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CAPITAL ACCUMULATION AND EFFICIENT ALLOCATION OF RESOURCES[1]

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1. *Introduction.* Among the many questions concerning the accumulation of capital the following has been said to be the most important.[2] According to which rules should choices between direct and indirect processes of production be determined, that is, when can we say that it is efficient to save today in order to increase future consumption? The present paper is devoted to this problem, which is clearly relevant for both the theory of capital and for welfare economics. The results given below are not essentially new. The author thinks, however, that his approach is likely to show in a more vivid light a few facts which, although obscurely felt, are not yet generally accepted in economic science.

The reader acquainted with welfare economics and the theory of efficient allocation of resources knows how some appropriate price system is associated with an efficient state. Loosely speaking, such a state would be an equilibrium position for a competitive economy using the given set of prices. The model introduced to prove this result does not allow explicitly for investment and capital accumulation. Thus one may wonder whether it can be extended to the case of capitalistic production. Admittedly, this is very likely. The introduction of time does not seem to imply any new principle. Choices between commodities available at different times raise essentially the same problem as choices between different commodities available at the same time. How can consumers' needs best be satisfied when the production of goods involves strong relations of interdependence?

[1] Based on Cowles Commission Discussion Paper, Economics, No. 2026 (hectographed), and a paper presented at the Minneapolis meeting of the Econometric Society in September, 1951. Acknowledgment is due staff members and guests of the Cowles Commission and those attending econometrics seminars in Paris. Their interest in the subject greatly helped me to bring the study to its present formulation. I am particularly indebted to M. Allais, T. C. Koopmans, and G. Debreu. Anyone acquainted with their work will discern their influence in this paper. But the reader might not know how much I owe to their personal encouragement and friendly criticism. I am also indebted to Mrs. Jane Novick who read my manuscript carefully and made many stylistic improvements.

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[2] Schneider [31].

However, one thing may not be clear: in a competitive economy there is a rate of interest that is used to discount future values both on the loan market and in business accounting. Is this rate a part of the price system associated with an efficient economic process? In particular, should prices of the same commodity available at different times stand in some definite ratio depending only on the time lag and not on the specific commodity considered?

In order to deal with this and related questions this paper is divided into four parts. In the first, the process of capitalistic production is analyzed. A general model is defined that may be given two equivalent presentations. An "extensive form" generalizes current capital-theory models, while a "reduced form" makes it possible to apply the usual welfare reasoning.

The second part is purely mathematical, the main result of the paper being proved there. It provides a somewhat straightforward generalization of what was already known for the timeless case, the only difficulty arising when the future is assumed not bounded by some given horizon. The economic meaning and implications of the main theorem are examined in Part III. As most of the previous work on the theory of capital was based on stationary economies, it is worth studying them carefully. This is attempted in Part IV.

Because this study is mainly concerned with formal results, heuristic comments are reduced as much as the subject permits. It is supposed that the reader is well acquainted with welfare economics.

2. *Notation*. The mathematical tools used here are primarily vectors and sets in finite-dimensional Euclidean spaces. A vector in m-dimensional Euclidean space is denoted by a Latin letter (x_t, for instance), with an index specifying the time considered. The components of x_t are denoted by x_{it}, the distinction between vectors and their components being shown by the placement of the index t.

The symbol $\{x_t\}$ represents a sequence of vectors $x_1, x_2, \cdots, x_t, \cdots$, where t takes all positive integral values. This sequence is also written more simply as $\mathbf{x}$, where the index is removed and the symbol is printed in bold-face type.

The inequality $x_t \leqq y_t$ (as well as $\mathbf{x} \leqq \mathbf{y}$) applied to vectors x_t and y_t (or to sequences $\mathbf{x}$ and $\mathbf{y}$) means that no component x_{it} of x_t (or x_{it} of $\mathbf{x}$) is greater than the corresponding component of y_t (or $\mathbf{y}$). The inequality $x_t \leqslant y_t$ (as well as $\mathbf{x} \leqslant \mathbf{y}$) means $x_t \leqq y_t$ and $x_t \neq y_t$ (or $\mathbf{x} \leqq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$).

A vector x_t (as well as a sequence $\mathbf{x}$) is said to be nonnegative if $x_t \geqq 0$ (or if $\mathbf{x} \geqq 0$).

Sets are denoted by bold-faced capitals. The addition of sets is defined as follows:

$\mathbf{V} = \mathbf{U}_1 + \mathbf{U}_2$ means: v is an element of $\mathbf{V}$ if and only if it can be written as $v = u_1 + u_2$, where u_1 and u_2 are elements of $\mathbf{U}_1$ and $\mathbf{U}_2$ respectively; that is,

$$u_1 \in \mathbf{U}_1, \qquad u_2 \in \mathbf{U}_2 .$$

u_0 is said to be a minimal element of $\mathbf{U}$ if there is no $u \in \mathbf{U}$ with $u \leqslant u_0$.

I. GENERAL MODEL OF CAPITALISTIC PRODUCTION

3. *Time, commodities, and capital goods.* Although time is usually considered as some continuous variable taking any value from minus infinity to plus infinity, it is given here as a succession of periods beginning at the present and going to infinity in the future. Indeed, since the past cannot be changed by any present economic decision, we may disregard it; moreover, there is little harm in assuming a decomposition in periods since their length may be made as short as one wishes.

Formally time appears as an index t that can take any positive integral value. $t = 1$ refers to the present moment, which is the beginning of the coming period, called period 1; $t + 1$ refers to the end of period t, or to the beginning of period $t + 1$.

The description of all economic activity proceeds in terms of commodities. Commodities, therefore, must be understood in a very general sense, and so as to cover in particular all services. The total number of commodities is supposed to be finite and equal to m.[3]

Formally, a set of given quantities of commodities is represented by a vector x_t in the m-dimensional Euclidean space. The component x_{it} of x_t defines which quantity of commodity i is included in x_t.

The concept of capital does not appear explicitly in our treatment and it is not needed. But for the interpretation of the following parts it may be better to define at least capital goods. Capital goods at time t include everything that has been made in preceding periods and is transferred to period t for further use in production. This definition is the old "produced means of production."[4] It stems from the essential character of capital. Indeed, it is made in order to make possible the use in future periods of goods or services that do not exist as natural resources or are not available in sufficient quantity.

[3] This assumption is not strictly necessary. All that follows remains true as long as there is only a finite number of commodities inside each period.

[4] This might be thought of as too inclusive. Indeed, there is little in our modern world that is not the result of previous economic activity. But the origin of existing wealth does not concern us here. The distinction between natural resources and produced means of production is not important as far as past activity is concerned. The only condition we need to keep in mind is the following: the available natural resources during all future periods must be independent of any present or future economic decision.

4. *"Chronics"—extensive form.* We shall mean by a "chronic"[5] a quantitative description of the economic activity occurring during all future periods. It is one of all possible courses of events. A chronic is completely determined when the quantities produced, traded, and consumed are known; i.e., it does not require the definition of any standard of value. Two different chronics, C^1 and C^2, are distinguished by their upper indexes; any vector written with an upper index 1, x_t^1 for instance, represents the value taken by the corresponding vector, x_t, in the chronic C^1.

More precisely, a chronic C provides the following picture. At the present time certain commodities are available and are represented by a vector $\bar{b}_1$. Parts of them are devoted to consumption during period 1, the rest being kept for further consumption or used in production. Let us call x_1^+ and c_1 these two parts:

$$\bar{b}_1 = x_1^+ + c_1 .$$

For production during the first period c_1 is used, together with natural resources z_1 and services x_1^- obtained from consumers (labor). If a_1 represents the total set of productive factors, then

$$a_1 = x_1^- + z_1 + c_1 ,$$

which is reminiscent of the familiar trilogy: labor, land, and capital.[6]

Productive activity transforms a_1 into some other vector, b_2, available at time 2.

The description of the second period will be similar to that of the first, with vectors b_2, x_2^+, c_2, x_2^-, z_2, a_2, b_3, and so on, for all periods. This defines the "extensive form" of chronics C.

The following equations hold:

$$b_t = x_t^+ + c_t \qquad \text{(for all } t\text{), and} \tag{1}$$

$$a_t = x_t^- + z_t + c_t \qquad \text{(for all } t\text{).} \tag{2}$$

If we define

$$x_t = x_t^+ - x_t^- , \tag{3}$$

[5] This neologism was introduced by G. Th. Guilbaud in his study on time series [11].

[6] The question of whether there are two or three primary factors of production has been much debated. However, the answer seems to be fairly clear. Considering any one period there are indeed three factors. But if economic development as a whole, past, present, and future, is considered, capital cannot be considered a primary factor.

then, we also have

(4) $$a_t = b_t - x_t + z_t .$$

C may be represented as in Figure 1.

Such a chronic is possible if and only if the transformation from a_t to b_{t+1} is technically possible and if the resources used, z_t, never exceed the resources available, given by vector $\bar{z}_t$. The second condition is formally expressed as

(5) $$z_t \leqq \bar{z}_t .$$

The condition that the transformation from a_t to b_{t+1} be technically possible may be translated into formal language by saying that the pair (a_t, b_{t+1}) must be in some set $\mathbf{T}_t$, given a priori from the state of technological knowledge at time t, or

(6) $$(a_t, b_{t+1}) \in \mathbf{T}_t .$$

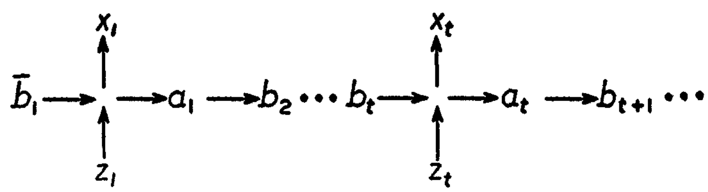

FIGURE 1

From this definition, $\mathbf{T}_t$ is clearly a set in the $2m$-dimensional Euclidean space.

5. *Assumptions concerning the sets of technological possibilities.* The theoretical results of the following sections make extensive use of some assumptions concerning the sets $\mathbf{T}_t$ of technological possibilities. The first assumption can hardly be objected to if one remembers that the limitation of resources is independently represented in the model.

ASSUMPTION 1. (additivity): *If from a_t^1 it is possible to obtain b_{t+1}^1 in period t, and from a_t^2 to obtain b_{t+1}^2 in the same period, then from $a_t^1 + a_t^2$ it is possible to obtain $b_{t+1}^1 + b_{t+1}^2$.*

Or, formally, if $(a_t^1, b_{t+1}^1) \in \mathbf{T}_t$ and $(a_t^2, b_{t+1}^2) \in \mathbf{T}_t$, then

$$(a_t^1 + a_t^2, b_{t+1}^1 + b_{t+1}^2) \in \mathbf{T}_t .$$

The second assumption is not so immediate and could be challenged by many readers. But it is taken as a crude first approximation to reality. Moreover, it is necessary in the proofs of the following sections. So it is justified in some way by its usefulness.

ASSUMPTION 2 (divisibility): *If from a_t it is possible to obtain b_{t+1}, then from αa_t it is possible to obtain αb_{t+1}, where α is any positive number less than* 1.

