SEQUENTIAL CONVERGENCE IN TOPOLOGICAL SPACES

ANTHONY GOREHAM

ABSTRACT. In this survey, my aim has been to discuss the use of sequences and countable sets in general topology. In this way I have been led to consider five different classes of topological spaces: first countable spaces, sequential spaces, Fréchet spaces, spaces of countable tightness and perfect spaces. We are going to look at how these classes are related, and how well the various properties behave under certain operations, such as taking subspaces, products, and images under proper mappings. Where they are *not* well behaved we take the opportunity to consider some relevant examples, which are often of special interest. For instance, we examine an example of a Fréchet space with unique sequential limits that is not Hausdorff. I asked the question of whether there exists in ZFC an example of a perfectly normal space that does not have countable tightness: such an example was supplied and appears below. In our discussion we shall report two independence theorems, one of which forms the solution to the Moore-Mrówka problem. The results that we prove below include characterisation theorems of sequential spaces and Fréchet spaces in terms of appropriate classes of continuous mappings, and the theorem that every perfectly regular countably compact space has countable tightness.

1. Introduction

MOTIVATION. For general topology, sequences are inadequate: everyone says so [Cartan(1937), Tukey(1940), Birkhoff(1937), Kelley(1955), Willard(1970), Engelking(1989)]. In Example 1.1 we shall see some evidence to support this claim. Undaunted, in the rest of this section I am going to set out how we shall proceed in our discussion of the rôle of sequences, and countable sets, in general topology. In Section 2 we review the properties of sequences in the class of first countable spaces, most of which are familiar. The work begins in earnest in Section 3 and Section 4, when we consider sequential spaces and Fréchet spaces. It is easily shown that a first countable space is Hausdorff if and only if its sequential limits are unique. However, this is no longer true for Fréchet spaces (even compact Fréchet spaces), as we shall see in Example 4.1. We are also going to prove characterisation theorems of the classes of sequential spaces and Fréchet spaces, in terms of appropriate continuous mappings (see Theorems 3.10, 4.4).

It is natural to ask what changes if, instead of studying the sequences in a space, one studies its countable subsets. This question leads us to consider the class of spaces of countable tightness. The relationship between the properties of having countable tightness and being a sequential space is of great interest, being the issue at the centre of the famous Moore-Mrówka problem. In one direction, it is a triviality to prove that every sequential space has countable tightness. The

Date: April 2001.

This survey was written as a dissertation for the Final Honour School of Mathematics, The Queen's College, Oxford University.

question of whether the converse is true for compact Hausdorff spaces occupied mathematicians for over twenty years (see Section 5).

The class of perfect spaces is not often studied in this context, and it does not immediately fit into the hierarchy formed by the others that we have named (see Figure 1.1). That a first countable normal space need not be perfectly normal is well known. Furthermore, [Arens(1950)] has given an example of a perfectly normal space that is not sequential. It seems, however, that the question of whether a perfect space need have countable tightness has never been properly considered. This may be partly because the cardinal invariant Ψ associated with the perfect property is rarely used (it does not even have a name; see page 22). A pertinent example was found by Mr. J. Lo, and it appears as Example 6.5.

When suitable compactness conditions are imposed the results are more encouraging. Indeed, we are going to prove that every perfectly regular countably compact space has countable tightness (Theorem 6.6). Like the other classes that we have defined, the class of perfect spaces contains all metric spaces. So I asked what follows if a space, like a metric space, is both perfectly normal and first countable. Of particular interest is the question of whether compactness and sequential compactness are equivalent for such spaces. The fact that this cannot be decided in ZFC is easily deduced from known results (see Corollary 6.12).

An Example. It is a familiar and useful fact that, in metric spaces, the crucial notions of closure, compactness and continuity can be described in terms of convergence of sequences. As a sample of the results available in metric spaces, we recall the following:

(1) *The closure of a set consists of the limits of all sequences in that set;*
(2) *The space is compact if and only if it is sequentially compact;*
(3) *A function from the space into a topological space is continuous if and only if it preserves limits of sequences.*

In general topological spaces, these results are no longer true, as the following example shows.

Example 1.1. Let $\beta\mathbb{N}$ denote the Stone-Čech compactification of the natural numbers. Then the only sequences in that converge in $\beta\mathbb{N}$ are those which are eventually constant, i.e., they always take the same value from some point on—see [Engelking(1989)] (Cor. 3.6.15). We see immediately that $\beta\mathbb{N}$ is a compact space that is not sequentially compact, which is a counter-example to item (2) above. Now let x be any point in $\beta\mathbb{N} \setminus \mathbb{N}$. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, we see that x belongs to $\overline{\mathbb{N}}$, the closure of $\mathbb{N}$. However, no sequence in $\mathbb{N}$ converges to x. This shows that statement (1) does not hold in $\beta\mathbb{N}$. Finally, let Y be the set of all points of $\beta\mathbb{N}$ in the discrete topology. Then the topology on Y is strictly finer than that on $\beta\mathbb{N}$, so the identity mapping $\iota\colon \beta\mathbb{N} \to Y$ is *not* continuous. However, since the only convergent sequences in $\beta\mathbb{N}$ are the trivial ones, we see that ι does preserve limits of sequences. This shows that (3) is not generally valid. □

One can give simpler examples of spaces in which (1), (3) do not hold; see for example [McCluskey and McMaster(1997)] (Sec. 3.1). However, $\beta\mathbb{N}$ is the canonical counter-example to (2). We shall return to this example of a space that is not discrete, but in which there are no non-trivial convergent sequences.

There will be more examples in subsequent sections of spaces in which sequences fail to describe important topological properties of the space. Thus it would appear that sequences are inadequate to the task of describing topology, in general. Why should we be worried by this? After all, there are many properties of metric spaces that fail to generalise to topological spaces; this is what makes topological spaces so interesting. The difficulty is that convergence of sequences is one of the most basic, and one of the most intuitive, aspects of topology. The first attempts by [Fréchet(1906)] to find a general class of spaces suitable for study of topological notions were centred around the convergence of sequences. The fact that open sets, closure, continuity and compactness cannot in general be described by sequential convergence is thus a great inconvenience.

How do we respond to this? There is more than one possibility. One approach is to stop thinking about convergence in terms of sequences, and to substitute some more general theory of convergence that is sufficient in any topological space. This is done in the twin theories of nets and of filters, in which a generalised convergence is developed that does describe open sets, closure, continuity and compactness in a satisfactory manner; see [Willard(1970)] (Chap. 4). Another possibility is to accept that sequences cannot always do everything, and to look to see what they *can* do, and when. This is the route that we shall follow.

MANIFESTO. We use the axiom of choice without question and without comment. The set of all natural numbers is denoted $\mathbb{N}$; we do not regard 0 as a natural number. A compact space is not necessarily Hausdorff, and in this regard our notion of compactness differs from that of [Bourbaki(1966)]. Likewise, we do not assume that a paracompact space is Hausdorff, nor that a Lindelöf space is regular. A regular or a normal space is not necessarily T_1; thus T_3 is a condition strictly stronger than regularity, and T_4 is strictly stronger than normality, for in both of these we insist upon T_1. A neighbourhood N of a point x need not necessarily be an open set; all that is required is that x be an interior point of N. We use the term 'subsequence' in the traditional sense, rather than in the sense of [Kelley(1955)]. Other terms occurring are either defined as they are used, or are thought to be unambiguous; in the latter case, the reader is referred to [Engelking(1989)] for a definition.

We shall examine, principally, four classes of spaces. Each class is defined either directly in terms of sequences or in terms of some sort of countability. These classes form a hierarchy in which the class of spaces under consideration becomes wider, while the results available become weaker. For an overview of this hierarchy, see Figure 1.1.

This diagram also indicates some of the properties of sequential convergence in each class of spaces—the results and counter-examples required to prove the various assertions in this diagram will appear in later sections.

In each section we are looking at the sequential convergence properties of the class of spaces under consideration. We also consider the standard questions: 'Is this class closed under taking products, or subspaces, or (inverse) images under appropriate continuous mappings?' In fact, the most useful continuous mappings in this regard are proper mappings, insofar as they are included in most of the results that we find. The answer to these questions is frequently 'no'—see Figure 1.2. This unfortunate fact nonetheless gives us the opportunity to consider some interesting examples.

$$\begin{tikzcd}
\text{metric space} \arrow[r] \arrow[d, Rightarrow] & \text{compactness} & \Longleftrightarrow & \text{sequential compactness} \\
\text{first countable space} \arrow[r] \arrow[d, Rightarrow] \arrow[urr, "/" marking] & \text{Hausdorff} & \Longleftrightarrow & \text{unique limits} \\
\text{Fréchet space} \arrow[r] \arrow[d, Rightarrow] \arrow[urr, "/" marking] & \text{closure = set of limits} & & \\
\text{sequential space} \arrow[r] \arrow[d, Rightarrow] \arrow[urr, "/" marking] & \text{continuity} & \Longleftrightarrow & \text{preservation of limits} \\
\text{space of countable tightness} \arrow[urr, "/" marking] & & &
\end{tikzcd}$$

FIGURE 1.1. The relations between some classes of spaces, and their properties.

Class of spaces	Hereditary?	Productive?	Proper mappings: Invariant?	Proper mappings: Inverse invariant?
First countable	Yes	No	No	No
Fréchet	Yes	No	Yes	No
Sequential	No	No	Yes	No
Countable tightness	Yes	No	Yes	No
Perfect	Yes	No	Yes	No

FIGURE 1.2

Concerning the characterisation of classes of spaces in terms of continuous images of metric spaces, the reader is referred to Figure 1.3. These results were achieved in the 1960s by [Ponomarev(1960)], [Arhangel′skiĭ(1963)] and [Franklin(1965)].

Space	+ Condition	= Image of a metric space under
first countable space	T_0	open onto mapping
$\Downarrow$		$\Downarrow$
Fréchet space	T_2	pseudo-open mapping
$\Downarrow$		$\Downarrow$
sequential space		quotient mapping

FIGURE 1.3. Characterisations of spaces in terms of continuous mappings.

Were I to include the proofs to all the relevant results mentioned above, then this survey would be twice as long. Therefore I have carefully selected which results and examples to consider in detail. I have been particularly concerned for variety, so where two or more examples or results use similar ideas, only one proof will be given. Moreover, very few of the lemmas are proved: in most cases the reader can supply a proof using the first idea that comes to mind.

Definitions. How do we interpret the statement that 'sequences describe the topology' in a space X? Our usual definition of 'topology' is that the topology on X is the family of open subsets of X. Hence the most natural interpretation of our statement is that we can give a necessary and sufficient condition for openness of a subset of X in terms of sequences. The following is a simple property of metric spaces:

(4) *A subset A of X is open if and only if whenever (x_i) is a sequence that converges to a point of A, then A contains all but finitely many of the members x_i of the sequence.*

We shall define the *sequential* spaces to be the spaces in which this statement holds. However, there are many other ways to define a topology on X, other than specifying the open sets. We could equally well specify the closed sets in X, or we could define a Kuratowski closure operation on X. In terms of closed sets, we note that metric spaces have the following property:

(5) *A subset A of X is closed if and only if whenever (x_i) is a sequence in A that converges to a point $x \in X$, then x belongs to A.*

We shall see that properties (4) and (5) are equivalent. It is perhaps surprising, therefore, that they are *not* equivalent to property (1), which relates sequences to the closure operator—see page 2. It turns out (see Example 3.5) that property (1) is *strictly* stronger than properties (4) and (5). Those spaces that satisfy (1) are the *Fréchet* spaces.

At this point we are going to define each of the main classes of spaces that will be considered in future sections. I have chosen to do this, rather than to withhold the definition of each class until we are ready to study it, in order to facilitate appropriate comparison of these classes as we go along.

