THE WORK OF EINSIEDLER, KATOK AND LINDENSTRAUSS ON THE LITTLEWOOD CONJECTURE

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Contents

This document is intended as a (slightly expanded) writeup of my (anticipated) talk at the AMS Current Events Bulletin in New Orleans, January 2007. It is a brief report on the work of Einsiedler, Katok and Lindenstrauss on the Littlewood conjecture [5].

It is not intended in any sense for specialists and is, indeed, aimed at readers without any specific background either in measure theory, dynamics or number theory.

Any reader with any background in ergodic theory will be better served by consulting either the original paper, or one of the surveys written by those authors: see [7] and [12].

1. The Littlewood conjecture

1.1. For $x \in \mathbb{R}$, let $\|x\|$ denote distance from x to the nearest integer.

It is not difficult to check that, for any $\alpha \in \mathbb{R}$, there exists integers p, q with $1 \le q \le Q$ and $|\alpha - \frac{p}{q}| \le \frac{1}{qQ}$. In other words, $\|q\alpha\| \le 1/Q$. The behavior of $\|q\alpha\|$, as q varies through integers, thereby reflects *approximation of α by rational numbers.*

The Littlewood conjecture concerns simultaneous approximation of *two* numbers α, β by irrationals. It asserts that:

$$\liminf n.\|n\alpha\|\|n\beta\| = 0, \tag{1}$$

whatever be α, β. In words it asserts (in a somewhat peculiar-seeming way)

(2) α, β may be simultaneously approximated, moderately well,
by rationals with the same denominator

My goal is to discuss, and give some of the context around, the following theorem of M. Einsiedler, A. Katok and E. Lindenstrauss in [5]:

1.1. **Theorem.** *The set of* α, β *for which* (1) *fails, has Hausdorff dimension* 0.

The Theorem is proved using ideas from dynamics: namely, by studying the action of coordinate dilations (e.g. $(x, y, z) \mapsto (\frac{x}{2}, 2y, z)$) on the space of *lattices* in $\mathbb{R}^3$. It is not important solely as a result about simultaneous Diophantine approximation, but because of the techiques and results in dynamics that enter into its proof.

Several applications of this type of dynamics are surveyed in [7]. For now it is worth commenting on two rather different contexts where exactly the same dynamics arise:

- In the study of analytic behavior of automorphic forms (see [17] for discussion and historical context)
- In the study of the analytic behavior of ideal classes in number fields, see [8].

1.2. **This document.** I will try to stress:

(1) Dynamics arises from a (not immediately visible) symmetry group; see §1.3; I will then discuss some historical context for this type of connection (§2, §3).
(2) The dynamics that is needed is similar to the simultaneous action of $x \mapsto 2x, x \mapsto 3x$ on $\mathbb{R}/\mathbb{Z}$; see §4.3 for a description of these parallels.
(3) A sketch of just one of the beautiful ideas that enters in proving Theorem 1.1 (see §5), which is to study the picture transverse to the acting group.

A massive defect of the exposition is that I will make almost no mention of *entropy.* This is an egregious omission, because the intuition which comes from the study of entropy underpins much of the recent progress in the subject. However, any serious discussion of entropy this would require more space and time and competence than I have, and better references are available. So, instead, I have given a somewhat *ad hoc* discussion adapated to the cases under consideration.

I will not come even close to sketching a proof of the main result.

Let us make two notes before starting any serious discussion:

(1) The Littlewood conjecture, (1), is quite plausible. Here is a naive line of heuristic reasoning that supports it. A consequence of what we have said in §1.1 is that there exists a sequence $q_k \to \infty$ of positive integers so that $q_k \|q_k \alpha\| \leq 1$. Barring some conspiracy to the

contrary, one might expect that $\|q_k\beta\|$ should be small *sometimes.* The problem in implementing this argument is that we have rather little control over the q_k.[1]

(2) Despite all the progress that I shall report on, we do not know that the statement (1) is true even for $\alpha = \sqrt{2}, \beta = \sqrt{3}$. The question of removing the exceptional set in Theorem 1.1 is related to celebrated conjectures (see Conjecture 4.1 and Conjecture 4.2) of Furstenberg and Margulis.

1.3. **Symmetry.** The next point is that the question (1) has a symmetry group that is not immediately apparent. This is responsible for our ability to apply dynamical techniques to it.

Pass to a general context for a moment. Let $f(x_1, \ldots, x_n)$ be an integral polynomial in several variables. An important concern of number theory has been to understand *Diophantine equalities:* solutions to $f(\mathbf{x}) = 0$ in integers $\mathbf{x} \in \mathbb{Z}^n$ (e.g. does $x^2 - y^2 - z^2 = 1$ have a solution? Does $x^3 + y^3 = z^3$ have a solution?)

A variant of this question, somewhat less visible but nonetheless (in my opinion) difficult and fascinating, concerns *Diophantine inequalities:* if f does not have rational coefficients, one may ask about the solvability of an equation such as $|f(\mathbf{x})| < \varepsilon$ for $\mathbf{x} \in \mathbb{Z}^n$ (e.g. does $|x^2 + y^2 - \sqrt{2}z^2| < 10^{-6}$ have a solution?)

In the most general context of an arbitrary f, our state of knowledge is somewhat limited. On the other hand, for special classes of f we know more: a typical class which is accessible to analytic methods is when the *degree of f is small compared to the number of variables.*

Another important class about which we have been able to make progress, consists of those f possessing symmetry groups. Both the examples $x^2 - y^2 - z^2 = 1$ and $x^2 + y^2 - \sqrt{2}z^2$ admit orthogonal groups in three variables as automorphisms.[2] The homogeneous equation $x^3 + y^3 = z^3$ has symmetry but not by a linear algebraic group (it defines an elliptic curve inside $\mathbb{P}^2$).

The Littlewood conjecture also has symmetry, although not immediately apparent. To see it, we note that $\|x\| = \inf_{m \in \mathbb{Z}} |x - m|$; consequently, we may rewrite (1) as the statement:

(3)
$$|n(n\alpha - m)(n\beta - \ell)| < \varepsilon \text{ is solvable, with } (n, m, \ell) \in \mathbb{Z}^3, n \neq 0, \text{ for all } \varepsilon > 0$$

[1] Amusingly, it is not even clear this heuristic argument will work. It may be shown that given a sequence q_k so that $\liminf q_{k+1}/q_k > 1$, there exists $\beta \in \mathbb{R}$ so that $\|q_k\beta\|$ is bounded away from 0. See [14] for this and more discussion.