Or, formally, if $(a_t, b_{t+1}) \epsilon \mathbf{T}_t$ and $0 < \alpha < 1$, then

$$(\alpha a_t, \alpha b_{t+1}) \epsilon \mathbf{T}_t .$$

When Assumptions 1 and 2 are made, $\mathbf{T}_t$, considered as a set in the $2m$-dimensional Euclidean space, is a convex cone with vertex at the origin.

In most of the demonstrations given below, only convexity of $\mathbf{T}_t$ plays an essential role. For the sake of clarity, it is better to assume convexity alone, although in practice such an assumption is probably as restrictive as Assumptions 1 and 2 together.

ASSUMPTION 3 (convexity): *If from a_t^1 it is possible to obtain b_{t+1}^1 and from a_t^2 to obtain b_{t+1}^2, then from any combination $\alpha a_t^1 + \beta a_t^2$ it is possible to obtain the corresponding $\alpha b_{t+1}^1 + \beta b_{t+1}^2$, where α is any positive number less than* 1 *and $\beta = 1 - \alpha$.*

Or, formally, if $(a_t^1, b_{t+1}^1) \epsilon \mathbf{T}_t$ and $(a_t^2, b_{t+1}^2) \epsilon \mathbf{T}_t$, with $0 < \alpha < 1$ and $\alpha + \beta = 1$, then

$$(\alpha a_t^1 + \beta a_t^2, \quad \alpha b_{t+1}^1 + \beta b_{t+1}^2) \epsilon \mathbf{T}_t .$$

The next and last assumption is trivial; it amounts to saying that production is not restricted if more of each good is available.

ASSUMPTION 4: *If from a_t^1 it is possible to obtain b_{t+1}^1, then it is also possible to obtain it from any vector a_t such that $a_t \geqq a_t^1$.*

Or, formally, if $(a_t^1, b_{t+1}^1) \epsilon \mathbf{T}_t$ and $a_t \geqq a_t^1$, then

$$(a_t, b_{t+1}^1) \epsilon \mathbf{T}_t .$$

6. *Decentralization of production.* In an actual economy production is not planned by a central bureau but is accomplished by many different firms, each having its own technology. The activity of the kth production unit during period t consists in a transformation of the vector a_{tk} into the vector $b_{t+1,k}$.[7] This transformation can be performed if and only

[7] The vectors a_{tk} and b_{tk} may be decomposed as follows:

$$a_{tk} = c_{tk} + q_{tk}, \qquad b_{tk} + g_{tk} = s_{tk} + c_{tk},$$

with c_{tk} representing capital equipment of firm k at time t; q_{tk}, current purchases of firm k at time t; g_{tk}, purchases of equipment of firm k at time t; and s_{tk}, sales of firm k at time t.

The following relations hold:

$$c_t = \sum_{k=1}^{n} c_{tk}, \qquad z_t + x_t^- = \sum_{k=1}^{n} q_{tk}, \qquad x_t^+ + \sum_{k=1}^{n} g_{tk} = \sum_{k=1}^{n} s_{tk}.$$

if $(a_{tk}, b_{t+1,k})$ is an element of some set of technological possibilities, $\mathbf{T}_{tk}$, or if

$$(a_{tk}, b_{t+1,k}) \in \mathbf{T}_{tk}. \tag{7}$$

For the economy as a whole the simultaneous operation of all production units, n in number,[8] results in a transformation of a_t into b_{t+1}, with

$$a_t = \sum_{k=1}^{n} a_{tk}, \qquad b_t = \sum_{k=1}^{n} b_{tk}. \tag{8}$$

Since (a_t, b_{t+1}) is in $\mathbf{T}_t$, it is clear that in all cases

$$\sum_{k=1}^{n} \mathbf{T}_{tk} \subset \mathbf{T}_t,$$

which only means that if some transformation is possible within the framework of given production units it is also possible a priori for society as a whole. However, the decomposition into production units could be inefficient, in the sense that it would make impossible some transformations that we know to be possible a priori. In the following pages it is supposed that some decentralization of production has been found that is efficient, or, in other words, that

$$\sum_{k=1}^{n} \mathbf{T}_{tk} = \mathbf{T}_t. \tag{9}$$

The technological possibilities for the kth firm are given by a sequence of sets, $\{\mathbf{T}_{tk}\}$. The assumptions on each $\mathbf{T}_{tk}$ are the same as those made on $\mathbf{T}_t$.

The decomposition of $\mathbf{T}_t$ may also be used to overcome the following difficulty. The inequality $z_t \leqq \bar{z}_t$ would introduce in the following Part II some complications that can be avoided by supposing the equality sign to hold, i.e., the utilized resources to be always equal to the available resources. This can easily be done by assuming the existence of some $(n + 1)$th activity which uses $\bar{z}_t - z_t$ but does not produce anything.

Formally, there is an activity characterized by the vectors

$$\begin{cases} a_{t,n+1} = \bar{z}_t - z_t, \text{ and} \\ b_{t,n+1} = 0. \end{cases} \tag{10}$$

[8] The reader might object that the decomposition into production units need not remain unchanged as time goes on. This is quite true. We do not want, however, to make the model too involved. From the treatment given below for consumption units the reader will see that our results hold true with little change as long as there is only a finite number of firms during each period.

The set associated with this activity is defined by

(11) $$(a_{t,n+1}, b_{t+1,n+1}) \in \mathbf{T}_{t,n+1} \text{ if } a_{t,n+1} \geqq 0,\ b_{t,n+1} = 0.$$

From Assumption 4, the following is obvious:

(12) $$\mathbf{T}_t + \mathbf{T}_{t,n+1} = \mathbf{T}_t .$$

Throughout the following pages we shall have

(13) $$z_t = \bar{z}_t \qquad \text{(for all } t).$$

The fictitious activity will be removed from the picture only when the final result is reached.

7. *Chronics—reduced form.* Let us now define the "input vector" y_t for time t as

(14) $$y_t = a_t - b_t .$$

From equalities (4) and (13), it follows that

(15) $$x_t + y_t = \bar{z}_t .$$

The "reduced form" of the chronic C is defined when the two sequences $\mathbf{x}$ and $\mathbf{y}$ are given, with the following necessary condition:

(16) $$\mathbf{x} + \mathbf{y} = \bar{\mathbf{z}}.$$

From the limitation on technological knowledge, $\mathbf{y}$ is a possible sequence of input vectors if and only if

(17) $$\mathbf{y} \in \mathbf{Y},$$

where $\mathbf{Y}$ may be defined from $\{\mathbf{T}_t\}$ in the following way:

$\mathbf{y} \in \mathbf{Y}$ *if and only if there are two sequences* $\mathbf{a}$ *and* $\mathbf{b}$ *such that*[9]

(18) $$\begin{cases} b_1 = \bar{b}_1 , \\ y_t = a_t - b_t , \\ (a_t , b_{t+1}) \in \mathbf{T}_t \end{cases} \qquad \text{(for all } t).$$

[9] The reader might find that the constraint $b_1 = \bar{b}_1$ does not pertain to technological knowledge and should not enter the definition of $\mathbf{Y}$. Nothing is changed in the following mathematical treatment and little in the economic interpretation if $\bar{z}_1$ is defined so as to include the services of natural resources *and* all existing commodities at time 1. As was pointed out in footnote 4, the exact content of initial capital has no real significance here; thus we are free to assume $\bar{b}_1 = 0$. If this is done, the first formula in (18) must be changed accordingly and the reasoning may proceed without any alteration.

From the convexity of $\mathbf{T}_t$, $\mathbf{Y}$ is convex. If $\mathbf{y}^1$ and $\mathbf{y}^2$ are in $\mathbf{Y}$, then there are $\mathbf{a}^1$, $\mathbf{b}^1$ and $\mathbf{a}^2$, $\mathbf{b}^2$ satisfying (18). Now, if $0 < \alpha < 1$ and $\alpha + \beta = 1$, then $\{\alpha a_t^1 + \beta a_t^2\}$, $\{\alpha b_t^1 + \beta b_t^2\}$ satisfies (18). Hence, $\alpha\mathbf{y}^1 + \beta\mathbf{y}^2$ is in $\mathbf{Y}$.

To the decomposition of $\mathbf{T}_t$ into convex sets $\mathbf{T}_{tk}$ corresponds a decomposition of $\mathbf{Y}$ into convex sets $\mathbf{Y}_k$. Each $\mathbf{y}$ in $\mathbf{Y}$ can be written as[10] $\mathbf{y} = \sum_{k=1}^{n+1} \mathbf{y}_k$ with $\mathbf{y}_k \in \mathbf{Y}_k$ and $y_{tk} = a_{tk} - b_{tk}$.

8. *Social choice among chronics.* According to principles first made clear by Pareto, it is sometimes possible to say that a chronic C^2 is "better" than some other chronic C^1. The exact definition of this preference may vary, but in all cases comparison is made only on the consumption vectors x_t. Indeed, economic organization aims at satisfying consumers' needs; hence, the technical process by which this is done is irrelevant to social choice.

The simplest possible criterion is undoubtedly the following: C^2 is said to be better than C^1 if the consumption sequences $\mathbf{x}^2$ and $\mathbf{x}^1$ fulfill the condition $\mathbf{x}^2 \geqslant \mathbf{x}^1$.

Loosely speaking, this means that there is at least as much of everything to consume in C^2 as in C^1 and that no more labor is required. This leads us to the concept of efficiency:[11]

DEFINITION 1: *A chronic C^1 is efficient if there is no possible chronic C leading to a consumption sequence $\mathbf{x}$ such that $\mathbf{x} \geqslant \mathbf{x}^1$.*

More generally, if there are any social preferences, then, attached to any given chronic C^1, there exists a set $\mathbf{X}$ of all $\mathbf{x}$ corresponding to chronics C that are preferred to C^1. The following assumption on $\mathbf{X}$ will be made:

ASSUMPTION 5: *$\mathbf{X}$ is convex and, if it contains $\mathbf{x}^2$, it also contains any $\mathbf{x}$ such that $\mathbf{x} \geqq \mathbf{x}^2$.*

C^1 may be said to be optimal if there is no possible C with $\mathbf{x} \in \mathbf{X}$. In the following pages we shall, however, restrict the meaning of optimality and deal only with the usual welfare criterion. According to this criterion social choices are determined from individual preferences in the following way:

There are present and future consumers,[12] each of whom is character-

[10] Using the definitions introduced in footnote 7, we may write $y_{tk} = q_{tk} + g_{tk} - s_{tk}$, so that the input vector for firm k at time t is the difference between purchases and sales.

[11] Because of its simplicity, this definition is not fully satisfactory. In particular, it does not provide for the existence of commodities that are not wanted for consumption. However, since we shall also deal with the most general criterion for social preferences, it is advisable to choose here the simplest possible definition of efficiency so as to make the treatment of this case easily understandable.

[12] It might seem strange to introduce those consumers who do not yet exist. But if we consider all the consequences of our present economic decisions, however distant they might be, we have to take account of future generations, at least in a crude fashion. If they are not taken into consideration, production of certain very durable equipment would never be profitable.

ized by an index j (a positive number). His activity is represented by a consumption sequence $\mathbf{x}_j$, which may also be written $\mathbf{x}_j = \mathbf{x}_j^+ - \mathbf{x}_j^-$.