To start with, it is well known that spaces satisfying the so-called first axiom of countability share most of the nice properties, with regard to sequences, of metric spaces.

Definition 1.2 (first axiom of countability)**.** Let X be a topological space. We say that X is *first countable*, or that X satisfies the *first axiom of countability*, if for every point x of X the neighbourhood system at x has a countable base.

It is clear that every metric space is first countable: a countable neighbourhood base at the point x is the set $\{S(x, 1/i) \mid i \in \mathbb{N}\}$ of open spheres of radius $1/i$ and centre x, as i runs through the natural numbers.

Definition 1.3 (Fréchet space)**.** Let X be a topological space. We say that X is a *Fréchet* space, or that it has the *Fréchet property*, if for every subset A of X, a point $x \in X$ belongs to $\overline{A}$ if and only if there exists a sequence in A that converges to x. Fréchet spaces are also known as *Fréchet-Urysohn* spaces.

In analysis one often uses the fact that metric spaces are Fréchet. More generally, any first countable space is Fréchet:

Lemma 1.4. *Let X be a first countable space. Then X is a Fréchet space.* □

The proof is easy to find; it appears in [McCluskey and McMaster(1997)] (Thm. 3.10), for instance.

It is convenient to adopt some terminology of [Franklin(1965)].

Definition 1.5 (sequentially open)**.** Let X be a topological space, and let A be a subset of X. We say that A is *sequentially open* if whenever (x_i) is a sequence in X that converges to a point in A, then there exists a natural number i_0 such that x_i belongs to A whenever $i \geqslant i_0$. (In these circumstances we often say that A contains *almost all* of the x_i, or that (x_i) is *eventually* in A.) Similarly, we say that A is *sequentially closed* if it contains the limits of all its sequences, i.e., if whenever there is a sequence in A that converges to some point $x \in X$, then x belongs to A.

From the definition of sequential convergence in a topological space, every open set is sequentially open. Similarly, it is a trivial matter to prove that every closed set is sequentially closed. That the converses are not true in general spaces we can see by again considering $\beta\mathbb{N}$, which is non-discrete but might be said to be 'sequentially discrete'. Note that a subset A of X is sequentially open iff $X \setminus A$ is sequentially closed. With this in mind, the next result is obvious.

Lemma 1.6. *Let X be a topological space. Then the following are equivalent:*

(1) *every sequentially open subset of X is open;*
(2) *every sequentially closed subset of X is closed.* □

Definition 1.7 (sequential space)**.** Let X be a topological space. We say that X is a *sequential* space if every sequentially open subset of X is open. By Lemma 1.6 this is the same as saying that every sequentially closed set is closed.

Remark 1.8 ([Arhangel′skiĭ and Ponomarev(1984)] (p. 44))**.** What is the difference between the present property and that of being a Fréchet space? We can see this more clearly by rephrasing the definition in a third way: a space X is sequential if for every non-closed subset A of X, there exists a sequence in A converging to some point x in $\overline{A} \setminus A$. Contrast this with the situation for a Fréchet space: for every non-closed subset A and *every* point x in $\overline{A} \setminus A$, there exists a sequence in A converging to x. For a Fréchet space, one can choose the point x in advance, whereas for a sequential space one cannot. Thus every Fréchet space is sequential, but not conversely (see Example 3.5).

We can say right now that none of our classes are invariant under inverse images of proper mappings. Indeed, $\beta\mathbb{N}$ is not sequential and yet $f\colon \beta\mathbb{N} \to \{0\}\colon x \mapsto 0$ is a proper mapping (i.e., it is onto, continuous, closed and has compact fibres). Likewise no class is closed with respect to images under continuous mappings: for if X denotes the set $\beta\mathbb{N}$ with the discrete topology, then X is metrisable and the identity mapping $\iota\colon X \to \beta\mathbb{N}$ is continuous.

We have defined some classes of spaces in which the topology (in particular, closed sets) may be described by sequences. We now generalise a little and look at spaces in which the closed sets may be described by countable sets, in the following sense.

Definition 1.9 (countable tightness)**.** Let X be a topological space. We say that X has *countable tightness* if whenever A is a subset of X that contains the closure of all its countable subsets, then A is closed.

The definition of countable tightness that we have given is analogous to that of sequential space, in the sense that if you replace 'countable subsets' by 'sequences' and 'closure' by 'limits' in the present definition, then you arrive at a definition of sequential space. Thus every sequential space, and hence every first countable space, has countable tightness.

There is an alternative characterisation of tightness, which is analogous in the same way to the definition of a Fréchet space. When the word *tightness* was first introduced (by [Arhangel′skiĭ and Ponomarev(1968)]), the class of spaces that we just defined would have been said to have countable *weak tightness*. The next result is a special case of the fact that tightness and weak tightness are one and the same thing. In the light of Remark 1.8, this is somewhat surprising.

Lemma 1.10. *Let X be a topological space. Then X has countable tightness if and only if whenever a point $x \in X$ belongs to the closure of some $A \subseteq X$, then there exists a countable subset C of A such that x belongs to $\overline{C}$.*

Proof. Suppose that X is a space such that for all $x \in \overline{A}$ there exists countable $C \subseteq A$ such that $x \in \overline{C}$. This implies that if A is a subset of X that contains the closure of all its countable subsets, then A contains all the points of its closure, so $A = \overline{A}$ and A is closed. Therefore X has countable tightness.

Now suppose conversely that X has countable tightness, and let A be a subset of X. Let

$$E := \bigcup \{ \overline{C} \,\big|\, C \text{ countable, } C \subseteq A \}.$$

We want to show that $\overline{A} \subseteq E$, because then every point in $\overline{A}$ belongs to E, so it lies in the closure of some countable subset of A, as desired. Clearly $A \subseteq E$, so to show that $\overline{A} \subseteq E$ it is sufficient to show that E is closed: to do this we shall use countable tightness. Let D be a countable subset of E, and let $x \in \overline{D}$. Observe that for every $y \in D$ we have $y \in \overline{C_y}$ for some countable subset C_y of A. Let $C := \bigcup_{y \in D} C_y$; since D is countable, C is countable, and it is certainly a subset of A. Now let U be any open neighbourhood of x. Then from the fact that $x \in \overline{D}$, there exists some y in $D \cap U$. Since U is open it is a neighbourhood of y, and since $y \in \overline{C_y}$ it follows that U meets C_y, hence U meets C. This shows that $x \in \overline{C}$, which implies that $x \in E$. Therefore $\overline{D} \subseteq E$, thus E contains the closure of all its countable subsets. Since X has countable tightness, it follows that E is closed. Now we see that E is closed and $A \subseteq E$, which implies that $\overline{A} \subseteq E$. It follows that, for any $x \in \overline{A}$, there exists some countable subset C of A such that $x \in \overline{C}$. $\square$

Lemma 1.10 was stated in a footnote by [Arhangel′skiĭ(1970)]. This proof is my answer to an exercise in [Engelking(1989)] (Prob. 1.7.13).

It is easy to show that every hereditarily separable space X has countable tightness: for any $A \subseteq X$, let C be a countable dense subset. This is dense in the sense that $\overline{C}^A = A$, i.e., $\overline{C} \cap A = A$, whence $A \subseteq \overline{C}$. Now if A contains the closure of all its countable subsets then $\overline{C} \subseteq A$, it follows that $A = \overline{C}$ and A is closed. Thus a hereditarily separable space has countable tightness. In particular, every countable space has countable tightness.

Another class of spaces can also be defined in terms of a 'countable' property of closed sets, as follows.

Definition 1.11 (perfect space)**.** Let X be a topological space, and let A be a subset of X. We say that A is a G_δ *set* if it can be written as the intersection of a countable family of open sets. We say that A is an F_σ *set* if it can be written as the union of a countable family of closed sets.

The topological space X is said to be *perfect* if every closed subset of X is a G_δ. A space that is both perfect and normal is called *perfectly normal*.

It is clear that a space is perfect iff every open set is an F_σ. We discuss perfect spaces in Section 6.

2. First Countable Spaces

The Basics. This section is another introductory one, in which we review the properties of first countable spaces. This in preparation for the real work of looking at sequential spaces in Section 3. There are two main points of interest. The first is that, in first countable spaces, the Hausdorff separation axiom is equivalent to the uniqueness of sequential limits (Proposition 2.4); this result is no longer true for the class of Fréchet spaces. The second point to note is that in passing from metric spaces to first countable spaces we lose the equivalence of compactness and sequential compactness. We give an example of a first countable sequentially compact space that is not compact in Example 2.5. Finally, for comparison with the main theorems of Sections 3–4, we shall state Ponomarev's characterisation of the T_0 first countable spaces in Theorem 2.6.

Naturally one asks what are the operations on first countable spaces that yield a first countable space. The operation of taking subspaces is one of them, according to the following result. We omit the (easy) proof.

Lemma 2.1. *Any subspace of a first countable space is again first countable.* □

Not every quotient of a first countable space is first countable; this will follow from results in Section 3.

We shall now give an example to show that the product of first countable spaces need not be first countable.

Example 2.2. Consider the space $\mathbb{R}^\mathbb{R}$, i.e., the Tychonoff product of $\mathfrak{c}$ copies of the real line. Since $\mathbb{R}$ is a metric space it is first countable; we shall show that $\mathbb{R}^\mathbb{R}$ is *not* first countable. Suppose for a contradiction that $\mathcal{B} = \{U^{(i)} \mid i \in \mathbb{N}\}$ is a countable base of open neighbourhoods of 0 in $\mathbb{R}^\mathbb{R}$. For each natural number i, since $U^{(i)}$ is open there exists a basic open set $\prod_{r\in\mathbb{R}} W_r^{(i)}$ such that $0 \in \prod_{r\in\mathbb{R}} W_r^{(i)} \subseteq U^{(i)}$. Since $\prod_{r\in\mathbb{R}} W_r^{(i)}$ is basic and open, there exists a finite subset $R^{(i)}$ of $\mathbb{R}$ such that $W_r^{(i)} = \mathbb{R}$ for every $r \in \mathbb{R} \setminus R^{(i)}$. Now $R := \bigcup\{R^{(i)} \mid i \in \mathbb{N}\}$ is a countable, hence proper, subset of $\mathbb{R}$. Therefore there exists $r \in \mathbb{R}$ with $r \notin R$, and this r satisfies $W_r^{(i)} = \mathbb{R}$ for every $i \in \mathbb{N}$. In particular $W_r^{(i)} \not\subseteq (-1,1)$, for each $i \in \mathbb{N}$. Hence

$$\prod_{s\in\mathbb{R}} W_s^{(i)} \not\subseteq (-1,1) \times \prod_{s\neq r} \mathbb{R} = {\pi_r}^{-1}(-1,1), \tag{1}$$

from which it follows that $U^{(i)} \not\subseteq {\pi_r}^{-1}(-1,1)$, for every $U^{(i)} \in \mathcal{B}$. But ${\pi_r}^{-1}(-1,1)$ is an open neighbourhood of 0 in $\mathbb{R}^\mathbb{R}$, which contradicts the fact that $\mathcal{B}$ is a neighbourhood base at 0. This shows that $\mathbb{R}^\mathbb{R}$ is not first countable. □

The argument above is adapted from [Kelley(1955)] (Thm. 3.6). Note that in this example we also see that an uncountable product of metric spaces need not be metrisable. However, it is easy to show that a *countable* product of metric spaces is metrisable. That result carries across to first countable spaces in the sense that a countable product of first countable spaces is first countable, as we now show.

