κ. That cardinals (other than $|V|$) have successors is a consequence of the Axiom of Choice, which implies that the natural order on cardinals is a well-ordering.*

The assumption that this function is actually identical to the exponential map, called the Generalized Continuum Hypothesis (GCH), is consistent with our set theory, but not a consequence of our axioms. Cantor's original Continuum Hypothesis is the assertion $\aleph_0+ = 2^{\aleph_0}$. Although we cannot prove that the successor operation on cardinals is equivalent to the exponential map, we can prove that it is equivalent (in the presence of the Axiom of Choice) to a certain natural operation on cardinals.

Definition. *Let κ be a cardinal. We define $\aleph(\kappa)$ as the result of applying T^{-2} to the cardinality of the collection of ordinals α such that $\mathrm{card}(\alpha) \leqslant \kappa$. The use of T^{-2} ensures that $\aleph$ is a function, known as the* Hartogs aleph function.

Theorem (Hartogs). *For all cardinals κ, the cardinality of the collection of ordinals α such that $\mathrm{card}(\alpha) \leqslant \kappa$ is $T^2\{\kappa\}+$.*

Corollary. *For all cardinals $\kappa \neq |V|$, $\kappa+ = \aleph(\kappa)$. The proof of the Corollary requires nothing more than the observation that the successor and Hartogs aleph operations commute with T.*

PROOF OF THE THEOREM. — $|\,\mathrm{seg}\{\alpha\}| = T^2\{\mathrm{card}(\alpha)\}$ (to see this, look at the discussion of order types of segments of the ordinals above). It is clear from this that the cardinality of the set of interest is $\geqslant T^2\{\kappa\}$. It is also clear that the cardinality of every proper initial segment of the set of interest (so every cardinal less than the cardinality of the set of interest) is $\leqslant T^2\{\kappa\}$.

The only thing that remains to be shown is that the cardinality of the set of interest is not equal to $T^2\{\kappa\}$. Suppose it were. Let β be the first ordinal not in the set of interest. We are assuming that $|\operatorname{seg}\{\beta\}| = T^2\{\kappa\}$, from which it follows that $\operatorname{card}(\beta) = \kappa$, which implies that β is in the set of interest, which is absurd. We need to consider the special case $\kappa = |V|$, for which the ordinal β does not exist; note that Ω is the first ordinal which is not the order type of a proper initial segment of the ordinals, and that the order types of proper initial segments of the ordinals are exactly the ordinals of the form $T^2\{\alpha\}$; this is enough to establish that $\operatorname{card}(\Omega)$ must be the first cardinal greater than all cardinals of the form $T^2\{\kappa\}$, that is, $T^2\{|V|\}+$. The proof of the Theorem is complete. $\square$

The Hartogs operation makes sense in the absence of the Axiom of Choice, but no longer coincides with the notion of successor. Without the Axiom of Choice, the argument above proves that $\aleph(\kappa)$ is greater than any cardinal of a well-orderable subset of a set of size κ; it cannot be $\leqslant \kappa$, but it may be incomparable with κ rather than $> \kappa$. It should be noted that $\aleph(\kappa)$ is certainly the cardinal of a well-orderable set!

We define the usual notation for cardinals, which also uses the letter $\aleph$:

Definition. *We define $\aleph_\alpha$ as the cardinal κ such that the natural order on infinite cardinal numbers restricted to the cardinals $< \kappa$ has order type α.*

Note that the type of α is two types higher than the type of $\aleph_\alpha$. If $\aleph$ were a symbol for the natural well-ordering on infinite cardinals (which it is not) this would be an example of the definition of ordinal indexed sequences.

The notation $\aleph_0$ already introduced for $|\mathcal{N}|$ is a special case of this. We see from Hartogs's theorem that $\aleph_1$ is the cardinality of the set of all order types of well-orderings of countable sets (countable ordinals): $\aleph_1 = \aleph_0+ =$ the image under T^{-2} of the cardinality of the set of countable ordinals; the latter set is Cantorian, so this is simply the cardinality of the set of countable ordinals. For each natural number n, we have that $\aleph_{n+1}$ is the cardinality of the set of order types of sets of cardinality $T^{-2}\{\aleph_n\} = \aleph_{T^{-2}\{n\}}$ (if this exists!). For each concrete natural number n, we can establish without an appeal to the Axiom of Counting or the Axiom of Small Ordinals that $T^{-2}\{\aleph_n\} = \aleph_n$. Without at least the Axiom of Counting we cannot conclude that $\aleph_n$ exists for each n; it is consistent with our other axioms that $\aleph_n = |V|$ for some n, in which case the cardinality of the set of order types of well-orderings of sets of size $|V|$ (card(Ω)) would be $\aleph_{T^2\{n\}+1}$, from which we would see clearly that $T^2\{n\} < n$. This argument in the presence of the Axiom of Counting (or of the Axiom of Small Ordinals) demonstrates that $|V|$ is not of the form $\aleph_n$ for any natural number n, and incidentally demonstrates the existence of $\aleph_\omega$.

Although such pathologies are averted at low levels by the Axiom of Small Ordinals, it is the case that $|V| = \aleph_\alpha$ for *some* ordinal α; card(Ω) = $\aleph_{T^2\{\alpha\}+1}$ for this ordinal α, which implies $T^2\{\alpha\} < \alpha$; this ordinal is a non-Cantorian ordinal. The Axiom of Small Ordinals allows us to deduce that for each Cantorian β, $\aleph_\beta$ exists, since we see that the index of the last cardinal is non-Cantorian, and we know that all Cantorian ordinals are less than any non-Cantorian ordinal.

We define another sequence of cardinals, which is identical with the sequence of alephs if the Generalized Continuum Hypothesis holds.

Definition. *We define $\beth_\alpha$ as W_α for W a certain well-ordering which we now describe (this use of W is a nonce notation!). W is the restriction of the natural well-ordering on cardinal numbers to the intersection of all sets B of cardinal numbers which contain $\aleph_0$ and are closed under the function* exp *(if $\kappa \in B$, $\exp(\kappa) \in B$) and the operation of taking suprema of subsets (if $A \subset B$ and A is bounded above in B, the least upper bound of A in the natural order on the cardinals is an element of B).*

This sequence looks like the sequence of $\aleph_\alpha$'s, except that the "successor" operation is exp instead of the Hartogs aleph. As is the case for any ordinal-indexed sequence, the type of $\beth_\alpha$ is two types lower than the type of α.

22.2. Classification of Cardinals and Solovay's Theorem on Inaccessibles

We classify cardinals. A cardinal of the form $\aleph_{\alpha+1}$ is called a *successor cardinal.* A cardinal of the form $\aleph_\lambda$, λ a limit ordinal, is called a *limit cardinal.*

The next step requires a

Definition. *Let $\leqslant$ be a well-ordering. A subrelation $\leqslant^*$ of $\leqslant$ which is also a well-ordering is said to be* cofinal *in $\leqslant$ iff for each element x of* $\mathrm{dom}(\leqslant)$ *there is an element y of* $\mathrm{dom}(\leqslant^*)$ *such that $x \leqslant y$. The* cofinality *of $\leqslant$ is the smallest order type of a subrelation cofinal in $\leqslant$.*

If κ is a cardinal, we define the cofinality *of κ, written* $\mathrm{cf}(\kappa)$ *as the smallest cardinal μ which contains the domain of some well-ordering cofinal in a well-ordering of order type* $\mathrm{init}(\kappa)$.

We use the notion of cofinality to further classify cardinals:

Definition. *A cardinal κ is said to be* regular *if* $\mathrm{cf}(\kappa) = \kappa$. *Cardinals which are not regular are said to be* singular.

Theorem. *Cofinalities of cardinals are regular cardinals.*

Proof. — Any well-ordering of a set of size κ must contain a sub-well-ordering of a set of size $\mathrm{cf}(\kappa)$ which must in turn contain a well-ordering of a set of size $\mathrm{cf}(\mathrm{cf}(\kappa))$. Since a well-ordered set of size κ must contain a well-ordered set of size $\mathrm{cf}(\mathrm{cf}(\kappa))$, it follows that $\mathrm{cf}(\mathrm{cf}(\kappa)) \geqslant \mathrm{cf}(\kappa)$, from which it obviously follows that $\mathrm{cf}(\mathrm{cf}(\kappa)) = \mathrm{cf}(\kappa)$, i.e., that $\mathrm{cf}(\kappa)$ is regular.

Theorem. *Successor cardinals are regular.*

Proof. — Given Hartogs's Theorem, it is sufficient to prove that for any κ, any sequence cofinal in the natural well-ordering on ordinals α with $\mathrm{card}(\alpha) \leqslant \kappa$ has the same cardinality as the set of such ordinals α (that is, $T^2\{\kappa+\}$). Suppose otherwise. We would then have a well-ordering with full domain of size $\leqslant T^2\{\kappa\}$ purportedly cofinal in this sequence of ordinals. But this is impossible: the size of the set of ordinals less than or equal to some element of this cofinal sequence is clearly $\leqslant T^2\{\kappa\} \times T^2\{\kappa\} = T^2\{\kappa\}$, which is incompatible with the assumption that the sequence is cofinal in a segment of the ordinals of size $T^2\{\kappa+\}$. The proof of the theorem is complete. □

An example of a singular limit cardinal is $\aleph_\omega$, which has cofinality $\aleph_0$.