[2] Although unimportant in the context of this paper, there is an important difference: while $x^2 + y^2 - \sqrt{2}z^2$ admits an action of the real Lie group $\mathrm{O}(2,1)$, the analysis of the form $x^2 + y^2 + z^2$ involves studying the action of the much larger *adelic* Lie group of automorphisms. In particular, this adelic group is noncompact, even though the real group $\mathrm{O}(3)$ is compact, and this is a point that can be fruitfully exploited; see [2].

But the function $L(n,m,\ell) = n(n\alpha - m)(n\beta - \ell)$ is a product of three linear forms and admits a two-dimensional torus as group of automorphisms.[3]

2. The Oppenheim conjecture

Here we pause to put the developments that follow into their historical context. The reader may skip directly to §4.

2.1. **Statement of the Oppenheim conjecture.** We briefly discussed above the form $x^2 + y^2 - \sqrt{2}z^2$. This is a particular case of a problem considered in the 1929: A. Oppenheim conjectured that if $Q(x_1, \ldots, x_n) = \sum_{i,j} a_{ij}x_ix_j$ is an *indefinite* quadratic form in $n \geq 3$ variables which is not a multiple of a rational form, then Q takes values which are arbitrary small, in absolute value.

In other words – note the analogy with (3) –

$$|Q(\mathbf{x})| < \varepsilon \text{ is solvable, with } \mathbf{x} \in \mathbb{Z}^n, \text{ for all } \varepsilon > 0 \tag{4}$$

When n is sufficiently large his conjecture was solved by Davenport (in 1956) by analytic methods. His paper required $n \geq 74$. This is an example of the fact, noted in §1.3, that purely analytic methods can often handle cases when the number of variables is sufficiently large relative to the degree.

On the other hand, the complete resolution of the conjecture[4] had to wait until G. Margulis, in the early 1980s, gave a complete proof using dynamical methods that made critical use of the group of automorphisms of Q.

2.2. **Symmetry.** Let $H = \mathrm{SO}(Q)$, the group of orientation-preserving linear transformations of $\mathbb{R}^n$ preserving Q. By definition $Q(\mathbf{x}) = Q(h.\mathbf{x})$. We wish to exploit[5] the fact that H is large.

In particular, in order to show (4), it suffices to show that Q takes values in $(-\varepsilon, \varepsilon)$ at a point of the form $h.\mathbf{x}\,(h \in H, \mathbf{x} \in \mathbb{Z}^n)$. *A priori*, this set might be much larger than $\mathbb{Z}^n$; certainly, if it were dense in $\mathbb{R}^n$, this would be enough to show (4).

For instance, if we could prove that

$$\text{The set } h.\mathbf{x} : h \in H,\ \mathbf{x} \in \mathbb{Z}^n \text{ contains 0 in its closure} \tag{5}$$

then (4) would follow immediately.

[3]The n-dimensional version of the Littlewood conjecture takes n linear forms $\ell_1, \ldots, \ell_n$ and asks: is the equation $0 < |\ell_1(\mathbf{x}) \ldots \ell_n(\mathbf{x})| < \varepsilon$ solvable? Conjecturally, this is so if $n \geq 3$ and $\ell_1 \ldots \ell_n$ is not a multiple of a rational polynomial. It is false for $n = 2$, see footnote 4.

[4]The analogous statement is false for $n = 2$: take, e.g. $Q(x,y) = (x - \sqrt{2}y)y$. To see that, write $Q(x,y) = \frac{(x^2-2y^2)y}{(x+\sqrt{2}y)}$.

[5]The idea that this should be exploitable was suggested by M. Raghunathan. It is also implicitly used in a paper of Cassels and Swinnerton-Dyer from the 1950s.

2.3. **Lattices.** (5) is rather nice, but a little unwieldy. We would rather deal with the H-orbit of a single point rather than an infinite collection. This can be done by "packaging" all $\mathbf{x} \in \mathbb{Z}^n$ into a single object: a *lattice*.

A *lattice* in $\mathbb{R}^n$ is simply a "grid containing the origin", i.e. a set of all *integral* combinations of n linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Every such lattice is of the form $g.\mathbb{Z}^n$ for some $g \in \mathrm{GL}(n, \mathbb{R})$.

Let $\widetilde{\mathcal{L}}_n$ be the set of lattices[6] and let $\widetilde{\mathcal{L}}_n[\varepsilon]$ be the set of lattices that contain $\mathbf{v} \in \mathbb{R}^n$ with Euclidean length $\|\mathbf{v}\| \leq \varepsilon$. So $\mathbb{Z}^n$ can be thought of as a point $[\mathbb{Z}^n] \in \widetilde{\mathcal{L}}_n$.

Then (5) would follow if:

$$H.[\mathbb{Z}^n] \cap \widetilde{\mathcal{L}}_n[\varepsilon] \neq \emptyset, \text{ for all } \varepsilon > 0 \tag{6}$$

This is a statement that fits cleanly into the context of dynamics: does the orbit of the point $\mathbb{Z}^n \in \widetilde{\mathcal{L}}_n$, under the group H, intersect the subset $\widetilde{\mathcal{L}}_n[\varepsilon]$? It is (6) which was proven by Margulis.

2.4. **Background on the space of lattices.** To each lattice we can assign a natural invariant, its *covolume*. This is the absolute value of the determinant of the matrix with rows $\mathbf{v}_1, \ldots, \mathbf{v}_n$; that is to say, the volume of a fundamental parallelpiped $\sum \lambda_i \mathbf{v}_i : \lambda_i \in [0, 1)$. For $g \in \mathrm{GL}(n, \mathbb{R})$ and $L \in \widetilde{\mathcal{L}}_n$, we observe that $\mathrm{covol}(g.L) = |\det g| \mathrm{covol}(L)$. In particular, because all $h \in H$ have determinant 1, all elements in $H.\mathbb{Z}^n$ have covolume 1. So $H.\mathbb{Z}^n$ belongs to the subset

$$\mathcal{L}_n = \{L \in \widetilde{\mathcal{L}}_n : L \text{ has covolume } 1.\} \tag{7}$$

The space $\mathcal{L}_n$ is more pleasant to work with than $\widetilde{\mathcal{L}}_n$. The map $g \mapsto g.[\mathbb{Z}^n]$ identifies $\widetilde{\mathcal{L}}_n$ with the quotient $\mathrm{GL}(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{Z})$ and $\mathcal{L}_n$ with the quotient $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$. These identifications give rise to topologies on $\widetilde{\mathcal{L}}_n$ and $\mathcal{L}_n$; indeed, they are given the structure of manifolds.