Proposition 2.3. *Suppose that X_n is a first countable space for each n in $\mathbb{Z}$, and let X denote the Tychonoff product $\prod_{n\in\mathbb{Z}} X_n$. Then X is a first countable space.*

Proof. Let x belong to X. For each $n \in \mathbb{Z}$, let $\{U_n^{(i)} \mid i \in \mathbb{N}\}$ be a countable base for the neighbourhood system at x_n. Let

$$N_n^{(i)} := U_n^{(i)} \times \prod_{m \neq n} X_m = {\pi_n}^{-1}(U_n^{(i)}), \qquad \text{for } i \in \mathbb{N},\, n \in \mathbb{Z},$$

$$\mathcal{B} := \left\{ \bigcap_{n \in F} N_n^{(i(n))} \,\middle|\, i(n) \in \mathbb{N},\, F \subseteq \mathbb{Z} \text{ is finite} \right\}.$$

Then $\mathcal{B}$ is a countable family of open subsets of X. Now, let N be a basic open neighbourhood of x. Then there exists a finite subset F of $\mathbb{Z}$ and a family $\{V_n \mid n \in \mathbb{Z}\}$ of sets such that V_n is open in X_n for all n, with $V_n = X_n$ for all $n \in \mathbb{Z} \setminus F$ and $N = \prod_{n\in\mathbb{Z}} V_n$. Pick any n in F; then V_n is a neighbourhood of x_n in X_n. So for each $n \in F$ there exists some natural number $i(n)$ such that $x_n \in U_n^{(i(n))} \subseteq V_n$. Therefore $x \in \bigcap_{n\in F} N_n^{(i(n))} \subseteq N$. This shows that $\mathcal{B}$ is a base for the neighbourhood system at x. Hence X is first countable. □

For First Countable Spaces Only. Now we present a result that is *not* available for Fréchet spaces (cf. Example 4.1). This characterisation of the Hausdorff property in the realm of first countable spaces is part of topological folklore (see for instance [Hu(1964)] (Chap. III, Ex. 3A)).

Proposition 2.4. *Let X be a first countable space. Then X is Hausdorff if and only if sequential limits in X are unique, that is whenever (x_i) converges to x and to y, then $x = y$.*

Proof. Suppose first that X is a Hausdorff space. Let (x_i) be a sequence converging to some point x, and let $y \neq x$. Then there exist disjoint open sets U, V such that $x \in U$ and $y \in V$. Since $x_i \to x$ there exists a natural number i_0 such that $x_i \in U$ whenever $i \geqslant i_0$. Since U, V are disjoint it follows that $x_i \notin V$ whenever $i \geqslant i_0$. Therefore (x_i) cannot possibly converge to y. This shows that sequential limits in X are unique.

Conversely, suppose that X is not a Hausdorff space. Then there exist two distinct points x, y in X such that every neighbourhood of x meets every neighbourhood of y. Let $\mathcal{B} = \{U_i \mid i \in \mathbb{N}\}$ be a base of open neighbourhoods at x, and let $\{V_i \mid i \in \mathbb{N}\}$ be a base of open neighbourhoods at y. Then for each $i \in \mathbb{N}$ we have $U_i \cap V_i \neq \varnothing$, so there exists some point x_i in $U_i \cap V_i$. Since $\mathcal{B}$ is a base at x, the sequence (x_i) converges to x. Similarly (x_i) converges to y. But by assumption $x \neq y$, thus sequential limits in X are not unique. □

We can see that the topology of first countable spaces is very largely determined by their convergent sequences. By Lemma 1.4 the closure operation can be described by sequences, and every first countable space is sequential, so the open sets are determined by sequences as well. In Lemma 3.1 we shall see that continuity too can be characterised by sequences, and Proposition 2.4 just proved shows that the same can be said for the Hausdorff property. Thus, most of the properties of first countable spaces are determined by knowledge of their convergent sequences. However, there is a substantial fly in the ointment: compactness.

An Ordinal Space. For sequential Hausdorff spaces, sequential compactness is equivalent to countable compactness (see Proposition 3.2). In particular, a first countable compact Hausdorff space is sequentially compact. However, this result has no converse: a first countable sequentially compact Hausdorff space need not be compact. The following example is standard, it appears in [Munkres(1975)] (Sec. 3.7), for instance.

Example 2.5. Let ω_1 be the least uncountable ordinal number, and define

$$W_0 := \{x \,|\, x \text{ is an ordinal and } x < \omega_1\}.$$

Then W_0 is a well-ordered set with ordinality ω_1; the topology on W_0 is the order topology. We are going to show that W_0 is a sequentially compact Hausdorff first countable space that is not compact. To show that W_0 is not compact we show that it is a dense proper subspace of a compact Hausdorff space. To this end, let $W := W_0 \cup \{\omega_1\}$, and let W have the order topology. First we establish some notation. Let $-\infty := \min W$, and given $x \in W$ write $[-\infty, x)$, $(x, \omega_1]$ for the sets $\{y \in W \mid y < x\}$, $\{y \in W \mid y > x\}$ respectively.

By definition these sets form a subbase for the order topology on W. When $x < y$ we define (x, y), $[x, y)$, $(x, y]$ and $[x, y]$ in the obvious way. Note that whenever $x < y$ we have $(x, y] = [-\infty, y + 1) \cap (x, \omega_1]$, so that $(x, y]$ is open. Now, every neighbourhood N of ω_1 contains a set of the form $(x, \omega_1]$ for some $x \in W_0$, hence N meets W_0. Therefore $\omega_1 \in \overline{W_0}$, so that W_0 is dense in W.

To show that W is compact, let $\mathcal{E}$ be an open cover for W and consider the set

$$S := \{x \in W \,|\, [-\infty, x] \text{ has a finite subcover in } \mathcal{E}\}.$$

Suppose for a contradiction that $S \neq W$. Then $W \setminus S$ is non-empty, so it has a least element, x say. Since $\mathcal{E}$ is a cover, there exists U_0 in $\mathcal{E}$ with $x \in U_0$. Clearly $-\infty \in S$, so $x \neq -\infty$. Since U_0 is open we have $x \in [-\infty, y) \cap (z, \omega_1] \subseteq U_0$ for some $y, z \in W$, whence $(z, x] \subseteq U_0$. Since $z < x$ we have $z \in S$, so $[-\infty, z]$ can be covered by a finite subcollection $\{U_1, U_2, \ldots, U_n\}$ of $\mathcal{E}$. But now $\{U_0, U_1, \ldots, U_n\}$ is a finite subcollection of $\mathcal{E}$ that covers $[-\infty, x]$, which contradicts the fact that $x \notin S$. Therefore $S = W$ after all, i.e., W has a finite subcover in $\mathcal{E}$. This shows that W is compact.

Now we show that W is Hausdorff: if x, y are distinct points of W, then we may assume without loss of generality that $x < y$. Then $U := [-\infty, x + 1)$ and $V := (x, \omega_1]$ are disjoint open sets with $x \in U$ and $y \in V$, whence W is a Hausdorff space. Hence W_0 is a non-closed subspace of the compact Hausdorff space W, so W_0 is not compact. As a subspace of W, W_0 is also a Hausdorff space.

Next we can show that W_0 is first countable. Pick $x \in W_0$. If $x = -\infty$ then $\{x\}$ is a countable neighbourhood base. Otherwise, the collection $\mathcal{B} := \{(y, x] \mid y < x\}$ is countable because $\mathcal{B}$ is well-ordered by inclusion, with ordinality $x < \omega_1$. Since $\mathcal{B}$ is a base for the neighbourhood system at x, this shows that W_0 is first countable.

We want to show that W_0 is sequentially compact. First we note the following fact: *If C is a countable subset of W_0, then C is bounded above.* For, let x be an element of C. Then since $x < \omega_1$, it follows that $[-\infty, x]$ is countable. Therefore the set $B := \bigcup_{x \in C} [-\infty, x]$ is a countable union of countable sets, hence countable. It follows that $B \neq W_0$, since W_0 is uncountable. Therefore there exists some $y \in W_0 \setminus B$. Since $y \notin B$ we must have $y > x$ for all $x \in C$, that is, y is an upper bound for C.

To show that W_0 is sequentially compact, let (x_i) be a sequence in W_0. Consider the set

$$F := \{x_i \,|\, x_i \geqslant x_j \text{ whenever } j \geqslant i\}.$$

If $F = \varnothing$, then let $I := \{1\}$. Otherwise, let $x := \min F$ and let $I := \{i \in \mathbb{N} \mid x_i = x\}$. In the case that I is infinite, we can arrange I into an increasing sequence $(i(r) \mid r \in \mathbb{N})$, and then we see that $(x_{i(r)})$ is a constant, hence convergent, subsequence of (x_i).

It only remains to consider the case that I is finite. Then let $i(1) := \max I + 1$, and consider $x_{i(1)}$. Then $x_{i(1)} \notin F$, so there exists some natural number $i(2) > i(1)$ with $x_{i(2)} > x_{i(1)}$. Continuing in this way we can pick out a strictly increasing subsequence $(x_{i(r)})$ of (x_i). Now the set $C = \{x_{i(r)} \mid r \in \mathbb{N}\}$ is countable, hence bounded above. Thus the set of all upper bounds of C is non-empty, let y be its least element. Let N be a neighbourhood of y; then N contains $(z, y]$ for some $z \in W_0$. Therefore $z < y$, so z is not an upper bound for C, whence $z < x_{i(s)} \leqslant y$ for some natural number s. Since the sequence $(x_{i(r)})$ is increasing it follows that $z < x_{i(r)} \leqslant y$, hence $x_{i(r)} \in N$, whenever $r \geqslant s$. This shows that $x_{i(r)} \to y$, thus (x_i) has a convergent subsequence. Therefore W_0 is sequentially compact. □

We finish off this section by stating a theorem that characterises the first countable spaces. Recall that a continuous mapping $f \colon X \to Y$ is said to be *open* if it maps open sets in X to open sets in Y.

Theorem 2.6 ([Ponomarev(1960)]). *If f is an open mapping from a metric space X onto a topological space Y, then Y is first countable. Conversely, if Y is a first countable space then for some metric space X there exists an open mapping f from X onto Y.*

This theorem will not be proved here. It is included mainly for comparison with our other characterisation theorems (Theorems 3.10 and 4.4).

3. Sequential Spaces

Sequences in Sequential Spaces. At this point we begin our work in earnest by examining the sequential spaces (recall Definition 1.7). Our ultimate goal in this section is Theorem 3.10, which states that every sequential space is the quotient of a metric space, and *vice versa*. This theorem is quite remarkable. First, quotient spaces of metric spaces are easy to form so we have a plentiful supply of sequential spaces. Indeed this shows that many of the most natural examples of topological spaces are sequential. Second, the theorem gives a characterisation of the class of sequential spaces in terms that do not involve sequential convergence.

How well do sequences describe the topology in a sequential space? Open sets and closed sets admit simple descriptions, by definition. The closure operation is not so well-behaved; we shall give an example of a sequential space that is not Fréchet in Example 3.5. However, the continuity of a function *can* be characterised in terms of sequences, as follows.

Lemma 3.1. *Let X be a sequential space, let Y be an arbitrary topological space and let $f \colon X \to Y$. Then f is continuous if and only if it preserves limits of sequences, i.e., whenever (x_i) is a sequence converging in X to x, then $(f(x_i))$ converges in Y to $f(x)$.*

Proof. Suppose that f is continuous, and let (x_i) be a sequence in X with $x_i \to x$. Let N be a neighbourhood of $f(x)$. Since f is continuous, $f^{-1}(N)$ is a neighbourhood of x. Therefore there exists some natural number i_0 such that $x_i \in f^{-1}(N)$ whenever $i \geqslant i_0$. This implies that $f(x_i) \in N$ whenever $i \geqslant i_0$, therefore $f(x_i) \to f(x)$.