For our next definition, we need a stronger notion of "limit cardinal".

Definition. *A cardinal κ such that for each $\mu < \kappa$, $2^\mu < \kappa$ is said to be a* strong limit *cardinal. It should be clear that a strong limit cardinal is a limit cardinal.*

Definition. *An uncountable regular strong limit cardinal is called an* inaccessible *cardinal. (Of course $\aleph_0$ is a regular strong limit cardinal!)*

Theorem. *A complete rank of Z_0 indexed by* $\mathrm{init}(\kappa)$, *κ inaccessible, is a model of ZFC (using the relation E to code membership as usual).*

PROOF. — The only axioms which require real attention are Power Set and Replacement. Power Set holds because the ordinal $\mathrm{init}(\kappa)$ is a limit ordinal (a set appearing at one stage has its power set appearing at the next stage, and there is no last stage). We see that Replacement holds by considering a set A of sets realized in the rank which can be placed in one-to-one correspondence with a set B realized in the rank; the common size of these sets will be less than κ, and the collection of ranks at which elements of A appear cannot be cofinal in the ordering of ranks below rank $\mathrm{init}(\kappa)$, because κ is regular; thus there is a rank below rank $\mathrm{init}(\kappa)$ at which all the elements of A are present, and the set A is realized at that rank. $\square$

The rest of this section is devoted to the proof of the following theorem of Robert Solovay:

Theorem (Solovay). *There is an inaccessible cardinal.*

The only consequence of the Axiom of Small Ordinals needed for the proof of Solovay's theorem is the assertion that each Cantorian set (resp. ordinal, cardinal) is strongly Cantorian. Note that the Axiom of Counting follows from this directly.

PROOF OF SOLOVAY'S THEOREM. — Fix a well-ordering $\leqslant^*$ of the set of sub-well-orderings of the natural well-ordering on the cardinals. For any sub-well-ordering $\leqslant_0$ of the natural order on the cardinals, we define $T[\leqslant_0]$ as $\{(T\{\kappa\}, T\{\lambda\}) \mid \kappa \leqslant_0 \lambda\}$. We define $T[\leqslant^*]$ as $\{(T[\leqslant_0], T[\leqslant_1]) \mid [\leqslant_0] \leqslant^* [\leqslant_1]\}$.

We define a function F with domain a subset of the set of pairs of cardinal numbers as follows:

(a) If α_0 and β_0 are distinct cardinals which are not strong limit, then $F(\alpha_0, \beta_0) = (\alpha_1, \beta_1)$, where α_1 and β_1 are the least cardinals such that $\exp(\alpha_1) \geqslant \alpha_0$ and $\exp(\beta_1) \geqslant \beta_0$.

(b) If α_0 and β_0 are distinct singular strong limit cardinals, and $\mathrm{cf}(\alpha_0) \neq \mathrm{cf}(\beta_0)$, define $F(\alpha_0, \beta_0)$ as $(\mathrm{cf}(\alpha_1), \mathrm{cf}(\beta_1))$.

(c) If α_0 and β_0 are distinct singular strong limit cardinals, and $\mathrm{cf}(\alpha_0) = \mathrm{cf}(\beta_0)$, we proceed as follows. Let $\leqslant^0$ be the first well-ordering of minimum possible length cofinal in the natural order on the cardinals less than α_0, where "first" is in the sense of the order $\leqslant^*$ fixed above. Let $\leqslant^1$ be the first well-ordering of minimum possible length cofinal in the natural order on the cardinals less than β_0, where "first" is in the sense of the order $T[\leqslant^*]$ defined above (if β_0 is not of the form $T\{\gamma\}$ for some cardinal γ, this clause of the definition does not succeed, and $F(\alpha_0, \beta_0)$ is undefined; this will not occur in cases of interest to us). These two well-orderings will be of the same order type $T^2\{\mathrm{init}(\mathrm{cf}(\alpha_0))\} = T^2\{\mathrm{init}(\mathrm{cf}(\beta_0))\}$ (the T^2 operator appears because cardinals and ordinals are two types higher than the objects they "count"), and will be distinct, since they are cofinal in the natural order on different initial segments of the cardinals. Let δ be the smallest ordinal such that $[\leqslant^0]_\delta \neq [\leqslant^1]_\delta$ (there must be such a δ by the preceding considerations); define $F(\alpha_0, \beta_0)$ as $([\leqslant^0]_\delta, [\leqslant^1]_\delta)$.

(d) In any other case, (α_0, β_0) is excluded from the domain of F. This includes in particular the case where α_0 and β_0 are inaccessible cardinals.

The function F has a stratified definition, and so is a set. By a slight modification of the Recursion Theorem for natural numbers, we can define a (possibly finite) sequence p of pairs of cardinals which has any desired pair of cardinals as p_0 and satisfies the condition that for each $i \in \mathcal{N}$ we have $p_{i+1} = F(p_i)$ if $p_i \in \mathrm{dom}(F)$ and undefined otherwise. We write p_i as (α_i, β_i).

To completely determine p, we need to choose its starting point $p_0 = (\alpha_0, \beta_0)$. Let α_0 be any non-Cantorian cardinal α, and let β_0 be $T\{\alpha\}$.

We want to argue by induction that the conditions $\beta_i = T\{\alpha_i\} \neq \alpha_i$ hold for each i in the domain of p. This condition is unstratified: however, an appeal to the Axiom of Counting shows that it is equivalent to the stratified condition $\beta_{T\{i\}} = T\{\alpha_i\} \neq \alpha_{T\{i\}}$, on which induction is permitted (strictly speaking, this condition is still not stratified; it is a substitution instance of a stratified sentence in which the same object $\alpha = \pi_1 \circ p$ replaces two parameters of different relative type; but this does not prevent it from defining a set).

The step that requires particular care is the step handling a pair of singular strong limit cardinals of the same cofinality.

(a) The condition clearly holds in the basis case.

(b) If α_i and β_i are cardinals which are not strong limit, and $\beta_i = T\{\alpha_i\} \neq \alpha_i$, it is clear that $\beta_{i+1} = T\{\alpha_{i+1}\} \neq \alpha_{i+1}$; the operation described clearly commutes with T, and so if the cardinals α_{i+1} and β_{i+1} chosen were equal, they would have to be Cantorian, so the images under exp of the cardinals would be Cantorian and dominate the non-Cantorian α_i and β_i, which is absurd.

(c) If α_i and β_i are singular strong limit cardinals, $\mathrm{cf}(\alpha_i) \neq \mathrm{cf}(\beta_i)$, and $\beta_i = T\{\alpha_i\} \neq \alpha_i$, it is clear that $\beta_{i+1} = T\{\alpha_{i+1}\} \neq \alpha_{i+1}$; all that is required is to note that cf commutes with T (i.e., $T\{\mathrm{cf}(\kappa)\} = \mathrm{cf}(T\{\kappa\})$ for any cardinal κ).

(d) If α_i and β_i are singular strong limit cardinals, $\mathrm{cf}(\alpha_i) = \mathrm{cf}(\beta_i)$, and $\beta_i = T\{\alpha_i\} \neq \alpha_i$, we choose the first well-orderings of shortest length cofinal in the natural order on cardinals below α_i and β_i respectively with respect to the orders $\leqslant^*$, $T[\leqslant^*]$, respectively; the operation described "commutes with T" in a suitable sense by the definition of $T[\leqslant^*]$. The common cofinality of α_i and β_i must be Cantorian, since the cf operator commutes with T, and $\mathrm{init}(\mathrm{cf}(\alpha_i)) = \mathrm{init}(\mathrm{cf}(\beta_i))$ will be the common length of the chosen well-orderings of the cardinals below α_i and β_i (the fact that the cofinality and thus its initial ordinal are seen to be Cantorian allows us to omit applying the T^2 operator to the order types as we had to do in the discussion in the corresponding case of the recursive definition). α_{i+1} and β_{i+1} are chosen to be the first pair of corresponding elements of the domain of these well-orderings that differ; suppose that they are the δth elements of these well-orderings (i.e., δ is the order type of the segments they determine). $T\{\alpha_{i+1}\}$ will be the $T\{\delta\}$th element of the cofinal well-ordering associated with β_i, because the cofinal well-ordering associated with $\beta_i = T\{\alpha_i\}$ is the image under T of the cofinal well-ordering associated with α_i (in the sense of image under T defined above for well-orderings); but $T\{\delta\} = \delta$, because δ is less than the initial ordinal associated with the Cantorian common cofinality of the two original cardinals and so is Cantorian. Thus $T\{\alpha_{i+1}\} = \beta_{i+1}$ as desired.

(e) In any other case, α_{i+1} and β_{i+1} will be undefined, which poses no problem for our induction.