Although $\mathcal{L}_n$ is not compact, it admits a natural $\mathrm{SL}(n, \mathbb{R})$-invariant measure which has finite volume, which is a reasonable substitute for compactness. Moreover, Mahler's criterion gives a precise description of in what way $\mathcal{L}_n$ fails to be compact:

2.1. **Theorem.** *A subset $K \subset \mathcal{L}_n$ is bounded (=precompact) if and only if it does not intersect $\mathcal{L}_n[\varepsilon]$, some $\varepsilon > 0$.*

In words, it asserts that the only way that a sequence of lattices $L_1, L_2, \ldots$ in $\mathcal{L}_n$ can degenerate (leave any compact set in $\mathcal{L}_n$) is if there exist vectors $\mathbf{v}_1 \in L_1, \mathbf{v}_2 \in L_2, \ldots$ so that $\|\mathbf{v}_i\| \to 0$.

We may therefore rephrase (6): The Oppenheim conjecture would follow if

$$H.[\mathbb{Z}^n] \text{ is unbounded in } \mathcal{L}_n. \tag{8}$$

[6]We will later work almost exclusively with the subset of $\mathcal{L} \subset \widetilde{\mathcal{L}}_n$ consisting of lattices of volume 1; therefore, for notational simplicity, we prefer to put a tilde for the whole space of lattices and omit it for the subset $\mathcal{L}_n$.

3. Unipotents acting on lattices.

Obviously, the statement (4) is false for Q positive definite, and, as observed in footnote 4, (less obviously) false for Q in two variables. How are we to detect this difference when considering the problem from the dynamical viewpoint of (6) or (8)?

3.1. **Unipotents from Margulis to Ratner.** An important difference is that the group H is isomorphic to $\mathrm{SO}(n) \subset \mathrm{GL}(n, \mathbb{R})$ in the first case, and $\mathrm{SO}(1,1) \subset GL(2, \mathbb{R})$ in the second case. In either case, *the group H consists entirely of semisimple elements.* Margulis' idea was to exploit the fact that, if Q is indefinite in $n \geq 3$ variables, the group H contains *unipotent* elements, i.e. $g \in \mathrm{GL}(n, \mathbb{R})$ for which all of the (generalized) eigenvalues of g are equal to 1.

At a vague level, the reason why these might be helpful is quite easy to state: if $u \in \mathrm{GL}(n, \mathbb{R})$ is unipotent, the matrix entries of u^n grow only *polynomially* in n. This contrasts sharply with the behavior of a "typical" element $g \in \mathrm{GL}(n, \mathbb{R})$, when these entries will grow expontially. This means that, when studying the trajectory $ux_0, u^2x_0, u^3x_0, \ldots$, we are able to "retain information" about it for much longer.

3.2. **Ratner's theorem.** We will not say anything about the specifics of Margulis' proof; see [1] for an elementary presentation. A far-reaching generalization of Margulis' result, which has been of fundamental importance for later work, is the following (special case of a) theorem of Ratner, see [15] and [16]: [7]

3.1. **Theorem.** *Let $H \subset \mathrm{SL}(n, \mathbb{R})$ be generated by one-parameter unipotent subgroups.*[8] *The closure of the orbit $\overline{H.[\mathbb{Z}^n]}$ inside $\mathcal{L}_n$ is of the form $H'.[\mathbb{Z}^n]$ for a closed subgroup $H' \geqslant H$. Moreover, there exists an H'-invariant probability measure on $H'.[\mathbb{Z}^n]$.*

This is a difficult theorem, which settled a conjecture of M. Raghunathan. The orbit $H.[\mathbb{Z}^n]$ can be extremely complicated. Ratner's theorem asserts that its closure is determined by a very simple piece of algebraic data: a subgroup intermediate between H and $\mathrm{SL}(n, \mathbb{R})$.

Let us see how this implies (6). The group $H = \mathrm{SO}(Q)$ is *maximal* inside $\mathrm{SL}(n, \mathbb{R})$. So Theorem 3.1 means that either $H.[\mathbb{Z}^n]$ is closed or $H.[\mathbb{Z}^n]$ is dense in $\mathcal{L}_n$. It may be seen that $H.[\mathbb{Z}^n]$ is closed only if the form Q is a multiple of a rational form. In this fashion, Theorem 3.1 implies the Oppenheim conjecture.

[7] Ratner's theorem is phrased not just about spaces like $\mathcal{L}_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$, but more general quotients of Lie groups by discrete subgroups.

[8] i.e. of the form $\exp(tX)$ where X is a nilpotent matrix.

3.3. **An idea from the proof of Theorem 3.1: Measures not sets.** Because our concern is not with unipotent dynamics here, we will not try to indicate any of the ideas of the proof of Theorem 3.1 that are specific to properties of the unipotent flows.

Instead, we will emphasize a more philosophical point from the proof of Theorem 3.1 that has been indispensable in later work.

(9) Measures are often easier to work with than sets.

To be a little more specific, let us comment on how Ratner's proof of Theorem 3.1 works. Let us take the simple case when H consists *entirely* of unipotent elements. (A comprehensive exposition of the proof is to be found in [18]).

Ratner begins by classifying the *probability measures* on $\mathcal{L}_n$ that are invariant under H. The topological statement of Theorem 3.1 is then *deduced* from the classification of H-invariant probability measures.