Now suppose conversely that $f(x_i) \to f(x)$ whenever $x_i \to x$. Let U be an open subset of Y. Let $x \in f^{-1}(U)$ and suppose that $x_i \to x$. Then $f(x_i) \to f(x)$, where $f(x) \in U$. Now U is open, so there exists a natural number i_0 such that $f(x_i) \in U$ whenever $i \geqslant i_0$. Hence $x_i \in f^{-1}(U)$ whenever $i \geqslant i_0$. This shows that $f^{-1}(U)$ is sequentially open. Since X is sequential, it follows that $f^{-1}(U)$ is open, thus f is continuous. □

In general compactness neither implies nor is implied by sequential compactness. We gave an example (Example 2.5) showing that, even for Hausdorff first countable spaces, sequential compactness does not imply compactness. The space $\beta\mathbb{N}$ is an example of a compact Hausdorff space that is not sequentially compact. However, it *is* true that a compact sequential space is sequentially compact. In order to prove this we shall actually show that, in sequential spaces, sequential compactness is equivalent to countable compactness. This result was first noted by [Franklin(1965)] (Prop. 1.10). Our proof is based on the sketch in [Engelking(1989)] (Thm. 3.10.31).

Proposition 3.2. *Let X be a sequential space. Then X is sequentially compact if and only if it is countably compact.*

Proof. Sequential compactness always implies countable compactness: use the obvious proof by contradiction. To prove the converse, let X be a countably compact sequential space. Let (x_i) be a sequence in X. Let $A := \{x_i \mid i \in \mathbb{N}\}$. Since X is countably compact, the sequence (x_i) has a cluster point x in X—see e.g., [Nagata(1985)] (IV.5.D)). Thus, for every neighbourhood N of x, and for all natural numbers j, there exists some $i \geqslant j$ such that x_i belongs to N. If $x_i = x$ for infinitely many i, then we have a convergent subsequence. If not, then x must be an element of $\overline{(A \setminus \{x\})}$. Therefore $A \setminus \{x\}$ is not closed. Since X is a sequential space, it follows that $A \setminus \{x\}$ is not sequentially closed. Hence there exists a sequence (y_r) in $A \setminus \{x\}$ that converges to some point y with $y \notin A \setminus \{x\}$. We may assume that all of the points y_r are distinct. Now $y_r \in A$ for each natural number r, so there exists some $j(r)$ such that $y_r = x_{j(r)}$. Now we delete those $x_{j(r)}$ that occur 'too late', by defining a function $i\colon \mathbb{N} \to \mathbb{N}$ in the following inductive manner:

$$i(1) := j(1), \qquad i(r+1) := j\bigl(\min\{s > r \,|\, j(s) > i(r)\}\bigr).$$

This is contrived so that i is monotone increasing, thus $(x_{i(r)})$ is a subsequence of (x_i). Also, it is subsequence of (y_r), so it converges to y. This shows that X is sequentially compact. □

Quotients of Sequential Spaces. Recall that a continuous onto mapping $f\colon X \to Y$ is called a *quotient mapping* if V is open in Y whenever $f^{-1}(V)$ is open in X. For example, the projection mapping from a product space onto one of its coordinate spaces is a quotient mapping, a fact that we shall use in Example 3.5.

Lemma 3.3 ([Franklin(1965)] (Prop. 1.2))**.** *Let X be a sequential space, and let f be a quotient mapping from X onto a topological space Y. Then Y is a sequential space.* □

This is proved by showing that if U is sequentially open in Y, then $f^{-1}(U)$ is sequentially open in X.

Corollary 3.4. *The class of sequential spaces is closed with respect to taking images under proper mappings.* □

With the help of Lemma 3.3 we can give an example of a sequential space that is not Fréchet. This shows that, in sequential spaces, sequences do not in general describe the closure operator.

Example 3.5 ([Franklin(1965)] (Eg. 2.3))**.** Let $L := \mathbb{R} \setminus \{0\}$ be the set of real numbers with 0 removed, and let $M := \{0\} \cup \{1/i \mid i \in \mathbb{N}\}$. Let $Y := (L \times \{0\}) \cup (M \times \{1\})$ have its usual (metric) topology as a subspace of $\mathbb{R}^2$. In particular, Y is a sequential space. Now let $f : Y \to X$ be the projection of Y onto its first coordinate space, the set $X := \{0\} \cup L$. Observe that X is the set of all real numbers. The topology on X is generated by the usual topology plus all sets of the form $\{0\} \cup U$ where U is open in $\mathbb{R}$ and contains $\{1/i \mid i \in \mathbb{N}\}$. As a quotient of a sequential space, X is sequential by Lemma 3.3.

Now we shall show that X is not a Fréchet space. Let $A := X \setminus M$. We show that $0 \in \overline{A}$ but that no sequence in A converges to 0. Let N be a neighbourhood of 0. Then N contains a point $1/i$ for some $i \in \mathbb{N}$. Away from 0, the topology of X is locally 'as usual'. At any rate, there exists some $\delta > 0$ such that $(1/i - \delta, 1/i + \delta) \subseteq N$. Now there exists some irrational x with $|1/i - x| < \delta$. Then $x \in N$ and, x being irrational, $x \in A$. Hence $A \cap N \neq \varnothing$. As this was true for any neighbourhood of 0, this shows that $0 \in \overline{A}$. Now suppose that (x_i) is any sequence in A. For each $i \in \mathbb{N}$, we have $x_i \neq 0$, and therefore $\inf_{j \in \mathbb{N}} |x_i - 1/j| > 0$. Consider the sequence (δ_i), where $\delta_i := \inf_{j \in \mathbb{N}} |x_i - 1/j|$, and let

$$N := \{0\} \cup \bigcup_{i \in \mathbb{N}} (1/i - \delta_i, 1/i + \delta_i).$$

Then N is a neighbourhood of 0 that contains none of the members of the sequence (x_i). Therefore (x_i) cannot possibly converge to 0. This shows that no sequence in A converges to 0, so X is not a Fréchet space. $\square$

We now look at how well the property of sequentiality behaves under the operations that we like to use to in forming new topological spaces. Granting that quotients of sequential spaces are sequential, the class of sequential spaces does not in fact possess many other nice closure properties. With Example 3.9 we shall show that the product of two sequential spaces need not be sequential. First, we are going to give an example of a sequential space with a non-sequential subspace.

Example 3.6 ([Franklin(1965)] (Eg. 1.8))**.** Let X be the space in Example 3.5, and let A be as above. Let $Z := A \cup \{0\}$. We show that Z is a subspace of X that is not sequential. Recall that no sequence in A converges to 0. Therefore the only sequences in Z that converge to 0 are those that are eventually constant. Hence the set $\{0\}$ is sequentially open in Z. However, $\{0\}$ is not open in Z, for $f^{-1}\{0\} = \{0\} \times \{1\}$ is not open in Y, as any neighbourhood of $\{0\} \times \{1\}$ contains some point $(1/i, 1)$. Thus Z is not a sequential space. $\square$

Although a general subspace of a sequential space might not be sequential, a closed subspace must be.

Lemma 3.7 ([Franklin(1965)] (Prop. 1.9))**.** *A closed subspace of a sequential space is again sequential.* $\square$

It should be noted that, in Example 3.6, the space X is a sequential space that is not Fréchet. This is not a coincidence; in fact any example of a non-hereditarily sequential space must be of this type:

Lemma 3.8 ([Franklin(1967)] (Prop. 7.2)). *Let X be a sequential space. If every subspace of X is sequential, then X is Fréchet.* □

Recalling Remark 1.8, the proof is easy: for a non-closed subset A, take a point $x \in \overline{A} \setminus A$, then consider the subspace $A \cup \{x\}$.

We conclude our short investigation into the closure properties of the class of sequential spaces with the following example. This nice argument is due to T. K. Boehme, cited in [Franklin(1966)].

Example 3.9. Let X be any non-Hausdorff Fréchet space with unique sequential limits. (By Example 4.1 below, such an X exists; at present we omit the construction.) We show that the space $X \times X$ is not sequential. Since X is not Hausdorff, the diagonal $\Delta := \{(x, x) \mid x \in X\}$ is not closed in $X \times X$. On the other hand, Δ is sequentially closed, because sequential limits are unique. Hence $X \times X$ is not a sequential space. □

The Main Theorem. We are now going to state and prove our characterisation theorem, that identifies every sequential space as the image of a metric space under a quotient mapping.

Theorem 3.10 ([Franklin(1965)]). *If f is a quotient mapping from a metric space X onto a topological space Y, then Y is sequential. Conversely, if Y is a sequential space, then for some metric space X there exists a quotient mapping f from X onto Y.*

The first half of Theorem 3.10 is immediate from Lemma 3.3. To prove the second half we must find a suitable metric space X. We construct X as a topological sum of convergent sequences.

Remark 3.11. By a *convergent sequence*, we mean a sequence (x_i) of points together with a single limit x (in this we are following [Franklin(1965)]). We consider the set $X := \{x_i \mid i \in \mathbb{N}\} \cup \{x\}$ in its natural topology: each of the points x_i is isolated, and the neighbourhoods of x are precisely the complements of the finite subsets of X. Topologised in this way, X is the Alexandroff one-point compactification of the discrete space $\{x_i \mid i \in \mathbb{N}\} \setminus \{x\}$. Since the family of finite subsets of a countable set is countable, we see that X is a second countable compact Hausdorff space, so by Urysohn's metrisation theorem, X is metrisable.

Proof of Theorem 3.10. Let Y be a sequential space, and let $\mathcal{S}$ be the set of all convergent sequences in Y. For each sequence $S\colon \mathbb{N} \to Y : i \mapsto x_i$ in $\mathcal{S}$, let $L_S := \{x \in Y \mid x_i \to x\}$. For each $x \in L_S$, consider the space $S(\mathbb{N}) \cup \{x\}$ endowed with the metrisable topology described above. We wish to define X to be the disjoint topological sum of all such spaces. However, the spaces $S(\mathbb{N}) \cup \{x\}$ as we have described them are not, *de facto*, disjoint. Before forming the sum we need disjoint homeomorphic copies of these spaces. This can be achieved (cf. [Engelking(1989)] (p. 75)) by taking the product with a one-point discrete space (twice): let $X_{(S,x)} := (S(\mathbb{N}) \cup \{x\}) \times \{S\} \times \{x\}$. Then the spaces $X_{(S,x)}$ are disjoint, so we may form their topological sum:

$$X := \bigoplus_{x \in L_S, S \in \mathcal{S}} X_{(S,x)}$$

As a sum of metric spaces, X is metrisable [Engelking(1989)] (Thm. 4.2.1). Define a mapping $f\colon X \to Y$ by $f(y, S, x) := y$. We wish to show that f is a quotient mapping. Since every constant sequence converges, f is onto.

To show that f is continuous, it is enough [Engelking(1989)] (Prop. 2.2.6) to show that each of the restrictions $f \restriction X_{(S,x)}$ is continuous. Let $S \in \mathcal{S}$, let $x \in L_S$ and consider the restricted mapping $g := f \restriction X_{(S,x)}$. Let U be an open subset of Y, and let (y, S, x) be a point of $g^{-1}(U)$. Suppose that $((y_i, S, x) \mid i \in \mathbb{N})$ is a sequence in $X_{(S,x)} \setminus \{(y, S, x)\}$ that converges to (y, S, x). Then $y_i \neq y$ for all $i \in \mathbb{N}$ and (y_i) converges to y in $S(\mathbb{N}) \cup \{x\}$. Now S converges in Y to x, hence so does its subsequence (y_i). Since U is open in Y it follows that (y_i) is eventually in U. Hence $((y_i, S, x))$ is eventually in $g^{-1}(U)$. This shows that $g^{-1}(U)$ is sequentially open, hence open, in $X_{(S,x)}$; thus g is continuous. Therefore f is continuous.