It is an immediate consequence of this result that each pair of cardinals (α_i, β_i) is a pair of cardinals of the same kind (both not strong limit, both singular strong limit, or both inaccessible), since the T operation preserves each of these properties of cardinals.

Since corresponding projections of the pairs of cardinals strictly decrease as the index increases, the sequence p must be finite. The only way it can terminate is with a pair of distinct non-Cantorian inaccessible cardinals. The proof of Solovay's theorem is complete. □

The theorem establishes that our set theory proves the existence of many inaccessible cardinals, and so is considerably stronger than ZFC.

22.3. Stronger Results. The Axiom of Large Ordinals

Solovay has proved more than this. We introduce some new concepts:

Definition. *A subset of the domain of a well-ordering is said to be* closed *if it contains the least upper bound of each of its non-cofinal subsets. A subset of the domain of a well-ordering is said to be a* club *if it is closed and cofinal in the well-ordering. A subset of the domain of a well-ordering is said to be* stationary *if it meets every club in the domain of the well-ordering.*

Definition. *A cardinal κ is said to be* Mahlo *if the set of inaccessible cardinals less than κ is stationary in the natural well-ordering of cardinals $< \kappa$. A Mahlo cardinal is also said to be* 1-Mahlo*; we define an* $(n+1)$-

Mahlo cardinal *(for each natural number n, by induction) as a cardinal κ such that the set of n-Mahlo cardinals less than κ is stationary in the natural well-ordering of the cardinals $< \kappa$.*

Theorem (Solovay). *The strength of the set theory whose axioms are Extensionality, Ordered Pairs, Stratified Comprehension, Choice, and "each Cantorian set is strongly Cantorian" is exactly the same as the strength of ZFC + "there is an n-Mahlo cardinal" for each n (notice that this is a list of axioms for each concrete n; the assertion "for each n, there is an n-Mahlo cardinal" is* not *a consequence of this theory). This means that each of these theories can interpret the other. It is also the case that the subset of our set theory described proves the existence of n-Mahlo cardinals for each concrete n.*

The proof of the Theorem is not given here.

We close by introducing and motivating a final axiom.

Definition. *A* T-sequence *is a finite sequence s of ordinals such that for each i, $s_{i+1} = T\{s_i\}$ iff s_{i+1} is defined. We define $T^n\{\alpha\}$ for any $n \in \mathcal{N}$ as the value of s_n for any T-sequence s such that $s_0 = \alpha$.*

This definition is needed because we have no guarantee that the natural numbers of our theory correspond to the concrete natural numbers (for which we already know how to define $T^n\{\alpha\}$, of course). The existence and uniqueness of $T^n\{\alpha\}$ is easily proved using mathematical induction (on horribly unstratified conditions).

Axiom of Large Ordinals. *For each non-Cantorian ordinal α, there is a natural number n such that $T^n\{\Omega\} < \alpha$.*

The Axiom of Large Ordinals asserts that the structure of the "large" ordinals is as simple and clean as we can manage. It clearly implies the assertion "each Cantorian ordinal is strongly Cantorian" and it turns out that it has exactly the same strength as this assertion in combination with our other axioms (exclusive of the Axiom of Small Ordinals).

An immediate corollary of the Axiom of Large Ordinals is

Corollary. *$T\{\alpha\} \leqslant \alpha$ for all ordinals α.*

The reason that we introduce this final axiom is that it allows a very nice treatment of proper classes of small ordinals.

Definition. *A* natural set *is a set of ordinals A with the property that $\alpha \in A$ iff $T\{\alpha\} \in A$ for all ordinals α.*

Theorem. *Two natural sets have the same elements iff they have the same Cantorian elements.*

PROOF. — Let $A \neq B$ be natural sets. Let α be the smallest element of $A \Delta B$ (the symmetric difference of the two sets). Suppose that α is non-Cantorian. This implies that $T\{\alpha\} < \alpha$ is not in $A \Delta B$, i.e., it is either in both sets or in neither. It follows by naturality of A and B that α is either in both sets or in neither, contrary to assumption. (the assumption that $T\{\alpha\} < \alpha$, a consequence of the Axiom of Large Ordinals, is *not* necessary for this proof; if we did not assume Large Ordinals and had $T\{\alpha\} > \alpha$, we would then consider the status of $T^{-1}\{\alpha\}$; it would exist and cause the same problem). □

Theorem. *For any sentence ϕ, there is a natural set A of ordinals such that the Cantorian ordinals x belonging to A are exactly the Cantorian ordinals x such that ϕ.*

PROOF. — By the Axiom of Small Ordinals, there is a set B satisfying all the required conditions except possibly naturality. Consider the smallest element x of the set $B \Delta T[B]$; x is obviously non-Cantorian. Let n be a natural number such that $T^{-n}\{x\}$ does not exist (the existence of such an n is a consequence of the Axiom of Large Ordinals). $T^{-n}[B]$ is the desired natural set. □

The effect of the two preceding Theorems is that there is a precise one-to-one correspondence between natural sets (which are objects of our theory) and definable proper classes of Cantorian ordinals, which are not objects of our theory! This means that we can define proper classes of small ordinals (and corresponding natural sets) in ways which involve quantification over all proper classes of small ordinals, by replacing the quantifications over proper classes with quantifications over the corresponding natural sets. Conditions involving quantification over all natural sets will of course be unstratified. This suggests that our set theory has considerable strength; the set theory QM (Quine–Morse set theory, also known as Morse–Kelley set theory) which adds quantification over proper classes to the machinery of ZFC is stronger than ZFC, for example.

We describe an interpretation of a version of ZFC with proper classes which turns out to have the precise strength of our set theory. The interpretation of the *sets* of ZFC will be the interpretation in the Cantorian ordinals which we gave above. Proper classes of ZFC will be interpreted using the natural sets extending the corresponding classes of Cantorian ordinals of our set theory. A technical point is that a set of the interpreted ZFC will be coded by a certain Cantorian ordinal, while the class with the same elements will be coded by a natural set which is actually a different object. It is convenient and harmless to pass over this distinction in the subsequent discussion.

There are von Neumann ordinals in the interpreted ZFC corresponding to each Cantorian ordinal of our set theory. There is in addition one proper class ordinal, the collection of all ordinals (which is implemented as the natural set of all ordinals coding von Neumann ordinals of ZFC). Let us refer to this "ordinal", the order type of the Cantorian ordinals of our theory, as κ. The universe of sets of our interpreted ZFC is the stage of the construction of the cumulative hierarchy of sets indexed by κ; call this V_κ. The classes which we admit in addition may be interpreted as adding one more stage to the cumulative hierarchy; we can view the universe of classes (including sets and proper classes) as representing $V_{\kappa+1}$. We do not have any objects constructed at later stages available to us.

Observe that we can define proper classes and sets in our interpreted ZFC using sentences which involve quantifiers not only over sets, but over proper classes. This is a consequence of our ability to code proper classes of Cantorian ordinals in our set theory as natural sets, which are objects in our theory; sentences involving quantification over natural sets can be used to define sets implementing classes of Cantorian ordinals (an application of the Axiom of Small Ordinals, since such sentences are clearly unstratified!). The axioms of Separation and Replacement, which concern sets rather than classes, are valid for sentences involving quantification over classes, because any class of Cantorian ordinals which is a subclass of a set or which can be placed in an external one-to-one correspondence with the elements of a set of Cantorian ordinals, however defined, is a set (this was shown above). An extension of ZFC which admits proper classes and allows quantification over proper classes in definitions of classes and of sets strengthens ZFC significantly.

Though we do not admit elements of the stage $V_{\kappa+2}$ into our interpreted ZFC, it is possible to code "sequences" of classes indexed by set ordinals as single classes. The trick is to code a sequence A of classes indexed by ordinals β less than some $\alpha \leqslant \kappa$ as $\{\langle\langle \beta, x \rangle\rangle \mid x \in A_\beta\}$, where the brackets signify the coding of pairs of ordinals into the ordinals introduced earlier. This trick can be carried out in our set theory to implement sequences of natural sets indexed by Cantorian ordinals as natural sets. It is useful to recall that our coding of pairs of ordinals into the ordinals "commutes with T", so will behave nicely in the context of reasoning about natural sets.

There is a further strengthening of ZFC possible. Consider the proper class Υ of natural sets which contain Ω as an element. We cannot introduce Υ as an object of our interpreted ZFC with classes, but we can introduce membership in Υ as a predicate of classes of ordinals. The "superclass" Υ over our interpreted ZFC will have the following properties:

(a) Any superclass of an element of Υ is an element of Υ (a natural set which includes a natural subset containing Ω contains Ω).

(b) Any element of Υ is of size κ (a natural set containing Ω, or any non-Cantorian element, has a proper class of Cantorian ordinal elements which can be placed in an external one-to-one correspondence with the class of all Cantorian ordinals).

(c) Any intersection of fewer than κ elements of Υ belongs to Υ (collections of fewer than κ elements of Υ are coded as described above; the intersection of natural sets implemented as "rows" in a Cartesian product with "rows" indexed by a Cantorian initial segment of the ordinals will be natural and contain Ω if each of the "rows" contains Ω).