Since the life of any consumer j is limited,[13] then necessarily $x_{tj} = 0$ except for a finite number of values of t. More precisely, let us suppose that the indexes j are so chosen that, for a given t, $x_{tj} = 0$ except for $j_t^0 \leqq j \leqq j_t^1$. (There is only a finite number of consumers living at any time.) For a given j we also have $x_{tj} = 0$ except for $t_j^0 \leqq t \leqq t_j^1$ and, for any j, $t_j^1 - t_j^0 \leqq \theta$.

With these assumptions we may write

$$\mathbf{x} = \sum_j \mathbf{x}_j. \tag{19}$$

Now, for each consumer j, there is a set X_j of all sequences $\mathbf{x}_j$ that are at least equivalent to $\mathbf{x}_j^1$, and a set $\mathbf{X}_j$ of all sequences $\mathbf{x}_j$ that are preferred to $\mathbf{x}_j^1$. According to the Pareto principle[14] we say that $\mathbf{x}$ is preferred to $\mathbf{x}^1$ (or $\mathbf{x} \in \mathbf{X}$) if it may be written as a sum of sequences $\mathbf{x}_j$ with

$$\mathbf{x}_j \in X_j \qquad \text{for all } j, \text{ and}$$

$$\mathbf{x}_j \in \mathbf{X}_j \qquad \text{for at least one } j.$$

In the following we shall suppose that X_j and $\mathbf{X}_j$ fulfill Assumption 5. We may now give the following definition.

DEFINITION 2: *A chronic C^1 is "optimal" if there is no possible chronic C such that* $\mathbf{x} \in \mathbf{X}$, *where* $\mathbf{X}$ *is defined according to the Pareto principle.*

It is not necessary to insist here on the meaning of such concepts as efficiency and optimality for practical economic policy. This has been done elsewhere.

II. PROPERTIES OF EFFICIENT AND OPTIMAL CHRONICS

In this part, general properties of efficient and optimal chronics are studied. Nothing is assumed regarding the rhythm of expansion in the economy. In particular, some chronics may be efficient although they

[13] It would also be possible to introduce consumption units with infinite life, such as a national army. This would not create much difficulty.

[14] One might think the Pareto principle is still too restrictive as soon as choices involving time are concerned. Old people often say they would have planned their lives differently "if they had known." Clearly, only present individual preferences are considered in this paper. Each consumer is supposed fully to appreciate the relative urgency of his present and future needs. However, should this hypothesis be rejected, it would still be possible to introduce a weaker principle for social choices. One may say C is better than C^1 if it is preferred by all consumers now, and will still be preferred by them given all their future preference patterns. The latter concept has been used extensively by M. Allais [3, Chapter VI].

include periods with low levels of consumption and high investment followed by periods of disinvestment and high consumption. As usual in welfare economics and the theory of efficient allocation of resources, the final theorem introduces a price vector and rules of decentralization very similar to those which would hold in a competitive economy.

In order to make the main proof easier to understand, it is given in full detail for efficient chronics. The generalization to optimal chronics is merely sketched in the last paragraph. The reader will probably better understand the process of deduction if we first consider the case in which there is an economic horizon.

9. *Case of a finite horizon.* A chronic C^1 is efficient if there is no chronic C fulfilling[15]

$$(20) \qquad \begin{cases} \mathbf{x} \geqslant \mathbf{x}^1, \\ \mathbf{x} + \mathbf{y} = \bar{\mathbf{z}}, \text{ and} \\ \mathbf{y} \in Y. \end{cases}$$

Suppose now that there is some finite economic horizon h; in other words suppose that the result of economic activity is no longer an infinite sequence of consumption vectors but that there are only consumption vectors x_t for the $h - 1$ coming periods and the final stock of commodities b_h for the last period. Thus, the economic output is given by the finite set

$$\mathbf{x} = \{x_1, x_2, \cdots, x_t, \cdots, x_{h-1}, b_h\}.$$

$\mathbf{x}$ is a vector in the mh-dimensional Euclidean space. In the same way,

$$\mathbf{y} = \{y_1, y_2, \cdots, y_t, \cdots, y_{h-1}, \bar{z}_h - b_h\},$$

and Y becomes a convex set in the mh-dimensional Euclidean space.

In this form the problem is mathematically the same as in the static case. From previous works it is known that an efficient state is associated with some price vector,

$$\mathbf{p} = \{p_1, p_2, \cdots, p_t, \cdots, p_h\}.$$

The reader will find, for instance, a complete treatment of this finite case in Debreu's paper [**8**]. The price vector $\mathbf{p}$ is introduced, and its meaning when several periods are considered is indicated. (See, in particular, [**8,** p. 282, lines 10 to 14].)

[15] Hence, we look for minimal elements in Y. From a mathematical viewpoint, Theorem 1 provides a characterization of a minimal element in a convex set embedded in the linear space obtained by the Cartesian product of an infinite sequence of m-dimensional Euclidean spaces.

The existence of a price sequence will also be the essential result of the next section. But, as it stands now, it is somewhat unsatisfactory because nothing implies that the final stock of commodities is economically efficient in any sense.

In order to remove this limitation the efficiency of a chronic C^1 will be determined by successive steps. First C^1 will be compared to all C that are analogous to it after some given period h. Then h will be moved farther and farther into the future. If in this process there is never found any C better than C^1, then C^1 is efficient. This is, indeed, the only way in which the problem can be handled in practice; hence, one may expect that it is also the only way in which economically meaningful results can be reached.

10. *Existence of a price vector.* To justify this procedure we need, however, to establish the following lemma:

LEMMA 1: *Under Assumption* 4, C^1 *is efficient if and only if, for all* h, *there is no possible* C *with*

$$\text{(21)} \qquad \begin{cases} \mathbf{x} \geqslant \mathbf{x}^1, \\ x_t = x_t^1 \qquad (\text{for } t > h). \end{cases}$$

PROOF: If C^1 is efficient, there is clearly no C fulfilling (21). Conversely, suppose there is some possible C fulfilling $\mathbf{x} \geqslant \mathbf{x}^1$. Then, for at least one h, $x_h \geqslant x_h^1$. Given such an h, consider $\mathbf{x}^2$ defined by

$$\begin{cases} x_t^2 = x_t \qquad (\text{for all } t \leqq h), \text{ and} \\ x_t^2 = x_t^1 \qquad (\text{for all } t > h). \end{cases}$$

Clearly $\mathbf{x}^2 \leqq \mathbf{x}$, so that, by Assumption 4, there is associated with $\mathbf{x}^2$ some possible chronic C^2. C^2 satisfies (21), which completes the proof.

Given a chronic C^1, suppose we now restrict our attention to the possible chronic C fulfilling

$$\text{(22)} \qquad \begin{cases} \mathbf{y} \in \mathbf{Y}. \\ x_t = x_t^1 \qquad (\text{for } t > h). \end{cases}$$

This leads to the following lemma:

LEMMA 2: *Under Assumptions* 3 *and* 4, *if* C^1 *is efficient among all* C *satisfying* (22), *then there are* h *nonnegative vectors* p_t, *not all zero, such that* $\sum_{t=1}^{h} p_t y_t$ *is minimum for* C^1 *among all* C *satisfying* (22).

PROOF: For all possible C satisfying (22) the following holds:

$$\text{(23)} \qquad \begin{cases} \mathbf{y} \in \mathbf{Y}, \text{ and} \\ y_t = \bar{z}_t - x_t^1 \qquad (\text{for all } t > h). \end{cases}$$

Thus, if C^1 is efficient among all C satisfying (22), $\mathbf{y}^1$ is minimal among all $\mathbf{y}$ satisfying (23).

Now, consider the following vector in the mh-dimensional Euclidean space: $\mathbf{y}_h = \{y_1, \cdots, y_t, \cdots, y_h\}$. $\mathbf{y}$ fulfills (23) only if the vector $\mathbf{y}_h$ obtained from it is in some set $\mathbf{Y}_h$ depending on C^1 and h. From the convexity of $\mathbf{Y}$ it follows that $\mathbf{Y}_h$ is convex. Thus, $\mathbf{y}_h^1$ has to be a minimal element in the convex set $\mathbf{Y}_h$. This implies the existence in $\mathbf{y}_h^1$ of a support plane to $\mathbf{Y}_h$ whose normal vector $\mathbf{p}_h$ is nonnegative;[16] or the existence of a nonnegative linear form $\sum_{t=1}^{h} p_t y_t$ which is minimal for $\mathbf{y}_h^1$. Lemma 2 follows from this.

More precisely, if there are several support planes, the normal vectors generate a convex closed cone in the mh-dimensional Euclidean space.[17] (See Figure 2.)

We are now able to prove the following natural generalization of the efficiency theorem:

THEOREM 1: *Under Assumptions* 3 *and* 4, *associated with an efficient chronic* C^1, *there is a nonnegative sequence* $\mathbf{p}$ *such that, for all* h, $\sum_{t=1}^{h} p_t y_t$ *is minimal for* C^1 *among all* C *satisfying* (22).

PROOF: This will be proved if we are able to determine all vectors p_t of $\mathbf{p}$ under the condition that C^1 is efficient. Suppose we are interested in the sequence up to some period h. By Lemma 2, there is in the mh-

[16] The following mathematical theorem is applicable here:

THEOREM: *In finite-dimensional Euclidean space, given a convex set* $\mathbf{A}$ *with a non-empty interior and a point* x *not interior to* $\mathbf{A}$, *there is a plane* $\mathbf{P}$ *containing* x *and such that* $\mathbf{A}$ *is entirely contained in one of the closed half-spaces limited by* $\mathbf{P}$.

(For proof of this one may, for instance, transpose a proof by Banach [4, p. 28]. The reader may notice that $\mathbf{A}$ need not be closed.)

$\mathbf{y}_h^1$, being minimal, is necessarily a boundary point of $\mathbf{Y}_h$. So there is a nonzero vector $\mathbf{p}_h$ fulfilling the conditions of Lemma 2. The fact that $\mathbf{p}_h \geqq 0$ follows directly from Assumption 4.

[17] If

$$\sum_{t=1}^{h} p_t(y_t - y_t^1) \geqq 0 \qquad \text{and} \qquad \sum_{t=1}^{h} p_t'(y_t - y_t^1) \geqq 0,$$

then, clearly

$$\sum_{t=1}^{h} (\alpha p_t + \beta p_t')(y_t - y_t^1) \geqq 0$$

for any $\alpha \geqq 0$ and $\beta \geqq 0$.