So far, everything that we have done could be done for a general space Y. Now finally we must show that f is a quotient mapping: here we need to use the fact that Y is sequential. Let V be a subset of Y such that $f^{-1}(V)$ is open in X. Let $x \in V$ and suppose that $S = (x_i \mid i \in \mathbb{N})$ is a sequence in Y that converges to x. Form the subspace $X_{(S,x)}$ of X, and note that $f^{-1}(V) \cap X_{(S,x)}$ is a neighbourhood of (x, S, x) in this subspace. By definition of the subspace, the sequence $((x_i, S, x))$ converges in $X_{(S,x)}$ to (x, S, x), so therefore this sequence is eventually in $f^{-1}(V) \cap X_{(S,x)}$. This implies that (x_i) is eventually in V. Therefore V is sequentially open in Y, so V is open because Y is a sequential space. This shows that f is a quotient mapping. □

4. Fréchet Spaces

Unique Limits. In this section we discuss the class of Fréchet spaces. We have seen (Lemma 1.4) that every first countable space is Fréchet. An example of a Fréchet space that is not first countable is the set $\mathbb{R}$ with the topology $\{\mathbb{R} \setminus F \mid \text{finite } F \subseteq \mathbb{R}\}$. Shortly we are going to define a non-Hausdorff Fréchet space with unique sequential limits. This will show that the result of Proposition 2.4 does not generalise to Fréchet spaces. Later on (Theorem 4.4) we are going to prove a characterisation theorem for the class of Hausdorff Fréchet spaces: the Hausdorff Fréchet spaces are precisely the images of the metric spaces under pseudo-open mappings. In particular, the image of a metric space under a closed mapping is Fréchet.

Example 4.1 ([Franklin(1967)] (Eg. 6.2))**.** Let X be the product $\mathbb{N} \times \mathbb{N}$ in the usual (discrete) topology, and let p, $q \notin X$ with $p \neq q$. Let $Y := X \cup \{p, q\}$, and define a topology on Y as follows. The topology will be the coarsest with a base consisting of the topology on X and:

(1) the sets $\{p\} \cup ((\mathbb{N} \setminus F) \times \mathbb{N})$, where F is a finite subset of $\mathbb{N}$;
(2) the sets $\{q\} \cup \bigcup_{i \in \mathbb{N}} (\{i\} \times (\mathbb{N} \setminus F_i))$, where (F_i) is a sequence of finite subsets of $\mathbb{N}$.

Let U, V be neighbourhoods of p, q respectively. Then U contains all but finitely many lines $\{i\} \times \mathbb{N}$, so we can pick a natural number i such that $\{i\} \times \mathbb{N} \subseteq U$. Then V contains (i, j) for all but finitely many $j \in \mathbb{N}$, so we can pick a natural number j such that (i, j) is in V. Then (i, j) is in U too, so $U \cap V \neq \varnothing$. This shows that every neighbourhood of p meets every neighbourhood of q, so Y is not Hausdorff.

Next we show that no sequence in Y may converge to more than one point. Since X is discrete the only sequences converging to points of X are the eventually constant ones. Thus we need only consider the possibility that there might exist a sequence in Y converging both to p and to q. Moreover, there exist neighbourhoods of p that do not include q, and vice versa, so we need only consider the case of a sequence $((i_r, j_r) \mid r \in \mathbb{N})$ in X. Since $((i_r, j_r))$ converges to p, every neighbourhood of p contains all but finitely many of the (i_r, j_r). For each $i \in \mathbb{N}$, the set $Y \setminus (\{i\} \times \mathbb{N})$ is a neighbourhood of p, so there are only finitely many r such that $i_r = i$. Let $F_i := \{j_r \mid i_r = i\}$. Since (F_i) is a sequence of finite subsets of $\mathbb{N}$, the set

$$N := \{q\} \cup \bigcup_{i \in \mathbb{N}} (\{i\} \times (\mathbb{N} \setminus F_i))$$

is a neighbourhood of q. This is a contradiction, because N contains none of the points of the sequence $((i_r, j_r) \mid r \in \mathbb{N})$ that supposedly converges to q. Therefore there exists no such sequence after all. This shows that sequential limits in Y are unique.

Finally we shall show that Y is a Fréchet space. Since X is discrete, it is certainly Fréchet. It remains to prove that, whenever one of p, q lies in the closure of a set, then it is the limit of a sequence in that set. First consider p. Suppose that A is a subset of Y such that $p \in \overline{A}$. For each natural number r, the complement of the set $\{i \in \mathbb{N} \mid i < r\} \times \mathbb{N}$ is a neighbourhood of p. It follows that for each $r \in \mathbb{N}$ there exists some (i_r, j_r) in A with $i_r \geqslant r$. Then $i_r \to \infty$ as $r \to \infty$. Since any neighbourhood of p contains almost all of the lines $\{i\} \times \mathbb{N}$, it must contain almost all of the points (i_r, j_r). Thus $((i_r, j_r) \mid r \in \mathbb{N})$ is a sequence in A that converges to p.

Now consider q. Let B be a subset of Y with $q \in \overline{B}$. If $q \in B$, then there is nothing to prove. Suppose $q \notin B$. Then for each $i \in \mathbb{N}$, let $F_i := \{j \in \mathbb{N} \mid (i, j) \in B\}$, and consider the sequence (F_i) of subsets of $\mathbb{N}$. If every F_i is finite, then the set

$$\{q\} \cup (X \setminus B) = \{q\} \cup \bigcup_{i \in \mathbb{N}} (\{i\} \times (\mathbb{N} \setminus F_i))$$

is of the form given in (2) above, so it is a neighbourhood of q that fails to meet B, contradicting $q \in \overline{B}$. Therefore there exists a natural number i such that F_i is infinite. Arrange the elements of F_i into an infinite sequence $(j_r \mid r \in \mathbb{N})$. Now any neighbourhood of q contains almost all points of the line $\{i\} \times \mathbb{N}$, in particular it contains almost all of the (i, j_r). Thus $((i, j_r) \mid r \in \mathbb{N})$ is a sequence in B that converges to q. This shows that Y is a Fréchet space. $\square$

It is possible to give a slightly simpler example of a non-Hausdorff Fréchet space with unique sequential limits [Franklin(1966)]. Note however that in Example 4.1, the space Y is compact. This is of interest, because recalling Example 3.9 we can deduce that the square of a compact Fréchet space need not be sequential. In fact, the situation is about as bad as it could possibly be: even the product of a Hausdorff Fréchet space with a compact metric space need not be Fréchet. For instance, if X is the quotient space of $\mathbb{R}$ obtained by identifying the integers to a point, then it is an easy exercise to show that X is a Fréchet space but $X \times \mathbb{I}$ is not (this example is due to [Franklin(1967)] (Eg. 7.4)). Products of spaces of countable tightness are a little better behaved: see Theorem 5.6.

Like first countability and unlike sequentiality, the Fréchet property is hereditary:

Lemma 4.2. *Any subspace of a Fréchet space is itself a Fréchet space.* $\square$

This result furnishes a converse to Lemma 3.8; so we can deduce that the Fréchet spaces are precisely the hereditarily sequential spaces.

$$\begin{tikzcd}
 & & \text{proper} \arrow[d] \\
\text{open onto} \arrow[dr] & & \text{closed onto} \arrow[dl] \\
 & \text{pseudo-open} \arrow[d] & \\
 & \text{quotient} &
\end{tikzcd}$$

FIGURE 4.1. Relations between some classes of continuous mappings.

CHARACTERISATION THEOREM. Before we state our characterisation theorem for Fréchet spaces, we need to define a new class of mappings.

Definition 4.3 (pseudo-open mapping)**.** Let X, Y be Hausdorff spaces, and let $f\colon X \to Y$ be continuous and onto. We say that f is *pseudo-open* if whenever U is an open set in X containing some fibre $f^{-1}\{y\}$, then $f(U)$ is a neighbourhood of y in Y.

It is clear that every open onto mapping is pseudo-open. Also, any closed onto mapping is pseudo-open: for if $f\colon X \to Y$ is closed, then let $y \in Y$. Suppose that U is open in X with $f^{-1}\{y\} \subseteq U$. Then $X \setminus U$ is closed in X, so $f(X \setminus U)$ is closed in Y. Since $f^{-1}\{y\}$ is contained in U, it follows that y is not an element of $f(X \setminus U)$. Hence y belongs to the open set $Y \setminus f(X \setminus U)$, which is a subset of $f(U)$ (because f is onto). Thus $f(U)$ is a neighbourhood of y and f is pseudo-open. Figure 4.1 summarises the situation.

Now we can present the characterisation theorem. This theorem is due to [Arhangel′skiĭ(1963)] (Thm. 2), although in our proof we follow [Franklin(1965)].

Theorem 4.4 (Arhangel′skiĭ)**.** *If f is a pseudo-open mapping from a metric space X onto a Hausdorff space Y, then Y is Fréchet. Conversely, if Y is a Hausdorff Fréchet space, then for some metric space X there exists a pseudo-open mapping f from X onto Y.*

To prove Theorem 4.4 we shall call upon Theorem 3.10. First, we note that every pseudo-open mapping is quotient:

Lemma 4.5. *Let X, Y be Hausdorff spaces, and let $f\colon X \to Y$ be pseudo-open. Then f is a quotient mapping.* □

The next result will allow us to give an easy proof of Theorem 4.4.

Proposition 4.6 ([Arhangel′skiĭ(1963)] (Thm. 4))**.** *Let X be a Hausdorff Fréchet space, let Y be a Hausdorff space and let $f\colon X \to Y$ be a quotient map. Then Y is a Fréchet space if and only if f is pseudo-open.*

Proof. (Franklin) Suppose first that Y is a Fréchet space. Let U be an open subset of X containing some fibre $f^{-1}\{y\}$. Suppose for a contradiction that $f(U)$ is not a neighbourhood of y. Then $y \in \overline{(Y \setminus f(U))}$, so as Y is Fréchet there exists a sequence

(y_i) in $Y \setminus f(U)$ that converges to y. Let $A := \{y_i \mid i \in \mathbb{N}\}$, and let $B := f^{-1}(A)$. Note that $\overline{A} = A \cup \{y\}$ because Y is Hausdorff and (y_i) converges to y. Therefore, since f is continuous,

$$\overline{B} = \overline{(f^{-1}(A))} \subseteq f^{-1}(\overline{A}) = f^{-1}(A \cup \{y\}) = B \cup f^{-1}\{y\} \tag{2}$$

However, we have $A \subseteq Y \setminus f(U)$, whence $U \cap B = \varnothing$. Since U is open, it follows that $U \cap \overline{B} = \varnothing$. Now we recall that $f^{-1}\{y\} \subseteq U$, so therefore $f^{-1}\{y\} \cap \overline{B} = \varnothing$. From (2) we now see that $\overline{B} \subseteq B$, i.e., that B is closed. Therefore $f^{-1}(Y \setminus A) = X \setminus B$ is open, and since f is a quotient mapping it follows that $Y \setminus A$ is open, i.e., that A is closed. Thus $A = \overline{A} = A \cup \{y\}$, so that $y \in A \subseteq Y \setminus f(U)$. This contradicts the fact that $f^{-1}\{y\} \subseteq U$. Therefore $f(U)$ is a neighbourhood of y after all, hence f is pseudo-open.