(d) Each class of ordinals or its complement belongs to Υ. (Each natural set either contains Ω or has a natural complement (relative to the set of all ordinals) which contains Ω).

(e) Membership of classes in Υ may be used freely in definitions of sets and classes.

Such a "superclass" Υ is called a *κ-complete nonprincipal ultrafilter* over κ. The existence of a κ-complete nonprincipal ultrafilter over an uncountable set ordinal κ is a very strong condition in ZFC (the existence of a measurable cardinal); the strength of the situation we describe here is not as great, because κ is not a set ordinal and we do not have access to levels of the hierarchy of sets above $V_{\kappa+1}$, but it is still considerable. It turns out that the extension of ZFC with proper classes and a nonprincipal ultrafilter over the proper class ordinal is sufficient to describe our set theory as well as to be described by it; there is an exact equivalence in strength. Since the existence of a measurable κ and the corresponding ultrafilter is a stronger condition, this implies that our theory can be interpreted in the presence of a measurable cardinal, but this is almost certainly more strength than is needed.

Some further technical observations for the non-naive reader: the construction of the constructible universe L can be carried out in Z_0; a subset of our set theory can be interpreted in L as defined in Z_0 in the same way we interpret our set theory in Z_0. We say "a subset" because there is no apparent reason why the full Axiom of Small Ordinals would be satisfied in the interpretation via L, though many of its consequences would be. Forcing can be carried out in this theory: we think the best way to do this is to build a structure analogous to Z_0 for well-founded relations in which each item is paired with an element of a fixed Cantorian partial order indicating the conditions under which it is to be an element of the set being constructed; the resulting structure would have a T operation which could be used to interpret our set theory as we interpreted it in Z_0. Neither constructibility nor forcing are showcases for the independence of our approach from that of the usual set theory; this is hardly surprising, as both are closely bound up with the cumulative hierarchy of sets in ZFC.

Cahiers du Centre de logique
Volume 10

Chapter 23

Taking “Function” Rather than “Set” as Primitive: a Stratified λ-Calculus

23.1. Axioms, Abstraction, and Stratified Comprehension

In this section, we introduce an approach to the foundations of mathematics based on different primitives. We will take the notions of *function* and *application* to be primitive, rather than the notions of *set* and *membership*.

For any two objects x and y we introduce the notation $x(y)$ for the application of x to y. Just as in our set theory, where we did not expect all objects to be sets, we regard some of our objects as *functions* and some of our objects as *atoms*.

We had a default extension to assign to a non-set; it is less clear what to do about defining the extension of a non-function. We commit ourselves to the proposition that any extension which is realized by any object is realized by some function:

Axiom of Functionality. Fn *is a specific object of our theory. For all* x, $\mathrm{Fn}(a)(x) = a(x)$, *and for all* a, $\mathrm{Fn}(\mathrm{Fn}(a)) = \mathrm{Fn}(a)$.

Definition. *An object* a *is said to be* a function *if* $\mathrm{Fn}(a) = a$. *An object which is not a function is said to be an* atom.

Notice that the axiom of functionality allows us to assert that $\mathrm{Fn}(a)$ is a function with the same extension as a for all a.

We have an

Axiom of Extensionality. *If f and g are functions, then $f = g$ iff for all x, $f(x) = g(x)$.*

Just as a set is completely determined by its elements, a function is completely determined by its values.

Just as in our set theory, we have a primitive ordered pair construction: if x and y are objects, we have a pair (x, y). We also have a primitive construction which gives us a constant function $K[x]$ for each object x; note that we do not introduce a function K. We state some axioms.

Axiom of Projections. *π_1 and π_2 are distinct functions such that for all x_1, x_2, $\pi_i(x_1, x_2) = x_i$ (where i stands for either 1 or 2).*

Axiom of Products. *For all x, y, z, $(x, y)(z) = (x(z), y(z))$ and $\mathrm{Fn}(x, y) = (\mathrm{Fn}(x), \mathrm{Fn}(y))$.*

Axiom of Identity. Id *is a function and for all x, $\mathrm{Id}(x) = x$.*

Axiom of Constant Functions. *For each x, $K[x]$ is a function and for all y, $K[x](y) = x$.*

Axiom of Abstraction. Abst *is an object of our theory. For all x, y, z, $\mathrm{Abst}(x)(y)$ is a function and $\mathrm{Abst}(x)(y)(z) = x(K[z])(y(z))$.*

Axiom of Equality. Eq *is a function and for all x, y, if $x = y$, $\mathrm{Eq}(x, y) = \pi_1$ and if $x \neq y$, $\mathrm{Eq}(x, y) = \pi_2$.*

We would like to be able to define functions using the function-builder notation $(x \mapsto T)$, where T is an arbitrary expression. Unfortunately, this is not possible in general.

Let us consider the "function" $R = (x \mapsto \mathrm{Eq}(x(x), \pi_2))$. It would follow from the existence of R that $R(R) = \mathrm{Eq}(R(R), \pi_2)$, so $R(R) = \pi_1$ iff $R(R) = \pi_2$ and $R(R) = \pi_2$ iff $R(R) \neq \pi_2$ (by the axiom characterizing Eq). This is absurd, of course. This contradiction, a version of Curry's paradox, is the result of a natural coding of Russell's paradox into a theory with functions as primitive.

Just as in our set theory, it turns out that we can make sense of $(x \mapsto T)$ for a large class of expressions T. The expressions T for which we can do this are called "stratified" expressions, and the notion of stratification we use is quite analogous to that found above, except that we work with expressions rather than sentences.

The usual notation for functions in a theory of the kind we are developing is not $(x \mapsto T)$ but $(\lambda x.T)$; such theories are usually called "λ-calculi". Another notational deviation traditional in this field is the use of (fx) rather than the more usual $f(x)$ to represent function application. However, we will continue to use the notation we have used in the rest of the book, which is probably familiar to more potential readers.

Definition. *Let T be an expression. We assign an integer to each occurrence of a subexpression of T which we call its* relative type. *The definition is inductive.*

(a) *The relative type of the occurrence of T is 0.*

(b) *If the relative type of an occurrence of (x, y) is n, the relative types of the obvious occurrences of x and y are also n.*

(c) *If the relative type of an occurrence of $x(y)$ is n, the relative types of the obvious occurrences of x and y are $n+1$ and n, respectively.*

(d) *If the relative type of an occurrence of $K[x]$ is n, the relative type of the obvious occurrence of x is $n - 1$.*

Relative typing in the stratified λ-calculus appears more rigid than in the set theory presented in the rest of the book; assigning a type to a whole expression forces the assignment of types to all of its subexpressions. We will introduce devices below which allow typing of subexpressions to have more freedom. The use of type 0 as the top type, with the attendant likelihood of using negative types, is merely a technical convenience: we could rephrase all the results in the chapter in such a way as to allow only non-negative types.

Definition. *Let T be an expression. We say that T is* stratified relative to x *iff x does not occur in T with any type other than 0.*

Abstraction Theorem. *Let T be an expression which is stratified relative to x. It follows that $(x \mapsto T)$ exists, i.e., there is a function f such that "for all x, $f(x) = T$" is a theorem.*

PROOF. — The proof is by induction on the structure of T.

We can assume without loss of generality that any subexpression $K[A]$ of T is of the form $K^n[z]$, with z an atomic expression (variable or atomic constant). $K[(U,V)] = (K[U], K[V])$ and $K[U(V)] = \text{Abst}(K[U])(K[V])$ follow immediately from the Axioms of Products or Abstraction, respectively, and Extensionality. These theorems can be used to eliminate any application of the K operator to a pair or application expression.

(a) If T is the variable x, we can let $(x \mapsto T) = \text{Id}$. Note that $(x \mapsto T)$ is a function.

(b) If T is an atomic expression a distinct from x (or any expression in which x does not occur), we can let $(x \mapsto T) = K[a]$. Note that $(x \mapsto T)$ is a function.

(c) If T is of the form (U, V), it should be clear that U and V are stratified with respect to x, so we can let $(x \mapsto T) = ((x \mapsto U), (x \mapsto V))$ (applying the Axiom of Products). Note that $(x \mapsto T)$ is a function.

(d) If T is of the form $K^n[z]$, z an atomic expression, $n > 0$, z must be distinct from x, because its relative type is wrong, so we can let $(x \mapsto T) = K^{n+1}[z]$. Note that $(x \mapsto T)$ is a function.

(e) If T is of the form $U(V)$, it should be clear that V is stratified with respect to x. It should also be clear that all occurrences of x in U must be in the context $K[x]$, since the relative type of x in U is -1, the K construction is the only type-lowering construction, and it is only applied (possibly repeatedly) to atoms in T. The type of any occurrence of $K[x]$ in U must be 0; thus, if we replace $K[x]$ with a new variable u in U, the resulting expression U' will be stratified relative to u. We can then let $(x \mapsto T) = \text{Abst}(u \mapsto U')(x \mapsto V)$. Note that $(x \mapsto T)$ is a function.