Also, if $\mathbf{p}_h^n$ is a sequence of vectors converging to $\mathbf{p}_h$, and if

$$\sum_{t=1}^{h} p_t^n(y_t - y_t^1) \geqq 0 \text{ for all } n,$$

then

$$\sum_{t=1}^{h} p_t(y_t - y_t^1) \geqq 0.$$

dimensional Euclidean space a closed convex cone $\mathbf{P}_0$ of price vectors $\mathbf{p}_h$ corresponding to the horizon h. For the horizon $h + 1$, there is also a similar cone in $m(h + 1)$-dimensional space. Let us call $\mathbf{P}_1$ the projection of this cone on the space of its mh first coordinates. $\mathbf{P}_1$ is closed. We may now assert:

$$\mathbf{P}_0 \supset \mathbf{P}_1 . \tag{26}$$

Indeed, if $\{p_1, \cdots, p_t, \cdots, p_h, p_{h+1}\}$ is a vector associated by Lemma 2 with C^1 for the horizon $h + 1$, then $\{p_t, \cdots, p_t, \cdots, p_h\}$ also is a vector associated by Lemma 2 with C^1 for the horizon h. Indeed, for all $\mathbf{y}_{h+1} \in \mathbf{Y}_{h+1}$, $\sum_{t=1}^{h+1} p_t(y_t - y_t^1) \geqq 0$. If, in particular, $y_{h+1} = y_{h+1}^1$, then $\sum_{t=1}^{h} p_t(y_t - y_t^1) \geqq 0$.

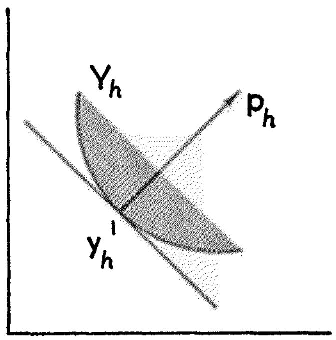

FIGURE 2

Pushing the horizon farther and farther in the future, we build up in the same way a sequence of closed convex cones in the mh-dimensional Euclidean space, with $\mathbf{P}_0 \supset \mathbf{P}_1 \supset \cdots \supset \mathbf{P}_{t-1} \supset \mathbf{P}_t \supset \cdots$.

It is known that the intersection of such a family is a nonempty closed cone. Thus, there is at least one sequence $\mathbf{p}$. If there are several such sequences, they generate a convex cone $\mathbf{P}$.

The following lemma will give the converse of Theorem 1:

LEMMA 3: *Under Assumption* 4, *a sufficient condition for the efficiency of a chronic* C^1 *is the existence of a positive*[18] *sequence* $\mathbf{p}$ *such that, for all h,* $\sum_{t=1}^{h} p_t y_t$ *is minimum for* C^1 *among all C satisfying* (22).

PROOF: Suppose C^1 is not efficient. Then, by Lemma 1, there exists

[18] The reader may notice we have $\mathbf{p} \geqq 0$ in Theorem 1 and $\mathbf{p} > 0$ in Lemma 3, so that the lemma is not exactly the converse of the theorem. However, it does not seem to be worth extending our investigations here in order to reduce the gap. This would lead us into a rather long study. It was done for the static case in Koopmans' work. Moreover, in dealing with optimality we shall presently give a more satisfactory treatment of the difficulty.

an h and a C satisfying (22) such that $\mathbf{x} \geqslant \mathbf{x}^1$, hence $\mathbf{y} \leqslant \mathbf{y}^1$. Since $p_t > 0$, then $\sum_{t=1}^{h} p_t(y_t - y_t^1) < 0$, contradicting the hypothesis.

Going back to the extensive form of the chronics, we may write

$$\sum_{t=1}^{h} p_t y_t = -p_1 \bar{b}_1 + \sum_{t=1}^{h-1} (p_t a_t - p_{t+1} b_{t+1}) + p_h a_h. \tag{27}$$

Since the sets $\mathbf{T}_t$ are defined independently of the values taken by the y_t, $\sum_{t=1}^{h} p_t y_t$ is minimal for C^1 among all C satisfying (22) if and only if:

(i) For all $t < h$, $p_t a_t - p_{t+1} b_{t+1}$ is minimal at (a_t^1, b_{t+1}^1) among all $(a_t, b_{t+1}) \in \mathbf{T}_t$;

(ii) $p_h a_h$ is minimal at a_h^1 among all a_h that make possible $y_t = \bar{z}_t - x_t^1$ for all $t > h$.

If Assumptions 1 and 2 hold (i.e., if $\mathbf{T}_t$ are convex cones), then condition (i) implies

$$A = p_t a_t^1 - p_{t+1} b_{t+1}^1 = 0.$$

Indeed, suppose we had, for instance, $A < 0$. Then, for $a_t = \alpha a_t^1$ and $b_{t+1} = \alpha b_{t+1}^1$, we would have

$$p_t a_t - p_{t+1} b_{t+1} < p_t a_t^1 - p_{t+1} b_{t+1}^1$$

as soon as $\alpha > 1$.

11. *Decentralization rule.* The preceding section provides a generalization of the first part of the efficiency theorem which was obtained in the static case. The second part of the same theorem specifies a rule of decentralization; more explicitly, it says that $\mathbf{py}$ is minimal for the society as a whole if and only if $\mathbf{py}_k$ is minimal for each firm. This will be the subject of Lemmas 4 and 5.

LEMMA 4: *Under Assumptions* 3 *and* 4, *if* C^1 *is efficient there is a nonnegative sequence* $\mathbf{p}$ *such that, for all* h *and* k, $\sum_{t=1}^{h} p_t y_{tk}$ *is minimal at* C^1 *among all* C *satisfying*

$$\begin{cases} \mathbf{y}_k \in \mathbf{Y}_k, \text{ and} \\ y_{tk} = y_{tk}^1 \qquad \text{(for } t > h). \end{cases} \tag{34}$$

PROOF: This follows directly from Theorem 1 because, if there were any C satisfying (34) such that $\sum_{t=1}^{h} p_t(y_{tk} - y_{tk}^1) < 0$ for some k, then we could find a chronic C^2 identical with C^1 except for the input vectors of firm k. For the latter we would choose $y_{tk}^2 = y_{tk}$. Hence, C^2 would satisfy (22) and $\sum_{t=1}^{h} p_t(y_t^2 - y_t^1) < 0$, implying by Theorem 1 that $\mathbf{p}$ is not an efficient sequence.

Let us first note as a consequence of Lemma 4 that C^1 is not efficient unless there is complete use of those resources which have a nonzero price. Indeed, for the $(n + 1)$th production unit we should have $\sum_{t=1}^{h} p_t(\bar{z}_t - z_t)$ at a minimum. Since p_t and $\bar{z}_t - z_t$ are nonnegative, the minimum is reached when $p_t(\bar{z}_t - z_t) = 0$ for all t.

Even if $\mathbf{p} > 0$, the converse of Lemma 4 does not necessarily hold.[19] The difficulty lies in the possibility of having some C^2 such that

$$\begin{cases} y_t^2 = y_t^1 & \text{(for } t > h\text{), but not necessarily } y_{tk}^2 = y_{tk}^1 \\ \sum_{t=1}^{h} p_t(y_t^2 - y_t^1) < 0, & \end{cases}$$

although there is no C such that

$$\begin{cases} y_{tk} = y_{tk}^1 & \text{(for } t > h\text{, and all } k\text{)} \\ \sum_{t=1}^{h} p_t(y_t - y_t^1) < 0. & \end{cases}$$

Such a case corresponds to an inadequate distribution of capita among firms, which cannot be detected when comparisons are limited to any finite horizon.

However, the possibility of this can be ruled out if $p_h a_h$ tends to zero when h tends to infinity, i.e., if the present value of capital for period h decreases to zero when h tends to infinity. This is the meaning of the following lemma.

LEMMA 5: *Under Assumption 4, a sufficient condition for the efficiency of C^1 is that there is a positive sequence* $\mathbf{p}$ *such that:*

(i) *for all h and k, $\sum_{t=1}^{h} p_t y_{tk}$ is minimal at C^1 among all C satisfying* (34);

(ii) *$p_t a_t^1$ tends to zero when t tends to infinity.*

[19] The following counter-example illustrates the point. Suppose there are two commodities and two firms with the same technological set:

$$(a_{tk}, b_{t+1,k}) \in T_{tk} \text{ if } \begin{cases} a_{\iota tk} \geqq 0,\ b_{\iota,t+1,k} \geqq 0; \text{ and} \\ a_{1tk} + a_{2tk} - b_{1,t+1,k} - b_{2,t+1,k} \geqq 0. \end{cases}$$

Consider C^1 defined by

$$C^1 \begin{cases} a_{1t} = 1, & a_{2t} = 2, & b_{1t} = 2, & b_{2t} = 1, & y_{1t} = -1, & y_{2t} = 1, \\ a_{1t1} = 0, & a_{2t1} = 2, & b_{1t1} = 2, & b_{2t1} = 0, & y_{1t1} = -2, & y_{2t1} = 2, \\ a_{1t2} = 1, & a_{2t2} = 0, & b_{1t2} = 0, & b_{2t2} = 1, & y_{1t2} = 1, & y_{2t2} = -1. \end{cases}$$

C^1 fulfills the condition of Lemma 4, with the price vector $p_t = (1, 1)$, but it is not efficient, as can be seen by comparison with the following

$$C \begin{cases} a_{1t} = a_{1t1} = 0, & a_{2t} = a_{2t1} = 1, & a_{\iota t2} = b_{\iota t2} = 0 & \text{(for all } t\text{)} \\ b_{2t} = b_{2t1} = 0, & b_{1t} = b_{1t1} = 1, & & \text{(for all } t > 1\text{)'} \\ y_{t2} = 0, & y_{1t1} = -1,\ y_{2t1} = 1 & & \text{(for all } t > 1\text{)'} \end{cases}$$

Indeed, C provides us with the same net output for all periods after the first one: $x_t = x_t^1 = \bar{z}_t + (1, -1)$ for $t > 1$. And it makes possible an increase in the first consumption vector: $x_1 = \bar{z}_1 + (2, 0)$, $x_t^1 = \bar{z}_1 + (1, -1)$.

PROOF: Suppose C^1 is not efficient. There is some h and some C^2 such that

$$\sum_{t=1}^{h} p_t(y_t^2 - y_t^1) < 0, \tag{35}$$

$$\begin{cases} y_t^2 = y_t^1 & \text{(for all } t > h), \\ \text{or } a_t^2 - b_t^2 = a_t^1 - b_t^1 & \text{(for all } t > h). \end{cases} \tag{36}$$

From Condition (i) and the remark at the end of the preceding section, it follows that $p_t a_{tk} - p_{t+1} b_{t+1,k}$ is minimal at $(a_{tk}^1, b_{t+1,k}^1)$ for all t and all $(a_{tk}, b_{t+1,k}) \in \mathbf{T}_{tk}$. Hence, for all t,

$$p_t(a_t^2 - a_t^1) \geqq p_{t+1}(b_{t+1}^2 - b_{t+1}^1). \tag{37}$$

(35) and (37) imply

$$p_h(a_h^2 - a_h^1) < 0. \tag{38}$$

Now, (36), (37), and (38) imply that the following is a nonincreasing sequence of negative vectors:

$$0 > p_h(a_h^2 - a_h^1) \geqq p_{h+1}(b_{h+1}^2 - b_{h+1}^1) = p_{h+1}(a_{h+1}^2 - a_{h+1}^1) \geqq \cdots .$$

But such a sequence cannot exist because $p_t a_t^2$ is nonnegative and $p_t a_t^1$ can be made smaller than any positive number, so that $p_h(a_h^2 - a_h^1)$ must be greater than any negative number.