Now suppose conversely that f is pseudo-open: we wish to show that Y is Fréchet. Let A be a subset of Y and suppose that $y \in \overline{A}$. Let $B := f^{-1}(A)$ and suppose for a contradiction that $f^{-1}\{y\} \cap \overline{B} = \varnothing$. Then there exists an open set U containing $f^{-1}\{y\}$ such that $U \cap B = \varnothing$, whence $f(U) \cap A = \varnothing$. Since f is pseudo-open, $f(U)$ is a neighbourhood of y, and this contradicts the fact that $y \in \overline{A}$. Therefore $f^{-1}\{y\} \cap \overline{B} \neq \varnothing$, so there exists some $x \in f^{-1}\{y\}$ such that $x \in \overline{B}$. Since X is a Fréchet space, there exists a sequence (x_i) in B that converges to x. Then $(f(x_i))$ is a sequence in A that converges to $f(x) = y$. This shows that Y is a Fréchet space. □

Corollary 4.7. *The class of Fréchet spaces is closed with respect to taking Hausdorff images under proper mappings.* □

At last we arrive at the proof of the main theorem.

Proof of Theorem 4.4. Suppose first that X is a metric space, that Y is a Hausdorff space and that $f: X \to Y$ is pseudo-open and onto. Since X is a metric space, it is Hausdorff and Fréchet. Therefore by Proposition 4.6, Y is a Fréchet space.

Conversely, suppose that Y is a Hausdorff Fréchet space. In particular Y is a sequential space, so by Theorem 3.10 there exists a metric space X and a quotient mapping $f: X \to Y$. As a metric space, X is Hausdorff and Fréchet, therefore f is pseudo-open by Proposition 4.6. □

5. Countable Tightness

Preamble. Countable tightness is of interest to us principally because of the Moore-Mrówka problem. The solution of the problem has been described as 'the main advance in the theory of compact spaces during [the 1980s],' by [Shakhmatov(1992)] (p. 574). This indicates the significance of tightness and sequentiality in analytic topology. One of the trends in topology since the 1960s has been the growing importance of typical example. These invariants are usually defined in general terms, but most of them are generalisations of preexisting concepts involving countability in some way. A summary of relations between some of the most important cardinal invariants appears in Figure 5.1. For more details (and proofs) see [Hodel(1984)] and [Juhász(1980), Juhász(1984)].

How about Sequences? We noted in Section 1 that every sequential space has countable tightness. The converse is false, however.

$$\begin{tikzcd}
 & {|X|} \arrow[d] & w(X) \arrow[dl] \arrow[dd] & \\
 & nw(X) \arrow[d] \arrow[drr] & & \\
 & hL(X) \arrow[dl] \arrow[d, "T_3"] & \chi(X) \arrow[ddl, "T_1"] \arrow[dd] & hd(X) \arrow[ddl] \arrow[d] \\
L(X) & \Psi(X) \arrow[d, "T_1"'] & & d(X) \arrow[d] \\
 & \psi(X) & t(X) & c(X)
\end{tikzcd}$$

Key.

Invariant	Name	If invariant $= \aleph_0$, space is called
$\lvert X \rvert$	cardinality of X	countable
$w(X)$	weight of X	second countable
$nw(X)$	network weight of X	
$\chi(X)$	character of X	first countable
$hL(X)$	hereditary Lindelöf number of X	hereditarily Lindelöf
$hd(X)$	hereditary density of X	hereditarily separable
$\Psi(X)$	index of perfection[a] of X	perfect
$L(X)$	Lindelöf number of X	Lindelöf
$d(X)$	density of X	separable
$t(X)$	tightness of X	
$\psi(X)$	pseudo-character of X	
$c(X)$	Souslin number of X	CCC (countable chain condition)

[a]This terminology is non-standard. See Section 6.

Figure 5.1. Relations between cardinal invariants.

Example 5.1 ([Franklin(1969)] (Thm. A)). We exhibit a non-sequential space with countable tightness. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of the natural numbers. Let $X := \beta\mathbb{N}$ and consider the topology on X generated by the topology on $\beta\mathbb{N}$ together with the sets $\{x\} \cup \mathbb{N}$ for all x in $X \setminus \mathbb{N}$. We have already noted that the only convergent sequences in $\beta\mathbb{N}$ are those that are eventually constant (Example 1.1). Clearly, in a finer topology the number of convergent sequences cannot increase, so the only convergent sequences in X are those that are eventually constant. Therefore every subset of X is sequentially open, but $X \setminus \mathbb{N}$ is not open, so X is not a sequential space. However, X has countable tightness. To see this, let A be any subset of X. Note that any subset of X containing $\mathbb{N}$ is open, hence any subset that is disjoint from $\mathbb{N}$ is closed. Therefore

$$\overline{A} = \overline{(A \setminus \mathbb{N})} \cup \overline{(A \cap \mathbb{N})} = (A \setminus \mathbb{N}) \cup \overline{(A \cap \mathbb{N})} \subseteq A \cup \overline{(A \cap \mathbb{N})}. \tag{3}$$

Now, if A contains the closure of all its countable subsets then in particular it contains the closure of $A \cap \mathbb{N}$. Then (3) shows that $\overline{A} = A$, so that A is closed. This shows that X has countable tightness. $\square$

Using the last example, we can show that there is no generalisation of Lemma 3.1 to spaces of countable tightness.

Example 5.2. Let X be the space of Example 5.1, and let Y be the same set ($\beta\mathbb{N}$) in the discrete topology. Then the topology on Y is strictly finer than the topology on X, so the identity mapping $\iota\colon X \to Y$ is *not* continuous. However, the only convergent sequences in X are those that are eventually constant, hence ι preserves limits of sequences. $\square$

With this example and previous results, all the assertions of Figure 1.1 are proved.

Countable Tightness is not Productive. How well does the class of spaces of countable tightness behave under the usual operations? First of all we note the elementary result that countable tightness is hereditary.

Lemma 5.3. *Let X be a topological space with countable tightness, and let Y be a subspace of X. Then Y has countable tightness.* $\square$

To show that countable tightness is an invariant of proper mappings, we use the following:

Lemma 5.4 ([Arhangel′skiĭ and Ponomarev(1968)] (Prop. 3))**.** *Let X be a topological space of countable tightness, and suppose that $f\colon X \to Y$ is a quotient mapping. Then Y has countable tightness.*

Proof. Let A be a subset of Y that contains the closure of all its countable subsets. Let C be a countable subset of $f^{-1}(A)$. Then $f(C)$ is a countable subset of A, so $\overline{f(C)} \subseteq A$. Using the fact that f is continuous, it follows that

$$\overline{C} \subseteq \overline{\left(f^{-1}f(C)\right)} \subseteq f^{-1}\left(\overline{f(C)}\right) \subseteq f^{-1}(A). \tag{4}$$

Therefore $f^{-1}(A)$ is a subset of X that contains the closure of each of its countable subsets. Since X has countable tightness, we can deduce that $f^{-1}(A)$ is closed in X. Since f is a quotient mapping, it follows that A is closed in Y. Hence Y has countable tightness. $\square$

Corollary 5.5. *The class of spaces with countable tightness is closed with respect to taking images under proper mappings.* $\square$

The product of two spaces of countable tightness need not in general have countable tightness. An example was given by [Arhangel′skiĭ(1972)]: we omit the details. Let X be the quotient space obtained by identifying the limit points in the disjoint sum of countably many convergent sequences, and let Y be the corresponding space constructed from uncountably many convergent sequences. Then it is easily verified that X and Y have countable tightness. However, their product $X \times Y$ has uncountable tightness (consider the point whose coordinates are the limit points in X, Y respectively).

If one of the factor spaces is compact then the product *does* have countable tightness, according to the following theorem. The proof we give is due to [Juhász(1980)] (5.9).

Theorem 5.6 ([Malyhin(1972)] (Thm. 4)). *Let X, Y be Hausdorff spaces with countable tightness, and suppose that Y is compact. Then $X \times Y$ has countable tightness.*

Proof. Let A be a subset of $X \times Y$ that contains the closure of all its countable subsets. We wish to show that A is closed. Suppose for a contradiction that it is not, so that there exists a point (x, y) in $\overline{A} \setminus A$. It is not obvious how to obtain a contradiction, so a few remarks are in order before we proceed. We know very little about the product $X \times Y$, so most of the work will have to be done in the factor spaces. The plan of attack is as follows: we are going to define two subsets B and W of A in such a way that they are obviously disjoint, and then we shall show that they have a point in common. More specifically, we define B and project it down to Y, then we use the separation properties of Y and the assumption $(x, y) \notin A$ to define W disjoint from B. Then we project W down to X and show that the assumption $(x, y) \in \overline{A}$ implies that W and B *cannot* be disjoint after all.

Since X is a T_1-space, $\{x\}$ is closed in X, hence $\{x\} \times Y$ is closed in $X \times Y$. The subspace $\{x\} \times Y$ is homeomorphic to Y, so it has countable tightness. Let $B := A \cap (\{x\} \times Y)$. Then B contains the closure in $\{x\} \times Y$ of all its countable subsets. Since $\{x\}\times Y$ has countable tightness, it follows that B is closed in $\{x\}\times Y$, hence in $X \times Y$. Now, $\{x\} \times Y$ is homeomorphic to Y, and B is closed in $\{x\} \times Y$, so therefore $\pi_Y(B)$ is closed in Y. Since $(x, y) \notin A$ we have $y \notin \pi_Y(B)$. Now Y is a regular space, so there exist disjoint open subsets U, V of Y such that $y \in U$ and $\pi_Y(B) \subseteq V$. So if we let $E := \overline{U}^Y$, then E is a closed neighbourhood of y in Y and $\pi_Y(B) \cap E = \varnothing$. Hence $X \times E$ is a closed neighbourhood of (x, y) in $X \times Y$.

Finally now, let $W := A \cap (X \times E)$. Let N be a neighbourhood of (x, y). Then $(X \times E) \cap N$ is another neighbourhood, and since (x, y) belongs to $\overline{A}$, the set $W \cap N = A \cap (X \times E) \cap N$ is non-empty. This shows that (x, y) is an element of the closure of W. Now W contains the closure of all its countable subsets, because it is the intersection of two sets with this property. Since Y is compact, the projection π_X is closed. Hence $\pi_X(W)$ is a subset of X that contains the closure of all its countable subsets. Since X has countable tightness, it follows that $\pi_X(W)$ is closed in X. Using the continuity of π_X, we have

$$\pi_X(W) = \overline{\pi_X(W)}^X \supseteq \pi_X(\overline{W}) \ni x. \tag{5}$$

Hence there exists some point (x, z) in $W = A \cap (X \times E)$. Then z belongs to E, and (x, z) belongs to $A \cap (\{x\} \times Y) = B$. Thus we arrive at the contradiction that z is an element of the set $\pi_Y(B) \cap E$, a set which is empty by our choice of E. □

THE MOORE-MRÓWKA PROBLEM. It is worth remarking that in Example 5.1 we had to adapt the topology of the non-sequential space $\beta\mathbb{N}$ in such a way that it was no longer compact, before we obtained countable tightness. The question

(MM) Is every compact Hausdorff space of countable tightness a sequential space?

is known as the *Moore-Mrówka problem*. The problem was posed in [Moore and Mrówka(1964)]. It is now known that the question cannot be answered in ZFC. The set-theory required to prove this means that it would take too long to go into the details here, so we must content ourselves with stating the following two-part theorem:

Theorem 5.7.

(1) *Assume $\diamondsuit$, and let X be Ostaszewski's space* [Ostaszewski(1976)]. *Let αX be the Alexandroff one-point compactification of X. Then αX has countable tightness but is not a sequential space.*
(2) *Assume PFA. Then every compact Hausdorff space of countable tightness is a sequential space—see* [Balogh(1989)].

Many other results connected to the Moore-Mrówka problem can be found in the survey by [Shakhmatov(1992)] (Sec. 2).