The proof of the Abstraction Theorem is complete. □

Definition. *We define $(x \mapsto T)$ as the expression whose construction is outlined in the proof of the Abstraction Theorem.*

Observation. We observe that each occurrence of an atomic subexpression y (other than x) of T with type n corresponds to an occurrence of the same atomic subexpression y with type $n-1$ in $(x \mapsto T)$. x does not occur in $(x \mapsto T)$. It is straightforward to verify this.

The observation allows us to add the following clause to the definition of relative type (and thus of stratification):

Definition. *If $(x \mapsto U)$ occurs in T with relative type n, we assign relative type $n-1$ to the obvious occurrence of U.*

This extension of the definition of stratification is safe because the notion of stratification depends only on the typing of atomic subexpressions.

A further theorem vital to the usefulness of the notation $(x \mapsto T)$ is

Theorem. *If $S = T$ is a theorem, so is $(x \mapsto S) = (x \mapsto T)$, if the functions exist.*

Proof. — Since $(x \mapsto S)(x) = S = T = (x \mapsto T)(x)$ will hold for any x, and since any object $(u \mapsto U)$ is a function, we have the desired equality by extensionality. □

This theorem allows substitutions of equals for equals in function abstraction expressions, for example.

Another refinement:

Definition. *We define $((x, y) \mapsto T)$ as $(u \mapsto T')$, where u is a variable not occurring in T and T' is obtained from T by replacing each occurrence of x with $\pi_1(u)$ and each occurrence of y with $\pi_2(u)$. It should be clear that $((x, y) \mapsto T)(x, y) = T$ is true for all x and y. Similar extensions of notation to more complex finite structures are defined in the obvious way.*

It would be possible to take the function abstraction construction as primitive (in which case the constant function construction would not be needed as a primitive). Function abstraction expressions would then be well-formed iff they were stratified, and the notion of stratification would have to be defined by a mutual recursion with the notion of well-formedness for function expressions. Such a system would properly be called "stratified λ-calculus"; a system like the one presented here in which the "λ-terms" (functional abstraction expressions $(x \mapsto T)$, which would traditionally be written $(\lambda x.T)$) are built from a finite set of basic constructions is called a "(synthetic) combinatory logic". We believe that the present approach is easier to motivate, and leaves one in the end with the same freedom to reason about abstraction expressions. It is also closely analogous to the way that we developed Stratified Comprehension from a finite set of basic constructions in the rest of the book.

In a development based upon the primitive notions of set and membership, we eventually found it necessary to define the notions of function and application; an analogous condition holds here! We will now define the notions of set and membership:

Definition. *A* set *is a function f such that for all x, $f(x) = \pi_1$ or $f(x) = \pi_2$.*

Definition. *For any objects x and y, we understand the sentence $x \in y$ to mean "y is a set and $y(x) = \pi_1$".*

Definition. *For ϕ any sentence in the language of set theory, we understand $\{x \mid \phi\}$ to be the set A such that for all x, $x \in A$ iff ϕ (ϕ cannot mention A).*

The basic idea is that we are using projection operators to represent truth values and representing a set as a function from the universe to truth values (a kind of characteristic function).

We make a crucial observation allowing us to relax our definition of stratification for expressions coding sentences:

Lemma. *The relative type of any occurrence of an expression T such that the value of T must be either π_1 or π_2 can be raised and lowered freely.*

PROOF. — $\pi_i = \pi_i(\pi_1, \pi_2)$, so $T = T(\pi_1, \pi_2)$; this raises the type of T. $\pi_i = \mathrm{Eq}(K[\pi_i], K[\pi_1])$, so $T = \mathrm{Eq}(K[T], K[\pi_1])$; this lowers the type of T. □

Note that an application of the Lemma raises or lowers the types of all subexpressions of the expression T by the same amount.

We now show that we have the same facility for constructing sets that we have in the theory of the rest of this book:

Stratified Comprehension Theorem. *For each stratified sentence ϕ, the set $\{x \mid \phi\}$ exists.*

Proof. — Our argument is inductive. We first describe a coding of stratified sentences of our set theory as expressions of value either π_1 or π_2. We assume that we are given a fixed assignment of types to variables witnessing the stratification of the sentence ϕ (in the sense of our set theory).

(a) We initially code $x\ \pi_i\ y$ as $\mathrm{Eq}(\pi_i(x), y)$ for $i = 1$,2. We code $x = y$ as $\mathrm{Eq}(x, y)$. We then use the Lemma to raise or lower relative types of x and y in these expressions to the types assigned to them in the stratified sentence ϕ being coded. This is possible because x and y have the same relative types in these expressions.

(b) If we have coded ψ and ξ by T and U, we code "ψ and ξ" by

$$\mathrm{Eq}((T, U), (\pi_1, \pi_1))$$

and "not ϕ" by $\mathrm{Eq}(T, \pi_2)$. This construction does not perturb relative types of variables.

(c) We initially code $x \in y$ by the code of

$$\text{“}y(x) = \pi_1 \text{ and } y = (u \mapsto \mathrm{Eq}(y(u), \pi_1))\text{”}.$$

We have already seen how to code equations and conjunctions; the second clause of the conjunction codes "y is a set". We then use the Lemma to raise or lower the types of x and y to the types assigned to them in the stratified sentence being coded. This is possible because x has type one lower than the type of y in the one (code of a) conjunct in which they occur together; the type of y in the other conjunct can be manipulated freely.

(d) If we have coded ψ by T, an expression in which y appears with the same relative type everywhere (not necessarily 0), then we can apply the Lemma (since T will have only projection operators as values) to change the expression T to an expression T' with the same value in which y has type 0 everywhere, and code "for all y, ψ" by

$$\mathrm{Eq}((y \mapsto T'), K[\pi_1]).$$

The existential quantifier can be defined in terms of the universal quantifier. Then use the Lemma to reset relative types of variables in the resulting expression to the types assigned them in the stratification of the sentence ϕ being coded.

It is straightforward to verify that the coding scheme described succeeds in coding true sentences by π_1 and false sentences by π_2. Now consider any stratified sentence ϕ. It will be coded by an expression T in which the variable x will appear with exactly one type, not necessarily 0. This expression will have only projection operators as values, so it is equivalent by the Lemma to an expression T' in which the types of all variables have been uniformly raised or lowered in such a way that the type of x becomes 0. $(x \mapsto T')$ is the desired function $\{x \mid \phi\}$. The proof of the Stratified Comprehension Theorem is complete. □

Because we have now coded the basic concepts of the rest of the book into our theory of functions, we can now see that the development here is an alternative foundation for the entire book.

We state a final axiom for this subsection which is technically useful, does not follow from our axioms, but does not strengthen the theory essentially:

Axiom of Definite Description. *If ϕ is any stratified sentence in which x and y can be assigned the same type such that for each x there is exactly one y such that ϕ, then there is a function f such that for all x, $\phi[f(x)/y]$.*

If we replace "there is exactly one y" with "there is some y", we obtain a form of the Axiom of Choice appropriate to this theory, but we do not officially adopt this axiom.

23.2. Strongly Cantorian Sets as Types

We redefine the notion of "strongly Cantorian" from above:

Definition. *Let A be a set. We say that A is* strongly Cantorian *if there is a function $(K\lceil A)$ such that for all $a \in A$, $(K\lceil A)(a) = K[a]$.*

It should be clear that this definition is essentially equivalent to the definition in the set theory of the previous chapters; the constant function construction, like the singleton construction, raises type by one.

Theorem. *The relative type of an occurrence of an expression whose value must belong to a fixed strongly Cantorian set A can be raised or lowered freely.*

Proof. — Such an expression T may be replaced by the expression $(K\lceil A)(T)(\mathrm{Id})$, in which it appears with type raised by one; this can be iterated as needed. The Axiom of Definite Descriptions can be used to prove that there is a function $(K\lceil A)^{-1}$ such that for each $a \in A$, $(K\lceil A)^{-1}(K[a]) = a$. The expression T may be replaced by $(K\lceil A)^{-1}(K[T])$, in which it appears with type lowered by one, and this may be iterated as needed. □

We pause to develop a version of the natural numbers appropriate to a theory of functions (the *Church numerals*).

Definition. *We define 0 as K[Id] and* Inc *(the successor operation) as $(n \mapsto (f \mapsto (x \mapsto f(n(f)(x)))))$. We then develop the definition of the set $\mathcal{N}$ exactly as we did earlier (it is the intersection of all sets containing 0 and closed under* Inc*).*

The effect is to define each concrete natural number n as $(f \mapsto (x \mapsto f^n(x)))$ (just as in the main part of the book we defined each concrete natural number n as the set of all sets with n elements).

We introduce the familiar (though differently phrased)

Axiom of Counting. *$\mathcal{N}$ is a strongly Cantorian set.*

The introduction of the Axiom of Counting essentially extends the system of the previous section; we introduce it here so that we will have nontrivial strongly cantorian sets to work with.

We now develop an alternative representation of sets, which we will use only for strongly Cantorian sets.

Definition. *A function τ such that $\tau(\tau(x)) = \tau(x)$ is called* a retraction.