The relation between probability measures and invariant sets is quite simple: an invariant probability measure has a support, which is a closed H-invariant set. Conversely, $Y \subset \mathcal{L}_n$ is an H-invariant closed set, it must support an H-invariant probability measure (average your favorite measure under H – note that this requires H to be amenable.) This relation is a good deal more tenuous than one would like – the support of the measure constructed this way may be strictly smaller than Y – and the deduction of statements concerning invariant sets from statements about probability measures is not formal.

Nonetheless, what is gained by going through measures? Measures have much better formal properties than sets. A particularly important difference is that an H-invariant probability measure can be decomposed into "minimal" invariant measures (ergodic decomposition). [9] That property does not seem to have a clean analogy at the level of H-invariant closed sets. In particular, an H-invariant closed set always *contains* a minimal H-invariant closed set, but cannot be decomposed into minimal H-invariant closed sets in any obvious way.

This is not to say that it is necessarily impossible to prove Theorem 3.1 by purely topological methods. Indeed, Margulis' original proof of (6) was purely topological (and utilized a study of minimal H-invariant closed sets). But, to my knowledge, no such proof has been carried out in the general case.

[9] The set of H-invariant probability measures forms, clearly, a convex set in the space of all probability measures. Any point in this convex set can be expressed as a convex linear combination of extreme points. These extreme points are called *ergodic* measures for H and are "minimal", in the sense that they cannot be expressed nontrivially as an average of two other H-invariant probability measures.

4. The dynamics of coordinate dilations on lattices, I: conjectures and analogies.

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We have seen that the assertion (4) about the values of the quadratic form $x^2 + y^2 - \sqrt{2}z^2$ can be converted to the assertion (6) about the orbit of $[\mathbb{Z}^n]$ under the group $H = \mathrm{SO}(Q)$. We now briefly carry through the corresponding reasoning in the case of the Littlewood conjecture. This will lead us to study the action of the diagonal group A_3 inside $\mathrm{GL}(3, \mathbb{R})$, on $\mathcal{L}_3$.

4.1. **Reduction to dynamics.** Let $P(x_1, x_2, x_3) = x_1(\alpha x_1 - x_2)(\beta x_1 - x_3)$. We have seen (see (3)) that the Littlewood conjecture is (almost, with a constraint $x_1 \neq 0$) equivalent to the assertion that $|P(\mathbf{x})| < \varepsilon$ is solvable. Let T be the *automorphism group of* P, that is to say, the set of $g \in \mathrm{GL}(3, \mathbb{R})$ such that $P(g.\mathbf{x}) = P(\mathbf{x})$. T contains a conjugated copy of the group of diagonal matrices. [10]

So $P(a.\mathbf{x}) = P(\mathbf{x})$ for $a \in T$. It would appear to be enough to show that $\{a.\mathbf{x} : a \in T, \mathbf{x} \in \mathbb{Z}^n\}$ approaches arbitrarily close to 0; or, repeating the line of implications (4) $\Longleftarrow$ (6) $\Longleftarrow$ (8), it seems to be enough to show that $T.[\mathbb{Z}^n]$ is unbounded in $\mathcal{L}_n$.

This is not quite right, though: $T.[\mathbb{Z}^n]$ being unbounded in $\mathcal{L}_n$ indeed would produce solutions to $|x_1(x_1\alpha - x_2)(x_1\beta - x_3)| < \varepsilon$, but, regrettably, provides no guarantee that $x_1 \neq 0$.

However, this can be avoided by replacing T with a certain subsemigroup $T^+ \subset T$ engineered specifically to avoid this. Moreover, T contains a conjugate copy of A_3 as a finite index subgroup, we can rephrase this assertion in terms of the dynamics of A_3, not of T.

We will not go through the details, but rather will explicate the result of going through this process: if $L_{\alpha,\beta} \subset \mathbb{R}^3$ is the lattice spanned by $(1, \alpha, \beta), (0, 1, 0)$ and $(0, 0, 1)$, the Littlewood conjecture for (α, β) is equivalent to:

(10)

$$A_3^+.L_{\alpha,\beta} \text{ is unbounded in } \mathcal{L}_n, A_3^+ = \{\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} : x \leq 1, y \geq 1, z \geq 1\}$$

The reader can easily verify (10) directly.

We are led to study the action of A_n on $\mathcal{L}_n$, and in particular, to seek an analogue of Theorem 3.1. The obstacle will be that the analogue of Theorem 3.1 *totally fails* for (conjugates of) A_2 acting on $\mathcal{L}_2$. There exists a plethora of orbit closures that do not correspond to closed orbits of intermediate subgroups $A_2 \leqslant H \leqslant \mathrm{SL}(2, \mathbb{R})$. (This corresponds roughly to the fact

[10] In a suitable coordinate system, P becomes $P(x_1, x_2, x_3) = x_1x_2x_3$. But the set of linear transformations that preserve $(x_1, x_2, x_3) \mapsto x_1x_2x_3$ consist of all permutation matrices whose determinant is ± 1, according to the sign of the permutation.

that there are many α for which $\liminf n\|n\alpha\| > 0$, i.e. the "one variable" Littlewood conjecture is false.)

4.2. **An analogy with $\times 2 \times 3$ on S^1.** Let us reprise: we are studying the action of the group A_n (diagonal matrices of size n, with determinant 1) on the space $\mathcal{L}_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$; or, geometrically, we are studying the action of *coordinate dilations* on grids in $\mathbb{R}^n$.

A very helpful analogy in studying the action of A_n on $\mathcal{L}_n$ is the following:

$$\text{Action of } A_2 \text{ on } \mathcal{L}_2 \text{ behaves like } x \mapsto 2x \text{ on } \mathbb{R}/\mathbb{Z}; \tag{11}$$

$$\text{Action of } A_3 \text{ on } \mathcal{L}_3 \text{ behaves like } x \mapsto 2x, x \mapsto 3x \text{ on } \mathbb{R}/\mathbb{Z} \tag{12}$$

Note that A_3 is a two-parameter (continuous) group, whereas $x \mapsto 2x, x \mapsto 3x$ generate a two-parameter (discrete) semigroup.