12. *Properties of optimal chronics.* In dealing with optimal chronics the mathematical technique will be essentially the same as in the two preceding sections. Detailed demonstrations will therefore be omitted and only the main steps given.

Let us recall Definition 2: C^1 is optimal if there is no possible chronic C such that $\mathbf{x} \in \mathbf{X}$; i.e., if there is no $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{Y}$ such that $\bar{\mathbf{z}} = \mathbf{x} + \mathbf{y}$.

Let us define the set $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$.

C^1 is optimal if and only if $\bar{\mathbf{z}}$ is not in $\mathbf{Z}$. From Assumptions 3, 4, and 5, $\mathbf{Z}$ is convex, and if it contains $\mathbf{z}^2$ it also contains any $\mathbf{z} \geqq \mathbf{z}^2$. Hence, the following may be proved:[20]

THEOREM 1′: *Under Assumptions* 3, 4, *and* 5, *if* C^1 *is optimal, there is a nonnegative sequence* $\mathbf{p}$ *such that*

$$\sum_{t=1}^{h} p_t(z_1 - \bar{z}_t) \geqq 0$$

[20] As in Section 10, one would define finite sequences $\mathbf{z}_h$ and the sets $\mathbf{Z}_h$ of all $\mathbf{z}_h$ such that $\mathbf{z}^2 \in \mathbf{Z}$ if $z_t^2 = z_t$ for $t \leqq h$, and $z_t^2 = \bar{z}_t$ for $t > h$. C^1 is optimal if and only if $\bar{\mathbf{z}}_h \notin \mathbf{Z}_h$ for all h. Hence, the existence of finite nonnegative sequences $\mathbf{p}_h$ such that $\mathbf{z}_h \in \mathbf{Z}_h$ implies $\sum_{t=1}^{h} p_t(z_t - \bar{z}_t) \geqq 0$ (cf. footnote 16), and hence, finally, the existence of an infinite non negative sequence $\mathbf{p}$.

for all h and all $\mathbf{z}$ *satisfying*

$$\begin{cases} \mathbf{z} \in \mathbf{Z}, & \\ z_t = \bar{z}_t & (\text{for } t > h) \end{cases} \tag{39}$$

The following trivial lemma goes in the opposite direction:

LEMMA 3′: *Under Assumptions* 4 *and* 5, *a sufficient condition for optimality of* C^1 *is the existence of a nonnegative sequence* $\mathbf{p}$ *such that*

$$\sum_{t=1}^{h} p_t(z_t - \bar{z}_t) > 0$$

for all h and all $\mathbf{z}$ *fulfilling* (39).

Theorem 1′ and Lemma 3′ may be summed up into a single theorem if the following weak assumption on $\mathbf{X}$ is made:

ASSUMPTION 6: *If* $\mathbf{x} \in \mathbf{X}$, *then there is* $\epsilon > 0$ *such that if* $| x_{it}^2 - x_{it} | < \epsilon$ *for all i and t, it is implied that* $\mathbf{x}^2 \in \mathbf{X}$.

This says that, if $\mathbf{x}$ is preferred to $\mathbf{x}^1$, then any sequence $\mathbf{x}^2$ sufficiently close to $\mathbf{x}$ is also preferred to $\mathbf{x}^1$.[21]

We may now formulate

THEOREM 2: *Under Assumptions* 3, 4, 5, *and* 6, C^1 *is optimal if and only if there is a nonnegative sequence* $\mathbf{p}$ *such that*

$$\sum_{t=1}^{h} p_t(z_t - \bar{z}_t) > 0$$

for all h and all $\mathbf{z}$ *satisfying* (39).

Along with the existence of a price vector, a scheme of decentralization may be introduced. This is included in Lemmas 4′ and 5′.

LEMMA 4′: *Under Assumptions* 3, 4, *and* 5, *if* C^1 *is optimal, there is a nonnegative* $\mathbf{p}$ *such that, for all h, k, and j,*

(i) $\sum_{t=1}^{h} p_t y_{tk}$ *is minimal at* C^1 *among all C satisfying* (34);

(ii) $\sum_t p_t(x_{tj} - x_{tj}^1) \geqq 0$ *for all* $\mathbf{x}_j \in \mathbf{X}_j$.

Since the vectors of the sequence $\mathbf{x}_j$ are null except for a finite number, the sum in Condition (ii) does make sense. Also, Condition (ii) may be written with the strict sign if Assumption 6 holds for the individual preference sets $\mathbf{X}_j$.

For the converse of Lemma 4′ one more assumption is needed:

ASSUMPTION 7: *For all t and j, there exist vectors* $\bar{u}_{tj}$ *such that* $\mathbf{x}_j \in \mathbf{X}_j$ *implies* $\mathbf{x}_{tj} \geqq \bar{u}_{tj}$.

Since $x_{tj}^+ \geqq 0$, Assumption 7 means essentially that there is some

[21] Although it is not satisfied by the efficiency concept, this assumption does not seem to be restrictive. It is clearly fulfilled if the individual preferences may be represented by continuous utility functions.

upper limit to the amount of labor $\bar{x_{tj}}$ that can be required from consumer j. For society as a whole we shall write $\bar{u}_t = \sum_j \bar{u}_{tj}$. Hence we have the following lemma:

LEMMA 5′: *Under Assumptions* 3, 4, 5, 6, *and* 7, *a sufficient condition for the optimality of* C^1 *is the existence of a nonnegative sequence* $\mathbf{p}$ *such that, for all h, j, and k,*

(i) $\sum_{t=1}^{h} p_t y_{tk}$ *is minimal for* C^1 *among all C satisfying* (34);

(ii) $\sum_t p_t(x_{tj} - x_{tj}^1) \geqq 0$ *for all* $\mathbf{x}_j \in X_j$;

(iii) $p_t a_t^1$ *and* $p_t(x_t^1 - \bar{u}_t)$ *tend to zero when t tends to infinity.*

As in the efficiency case, Condition (iii) means that the present values of future capital and future consumption tend to zero when we consider periods that are farther and farther away in the future. Conditions (ii) in Lemmas 3′ and 4′ are not exactly the same. But they are equivalent if X_j is contained in the closure of $\mathbf{X}_j$, or if, for any $\mathbf{x}_j^2 \in X_j$ and any positive sequence $\mathbf{u}$, there is some $\mathbf{x}_j \in \mathbf{X}_j$ such that $\mathbf{x}_j - \mathbf{x}_j^2 \leqq \mathbf{u}$. This amounts to saying that there does not exist any complete saturation of all consumers' needs. By increasing the quantity of some conveniently chosen commodity, the consumer may be made better off, however small the increase might be.

III. EFFICIENCY AND THE RATE OF INTEREST

13. *Efficiency in actual societies.* The results of the last part were concerned mainly with the general properties of efficient and optimal chronics. They merely extended what was already known about the static case. It is, however, of paramount interest to study the extent to which these requirements are fulfilled in a real society. This is the purpose of the present part, in which we shall try to move closer to reality, introducing some institutional rules together with the general scheme of production and consumption. This inquiry aims at showing which restrictions are necessary in order to interpret the preceding formal lemmas as a justification of a competitive economic system.

We shall first rule out uncertainty in its two-fold aspect. Any firm will be supposed to know exactly which technical transformations are, and will be, possible; that is, firm k knows perfectly the sets $\mathbf{T}_{tk}$ for all values of t. In addition every economic unit, whether firm or consumer, also knows the present and future conditions of the market; i.e., prices and interest rates.

A second hypothesis concerns money. We shall suppose that firms and consumers do not hold money, either because they are not allowed to or because they do not want to. Once uncertainty is removed, this amounts to supposing that interest rates are positive and services of the banking system free (by which we mean only the fixed costs for any transfer from one account to another—not the normal interest discounts, which are, indeed, retained in the model). Thus, money will be a value unit only.

With these hypotheses we shall proceed to show, first, how interest rates do appear in the price system and, second, how the usual profit-maximizing principles coincide with the preceding decentralization rules. Then we shall be able to exhibit very simply some relations between private and national accounting. Finally, we shall deal with the question of why interest rates should be positive.

14. *Interest rates in actual economies.* As in Sections 11 and 12, the price system **p** apparently does not include a rate of interest. This may seem strange since, in society as we know it, interest rates are used on the loan market and in business accounting for discounting future values. The point may be made clearer by the following remark.

In the static case the efficiency theorem leads to a set of prices that are determined up to some common multiplicative scalar. Thus only a set of relative prices is given. Absolute prices may be fixed at any level in accordance with monetary conditions. Our result in the dynamic case is formally similar but entails a different interpretation: The whole set of present and future prices is still determined up to a multiplicative scalar; this, however, determines not only the relative prices for each period but also all future absolute prices, given the present ones. If, as is usually the case, the institutional structure is such that the absolute prices must satisfy some normalization condition within each period, then our lemmas must be modified.

A normalization rule states which multiple of p_t should be taken as the absolute price vector for period t. To avoid confusion, let us denote by p'_t the normalized price vector associated with p_t :

$$p_t = \beta_t p'_t, \tag{41}$$

where β_t is some convenient positive scalar.[22] Let us call it the discount coefficient for period t. Since the sequence **p** is determined up to a multiplicative constant, we shall suppose $\beta_1 = 1$.

In the following, the sequence **p** will be replaced by two sequences, one of nonnegative normalized price vectors **p**′ and the other of the positive discount coefficients β_t . However, it should be clear that neither, taken alone, has any intrinsic meaning. This is provided only when the normalization rule is given.

Let us define

$$1 + \rho_t = \beta_t/\beta_{t+1} . \tag{42}$$

ρ_t will appear as a rate of interest in the next section and later on in the treatment of stationary cases.

[22] For the sake of simplicity, it is supposed that $p_t \neq 0$.

15. *Rules of behavior for consumers and firms.* As we have seen, the firm k should maximize in each period $p_{t+1}b_{t+1,k} - p_t a_{tk}$ subject to $(a_{tk}, b_{t+1,k}) \epsilon \mathbf{T}_{tk}$. This is equivalent to maximizing

$$(43) \quad B_{tk} = p'_{t+1}(b_{t+1,k} - a_{tk}) + (p'_{t+1} - p'_t)a_{tk} - \rho_t p'_t a_{tk} .$$

B_{tk} is the usual net profit concept[23] for period t. It is computed as the sum of

+ value of net production,	$p'_{t+1}(b_{t+1,k} - a_{tk})$
+ capital gains,	$(p'_{t+1} - p'_t)a_{tk}$
− interest costs,	$\rho_t p'_t a_{tk}$

One may also note that if $a^1_{tk} = b^1_{tk} = 0$ after some horizon h, maximizing $\sum_{t=1}^{h} p_t y_{tk}$ is equivalent to maximizing

$$(45) \qquad F_{1k} - p'_1 c_{1k} = \sum_t \beta_{t+1} B_{tk} .$$

F_{1k}, so defined, may be interpreted as being the present value of the firm.

Formulas (42), (43), and (45) show that the theory of allocation of resources justifies the usual accounting procedures. The interest rates here introduced play the same role as they do in business accounting.