Theorem. *Every nonempty set A is the range of some retraction σ.*

PROOF. — Choose an element a of A: σ sends each element of A to itself and each non-element of A to a. It is easy to see that σ must exist (it is possible to write a function abstraction expression representing it), and also that σ is a retraction. □

We adopt a special notation in connection with retractions with strongly Cantorian range:

Definition. *If τ is a retraction with strongly Cantorian range, we define $\tau : x$ as $\tau(x)$.*

We interpret the retraction τ written in this way as a "type label". Observe that an expression $\tau : x$, since its value must belong to the strongly Cantorian set $\mathrm{rng}(\tau)$, can have its type freely raised or lowered as desired for purposes of stratification. This allows us to regard a wider class of expressions as stratified.

Strongly Cantorian sets are closed under certain basic constructions; we exhibit these constructions, expressed in terms of retractions.

Definition. *If σ and τ are retractions with strongly Cantorian range, we define $\sigma \times \tau$ as $((x, y) \mapsto (\sigma : x, \tau : y))$.*

Observation. $\sigma \times \tau$ is a retraction and its range is the Cartesian product of the ranges of σ and τ, and is strongly Cantorian, so the notation $(\sigma \times \tau) : (x, y)$ is justified.

Definition. *If σ and τ are retractions with strongly Cantorian range, we define $[\sigma \to \tau]$ as $(f \mapsto (x \mapsto (\tau : f(\sigma : x))))$.*

Observation. $[\sigma \to \tau]$ is a retraction and its range is the set of all functions from the range of σ to the range of τ, which is strongly Cantorian, so the notation $[\sigma \to \tau] : f$ is justified.

Definition. *We define $((\tau : x) \mapsto T)$ as $(x \mapsto T[(\tau : x)/x])$.*

Observation. It should be clear that an expression of the form $((\tau : x) \mapsto \sigma : T)$ is in the range of $[\sigma \to \tau]$. It should also be clear that the formation of a function $((\tau : x) \mapsto T)$ will not be restricted by stratification; all occurrences of x are in the context $\tau : x$, whose type can be freely raised or lowered as desired. Note that this is only true if τ itself is a constant, or at least does not contain any variable parameter whose relative type needs to be taken into account; $K[x]$, for example, is a retraction with strongly Cantorian range, and so $K[x] : x$ is a sensible piece of notation, but it would be disastrous to allow the type of this expression (whose value is just x) to be freely raised and lowered!

If we provide ourselves (as we easily can) with retractions β and ν onto the two-element set of projection operators (which we view as "truth values") and the set of natural numbers, respectively, then we have provided ourselves with a system of data types which is adequate for most purposes of theoretical computer science. The Axiom of Counting is needed to ensure that the range of ν is strongly Cantorian. The subset of our system in which all expressions are decorated with type labels looks like the "typed λ-calculus" used in theoretical computer science. More refined type systems can be constructed, including recursive and polymorphic type systems, but this seems like an adequate sample of the possibilities.

We close with a warning: before attempting to investigate more advanced type theories which involve abstraction over types, notice that expressions of the form $(\tau \mapsto T)$ where τ is a type label (retraction onto a strongly Cantorian set) are *not* justified by the development so far, because the class of retractions onto strongly Cantorian sets which would be the domain of such a "function" is not a set! One way one can go in this direction is to restrict the type labels used to be elements of *sets* of retractions closed under suitable operations; for example, one might postulate the existence of a set of "types" containing ν and β and closed under the product and power constructions. The existence of such a set of retractions does follow from the Axiom of Counting, but further strength is needed to ensure that all retractions in such a set have strongly Cantorian range. The strong axioms we adjoin to our set theory provide more than enough strength to justify such constructions!

We have designed and implemented an automated reasoning system whose higher-order logic is based on stratified function abstraction, with built-in support for subversion of stratification for expressions decorated with type labels standing for retractions with strongly Cantorian range.

23.3. Programming and Our Metaphor

In an earlier chapter, we developed a metaphor for set theory in an attempt to motivate stratified comprehension. Here, we will briefly develop a metaphor for a theory of functions with the applications to computer science suggested in the last subsection in mind.

We develop an extremely abstract model of programming. We have a set $\mathcal{D}$ of objects which we view as data or addresses (indifferently). An abstract program in our model is a function sending data to data (or, equivalently addresses to addresses), i.e., an element of $[\mathcal{D} \to \mathcal{D}]$.

A state of our machine is a way of storing abstract programs in addresses: a state is a function $\sigma : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]$; for each $d \in \mathcal{D}$, $\sigma(d)$ is the program stored in d. To complete the analogy with our system of this chapter, we suppose that we have an association of pairs of addresses with addresses: with each pair of addresses d, e, we associate another address $\langle d, e\rangle$ from which d and e can be uniquely determined.

An operation of application on addresses can be defined: $d(e)$ means "the result of applying the program stored in d to e", more briefly $\sigma(d)(e)$, where σ is the state of our machine.

The "data type security" motivation suggested above for stratification in set theory manifests itself now as the observation that an operation on the data type "program" should not use details of the implementation of abstract programs (functions in $[\mathcal{D} \to \mathcal{D}]$) as concrete programs (functions in $[\mathcal{D} \to \mathcal{D}]$ associated with specific elements of $\mathcal{D}$).

For example, a program which we might denote as $(d \to d(d))$, which would send each address d to the result $\sigma(d)(d)$ of applying the program stored in d to d itself, should not be regarded as legitimate (although it is not paradoxical). This is clearly intended as an operation on the program represented by d, but the actual address d is not a feature of the abstract program implemented by d and should not be accessed by an operation on programs. The same stricture against self-application in function definitions rules out the paradoxical function of Curry's paradox.

A hierarchy of ways of viewing addresses is seen: an address can be interpreted as a bare object, as a function from objects to objects (an abstract program), but also as a function from abstract programs to abstract programs, and so on through a hierarchy of levels indexed by the natural numbers, in which level $n+1$ is the set of functions from level n to level n. This is precisely analogous to the hierarchy of "roles" seen in our motivation of stratified comprehension in set theory. A reasonable criterion for "type safety" of function definitions is that each object mentioned in a specification should appear at a unique level, and this leads to the criterion of stratification expressed by the Abstraction Theorem.

The subversion of stratification requirements on strongly Cantorian domains has an interesting interpretation in terms of this metaphor. The function $K\lceil A$ can be viewed as allowing us to get access to the restriction of the state σ to A. Knowledge of the details of σ (how abstract programs are assigned to addresses) is exactly what stratification is designed to prevent. The function $K\lceil A$ gives us this information on the domain A; in terms of our metaphor, we know how programs in A are "stored in memory", and this seems a reasonable description on this level of abstraction of what it means for A to be a "data type". Our suggestion is that "strongly Cantorian set" translates to "data type" via our metaphor; this appears reasonable, given the results of the previous subsection.

Finally, it seems striking that two notions of type, both historically related to Russell's original notion of type, the "relative types" of stratification and the "data types" of computer science, turn out to be orthogonal in this interpretation: a data type is a domain on which relative types can be ignored!

Chapter 24

Acknowledgements and Notes

I wish to acknowledge the help of Robert Solovay, who has shown me how strong the system of this book is, and who has allowed me to include his results in this area and one of his proofs. I also wish to acknowledge the patience of Tomek Bartoszynski, Alex Feldman, Steve Grantham, and Marion Scheepers, who have listened to various talks about this material in the logic seminar at Boise State University. Thanks to G. Aldo Antonelli for using an early draft of the book in a graduate seminar at Yale and making useful remarks. Last but by no means least, I appreciate the careful and repeated reading of the text by Paul West.

I especially acknowledge the assistance of my editor, Daniel Dzierzgowski, and of the referees, who made many helpful suggestions.

References are excluded from the body of the text. They may be found here, organized by chapter.

The entire book owes a systematic debt to Halmos's *Naive Set Theory* ([6]). The first draft was written by the author in parallel with Halmos's book, with no thought of publication, as training in carrying out elementary set theoretical constructions in NFU. Of course, no work in this area can ignore Quine's original paper [21], in which "New Foundations" was introduced, or Jensen's paper [17] in which NFU was introduced and shown to be consistent.

We appreciate Rosser's *Logic for Mathematicians*, [24], the best full-length treatment of foundations of mathematics based on NF (the only other one, Quine's [22], while interesting, contains some serious and distracting errors), and Thomas Forster's *Set Theory with a Universal Set*, [3], the only current book about this kind of set theory (with an exhaustive bibliography of the subject).

We acknowledge the support of the U. S. Army Research Office (grant no. DAAH04-94-0247); this grant supported the development of the Mark2 theorem prover alluded to in chapter 23, and chapter 23 is included to provide guidance to users of the prover on the relationship between the higher order logic of Mark2 and the system of the main part of the book.

Chapter 2

The introduction of a distinction between sets and atoms as a primitive notion in the context of NFU is due to Quine himself in his note accompanying Jensen's original paper [17] on NFU (of course this distinction has been made in other set theories!). Jensen himself simply restricted extensionality to nonempty sets. This approach leaves us with no way to distinguish the empty set from the atoms.