These analogies appear to be quite strong, although I do not know of any entirely satisfying "reason" for them. The analogy between $(A_2, \mathcal{L}_2)$ and $(\times 2, \mathbb{R}/\mathbb{Z})$ is particularly strong: in a fairly precise sense[11], the action of a suitable element $a \in A_2$ on $\mathcal{L}_2$ behaves like a shift on $\{0,1\}^{\mathbb{Z}}$, whereas the action of $x \mapsto 2x$ behaves like a shift on $\{0,1\}^{\mathbb{N}}$. We will list in the next section some results and questions in both the $\mathcal{L}$ and $\mathbb{R}/\mathbb{Z}$ cases and see they are quite analogous.

For the moment, let us just observe that the action of $x \mapsto 2x$ on $\mathbb{R}/\mathbb{Z}$ is fundamentally different to the simultaneous action of $x \mapsto 2x, x \mapsto 3x$. Indeed, the trajectory $\{2^n x\}$ of a point under $x \mapsto 2x$ essentially encodes the binary expansion of x, which can be arbitrarily strange (cf. Lemma 4.1). For instance, there exist uncountably many possibilities for the closure $\overline{\{2^n x\}}$. On the other hand, it is much more difficult to arrange that the binary and ternary expansions of a given x be *simultaneously* strange. This means it is much harder to arrange that the orbit of x under $x \mapsto 2^n 3^m x$ be strange, and indeed it is known that the possibilities for the closure $\overline{\{2^n 3^m x\}}$ are very simple (see Theorem 4.1).

Correspondingly, one might hope that the fact that Theorem 3.1 fails for $(A_2, \mathcal{L}_2)$, as commented at the end of §4.1, might be a phenomenon that vanishes when one passes to $(A_n, \mathcal{L}_n)$ for $n \geq 3$. Indeed, this is believed to be largely the case.

4.3. **Conjectures and results for $\times 2 \times 3$ and for A_n.** Recall that a probability measure ν invariant under a group G is said to be G-ergodic if any G-invariant measurable subset S has either $\nu(S) = 1$ or $\nu(S) = 0$. An equivalent definition is found in footnote 9. We observe that a classification of G-invariant ergodic probability measures is as good as a classification of G-invariant probability measures, for any G-invariant probability measure can be expressed as a convex combination of G-invariant ergodic probability measures.

[11] e.g. the systems are measure-theoretically isomorphic

We shall also make use in this section of the notion of *positive entropy*; for a definition see (19), but the reader might be better served by simply treating it as a black-box notion for the moment and reading on.

Formalizations of some of the intuitions we suggested in the previous section are to be found in the following results. They state, in that order, that:

- There are a huge number of closed invariant sets for $x \mapsto 2x$.
- There are very few closed invariant sets for $x \mapsto 2x, x \mapsto 3x$ simultaneously (and a clean classification)
- Conjecturally, there are very few invariant probability measures under $x \mapsto 2x, x \mapsto 3x$;
- One can prove the third assertion under an additional assumption on the measure, positive entropy.

4.1. **Lemma.** *There exists orbit closures $\{\overline{2^n x}\}_{n\geq 0}$ of any Hausdorff dimension between* 0 *and* 1.

Similarly, there exist "very many" probability measures on $\mathbb{R}/\mathbb{Z}$ invariant under $x \mapsto 2x$. (A measure is said to be invariant under $x \mapsto 2x$ if the integral of $f(x)$ and $f(2x)$ is the same, for f a continuous function).

4.1. **Theorem.** *(Furstenberg) The orbit closure $\overline{\{2^n 3^m x\}}_{n,m\geq 0}$ is $\mathbb{R}/\mathbb{Z}$ or finite, according to whether x is irrational or rational.*

4.1. **Conjecture.** *(Furstenberg) Let μ be a probability measure on $\mathbb{R}/\mathbb{Z}$ that is invariant under $x \mapsto 2x$ and $x \mapsto 3x$ and ergodic w.r.t. $x \mapsto 2x, x \mapsto 3x$. Then μ is either Lebesgue measure, or supported on a finite set of rationals.*

4.2. **Theorem.** *(Rudolph) Let μ be a probability measure on $\mathbb{R}/\mathbb{Z}$ that is invariant under $x \mapsto 2x$ and $x \mapsto 3x$ and ergodic w.r.t. $x \mapsto 2x, x \mapsto 3x$, and so that either $\times 2$ or $\times 3$ acts with positive entropy. Then μ is Lebesgue measure.*

Now let us enunciate the analogues of these statements for A_n acting on $\mathcal{L}_n$. They state, in this order, that:

- There are a huge number of orbit closures and invariant measures for A_2 acting on $\mathcal{L}_2$.
- Conjecturally, there are very few closed sets for A_n acting on $\mathcal{L}_n$, when $n \geq 3$. The statement here is not as satisfactory as in the $(\times 2 \times 3, \mathbb{R}/\mathbb{Z})$ case.
- Conjecturally, there are very few invariant probability measures on $\mathcal{L}_n$ under A_n, when $n \geq 3$.
- One can prove the third assertion under an additional assumption on the measure, positive entropy.

4.2. **Lemma.** *There exists orbit closures $\overline{A_2.x}$ of any Hausdorff dimension between* 1 *and* 3.

Similarly, there exists "very many" probability measures on $\mathcal{L}_2$ invariant by A_2.

The following conjectures are stated (in a considerably more general form) in [13].

4.2. **Conjecture.** *(Margulis) The orbit closure $\overline{A_n.x}$ (for $n \geq 3$ and $x \in \mathcal{L}_n$) is,* if compact*, a closed A_n-orbit.* [12]

4.3. **Conjecture.** *(Margulis) Let μ be a probability measure on $\mathcal{L}_n$ that is invariant under A_n and ergodic w.r.t. A_n. Then*[13] *μ coincides with the H'-invariant measure on a closed orbit $H'x_0$, for some subgroup $A_n \leqslant H' \leqslant \mathrm{SL}(n, \mathbb{R})$.*

4.3. **Theorem.** *(Einsiedler-Katok-Lindenstrauss) Let μ be a probability measure on $\mathcal{L}_n$ that is invariant under A_n and ergodic w.r.t. A_n, so that* some element of A_n acts with positive entropy*. Then μ is algebraic.*

Theorem 4.3 is the main theorem of [5]. The result concerning Littlewood's conjecture is deduced from it. It should be noted that, while Theorem 4.3 is closely analogous to Theorem 4.2, the technique of proof is quite different.