Let us also remember that if additivity of the technical processes holds, together with divisibility, then necessarily

$$(46) \qquad B_{tk} = 0 \quad \text{and} \quad F_{1k} = p'_1 c_{1k} .$$

In interpreting the rule of behavior for consumers, suppose that they can receive or make loans. An account of their assets and liabilities is kept at some bank, and the net assets at the beginning of period t for consumer j is equal to A_{tj}. The consumer will be paid interest on it equal to

$$(47) \qquad K_{tj} = \rho_{t-1} A_{t-1,j} .$$

K_{tj} may be called the consumer's capitalist income. During period t he will save

$$S_{tj} = A_{tj} - A_{t-1,j} .$$

[23] It has not always been clear in economic literature which quantity the entrepreneur ought to maximize. (See, for instance, Boulding [7], Samuelson [**28**], Lutz [**21**], Rottier [**27**].) In any case, maximization of B_{tk} is a necessary but not a sufficient condition.

If we write

$$C_{tj} = p'_t x^+_{tj} \quad \text{and} \quad W_{tj} = p'_t x^-_{tj} ,$$

the budget equation for j will be

$$C_{tj} + S_{tj} = Y_{tj} = W_{tj} + K_{tj} , \tag{48}$$

which may be read as *consumption* + *savings* = *income*. Formally it may be written

$$p_t x_{tj} = \beta_{t-1} A_{t-1,j} - \beta_t A_{tj} . \tag{49}$$

Minimizing $\sum_t p_t x_{tj}$ subject to $\mathbf{x}_j \in \mathbf{X}_j$ amounts to maximizing the final assets $A_{t^1_j,j}$ under the constraints $\mathbf{x}_j \in \mathbf{X}_j$, and $p_t x_{tj} = \beta_{t-1} A_{t-1,j} - \beta_t A_{tj}$.

Thus, roughly speaking, the rule advises us to choose C^1 if, among all chronics that are at least as good, it is associated with the greatest final assets.[24] The budget equation, together with this last rule, shows how the interest rates we have introduced play the usual role on the loan market.[25]

In fact, we shall introduce a somewhat different definition, which is a little more involved but makes the following section simpler. We shall suppose wages to be paid at the end of the period and to include conveniently the interest earned thereon. Thus the budget equation becomes

$$Y_{tj} = K_{tj} + (1 + \rho_{t-1}) W_{t-1,j} = C_{tj} + S_{tj} . \tag{50}$$

Accordingly, formula (49) becomes

$$p_t x_{tj} = \beta_{t-1}(A_{t-1,j} + W_{t-1,j}) - \beta_t(A_{tj} + W_{tj}). \tag{51}$$

Since consumer j disposes of initial assets $A_{t^0_j,j}$ but does not get any wage before the end of period t^0_j, the intuitive meaning of the behavior rule is still to maximize the final assets while enjoying a given level of utility.

16. *Real capital and assets; private and national accounting.* As was shown by Fetter [**9**], there are essentially two concepts of capital given in the economics literature. According to the first, capital includes all "owned sources of income"; thus, it may be defined as the totality of assets: $A_t = \sum_j A_{tj}$. According to the other definition it is a "stock of physical goods used as means of production." The latter concept is sometimes called "real capital" and could be written $p'_t c_t$.

[24] This is clearly only one among many possible rules which would bring about a minimum of $\sum_t p_t x_{tj}$ subject to $\mathbf{x}_j \in \mathbf{X}_j$.

[25] Thus here, as in classical economics, interest rates appear in their two-fold aspect—as a margin of technical profit and as a price for loans.

But assets and real capital are not independent of each other. Indeed, if any net assets exist, they represent some "real" values. If, to simplify, we deal directly with aggregates and write $B_t = \sum_k B_{tk}$, we may define:

$$A_t = L_t + F_t, \tag{52}$$

where L_t and F_t are the values of natural resources (land) and of firms, respectively.

$$L_t = \frac{1}{\beta_t} \sum_{\theta=t}^{\infty} \beta_\theta p'_\theta z_\theta, \tag{53}$$

$$F_t = p'_t c_t + \frac{1}{\beta_t} \sum_{\theta=t}^{\infty} \beta_{\theta+1} B_\theta, \tag{54}$$

supposing that the infinite sums are meaningful.

From these definitions it is possible to give an expression for capitalist income:

$$K_{t+1} = \rho_t A_t = B_t + \rho_t p'_t c_t + (1 + \rho_t) p'_t z_t + G'_t, \tag{55}$$

where $G'_t = G_t - (p'_{t+r} - p'_t)c_t$, $G_t = (L_{t+1} - L_t) + (F_{t+1} - F_t)$. Formula (55) shows that capitalist income is the sum of

+ *profits of firms*	B_t
+ *interest from real capital,*	$\rho_t p'_t c_t$
+ *rents from land,*	$(1 + \rho_t) p'_t z_t$
+ *capital gains,*	G'_t

The capital gains on real capital, which are not included in G'_t, are part of profits.[26]

It is now possible to show very simply how national production is related to national income. Let us define the latter by

$$Y_t = \sum_j Y_{tj}, \tag{56}$$

and net national production by

$$P_t = p'_{t+1}(b_{t+1} - c_t). \tag{57}$$

[26] We considered firms and consumers as different units and found some behavior rules for them separately. But actually many consumers do perform productive activities; there is no such sharp distinction in reality between production and consumption units. It is therefore important to notice here that the two behavior rules are consistent. Nobody is faced with the difficult problem of choosing between a maximum of B_t and a maximum of A_t.

Let us also define net national investment as

$$I_t = p'_{t+1}(c_{t+1} - c_t). \tag{58}$$

The reader may check that the following relations hold:

$$P_t = C_{t+1} + I_t , \tag{59}$$

$$Y_t = C_t + S_t , \tag{60}$$

$$S_{t+1} = I_t + G_t , \qquad Y_{t+1} = P_t + G_t . \tag{61}$$

These relations bear a strong resemblance to the usual national accounting equations. However, the matter of capital gains seems to introduce some difficulty.[27] Needless to say, our relations are not directly transposable to actual societies because money and international trade have been deliberately excluded.

17. *Why should interest rates be nonnegative?* Interest theory, if not capital theory, has often been thought of as dealing only with one question: Why does competition not bring the rate of interest down to zero? The emphasis on this point seems to have been a little misplaced. Once it is understood that two equal quantities of the same thing available at two different moments are not economically equivalent, there is no a priori reason for the interest rate to be zero. However, we do observe in fact that interest rates have always been positive; thus, we may wonder why this is so. The following remarks are intended to reformulate a few reasons that seem to be important in this respect.

First, in a monetary economy, consumers may always hold money, so that there would not be any loans unless the interest rate were

[27] Of course, this difficulty could be avoided by changing our definitions. But there are good reasons for our choice. If income did not include capital gains, the behavior rules could no longer be interpreted in the frame of a competitive economy. If capital gains were included in national production and investment, these aggregates would no longer be evaluated from real physical net output and investment by using a unique set of prices. Concepts like the investment schedule would also be much more difficult to define.

The above equations should not, however, lead the reader to think that the whole of the present national accounting analysis is not well founded. If one considers what would happen in times of inflation, he will find that net national production is the very concept people have in mind when they speak of national income. As we have defined the latter it would include large capital gains which should be saved and invested on the loan market if capitalists wanted to keep constant the *real* value of their assets. Thus, both income and savings might seem to be largely overrated by our definitions.

One should also notice that the equation $S = I + G$ is not an equilibrium relation on any market but rather a necessary identity as soon as net assets are supposed to equate the value of firms and natural resources.

positive. This reason, however, important as it is, does not provide a complete answer. It has been argued that not only monetary but also real interest rates[28] are always positive. We also want to see if positive interest rates in a nonmonetary economy can be explained.

Note that in such an economy interest rates alone do not have any intrinsic meaning, so that the question does not make sense unless one specifies the normalization rule on the price vector p'_t. This must be kept in mind to understand the following remarks.

1. Suppose first that the prices p'_t are such that

$$p'_t\bar{z}_t = p'_{t+1}\bar{z}_{t+1} \qquad \text{(for all } t\text{)}$$

so that ρ_t may be computed by

$$1 + \rho_t = \frac{p_t\,\bar{z}_t}{p_{t+1}\,\bar{z}_{t+1}}.$$

If the natural resources are privately owned, they must have some value. Formula (53) defining L_t must have meaning. This implies that $\lim_{h\to\infty} \sum_{\theta=t+1}^{h} \beta_\theta$ exists for all t. This cannot be so unless

$$lim_{t\to\infty}(\beta_t/\beta_{t+1}) \geqq 1,$$

or, equivalently, unless $lim_{t\to\infty}\rho_t \geqq 0$. Such was the idea behind Turgot's theory of fructification.

2. Suppose now that the price of some commodity i_0 is kept constant: $p'_{i_0t} = p'_{i_0.t+1}$ for all t, so that ρ_t may be computed by

$$1 + \rho_t = \frac{p_{i_0t}}{p_{i_0,t+1}}.$$

If commodity i_0 may be stored without any cost, we may write $(a_t^2, b_{t+1}^2) \in \mathbf{T}_t$ with $a_{it}^2 = a_{it}^1$, $b_{i,t+1}^2 = b_{i,t+1}^1$, for $i \neq i_0$; $a_{i_0t}^2 = a_{i_0t}^1 + \alpha_t$ with $\alpha_t > 0$; $b_{i_0,t+1}^2 = b_{i_0,t+1}^1 + \alpha_t$. If $p_ta_t - p_{t+1}b_{t+1}$ is minimum for C^1, then necessarily $(p_{i_0t} - p_{i_0\,t+1})\alpha_t \geqq 0$; hence $\rho_t \geqq 0$.

3. With the same normalization rule as in 2, we may suppose $\mathbf{x}^2 \in \mathbf{X}$ when $\mathbf{x}^2$ is defined by $x_t^2 = x_t^1$ for $t > 2$; $x_1^2 = x_1^1 + a$; and $x_2^2 = x_2^1 - \beta a$, with $a_i = 0$ for $i \neq i_0$ and with $a_{i_0} > 0$ and $\beta > 1$. Thus, $\mathbf{x}^2$ is analogous to $\mathbf{x}^1$ except that it allows for a greater consumption of i_0 in the first period and a smaller consumption of i_0 in the second period, the total

[28] Real interest rates are defined once the effect of changes in the general level of prices is removed. Here, the real interest rate associated with chronic C^1 appears if the normalization rule is such as to make invariant the value of some representative bundle of goods. Usually this concept is defined as follows: Let r be the monetary interest rate and P the general level of prices. The real interest rate is then given by the formula: $\rho = r - (1/P)\cdot(dP/dt)$.

quantity of i_0 for both periods being smaller than in $\mathbf{x}^1$. And it it supposed that C^2 is preferred to C^1.

If C^1 is optimal, then $p_1(^2_1 - x^1_1) + p_2(x^2_2 - x^1_2) \geqq 0$. Hence

$$\rho_1 \geqq \beta - 1 > 0.$$

This is the usual theory of preference for present commodities.