6. Perfect Spaces

Outline. We are now going to consider perfect spaces: we are particularly interested in compactness in perfect spaces. We start off by considering the relationship between the class of perfect spaces and other important classes of spaces. In general, the perfect spaces are not directly related to those that we have considered up until now (Examples 6.3 and 6.5). However, when suitable compactness properties and separation axioms are assumed, then some useful results are achieved. In particular, we are going to prove that every countably compact regular perfect space has countable tightness (Theorem 6.6).

The reader is asked to look again at Figure 1.1. Of the many useful properties of sequences in metric spaces, we have generalised all except the following: 'compactness and sequential compactness are equivalent'. The statement is true for first countable paracompact spaces. However, paracompactness is a very strong condition. Looking to strengthen the result, it is noteworthy that every perfectly normal space is *countably* paracompact [Nagata(1985)] (Cor. to V.5). Since every metric space is both first countable and perfectly normal, an attractive conjecture is that compactness and sequential compactness coincide for these spaces—in fact this is independent of ZFC (Corollary 6.12).

When is a Space Perfect? The following result appears in [Arhangel′skiĭ and Ponomarev(1984)] (Chap. 3, Prob. 206).

Lemma 6.1. *Suppose that X is hereditarily Lindelöf and regular. Then X is a perfect space.*

Proof. Let U be any open subset of X. Since X is regular, for every $x \in U$ there exists an open set V such that $x \in V \subseteq \overline{V} \subseteq U$. Then the family consisting of all the sets V is an open cover for U. Since X is hereditarily Lindelöf, U is Lindelöf. So there exists a countable subcover $\{V_i \mid i \in \mathbb{N}\}$. But by construction $\overline{V_i} \subseteq U$ for every natural number i, and since $\{V_i\}$ is a cover, it follows that $U = \bigcup_{i \in \mathbb{N}} \overline{V_i}$. This shows that U is an F_σ. Hence X is a perfect space. □

Corollary 6.2. *A second countable regular space is perfect.* □

Of course, by Urysohn's metrisation theorem a second countable regular space is pseudo-metrisable, and pseudo-metric spaces are always perfect. However, our argument generalises to arbitrary cardinalities. Thus for regular spaces, $\Psi(X) \leqslant hL(X)$, where $hL(X)$ is the hereditary Lindelöf number of the space, and $\Psi(X)$ is the *index of perfection*, that is, the least infinite cardinal $\mathfrak{m}$ such that every closed set can be written as an intersection of $\leqslant \mathfrak{m}$ open sets.

It is well known that every regular Lindelöf space is normal ([Kelley(1955)] (Lem. 4.1) calls this *Tychonoff's lemma*). Thus Lemma 6.1 implies that a regular hereditarily Lindelöf space is perfectly normal.

It should be noted in this context that there exist first countable compact Hausdorff spaces that are not perfect:

Example 6.3 (Alexandroff double circle [Alexandroff and Urysohn(1929)])**.** Consider the two concentric circles in $\mathbb{R}^2$ defined by $C_1 := \{(r\angle\theta) \in \mathbb{R}^2 \mid r^2 = 1\}$, and $C_2 := \{(r\angle\theta) \in \mathbb{R}^2 \mid r^2 = 2\}$. For each natural number i and each point $z = (1\angle\theta)$ in C_1, define two points z_{i+} and z_{i-} by $z_{i\pm} := (1\angle\theta \pm 1/i)$. Define a map $p\colon C_1 \to C_2\colon z \mapsto \sqrt{2}z$, so that p is the projection of C_1 onto C_2 through 0. The *Alexandroff double circle* is the set $X := C_1 \cup C_2$, with the topology on X generated by all the sets $\{z\}$ for z in C_2 and the sets

$$U_i(z) := V_i(z) \cup p(V_i(z) \setminus \{z\}), \qquad \text{for } z \in C_1 \text{ and } i \in \mathbb{N},$$

where $V_i(z)$ is the open arc of C_1 between z_{i-} and z_{i+}, i.e.,

$$V_i(r\angle\theta) := \left\{ (r\angle\theta + \delta) \,\middle|\, -\frac{1}{i} < \delta < \frac{1}{i} \right\}.$$

It is clear that X, so topologised, is a first countable Hausdorff space.

We shall show that X is compact. Let $\mathcal{E}$ be an open cover for X, consisting of open sets from the subbasis described above. Now, $f\colon [0, 2\pi] \to C_1\colon \theta \mapsto (1\angle\theta)$ is a continuous mapping of $[0, 2\pi]$ onto C_1 in the subspace topology. Hence C_1 is a compact subspace of X. So there exists a finite subset $\{W_1, \ldots, W_n\}$ of $\mathcal{E}$ that covers C_1. Then as the elements of $\mathcal{E}$ are subbasic, each W_k is of the form $U_{i(k)}(z_k)$ for some $i_k \in \mathbb{N}$ and $z_k \in C_1$. Clearly then, $\{W_1, \ldots, W_n\}$ covers all of C_2 except for possibly the points $p(z_1)$, $p(z_2)$, $\ldots$, $p(z_n)$. For each $k = 1, 2, \ldots, n$, let Z_k be an element of $\mathcal{E}$ such that $p(z_k)$ belongs to Z_k. Then $\{W_k, Z_k \mid k = 1, \ldots, n\}$ is a finite subcover of $\mathcal{E}$. Thus every cover of $\mathcal{E}$ consisting of subbasic open sets has a finite subcover. By the Alexander theorem [Kelley(1955)] (Thm. 5.6) this is enough to show that X is a compact space. Of course, the Alexander theorem is not required to prove this, but it does simplify the proof.

Finally we show that X is not a perfect space. The set $C_1 = X \setminus \{\{z\} \mid z \in C_2\}$ is closed, but it is not a G_δ set. For if W is an open neighbourhood of C_1, then it must contain all but finitely many points of C_2. Hence any countable intersection of neighbourhoods of C_1 meets C_2, and it follows that C_1 is not a G_δ. This shows that X is a compact Hausdorff first countable space that is not perfect. □

What closure properties are possessed by the class of perfect spaces? The inverse image of a perfect space under a proper mapping need not be perfect, as we see if we map the Alexandroff double circle to a single point. On the other hand, the following results can be proved in the obvious fashion:

Lemma 6.4.

(1) *The image of a perfect space under a closed onto mapping is again a perfect space. In particular, the class of perfect spaces is closed with respect to taking images under proper mappings.*
(2) *Every subspace of a perfect space is perfect.* □

We shall complete the task of showing that perfect spaces do not fit into the hierarchy of Figure 1.1. Assuming CH, [Hajnal and Juhász(1974)] (Thm. 1) gave an example of an L-space in which every countable subset is closed.[1] It follows that their space is perfectly normal with uncountable tightness. I posed the question of whether there exists such a space in ZFC; it cannot be compact, as we shall see in Theorem 6.6. The [Arens(1950)] example of a perfectly normal space that is not sequential cannot be used here, because it is countable, hence automatically has countable tightness. Jo Lo has supplied me with the following example, which is a modification of the 'Radial Interval' topology described in [Steen and Seebach(1996)] (141). With hindsight it appears that Lo's example and Arens's are actually quite similar.

Example 6.5. Let $\mathbb{I}$ denote the unit interval in the usual topology, and let L be the disjoint topological sum $\bigoplus_{r\in\mathbb{R}} \mathbb{I} \times \{r\}$. Let M be the quotient space of L in which all the points $(0, r)$ are identified (the resulting point is henceforth denoted 0). Let $q\colon L \to M$ be the canonical mapping. Let X be the set M with the topology whose closed sets are the closed sets in M together with all countable unions of sets of the form $q((0, 1] \times \{r\})$. Then X does not have countable tightness. For, the set $X \setminus \{0\}$ is not closed, but $0 \notin \overline{C}$ for any countable $C \subseteq X \setminus \{0\}$. To see this, note that for at most countably many values of r can the set $C \cap q((0, 1] \times \{r\})$ be non-empty. If these values of r are enumerated $\{r_i \mid i \in \mathbb{N}\}$, then we have

$$\overline{C} \subseteq \bigcup_{i\in\mathbb{N}} q((0, 1] \times \{r_i\}), \tag{6}$$

so that $0 \notin \overline{C}$. Thus X does not have countable tightness.

Now we show that X is a perfectly normal space. First, note that L is a topological sum of metric spaces, so it is metric. The mapping q is closed, because $q^{-1}\{0\} = \{(0, r) \mid r \in \mathbb{R}\}$ is closed in L. It follows that $M = q(L)$ is perfectly normal; combine Lemma 6.4(1) with the elementary fact that closed mappings preserve normality. Now, X has mostly the same closed sets as M, and the extra ones of the form $q((0, 1] \times \{r\})$ are both open and closed. It is easy to see that perfect normality is preserved under such an extension of the topology. So X is perfectly normal with uncountable tightness, as claimed. □

On Compactness. Now we begin our consideration of compactness in perfect spaces. First we are going to see how its presence affects matters, and later on we shall consider when its presence may be guaranteed by weaker covering properties.

We are going to start by proving the next theorem, which is noted in [Weiss(1978)] (Prop. 3). It is used in a crucial way by [Weiss(1978)] in proving Theorem 6.11(2) below.

Theorem 6.6. *Every perfectly regular countably compact space has countable tightness.*

We shall prove this using two subsidiary results. The first is a proposition of [Ostaszewski(1976)] (Prop. 2.1), which states that perfect countably compact spaces have *countable spread*.

Proposition 6.7. *Let X be a perfect countably compact space. Whenever Y is a subset of X that is discrete in the subspace topology, then Y is countable.*

[1]An *L-space* is a regular space that is hereditarily Lindelöf but not hereditarily separable.

Proof. Suppose for a contradiction that Y is uncountable. Without loss of generality we may assume that $|Y| = \aleph_1$, so we can index Y by the countable ordinal numbers: $Y = \{x_\alpha \mid \alpha < \omega_1\}$. Since Y is discrete, for every $\alpha < \omega_1$ we can choose an open subset U_α of X such that $U_\alpha \cap Y = \{x_\alpha\}$. Now let $U := \bigcup\{U_\alpha \mid \alpha < \omega_1\}$. Then U is an open set, and since X is perfect it follows that U is an F_σ. Let (F_i) be an increasing sequence of closed sets such that $U = \bigcup_{i\in\mathbb{N}} F_i$. Then

$$Y = U \cap Y = \bigcup_{i\in\mathbb{N}} (F_i \cap Y). \tag{7}$$

Since Y is uncountable, there is some natural number i such that $F_i \cap Y$ is uncountable. Let $\alpha < \omega_1$ be such that x_α belongs to F_i. Then the set $\{x_\beta \mid \beta \leqslant \alpha\}$ is countable, so it does not exhaust $F_i \cap Y$. This fact allows us to pick a strictly increasing sequence $(\alpha(r))$ of ordinal numbers $< \omega_1$ such that $x_{\alpha(r)}$ belongs to F_i for all $r \in \mathbb{N}$.

The rest of the proof is book-keeping, as we show that the existence of the sequence $(x_{\alpha(r)})$ is incompatible with countable compactness. For each $r \in \mathbb{N}$, let $A_r := \{x_{\alpha(s)} \mid s \in \mathbb{N}, s \geqslant r\}$, and let $E_r := \overline{A_r}$. Then (E_r) is a decreasing sequence of closed subsets of the countably compact space X, so the intersection $E := \bigcap_{r\in\mathbb{N}} E_r$ is non-empty. Let x be a point in E. Since F_i is closed we have $E \subseteq F_i$, and recalling that $F_i \subseteq U$ it follows that x belongs to U_β for some $\beta < \omega_1$. Since U_β is open, it is a neighbourhood of x.