In the remainder of this article, we shall indicate one key idea that enters not only into the proof of 4.3, but into the proof of all results in that line proved so far, including [4] and [11].

5. Coordinate dilations acting on lattices, II: the product lemma of Einsiedler-Katok

The contents of this section are sketchy and impressionistic! For concreteness, we will primarily confine ourselves to the action of A_3 on $\mathcal{L}_3$.

The main thing which the reader might come away with is the importance and naturality of *conditional measures.* The study and usage of conditional measures is a formalization of the following natural idea: given an A_3-invariant measure μ on $\mathcal{L}_3$, study μ along slices transverse to A_3. Note that the action of A_3 contracts part of these slices and dilates other parts.

The ideas we will discuss in this section are contained in the important paper [4] of Einsiedler and Katok; we will not discuss the new ideas introduced in [5]. Those new ideas stem from [11] and are, indeed, essential to get the main result on the Littlewood conjecture. On the other hand, the ideas from [4] that we now discuss have been fundamental in all the later work in this topic.

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[12] This is not quite as good as a complete classification of orbit closures, and, indeed, [13] posits a more precise classification. Conjecture 4.2 is just a simple clean statement that can be extracted from this classification.

[13] i.e. "the measure-theoretic analogue of Theorem 3.1 holds for A_n acting on $\mathcal{L}_n$."

5.1. **Closed sets.** Before we embark on describing some of the ideas in [4], we begin by explaining how one might try to approach the analysis of A_3-invariant closed sets. We then explain – in the spirit of §3.3 – why it might be helpful to switch to measures.

Suppose $\sigma \subset \mathcal{L}_3$ is an A_3-invariant closed set.

We wish to study the behavior of σ in directions transverse to A_3. Let e_{ij} be the elementary matrix with a 1 in the (i,j) position and 0s everywhere else; for $i \neq j$ let $n_{ij}(x) = \exp(x.e_{ij})$. Then $N_{ij} = \{n_{ij}(x) : x \in \mathbb{R}\}$ is a subgroup of $SL_3(\mathbb{R})$.

A natural way of studying, then, of how σ behaves *transverse to* A_3 are the subsets:

$$\sigma_x^{ij} := \{t \in \mathbb{R} : n_{ij}(t)x \in \sigma\} \subset \mathbb{R}$$

This set is a closed subset of $\mathbb{R}$ and is defined for all $x \in \mathcal{L}_3$.

Now, we wish to use the fact that a typical element $a \in A_3$ can contract some N_{ij}s and expand others.

Let us take an explicit example: the matrix $a = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. It centralizes N_{12} but it *shrinks* N_{23}, that is to say:

$$a.n_{23}(x).a^{-1} = n_{23}(x/8)$$

Now consider two points $x_1, x_2 \in \sigma$ which lie along the N_{23} direction from one another, i.e. $x_2 \in N_{23}x_1$. Let us compare $\sigma_{x_1}^{12}$ and $\sigma_{x_2}^{12}$. Because our element a centralizes N_{12},

$$\sigma_{x_1}^{12} = \sigma_{ax_1}^{12} = \sigma_{a^2x_1}^{12} = \dots \text{ AND } \sigma_{x_2}^{12} = \sigma_{ax_2}^{12} = \sigma_{a^2x_2}^{12} = \dots. \tag{13}$$

But $a^k x_1$ and $a^k x_2$ are becoming very close as $k \to \infty$ – because a shrinks the direction N_{23}. Therefore, if we had some version of the statement

$$\text{Wishful thinking: as } x \text{ approaches } y,\ \sigma_x^{12} \text{ approaches } \sigma_y^{12} \tag{14}$$

we could deduce from (13) – by considering $a^k x_1, a^k x_2$ as $k \to \infty$ – the following surprising fact:

$$\sigma_{x_1}^{12} = \sigma_{x_2}^{12} \quad \text{(NOT proved, based on wishful thinking!)} \tag{15}$$

In other words, were some version of (14) true: we would have a rather weak version of the following statement: the behavior of a closed set σ in the N_{12}-direction, is constant along the N_{23}-direction. It is not immediate how to *use* this, but nonetheless it is an important structural fact. (See discussion after Lemma 5.1 for an indication of how the measure-theoretic version of this fact is used). It is quite surprising, because we assumed nothing about the behavior of σ besides A_3-invariance.

In order to get any mileage, of course, we need to be able to find points x_1, x_2 which differ in the N_{23} directions; or equivalently, the sets $\sigma_x^{N_{23}}$ should have more than one point. So in order to have any hope of using this entire setup, we should also have:

(16) The sets $\sigma_x^{N_{ij}}$ should not always be singletons.

Let me emphasize that the above is, indeed, essentially wishful thinking, and is based on the rather baselessly optimistic (14). The surprising fact is that, by working with measures, we can salvage a version of (14).

5.1. **Example.** *Take a closed subset S of the square $[0,1]^2$. Let $\pi : [0,1]^2 \to [0,1]$ be the projection. For each $x \in [0,1]$, we can consider the set $S_x = \pi^{-1}(\{x\}) \cap S$. There is no reason that nearby xs should have similar S_xs; this is the failure of* (14).

However, a measure-theoretic version of this is valid. *If μ is a probability measure on $[0,1]^2$, we can disintegrate it along fibers: we can write $\mu = \int_{x\in[0,1]} \mu_x d\nu(x)$, where $\nu = \pi_*\mu$ is the pushed-down measure on $[0,1]$, and μ_x is a probability measure supported on the fiber $\pi^{-1}(\{x\})$. The μ_xs are the measure-theoretic analogue of S_x; and:*

(17) *On a set of measure* 0.999999 *the function $x \mapsto \mu_x$ is continuous.*

In other words, throwing away a set of small measure, we can think of the μ_xs as satisfying a version of (14).

5.2. **What comes next.** Einsiedler and Katok implement the strategy discussed in §5.1, but in the world of measures, not sets.