Chapters 3–7

In these chapters, the aim is to develop a finite axiomatization of NFU + Infinity and prove that the full Stratified Comprehension Theorem follows from it. The first finite axiomatization of NF, which can be adapted to NFU, was due to Hailperin in [5]; it is quite difficult to understand and use, because Hailperin uses the Kuratowski ordered pair, whose projections have type two lower than the type of the pair. In NF, one can *define* a type-level pair (Quine did this). This cannot be done in NFU, both because of the lack of structure on atoms and because the Axiom of Infinity is not a theorem in NFU as it is in NF (the existence of a type-level ordered pair implies that the universe is infinite).

We do know how to define a pair *one* type higher than its projections in

NFU without infinity: define $\langle x, y\rangle$ as

$$\{\{x', 0, 1\}, \{x', 2, 3\}, \{y', 4, 5\}, \{y', 6, 7\} \mid x' \in x, y' \in y\},$$

where the objects denoted by 0–7 are any eight distinct objects (iterated singletons of the empty set would serve). So far as we know, this definition is ours, but we would not be surprised if such pairs had been defined earlier.

The approach of taking the type-level ordered pair as a primitive is original with us, so far as we know. We presented the finite axiomatization described here in [13]. The use of relation algebra and an ordered pair to give the expressive power of first order logic is inspired by the book [27] of Tarski and Givant.

A finite axiomatization of NFU + Infinity of an entirely different nature, also due to us, is described in the last chapter of the book.

For the work of Bertrand Russell (the theory of descriptions and his famous paradox) we refer the reader to [25].

The proof of the Stratified Comprehension Theorem given here follows ideas of Grishin found in [4].

Chapter 8

We are entirely to blame for the speculations found in Chapter 8. It has come to our attention that David Lewis's approach to set theory in his book [19] is very similar in important respects.

Chapters 9–11

The content of these chapters is motivated by Halmos's development in [6]. This is somewhat masked by the fact that the constructions here involve stratification considerations that Halmos did not have to deal with. The natural introduction of the Axiom of Choice to deal with the type differential between elements of partitions and representatives of the elements seems serendipitous. I use f^{-1} in its usual sense, not in the sense in which Halmos uses it.

We have been criticized for our eccentric use of parentheses, braces and brackets, but those who live in glass houses should not throw stones. The notation used by current workers in set theory with stratified comprehension is peculiar (and inconsistent from one worker to another). Traditional notation derived from Russell's *Principia Mathematica* ([25]) or from Rosser's *Logic for Mathematicians* can be quite confusing to a newcomer to this area. Our aim has been to adopt a notation somewhat more similar to that usually found in set theory; of course, the actual effect may be only to multiply confusion!

Whether our particular notational conventions are adopted or not, we do support the development of a common notation to be used by all who work with this kind of set theory!

Chapter 12

The approach to the natural numbers found here is ultimately due to Frege, and may be found (adorned with types) in Russell's [25]. The special feature that Infinity is deduced from the type-level nature of the ordered pair is original with us (if not especially laudable).

The Axiom of Counting was first introduced by Rosser in [24], in the form of the theorem $|\{1,\ldots,n\}| = n$. It has been shown in by Orey in [20] to be independent of NF (if NF is consistent). It is known to strengthen NFU essentially.

Chapter 13

The development of the reals here is very similar to that of Quine in [22].

Chapter 14

The specific equivalents of the Axiom of Choice proven here are motivated by Halmos's development. Of course, it is only because we are in NFU that we can well-order the universe (the alternatives being NF, with no well-ordering, or ZFC, with no universe)!

Chapters 15–16

The development of the ordinal and cardinal numbers in this kind of set theory is entirely different from that in the usual set theory. The basic ideas are ultimately due to Frege and are found (typed) in [25]. The description of how NF avoids the Burali-Forti paradox is found in the original paper [21] of Quine.

The Axiom of Small Ordinals is an extravagance due to the author. The weaker but still very strong axiom that all Cantorian sets are strongly Cantorian is due to C. Ward Henson in [7]. The precise determination of the strength of Henson's axiom (which also applies as a lower bound to the strength of NF with Henson's axiom) was made very recently by Robert Solovay (unpublished). The precise consistency strength of the full system of this book has not been determined; its consistency follows from the existence of a measurable cardinal.

The theorems about cardinal numbers in chapter 16 are classical; the development follows Halmos.

Chapter 17

The definition of exponentiation is extended from the original proposal of Specker in [26] following a suggestion of Marcel Crabbé. The definitions of infinite sums and products of cardinals are the trickiest exercises in stratification in the whole book; we hope that the reader enjoys them (and their application in the proof of the classical theorem of König).

The original discussion of the resolution of Cantor's Paradox in this kind of set theory is found in [21]. Specker's disproof of the Axiom of Choice in *NF* (originally given in [26]) converts itself in the context of NFU with Choice to a proof of the existence of atoms. It loses none of its surprising character. The proof of the theorem is simplified by using the Axiom of Counting; we do not know whether this form of the proof has been published by earlier workers.

The comments on Cantorian and strongly Cantorian sets are standard in this field.

Chapter 18

The topological section is closely parallel to the discussion of this subject in Hrbacek and Jech's excellent [16] (except for any defects, of course).

The very abstract definition of *filter* and related notions is cribbed from Kunen's development of forcing in [18]. The implementation of the reals as ultrafilters over the closed intervals in the rationals is not standard, but surely not novel.

The treatment of ultrapowers is standard, except for the remarks about stratification restrictions.

For nonstandard analysis one can see Abraham Robinson's [23].

Chapter 19

All results in this section are due to the author. Of course, the idea of coding set theory into the ordinals is not original!

Chapter 20

All results in this section can be found in the work of Roland Hinnion, except for the discussion of the Axiom of Endomorphism, originally found in our [12], and any discussion of the consequences of our Axiom of Small Ordinals. See Hinnion's Ph. D. thesis, [8], and, for the use of relations $[\sim^{+}]$ in the construction of extensional relations, [9]. What we call a "nice relation" in the proof of the Collapsing Lemma is called a "bisimulation" elsewhere.

Chapter 21

This section is shamelessly parasitic on John Horton Conway's wonderful *On Numbers and Games* ([1]). It was motivated by the desire to tackle a mathematical structure whose original definition was clearly horribly unstratified!

Chapter 22

For the discussion of the Hartogs operation, I am indebted to Forster ([3]). All results are classical, except the major recent result of Solovay (unpublished) that there are inaccessible cardinals, which he has graciously allowed us to present here. Solovay's original argument is given entirely in terms of ZFC with an external automorphism, rather than being framed in NFU terms as we have essayed to do here. This result is in a sense superseded by the stronger result of Solovay that there are n-Mahlo cardinals for each concrete n. The Axiom of Large Ordinals and the discussion of the coding of proper classes of small ordinals into our theory and the codings between our theory and second order ZFC with a measure on classes are due to us (and any errors are our responsibility!)

Chapter 23

Any work in combinatory logic should cite H. B. Curry's masterful [2].

This section is based on our Ph. D. thesis [10] and the subsequent paper [11] of the same title. It is further discussed in our more recent [14]. The interpretation of strongly Cantorian sets as data types is a perhaps silly idea of ours; it appeared in our [13]. The adaptation of our "metaphor" to functions is also discussed there.

For more about the Mark2 automated reasoning system, see [15].

Cahiers du Centre de logique
Volume 10

Appendix

Axioms and Selected Theorems by Chapter

Chapter 2

Axiom of Extensionality. *If A and B are sets, and for each x, x is an element of A if and only if x is an element of B, then $A = B$.*

Axiom of Atoms. *If x is an atom, then for all y, $y \notin x$. (read "y is not an element of x")*

Chapter 3

Axiom of the Universal Set. $\{x \mid x = x\}$*, also called V, exists.*

Axiom of Complements. *For each set A, the set $A^c = \{x \mid x \notin A\}$, called the* complement *of A, exists.*

Axiom of (Boolean) Unions. *if A and B are sets, the set $A \cup B = \{x \mid x \in A$ or $x \in B$ or both$\}$, called the (Boolean)* union *of A and B, exists.*

Axiom of Set Union. *If A is a set all of whose elements are sets, the set $\bigcup[A] = \{x \mid$ for some B, $x \in B$ and $B \in A\}$, called the (set) union of A, exists.*

Chapter 4

Axiom of Singletons. *For every object x, the set $\{x\} = \{y \mid y = x\}$ exists, and is called the* singleton *of x.*

Axiom of Ordered Pairs. *For each a, b, the* ordered pair of a and b, (a, b), *exists; $(a, b) = (c, d)$ exactly if $a = c$ and $b = d$.*

Axiom of Cartesian Products. *For any sets A, B, the set $A \times B = \{x \mid$ for some a and b, $a \in A, b \in B$, and $x = (a, b)\}$, briefly written $\{(a, b) \mid a \in A$ and $b \in B\}$, called the* Cartesian product *of A and B, exists.*