4. It is sometimes said that the rate of interest is, or should be, equal to the rate of expansion of the economy. More precisely, suppose that the rate of interest is computed from nonnormalized prices by

$$1 + \rho_t = \frac{p_t c_t}{p_{t+1} c_{t+1}},$$

or, equivalently, that the normalization rule specifies $p'_{t+1}c_t = p'_t c_t$. Let us define the rate of capital accumulation δ_t as

$$1 + \delta_t = \frac{p'_{t+1} c_{t+1}}{p'_{t+1} c_t}.$$

Then $\rho_t - \delta_t$ is of the same sign as $-p'_{t+1}(c_{t+1} - c_t) + \rho_t p'_t c_t$. In particular, if the $\mathbf{T}_t$ are cones, this is also the sign of

$$p'_{t+1}x^+_{t+1} - (p'_{t+1} + \rho_t p'_t)(x^-_t + z_t).$$

There does not seem to be, in general, any definite sign for this expression. However, if we suppose $x^+_{t+1} = x^-_t = z_t = 0$, then, clearly, $\rho_t = \delta_t$. Such was the case in von Neumann's model of 1937 [24].

IV. STATIONARY ECONOMIES

Usually in capital theory "production is defined in relation to economic equilibrium . . . in the form of a stationary economy."[29] Indeed, if such an assumption is made, the interest rate appears quite naturally in the requirements for efficiency, along with the "marginal productivity of capital." In this part we shall deal first with the properties of efficient stationary chronics,[30] second with the marginal productivity of capital, and third with the concept of the optimum amount of capital. A last section will be devoted to some historical comments.

18. *Properties of efficient stationary chronics.* We shall now assume the set of technological possibilities and the available resource vector to be identical to a set $\mathbf{T}$ and to a vector $\bar{z}$ independent of time. The chronic C^1 is said to be stationary if the vectors characterizing the economic

[29] Knight [15].

[30] Throughout this part we shall study efficiency alone. The introduction of consumers' preferences would make the whole treatment unnecessarily involved.

activity remain unchanged from one period to another. Thus, C^1 is fully described by the four m-dimensional vectors, a, b, z, and x, with the conditions

$$\text{(62)} \qquad \begin{cases} a = b + z - x, \\ (a, b) \in \mathbf{T}, \\ z \leqq \bar{z}. \end{cases}$$

According to Theorem 1, if the stationary chronic C^1 is efficient, there is some nonnegative sequence $\mathbf{p}$ such that, for all t,

(i) $p_t a - p_{t+1} b$ is minimal at (a^1, b^1) among all $(a, b) \in \mathbf{T}$;

(ii) $p_t a$ is minimal at (a^1, b^1) among all $(a, b) \in \mathbf{T}$ such that $a - b = a^1 - b^1$.

Conversely, if there is a positive sequence $\mathbf{p}$ such that C^1 fulfills conditions (i) and (ii), then C^1 is efficient. More precisely, we state

LEMMA 7: *Under Assumptions* 3 *and* 4, *if a stationary chronic* C^1 *is efficient, there exists a nonnegative vector* p *and a scalar* $\rho > -1$ *such that*

(i) $p(b - a) - \rho pa$ *is maximal at* (a^1, b^1) *among all* $(a, b) \in \mathbf{T}$;

(ii) pa *is minimal at* (a^1, b^1) *among all* $(a, b) \in \mathbf{T}$ *such that* $a - b = a^1 - b^1$.

Conversely, if their is a positive vector p *and a scalar* $\rho > -1$ *such that the stationary chronic* C^1 *fulfills conditions (i) and (ii), then* C^1 *is efficient.*

PROOF: The second statement of the lemma follows directly from Lemma 3 if we define the sequence $\mathbf{p}$ by $p_t = p/(1 + \rho)^{t-1}$.

Conversely, if C^1 is efficient, there is, by Theorem 1, a nonnegative sequence $\mathbf{p}$ such that $(p_t, -p_{t+1})$ is in the closed convex cone of normals to $\mathbf{T}$ at (a^1, b^1). This implies that this cone contains some vector of the form $(p, -\beta p)$ with $\beta > 0$.[31] Lemma 7 follows, with $1 + \rho = 1/\beta$.

[31] This is obvious if the cone of normals is just a halfline. In general, the proof is somewhat more difficult. It is given here for completeness. We want to prove

LEMMA: *Given a sequence* $\mathbf{p}$ *of vectors in the m-dimensional Euclidean space, with* $p_t \geqq 0$, *and the convex closed cone* $\mathbf{\Gamma}$ *generated by* $(p_t, -p_{t+1})$ *in the 2m-dimensional space, there is some vector* $p \gg 0$ *and some positive* β *such that* $(p, -\beta p) \in \mathbf{\Gamma}$.

PROOF: Define

$$p_t^{(1)} = p_t, \; p_t^{(h)} = p_t^{(h-1)} + p_{t+1}^{(h-1)} \qquad \text{(for } h > 1\text{)}.$$

Let $\mathbf{C}^{(h)}$ be the convex closed cone generated by $p_t^{(h)}$ in m-dimensional space.

$$\mathbf{C}^{(h)} \subset \mathbf{C}^{(h-1)}.$$

Define

$$\mathbf{C}^{\infty} = \textstyle\bigcap_{h=1}^{\infty} \mathbf{C}^{(h)}.$$

$\mathbf{C}^{\infty}$ is a nonempty closed convex cone.

By definition of $\mathbf{C}^{(h)}$, for any $u \in \mathbf{C}^{(h)}$ there is a sequence $\{u_n\}$ of vectors fulfilling the following:

$$\text{(a)} \qquad \lim_{n \to \infty} u_n = u,$$

$$\text{(b)} \qquad u_n = \sum_t \alpha_{tn} p_t^{(h)},$$

Thus, associated with any efficient stationary chronic, there is some set of relative prices and some rate of interest. This seems to contradict the preceding result, according to which interest rates appear only when some monetary rule is given. But this last condition is in fact implicitly included in Lemma 7. Indeed, when prices are used in the computation of $p(b - a) - \rho pa$, it is supposed that absolute prices remain the same in all periods; or, in other words, that the normalization rule does not change.[32]

19. *Marginal productivity of capital.* It is a much debated question to know whether the interest rate is, or ought to be, equal to the marginal productivity of capital. As we shall see, the whole controversy boils down to the definition given to marginal productivity. Following Knight **[15]**, we shall adopt here the most usual concept.

Given the efficient stationary chronic C^1, let us consider the class $\mathcal{C}$ of all possible stationary chronics for which the inputs x^- and z take the same values as in C^1. These chronics differ by their capital vector c and their consumption vector x^+. Let p^1 be an efficient price vector

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with scalars $\alpha_{tn} \geqq 0$, all zero except for a finite number.

Define

$$v_n = \sum_t \alpha_{tn}\, p_t^{(h-1)}, \qquad w_n = \sum_t \alpha_{tn}\, p_{t+1}^{(h-1)}.$$

Clearly, $u_n = v_n + w_n \cdot \{v_n\}$ and $\{w_n\}$ are two bounded nondecreasing sequences of vectors in $\mathbf{C}^{(h-1)}$. They have limits v and w in $\mathbf{C}^{(h-1)}$, with $u = v + w$. It is trivial to note that $(v, -w) \in \mathbf{\Gamma}$.

Hence, for any $u \in \mathbf{C}^{(h)}$, there are two v and $w \in \mathbf{C}^{(h-1)}$, with

(a) $$u = v + w,$$

(b) $$(v, -w) \in \mathbf{\Gamma}.$$

It follows that, for any $u \in \mathbf{C}^{\infty}$, there are two v and $w \in \mathbf{C}^{\infty}$ such that (a) and (b) above are satisfied (Indeed, for all h, $u \in \mathbf{C}^{(h+1)}$. Hence, there are $v^{(h)}$ and $w^{(h)}$ in $\mathbf{C}^{(h)}$ with $u = v^{(h)} + w^{(h)}$ and $(v^{(h)}, -w^{(h)}), \in \mathbf{\Gamma}$. $\{v^{(h)}\}$ is a sequence of positive bounded vectors; it has a limit point v which is in all $\mathbf{C}^{(h)}$. $w = u - v$ is a limit point of $\{w^{(h)}\}$; it is in all $\mathbf{C}^{(h)}$; and $(v, -w) \in \mathbf{\Gamma}$).

Now, there is in $\mathbf{C}^{\infty}$ an extreme element, i.e., an element u such that $u = v + w$ with v and w in $\mathbf{C}^{\infty}$ implies $w = \alpha u = \beta v$ with positive scalars α and β.

Hence, for this element u, $(v, -w) = (v, -\beta v) \in \mathbf{\Gamma}$; which completes the proof.

[32] Similar results may be obtained by an approach more in accordance with the usual technique in capital theory. One can say that a stationary chronic is efficient if it is possible, without any present loss, to pass to some other stationary chronic allowing for a higher consumption.

Or, formally, C^1 is not efficient if there is some possible C such that:

(i) $x \geqslant x^1$; and

(ii) there are some b^2 and z^2 such that $(a^1, b^2) \in \mathbf{T}$, $z^2 \leqq \bar{z}$, $a = b^2 - x^1 + z^2$.

Condition (ii) says that it is possible to go from C^1 to C in one period with a consumption vector equal to x^1.

associated with C^1. The marginal productivity of capital for C^1 is defined as

$$\mu = \operatorname*{Sup}_{c \epsilon \mathfrak{C}} \frac{p^1(x - x^1)}{p^1(c - c^1)}. \tag{63}$$

This formula relates the gain in consumption, $p^1(x - x^1)$, to the corresponding increase of social capital, $p^1(c - c^1)$, both being evaluated from the set of prices p^1; μ is the maximum value taken by this ratio.

Now, from Lemma 7 it directly follows that

$$\rho^1 \geqq \mu, \tag{64}$$

where ρ^1 is the efficient interest rate associated with C^1. One might; moreover, see that the equality holds if **T** is bounded by a differentiable surface.

On the other hand, a long line of economists[33] define marginal productivity of capital as the ratio between the increase in value of consumption, $px - p^1x^1$, to the increase in value of real capital, $pc - p^1c^1$. Or, in our present terminology,

$$\mu' = \operatorname*{Sup}_{c \epsilon \mathfrak{C}} \frac{px - p^1x^1}{pc - p^1c^1}. \tag{65}$$

Clearly, μ' is not related by any definite formula to ρ^1. Thus there is no reason why they should be equal.

There remains the question which of the two definitions should be adopted in economic theory. There seem to be at least three reasons for choosing formula (63). First, it makes the marginal productivity of capital just equal to the interest rate. Second, it is the right measure for the ratio between the permanent future increase in national consumption and the necessary present savings, as one might easily see from our model. From this viewpoint, it provides welfare economics with a concept that has a much more profound meaning than the alternative, μ'. Finally, the definition of μ coincides with the general definition of marginal productivity, while formula (65) does not. Indeed, marginal productivity is always computed with a single set of prices. This may be made clearer if we suppose that $c = c^1 + \gamma_1$, where γ_1 is a given quantity of commodity 1, while the corresponding increase in consumption affects only commodity 2: $x = x^1 + \xi_2$. Formula (63) gives $\mu p_1^1/p_2^1 \geqq \xi_2/\gamma_1$, so that the ratio on the left-hand side is directly related to physical conditions of production, like any other substitution ratio in an efficient position. A similar result does not hold with formula (65).