- Rather than an A_3-invariant *closed set* σ, we start with an A_3-invariant probability measure μ.
- The analogue of $\sigma_x^{N_{ij}} \subset \mathbb{R}$ is played by *conditional measures* $\mu_x^{ij} \in$ Measures($\mathbb{R}$) discussed in §5.3. (Note that these are *not* probability measures in general, and may have infinite mass.)
- The assumption (16) that σ_x^{ij} not be singletons is replaced by the assumption that μ_x^{ij} not be *atomic* (a multiple of a point mass), which will be needed in both Theorem 5.1 and Theorem 5.2.
- One can prove the analogue of (15): it is the *product-lemma*, Lemma 5.1.

5.3. **Conditional measures: the analogue of the σ_x^{ij} for measures.** Let a nice group G (e.g. $G = N_{ij}$) act on a nice space X (e.g. $X = \mathcal{L}_3$).

Given a closed subset $S \subset X$, we can define the sets $\sigma_x^G = \{g \in G : gx \in S\}$, which isolates behavior of S along the G-direction. Now we want to define a similar concept but with the set S replaced by a probability measure μ, and replace the closed subset $\sigma_x^G \subset G$ by a measure μ_x^G (or just μ_x) on G.

This can indeed be done in a canonical way, except that the measures μ_x are defined only *up to scaling by a positive number.* In other words, there exists an association $x \mapsto \mu_x$ from points of X to measures on G, referred to as *conditional measures* along G, with the following properties:

(1) The map $x \mapsto \mu_x$ (thought of as a map from X to measures on G) is itself measurable.
(2) For $g \in G$ and $x \in X$ so that both μ_{gx} and μ_x are defined, the measures $\mu_{g.x}$ and $g.\mu_x$ are proportional[14] (one would like to say "equal" but everything is defined only up to a positive scalar).
(3) Let B be any open ball containing the identity in G. Then $\mu_x(B) > 0$ for almost all $x \in X$.
(4) μ is invariant under the G-action if and only if μ_x is a Haar measure on G for almost all $x \in X$.

Let's briefly describe how to do this when X is general but G is finite. In that case, one can normalize the μ_x canonically by requiring them to be probability measures on the finite set G. We will just describe the function $x \mapsto \mu_x(\{1\})$; then (2) determines μ_x totally (in this case, after normalizing the μ_x, the $\propto$ of (2) becomes equality).

Average μ under G to get a measure ν, w.r.t. which μ is absolutely continuous. Therefore, by the theorem of Radon and Nikodym, there exists a function $f \in L^1(\nu)$ so that $\mu = f.\nu$, i.e. $\mu(S) = \int_S f d\nu$. Then $f(x) = \mu_x(\{1\})$ almost everywhere, when matters are normalized so that μ_x is a probability measure.

Returning to the context of an A_3-invariant measure μ on $\mathcal{L}_3$, we denote by μ_x^{ij} the measure on $N_{ij} \cong \mathbb{R}$ defined by the process described above, applied to the action of N_{ij} on $\mathcal{L}_3$.

5.4. **From product lemma to unipotent invariance.** Let μ be an A_3-invariant measure on $\mathcal{L}_3$. The following is established in [4], Corollary to Proposition 5.1.

5.1. **Lemma.** *[Product lemma] Let μ be an A_3-invariant measure on $\mathcal{L}_3$.*

Then, for $(k,\ell) \neq (i,j),(j,i)$ we have $\mu^{ij}_{n^{k\ell}(t)x} \propto \mu^{ij}_x$, for $\mu^{k\ell}_x$-almost all $t \in \mathbb{R}$, and for μ-almost every $x \in X$.

The reasoning is a measure-theoretic version of that already discussed in (5.1). Thus Lemma 5.1 is "just" a consequence of the fact that it is possible to "shrink" the $N_{k\ell}$ while leaving N_{ij} unchanged.

We say that μ_x^{ij} is *trivial* if it is proportional to the Dirac measure supported at 0, i.e. if $\mu_x^{ij}(f) \propto f(0)$ for every continuous function f on the real line. To make usage of the μ_x^{ij}s, one really needs them to be *nontrivial* for almost all x. This is the analogue of (16).

Now let us briefly – and very heuristically – indicate how one might use Lemma 5.1. The assertion (5.1) says, in particular, that the value of $x \mapsto \mu_x^{13}$ is "the same" (at least, proportional) at x and at $n_{12}(t)x$, except for a set of t of $\mu_x^{(12)}$-measure 0. If μ_x^{12} is far from being atomic, we can find plenty of $t \neq 0$ for which this will be true. Similarly, if μ_x^{23} is far from being atomic,

[14]Here $g.\mu_x$ is the measure defined as $g.\mu_x(S) = \mu_x(Sg)$ for a subset $S \subset G$.

we can find plenty of s for which the value of $x \mapsto \mu_x^{13}$ is the same at x and at $n_{23}(s)x$.

Applying this argument repeatedly, we may hope to find t, s so that μ^{13} takes proportional values at x and $n_{12}(t)n_{23}(s)n_{12}(-t)n_{23}(-s)x$. But the groups N_{12} and N_{23} do not commute: indeed $n_{12}(t)n_{23}(s)n_{12}(-t)n_{23}(t) = n_{13}(ts)$. This shows that $\mu_x^{(13)}$ is proportional at x_0, and at $n_{13}(ts)x$.

This says something quite strong: the measure $\mu_x^{(13)}$ on the real line is proportional to its translate under ts! A simple auxiliary argument shows that we can find enough (t, s) to force $\mu_x^{(13)}$ to be Lebesgue measure on $\mathbb{R}$; so (by property (4) of conditional measures) μ is *invariant by* N_{13}. At this point we have invariance in a *unipotent* direction; and one may apply the measure-theoretic version of Ratner's theorem (Theorem 3.1, see also discussion of §3.3) to classify possibilities for μ.[15]

In words, (5.1) combines with the noncommutativity of the subgroups N_{ij} to show that μ is invariant in a unipotent direction.

The conclusion of this line of reasoning is the following, part of [4, Theorem 4.2]:

5.1. **Theorem.** *Suppose μ is an A_3-ergodic measure on $\mathcal{L}_3$ so that, for every $i \neq j$ and for a positive measure set of $x \in X$, the measure μ_x^{ij} is nontrivial. Then μ is Haar measure.*

Here *Haar measure* refers to the unique $\mathrm{SL}_3(\mathbb{R})$-invariant probability measure on $\mathcal{L}_3$. New ideas introduced by Lindenstrauss (based on his earlier work [11]) allowed this to be refined to the following result, which is (as we briefly discuss in §5.6) equivalent up to rephrasing to Theorem 4.3.