Chapter 5

Axiom of Converses. *For each relation R, the set $R^{-1} = \{(x, y) \mid yRx\}$ exists; observe that $x\, R^{-1}\, y$ exactly if yRx.*

Axiom of Relative Products. *If R, S are relations, the set*

$$(R|S) = \{(x, y) \mid \text{for some } z, x\ R\ z \text{ and } z\ S\ y\},$$

called the relative product *of R and S, exists.*

Axiom of Domains. *If R is a relation, the set*

$$\mathrm{dom}(R) = \{x \mid \text{for some } y,\ x\ R\ y\},$$

called the domain *of R, exists.*

Axiom of Singleton Images. *For any relation R, the set*

$$\mathrm{SI}\{R\} = \{(\{x\}, \{y\}) \mid x\ R\ y\},$$

called the singleton image *of R, exists.*

Axiom of the Diagonal. *The set* $[=] = \{(x, x) \mid x \in V\}$ *exists (this is the equality relation).*

Axiom of Projections. *The sets* $\pi_1 = \{((x, y), x) \mid x, y \in V\}$ *and* $\pi_2 = \{((x, y), y) \mid x, y \in V\}$ *exist.*

Chapter 6

Representation Theorem. *For any sentence* ϕ *in which all predicates mentioned in atomic sentences are realized, the set* $\{x \mid \phi\}$ *exists.*

Chapter 7

Axiom of Inclusion. *The set* $[\subseteq] = \{(x, y) \mid x \subseteq y\}$ *exists.*

Stratified Comprehension Theorem. *For each stratified sentence* ϕ*, the set* $\{x \mid \phi\}$ *exists.*

Chapter 9

Axiom of Choice. *For each set* P *of pairwise disjoint non-empty sets, there is a set* C*, called a* choice set *from* P*, which contains exactly one element of each element of* P*.*

Chapter 12

Theorem of Infinity. $\{\ \}$ *is not a natural number.*

Axiom of Counting. *For all natural numbers* n*,* $T\{n\} = n$*.*

Restricted Subversion Theorem. *If a variable* x *in a sentence* ϕ *is restricted to* $\mathcal{N}$*, then its type can be freely raised and lowered; i.e., such a variable can safely be ignored in making type assignments for stratification.*

Chapter 14 (equivalents of the axiom of choice)

Theorem. *The Cartesian product of any family of non-empty sets is non-empty.*

Zorn's Lemma. *If $\leqslant$ is a partial order such that each chain in $\leqslant$ has an upper bound relative to $\leqslant$, then the domain of $\leqslant$ has a maximal element relative to $\leqslant$.*

Theorem. *There is a well-ordering of V.*

Chapter 15

Transfinite Recursion Theorem. *Let W be the domain of a well-ordering $\leqslant$. Let F be a function from the collection of functions with domains segments of $\leqslant$ to the collection of singletons. Then there is a unique function f such that $\{f(a)\} = F(f\lceil \operatorname{seg}\{a\})$ for each $a \in W$.*

Theorem. *$T^2\{\Omega\} < \Omega$; i.e, $\leqslant$ on $\operatorname{seg}_{\leqslant}\{\Omega\}$ is not similar to $\leqslant$ on all ordinals.*

Axiom of Small Ordinals. *For any sentence ϕ in the language of set theory, there is a set A such that for all x, x is a small ordinal such that ϕ iff ($x \in A$ and x is a small ordinal).*

Chapter 16

Theorem (Dedekind). *A set is infinite exactly if it is equivalent to one of its proper subsets.*

Schröder–Bernstein Theorem. *If X and Y are sets, and f is a one-to-one map from X into Y and g is a one-to-one map from Y into X, then X and Y are equivalent.*

Chapter 17

Theorem (Cantor). $T\{\kappa\} < \exp(T\{\kappa\})$.

König's Theorem. *Let F and G be functions with the same nonempty domain I all of whose values are cardinal numbers. Suppose further that $F(x) < G(x)$ for all $x \in I$, where the order is the natural order on cardinals. It follows that $\sum[F] < \prod[G]$.*

Theorem (Specker). $|\mathcal{P}\{V\}| < |V|$, *and thus there are atoms.*

Subversion Theorem. *If a variable x in a sentence ϕ is restricted to a strongly Cantorian set A, then its type can be freely raised and lowered; i.e., such a variable can safely be ignored in making type assignments for stratification.*

Chapter 20

Axiom of Endomorphism. *There is a one-to-one map* Endo *from $\mathcal{P}_1\{V\}$ (the set of singletons) into $\mathcal{P}\{V\}$ (the set of sets) such that for any set B,* $\mathrm{Endo}(\{B\}) = \{\mathrm{Endo}(\{A\}) \mid A \in B\}$.

The Axiom of Endomorphism is not an axiom of our theory, but it is shown to be consistent with our theory in chapter 20.

Chapter 23

Theorem (Solovay). *There is an inaccessible cardinal.*

Axiom of Large Ordinals. *For each non-Cantorian ordinal α, there is a natural number n such that $T^n\{\Omega\} < \alpha$.*

Cahiers du Centre de logique
Volume 10

Bibliography

[1] J. H. Conway, *On Numbers and Games*, Academic Press, London, 1976.

[2] H. B. Curry and R. Feys, *Combinatory Logic*, Vol. I, North Holland, Amsterdam, 1958.

[3] T. E. Forster, *Set Theory with a Universal Set*, second edition, Oxford logic guides, no. 31, Cambridge University Press, 1995.

[4] V. N. Grishin, "Consistency of a fragment of Quine's *NF* system," *Sov. Math. Dokl.*, 10 (1969), pp. 1387–90.

[5] T. Hailperin, "A set of axioms for logic". *Journal of Symbolic Logic*, 9 (1944), pp. 1–19.

[6] Paul Halmos, *Naive Set Theory*, van Nostrand, Princeton, 1960.

[7] C. W. Henson, "Type-raising operations in *NF*", *Journal of Symbolic Logic* 38 (1973), pp. 59-68.

[8] Roland Hinnion, "Sur la théorie des ensembles de Quine", Ph.D. thesis, ULB Brussels, 1975.

[9] Roland Hinnion, "Extensional quotients of structures and applications to the study of the axiom of extensionality", *Bulletin de la Societé Mathématique de Belgique série B*, vol. 31 (1981), pp. 3-11.

[10] M. Randall Holmes, "Systems of Combinatory Logic Related to Quine's 'New Foundations',", Ph. D. thesis, SUNY Binghamton (now Binghamton University), 1990.

[11] M. Randall Holmes, "Systems of Combinatory Logic Related to Quine's 'New Foundations'," *Annals of Pure and Applied Logic*, 53 (1991), pp. 103–33.

[12] M. Randall Holmes, "The Axiom of Anti-Foundation in Jensen's 'New Foundations with Ur-Elements'," *Bulletin de la Société Mathématique de Belgique, série B*, 43 (1991), no. 2, pp. 167–79.

[13] M. Randall Holmes, "The set-theoretical program of Quine succeeded, but nobody noticed". *Modern Logic*, vol. 4, no. 1 (1994), pp. 1-47.

[14] M. Randall Holmes, "Untyped λ-calculus with relative typing", in *Typed Lambda-Calculi and Applications* (proceedings of TLCA '95), Springer, 1995, pp. 235-48.

[15] M. Randall Holmes, "Disguising recursively chained rewrite rules as equational theorems", in Hsiang, ed., *Rewriting Techniques and Applications*, the proceedings of RTA '95, Springer, 1995.

[16] K. Hrbacek, and T. Jech, *Introduction to Set Theory*, second edition, Marcel Dekker, New York, 1984.

[17] Ronald Bjorn Jensen, "On the consistency of a slight (?) modification of Quine's 'New Foundations'," *Synthese*, 19 (1969), pp. 250–63.

[18] K. Kunen, *Set Theory: an introduction to independence proofs*, North-Holland, Amsterdam, 1980.

[19] David Lewis, *Parts of Classes*, Basil Blackwell Ltd., Oxford, 1991.

[20] S. Orey, "New Foundations and the axiom of counting," *Duke Mathematical Journal*, 31 (1964), pp. 655–60.

[21] W. V. O. Quine, "New Foundations for Mathematical Logic," *American Mathematical Monthly*, 44 (1937), pp. 70–80.

[22] W. V. O. Quine, *Mathematical Logic*, second edition, Harvard, 1951.

[23] Abraham Robinson, *Nonstandard Analysis*, North-Holland, Amsterdam, 1966.

[24] J. Barkley Rosser, *Logic for Mathematicians*, second edition, Chelsea, New York, 1973.

[25] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica (to *56)*, Cambridge University Press, 1967.

[26] E. P. Specker, "The axiom of choice in Quine's 'New Foundations for Mathematical Logic'," *Proceedings of the National Academy of Sciences of the U. S. A.*, 39 (1953), pp. 972–5.

[27] Alfred Tarski and Steven Givant, *A Formalization of Set Theory Without Variables*, American Mathematical Society, Providence, 1988.

Index