[33] Cf., for instance, Wicksell [**33**]. For more detailed references the reader may consult Metzler [**23**].

μ is also equal to the marginal productivity of capital such as it is sometimes defined by considering a lengthening of a production or investment period. Indeed, let us compare C^1 with a stationary chronic C absolutely similar except for a one-unit increase of the investment period of commodity 1. If the invested quantity of commodity 1 in C^1 is equal to γ, $c - c^1 = \gamma$ and $\mu \geqq p^1(x - x^1)/p_1^1\gamma$. Thus, μ is also at least equal to the ratio between the increase in the product from a one-unit lengthening of the investment period of some commodity to the value of the quantity annually invested of the same commodity, or, equivalently, to the value which is to be saved on consumption during the present period in order to realize the given lengthening of the investment period. Such was the essential idea behind the Jevonian analysis.

20. *Optimum amount of capital.* The concept of an optimum amount of capital is given in a few places in economic literature.[34] It appears in such situations as the following. The government thinks some sacrifice should be made in order to accumulate enough capital to raise consumption above its present level. The rate of accumulation is not required to be in accordance with present consumers' preferences; these could be neglected if necessary in order to ensure a better future for the community. Is it always profitable for this purpose to increase the quantity of capital? Or is there any optimum beyond which one should rather disinvest than invest?

Indeed, as long as some increase in a_t leads to some increase in b_{t+1}, consumption may be made larger during the next period if it is reduced during the present one. However, it would not be reasonable to impose any given decrease in x_t if the corresponding increase in b_{t+1} becomes too small. This may be better formulated for stationary chronics. For these an increase of the capital vector will be said to be advantageous if it results in a permanent improvement in the future; or, in other words, if the stationary chronic associated with the new capital vector is preferable to that associated with the former one. It may seem likely a priori that the greater the capital vector, the higher the consumption level. This is not necessarily true because in stationary chronics provision must be made for capital replacement. The latter may become so heavy as to exceed the increase in production.

We shall adopt the following formal definition:

DEFINITION: *The efficient stationary chronic C^1 is associated with an optimal capital vector if there is no possible stationary chronic C such that $x \geqq x^1$, whatever the value taken by the capital vector.*[35]

[34] Cf., for instance, Wicksell [**33**, p. 209], Ramsey [**25**], Meade [**22**], Knight [**16**, p. 402], Allais [**3**].

[35] The objection that an optimal capital vector could not conceivably exist has frequently been raised against this concept; i.e., that a complete saturation of all capital needs can never occur, even under ideal conditions (cf., for instance,

We shall show that if some optimal capital vector exists, it is associated with a zero interest rate. By comparison with Lemma 7 this is included in the following:

LEMMA 8: *Under Assumptions* 3 *and* 4, *if* C^1 *is an efficient stationary chronic associated with an optimal capital vector, then there is a nonnegative price vector* p *such that* $p(b - a)$ *is maximum at* (a^1, b^1) *among all* $(a, b) \in \mathbf{T}$.

PROOF: If C^1 is an efficient stationary chronic associated with an optimal capital vector, there is no $(a, b) \in \mathbf{T}$ with $b - a \geqq b^1 - a^1$. Indeed, suppose there is such an (a, b); there would exist a possible stationary chronic C such that

$$x = b - a + \bar{z} \geqq b^1 - a^1 + z^1 = x^1.$$

Consider now the set $\mathbf{U}$ of all $u = b - a$ where $(a, b) \in \mathbf{T}$. $\mathbf{U}$ is convex and has u^1 as a maximal element; hence, there is a nonnegative vector p such that pu is maximum at u^1 among all $u \in \mathbf{U}$.

As we noticed earlier, the rate of interest in a stationary chronic provides a measure of the marginal productivity of capital. It is therefore not surprising to find that it is equal to zero when the capital vector is optimal.

Finally, we must insist on the very restricted meaning of the concept of the optimal amount of capital and, hence, on the restricted applicability of Lemma 8. Indeed, as we have seen, optimal capital vectors cannot be defined except for stationary chronics whose practical significance could be disputed.

21. *Historical note on the theory of capital.*[36] Throughout the preceding pages the traditional theory of capital has been related to the new welfare economics. But this attempt is not new. In economic literature, any sound approach to the analysis of capital formation stemmed from the theory of value whose connection with welfare economics is obvious

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Knight [16]). In the author's view this is not correct. It is indeed true that we shall probably never reach a state of complete saturation of all capital needs, but the reason is psychological or institutional and not technological.

The question of the existence of a stationary chronic C^1 associated with an optimal capital vector would be worth studying. Our present formulation, however, is not suitable for dealing with existence problems in a sufficiently precise way. The reader might find it interesting to consider the following example:

Suppose an economy with three commodities, the available resources vector $\bar{z} = (\bar{z}_1, \bar{z}_2, 0)$, and the technological set defined by $(a, b) \in \mathbf{T}$ if $b_1 = 0$, $(b_2)^2 + (b_3)^2 \leqq 8a_1a_2$. The following stationary chronic is associated with an optimal capital vector:

$$a = (\bar{z}_1, 2\bar{z}_1 + \bar{z}_2, 0) \qquad b = (0, 4\bar{z}_1, \sqrt{8\bar{z}_1\bar{z}_2}), \qquad x = (0, 2\bar{z}_1, \sqrt{8\bar{z}_1\bar{z}_2}).$$

[36] We shall not consider the theories dealing with welfare economics or efficient allocation of resources. For a short analysis of these subjects and references, see Debreu [8].

Thus, it may be worthwhile to compare the main expositions of the theory of capital and interest with the model presented here.

For this purpose we need not consider whether the authors were concerned with problems of equilibrium or with welfare, nor whether they took into account consumers' preferences. Moreover, we need not consider production or distribution theories that take capital as given; indeed, from our viewpoint they miss the essential problem, which is how choices are, or should be, made between direct and indirect processes of production.

We shall examine the principal theories of capital according to two criteria: first, the descriptive scheme of the productive process, and second, the author's solution.

Broadly speaking, the models describing capitalistic production may be classified under four main headings:

First, some theories start from a law, given a priori, of substitution between present and future commodities. This is made quite clear, for instance, in Irving Fisher's theory of interest **[10]**. In this approach the real nature of the substitution is not explored except for some heuristic comments. Thus, the theory is bound either to consider only a particular aspect of production (as, for instance, the growing of trees) or to assume the prices for each period to be independently determined. In this way, the substitution law must be interpreted as relating present to future income. This procedure, used extensively by Fisher, will be examined below.

Second, most theories of capital describe production as the result of the simultaneous operation of numerous elementary processes,[37] each of them specialized in the production of a particular commodity from labor and natural resources. Most often, roundabout methods are introduced so that the final product may be obtained after a very long time. But, in any case, labor and natural resources are considered as the only inputs in the process. Capital goods do not exist as such; they are expressed in terms of the original services invested in them at the time of their production. These services are said to "mature" when the final product is delivered for consumption. Such is the scheme underlying the theories of John Rae [**26**], Jevons [**14**], Böhm-Bawerk [**5, 6**], Wicksell [**33**], Åkerman [**1**], Lindahl [**19**], and Hayek [**12, 13**]. Sometimes it is also supposed that present and future prices are determined independently, so that somewhat less care is required in setting the problem.

To these theories is often attached the concept of the production or investment period. But, although it might be very helpful from an expository viewpoint, it is not at all necessary and could be deleted

[37] Usually the models were not as general as they could have been. Many unnecessary restrictions, which were intended to simplify the theoretical exposition, in fact often resulted in making the subject more abstruse.

altogether. Furthermore, as has been shown repeatedly, the definition of these periods raises innumerable difficulties.

In fact, the fundamental shortcoming of this approach follows from the assumption that it is possible to impute the services of capital goods to the original factors, land and labor. This is surely not the case except in some particular instances. Thus, the whole theoretical construction is dangerously weakened.

As a third alternative one may consider the services used in production as originating either from original sources or from existing equipment. Accordingly, the commodities produced include new durable equipment as well as consumption goods. This approach was used first by Walras [**22**] and more recently by Allais [**2**]. In order to arrive at manageable equations, both supposed that any capital good, once produced, provides a series of services that cannot be altered by more or less intensive utilization. Even so, this third approach seems to provide a good approximation to the conditions of the real world, as was rightly pointed out by Lindahl [**19**] in his penetrating essay.

It is apparent that the theory we have built throughout this paper proceeded from an attempt to give to Walras' model a more general content and to explain how a substitution law may be obtained from it.

Finally, it is also possible to give a simple and completely general description of production if the economy is assumed to be stationary. In this case there is a law relating capital equipment to the permanent consumption which it makes possible. This is the idea underlying most of Professor Knight's writings [**16**]. One may wonder, however, whether his analysis can provide an answer to the question: Why should the study of stationary, and therefore artificial, economies enable us to understand the conditions of production in our changing world? Moreover, as we have seen, the efficiency of any stationary chronic cannot be determined except by comparison with other chronics that are not stationary.

It may be noted also that a stationary economy has often been assumed in theories classified under the second heading (such as those of Jevons and Wicksell), but it does not play there the essential part it does in Professor Knight's treatment.

What sort of answers do the theories give? Here again we may group them under three headings.[38]

First, a few of them try to determine which relations must hold for a firm in a competitive economy. They more or less implicitly assume that these also hold for the whole economy. This is particularly clear in

[38] To do full justice to earlier theories, we should mention that they also wanted at times to study the effects of capital increments on wages, or similar questions related to distribution theory. But this does not concern us here.

papers by Åkerman [1], Leontief [20], Schneider [30], and Boulding [7]. The approach is, indeed, quite successful because it provides a simple answer to a difficult problem. However, a doubt may remain as to the generality of the results. Clearly, also, it is not suitable for dealing with efficiency or welfare.

Second, most theories aim at determining the interest rates, assuming the prices for all periods given a priori.[39] Although this method may bring sound results, there are strong objections to it. In the first place, prices are determined at the same time as interest rates; it is just in the philosophy of capitalistic production that no simple dichotomy exists between the markets for present and future goods. One may wonder, moreover, whether it has always been realized that interest rates to be associated with chronics do not exist independently of the monetary conditions ruling the economy. If any misunderstanding arose on that point, it should surely be attributed to those writers who studied interest formation independently of price formation.

Finally, a few writers did show how prices and interest rates were simultaneously determined. They made quite clear the connection between interest and the general theory of value. To the author's knowledge, Böhm-Bawerk [5], Wicksell [33], Landry [18], Lindahl [19], and Allais [2] provided us with valuable theories of capital. Unfortunately, their writings were largely misunderstood, if not unknown. The diffusion of their main ideas was greatly hampered by endless discussions on details in their exposition. It was the purpose of the present paper to make the analysis more general, and it is hoped in this way to help avoid in the future such lengthy debates as have occurred on the theory of capital in the past.

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