5.2. **Theorem.** *Suppose μ is an A_3-ergodic measure on $\mathcal{L}_3$ so that, for* at least one pair *$i \neq j$ and for a positive measure set of $x \in X$, the measure μ_x^{ij} is nontrivial. Then μ is Haar measure.*

Suitable analogues of these theorems are true replacing $(A_3, \mathcal{L}_3)$ by $(A_n, \mathcal{L}_n)$. In that case there are, in general, more possibilities for μ besides Haar measure, as in the statement of Theorem 4.3.

The question of removing the assumption in Theorem 5.2 seems to be a very difficult and fundamental one. If one could do so, the Littlewood conjecture (without any set of exceptions) would follow.

5.5. **Back to Theorem 1.1.** Now let's return to Theorem 1.1, which can be attacked using Theorem 5.2 and the relation between sets and measures.

We claim that for any fixed positive δ,

$$\text{BoxDimension}\,\{(\alpha, \beta) : \inf n.\|n\alpha\|.\|n\beta\| \geq \delta\} = 0 \tag{18}$$

from this it is easy to deduce Theorem 1.1.

[15] In fact, in [4], the use of Ratner's theorem was avoided by applying this argument repeatedly, with 13 replaced by various ij.

We saw that in the discussion preceding (10) that the failure of the Littlewood conjecture for a fixed pair (α, β) would correspond to the A_3^+-orbit of a certain $L_{\alpha,\beta} \in \mathcal{L}_3$ being bounded. If (18) fails, indeed, there exists a set of lattices $L_{\alpha,\beta}$ of box dimension ≥ 0.01 (say) whose A_3^+-orbits all remain within a fixed bounded set inside $\mathcal{L}_3$.

So the closure Y of $A_3^+.\{L_{\alpha,\beta}\}$ is a bounded, A_3^+-invariant closed set on $\mathcal{L}_3$ with box dimension ≥ 2.01 ((the extra 2 comes from taking the A_3-orbit; in words it means that Y has thickness transverse to the A_3-direction).

In what follows, let us ignore the distinction between A_3^+ and A_3 for simplicity. The necessity of dealing with A_3^+ complicates the argument slightly. So, let us assume that Y was actually A_3-invariant.

We construct a A_3-invariant measure μ supported on Y. It turns out that the fact that Y has thickness transverse to the A_3-direction translates into the fact that it is possible to choose μ so that at least one of the conditional measures μ_{ij}^x is nontrivial for almost all x. But then Theorem 5.2 shows that μ has to be Haar measure. So the support of μ is all of $\mathcal{L}_3$ and μ cannot be supported on the bounded set Y – a contradiction.

We observe that the need to allow a set of exceptions in Theorem 1.1 arises from the condition in Theorem 5.2 concerning conditional measures (equivalently, the positive entropy condition – see below). Removing that condition would settle the Littlewood conjecture in whole.

5.6. **Positive entropy.** The theorems 5.1 and 5.2 are not useful without a reasonable way to verify the conditions on μ_x^{ij}. The utility of these results stem, in enormous part, from the fact that there *is* a very usable way to verify the conditions. This is provided by the theory of entropy, and, in many other applications, it is through entropy that these conditions have been verified. Indeed, even the discussion in §5.5 is made rigorous using entropy.

The importance of entropy justifies ending this paper with a brief discussion. For more, see [7, Section 3].

The theory of metric entropy assigns to a measure-preserving transformation T of a probability space (X, μ) a non-zero number, the *entropy* $h_\mu(T)$ of T. We briefly reprise the definition, which, of course, is rather little use without motivation. If $\mathcal{P}$ is a partition of the probability space (X, μ), the entropy of $\mathcal{P}$ is defined as $h_\mu(\mathcal{P}) := \sum_{S \in \mathcal{P}} -\mu(S) \log \mu(S)$. We define the entropy of T as:

$$h_\mu(T) = \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{h_\mu(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-(n-1)}\mathcal{P})}{n} \tag{19}$$

where the supremum is taken over all finite partitions of X.

Roughly speaking, this means the following. Suppose for simplicity that there exists a finite partition $\mathcal{P}$ attaining the supremum on the right-hand side of (19). The entropy measures, in a suitable average sense, the amount of extra information required to specify which part $x \in X$ belongs to, *given* that one knows which part $Tx, T^2x, T^3x, \dots$ belong to. One bit of information corresponds to entropy $\log 2$.

(Clearly the above is not a complete description, because, besides being very ill-defined, it made no mention of the measure μ!)

For example, if $X = \mathbb{R}/\mathbb{Z}, T(x) = 2x, \mathcal{P} = \{[0, 1/2), [1/2, 1)\}$, the knowledge of $T^k x$ specifies the $(k+1)$st binary digit of x. So it requires one extra bit to specify x given $\{Tx, T^2x, \dots, \}$, which corresponds to entropy $= \log 2$

On the other hand, if $X = \mathbb{R}/\mathbb{Z}, T(x) = x + \sqrt{2}, \mathcal{P} = \{[0, 1/2), [1/2, 1)\}$, the entropy is 0: if we know the first binary digit of $\{Tx, T^2x, \dots\}$, we also know the first binary digit of x.

Thus, to a very crude approximation, positive entropy arises from the possibility of different $x, x' \in X$ having "similar" forward trajectories $\{Tx, T^2x, T^3x, \dots\}$. But, in the context of $(A_n, \mathcal{L}_n)$ there is a simple reason this could happen: if $x = n_{ij}x'$ and $a \in A_n$ contracts n_{ij} (cf. discussion near (13)), then the points $a^k x, a^k x'$ become very close as $k \to \infty$.

Formalizing this reasoning gives:

5.3. **Theorem.** *Let μ be an A_3-invariant probability measure on $\mathcal{L}_3$. Then $h_\mu(a) = 0$ for all $a \in A_3$ if and only if, for almost all $x \in \mathcal{L}_3$, the conditional measures μ_x^{ij} are trivial.*

Thus Theorem 4.3 and Theorem 5.2 are indeed the same.

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