There are now two cases to consider. Suppose first that $\beta \geqslant \alpha(r)$ for all $r \in \mathbb{N}$. Since $(\alpha(r))$ is increasing we must have $\beta > \alpha(r)$ for all $r \in \mathbb{N}$, therefore $x_\beta \notin A_1$. Now $U_\beta \cap Y = \{x_\beta\}$, and A_1 is a subset of Y, so it follows that $U_\beta \cap A_1 = \varnothing$. But this contradicts the fact that x belongs to $E_1 = \overline{A_1}$. Finally, suppose that $\beta < \alpha(r)$ for some natural number r. Then we have $x_\beta \notin A_r$. Now $U_\beta \cap Y = \{x_\beta\}$, and A_r is a subset of Y, so it follows that $U_\beta \cap A_r = \varnothing$. Therefore $x \notin E_r = \overline{A_r}$, again a contradiction. □

The remaining result is my own combination of arguments from [Juhász(1980)] (3.12), [Arhangel′skiĭ and Ponomarev(1984)] (Prob. III.68) and [Arhangel′skiĭ(1971)] (Lem. 4).

Proposition 6.8. *Let X be a regular countably compact space of uncountable tightness. Then X contains an uncountable subset Y which is discrete in the subspace topology.*

Proof. The proof is split into two steps. In Step 2 we are going to construct an uncountable discrete subspace Y by transfinite induction. In order to facilitate this construction, we shall find in Step 1 a point x and a subset A such that x belongs to $\overline{A} \setminus A$, and A meets every G_δ set that contains x.

Step 1 Since X has uncountable tightness, there exists some non-closed subset of A that contains the closure of all its countable subsets. Pick any point x in $\overline{A} \setminus A$, and let Q be a G_δ set that contains x. We shall show that Q meets A by the device of showing that Q meets the closure of a countable subset of A.

Let $\{U_i \mid i \in \mathbb{N}\}$ be a countable family of open sets such that $Q = \bigcap_{i\in\mathbb{N}} U_i$. Now, X is a regular space, so for every $i \in \mathbb{N}$ there exists an open set V_i such that $x \in V_i \subseteq \overline{V_i} \subseteq U_i$. Let $\mathcal{B}$ be the family of all finite intersections of $V_1, V_2, \ldots$, and let $P := \bigcap_{i\in\mathbb{N}} \overline{V_i} \subseteq Q$. Note that $\mathcal{B}$ is countable. Now, let $B = \bigcap_{i \in F} V_i$ be a typical element of $\mathcal{B}$, where $F \subseteq \mathbb{N}$ is finite. For each natural number i, we have $x \in V_i$, therefore x belongs to B. As a finite intersection of open sets, B is open, and since $x \in \overline{A}$ it follows that $B \cap A$ is non-empty. For each $B \in \mathcal{B}$, let x_B be a point in $B \cap A$, and let $C := \{x_B \mid B \in \mathcal{B}\}$. Then C is a countable subset of A, so therefore $\overline{C} \subseteq A$. We are going to show that $P \cap \overline{C} \neq \varnothing$, and it will follow that Q meets A. Let N be an open neighbourhood of P; then $X \setminus N$ is a closed subset of X, so it is countably compact. Now, recall that $P = \bigcap_{i \in \mathbb{N}} \overline{V_i}$, therefore the family $\{X \setminus \overline{V_i} \mid i \in \mathbb{N}\}$ is an open cover for $X \setminus P$. The countably compact set $X \setminus N$ is contained in $X \setminus P$, so there exists a finite subfamily $\{X \setminus \overline{V_1}, \ldots, X \setminus \overline{V_n}\}$ such that $X \setminus N \subseteq \bigcup_{i=1}^n (X \setminus \overline{V_i})$. Therefore, letting $B := \bigcap_{i=1}^n V_i$, we see that $B \subseteq \bigcap_{i=1}^n \overline{V_i} \subseteq N$. It follows that $x_B \in N$, so that $N \cap C \supseteq \{x_B\} \neq \varnothing$. We can do this for every neighbourhood N of P, hence $P \cap \overline{C}$ is non-empty. This shows that Q meets A, which completes Step 1.

Step 2 We are going to define a family $Y = \{y_\alpha \mid \alpha < \omega_1\}$ of distinct points of X such that Y is discrete as a subspace of X. This will complete the proof of the proposition, because Y is uncountable.

Let x, A be as in Step 1. Pick any point $y_{-\infty}$ in A, and let $W_{-\infty} := X$. Let $\beta < \omega_1$, and suppose inductively that for all $\alpha < \beta$, we have found a point y_α in A and an open set W_α such that

$$(\mathrm{H}_\alpha) \quad \begin{cases} x \in W_\alpha, \\ \overline{W_\alpha} \cap \{y_\gamma \mid \gamma < \alpha\} = \varnothing, \\ y_\alpha \in W_\gamma \qquad \text{for all } \gamma \leqslant \alpha. \end{cases}$$

Let $Y_\beta := \{y_\alpha \mid \alpha < \beta\}$. Then Y_β is a countable subset of A, so $\overline{Y_\beta} \subseteq A$. Therefore the point $x \notin A$ from Step 1 does not belong to $\overline{Y_\beta}$. So there exists an open neighbourhood N of x such that $N \cap Y_\beta = \varnothing$. Now X is regular, so we can find an open set W_β such that $x \in W_\beta \subseteq \overline{W_\beta} \subseteq N$. Therefore $\overline{W_\beta} \cap \overline{Y_\beta} = \varnothing$, that is $\overline{W_\beta} \cap \{y_\alpha \mid \alpha < \beta\} = \varnothing$. Now let $Q_\beta := \bigcap \{W_\alpha \mid \alpha \leqslant \beta\}$. Then Q_β is a G_δ set that contains x, so by Step 1 above it has non-empty intersection with A. Let y_β be any point in $Q_\beta \cap A$. Then y_β belongs to A and we have $x \in W_\beta$ and $y_\beta \in W_\alpha$ for all $\alpha \leqslant \beta$. This shows that (H_β) is satisfied by our choice of y_β, W_β. So by the principle of transfinite induction, we can define y_α, W_α in line with (H_α) for all $\alpha < \omega_1$. Then $Y := \{y_\alpha \mid \alpha < \omega_1\}$ is an uncountable subset of X. Now, by construction we have $y_\alpha \in W_\beta$ for all $\alpha \geqslant \beta$ but $y_\alpha \notin \overline{W_\beta}$ whenever $\alpha < \beta$. It follows that, whenever $\alpha < \omega_1$, the set $Z := W_\alpha \setminus \overline{W_{\alpha+1}}$ is open in X with the property that $Z \cap Y = \{x_\alpha\}$. This shows that Y is an uncountable discrete subspace of X, as required. □

Combining Propositions 6.7–6.8 we obtain Theorem 6.6.

One can also show that every compact perfectly normal space is first countable. The proof will be omitted because it uses no new ideas.

On Sequential Compactness. When are compactness and sequential compactness equivalent? We saw in Section 3 that if the space is sequential, then sequential compactness is equivalent to countable compactness. Since compactness always implies countable compactness, it is then natural to consider the question of when compactness is equivalent to countable compactness. Trivially this is true in every Lindelöf space. We also know that compactness and countable compactness are equivalent in metric spaces; the following result is a generalisation. The proof is based on that given in [Engelking(1989)] (Thm. 5.1.20).

Lemma 6.9. *A regular countably compact paracompact space is compact.*

Proof. This is true because all locally finite families in a countably compact space are in fact finite. If a space X is paracompact, then every open cover for X has a locally finite closed refinement, $\mathcal{E}$ say. Suppose, to get a contradiction, that $\mathcal{E}$ is not finite. Then $\mathcal{E}$ has a countable infinite subfamily $\{E_i \mid i \in \mathbb{N}\}$. We shall show that X is not countably compact. Since $\mathcal{E}$ is locally finite it is closure-preserving, so that $\bigcup \mathcal{D}$ is closed for every subfamily $\mathcal{D} \subseteq \mathcal{E}$. Therefore if we let $F_1 := \bigcup_{i=1}^{\infty} E_i$, then F_1 is closed. Now let $F_2 := \bigcup_{i=2}^{\infty} E_i$. Then F_2 is also closed, and $F_2 \subseteq F_1$. Continuing in this way we can define a decreasing sequence (F_i) of closed subsets of X. Since $\mathcal{E}$ is locally finite, no point of X is contained in infinitely many elements of $\mathcal{E}$, and it follows that no point of X is contained in all of the F_i. Therefore $\bigcap_{i=1}^{\infty} F_i = \varnothing$. Thus $\{F_i \mid i \in \mathbb{N}\}$ is a countable family of closed subsets of X with the finite intersection property, but with empty intersection; this shows that X is not countably compact. □

Of course this result may be strengthened. Indeed, the hypothesis 'paracompact' in the above can be replaced by 'metacompact' as shown by [Arens and Dugundji(1950)] (Thm. 2.4), or even by 'pure' (see [Arhangel′skiĭ(1980)] (Thm. 5)).

We may combine Lemma 6.9 and Proposition 3.2, to obtain a generalisation of the result that compactness and sequential compactness are equivalent in metric spaces.

Corollary 6.10. *Let X be a paracompact sequential space. Then X is compact if and only if it is sequentially compact.* □

As stated earlier, there are reasons why one might speculate that compactness and countable compactness are equivalent for perfectly normal spaces. In fact, it is known that this cannot be decided within ZFC. [Vaughan(1984)] (p. 597) describes this as 'one of the major results in set-theoretic topology.'

Theorem 6.11.

(1) *Assume $\diamondsuit$, and let X be Ostaszewski's space* [Ostaszewski(1976)]. *Then X is perfectly normal, first countable and countably compact, but not compact.*
(2) *Assume MA and $\neg$CH. Then every perfectly regular countably compact space is compact* (*this result is due to* [Weiss(1978)] *(Thm. 3)*).

The whole question of when compactness and countable compactness are equivalent is considered in more detail in the survey by [Vaughan(1984)]. In particular the two parts of Theorem 6.11 are there proved.

To get our final result on compactness in perfect spaces, we combine Theorem 6.11 with Proposition 3.2.

Corollary 6.12. *The statement 'Compactness and sequential compactness are equivalent for perfectly regular sequential spaces' is independent of ZFC.* □

References

[Alexandroff and Urysohn(1929)] P. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiques compacts* (Koninklijke Akademie van Wetenschappen, Amsterdam, 1929).

[Arens(1950)] R. Arens, "Note on convergence in topology," *Mathematics Magazine*, volume 23, (1950), pp. 229–234.

[Arens and Dugundji(1950)] R. Arens and J. Dugundji, "Remark on the concept of compactness," *Portugaliae Mathematica*, volume 9, (1950), pp. 141–143.

[Arhangel′skiĭ(1963)] A. V. Arhangel′skiĭ, "Some types of factor mappings and the relations between classes of topological spaces." *Soviet Mathematics, Doklady*, volume 4, (1963), pp. 1726–1729, translated by R. Roper from the Russian original.

[Arhangel′skiĭ(1970)] ———, "The Suslin number and cardinality. Characters of points in sequential bicompacta," *Soviet Mathematics, Doklady*, volume 11, (1970), pp. 597–601, translated by J. Ceder from the Russian original.

[Arhangel′skiĭ(1971)] ———, "On bicompacta hereditarily satisfying Suslin's condition. Tightness and free sequences," *Soviet Mathematics, Doklady*, volume 12, no. 4, (1971), pp. 1253–1257, translated by Z. Skalsky from the Russian original.

[Arhangel′skiĭ(1972)] ———, "The frequency spectrum of a topological space and the classification of spaces," *Soviet Mathematics, Doklady*, volume 13, no. 5, (1972), pp. 1185–1189, translated by B. Silver from the Russian original.

[Arhangel′skiĭ(1980)] ———, "The star method, new classes of spaces and countable compactness," *Soviet Mathematics, Doklady*, volume 21, no. 2, (1980), pp. 550–554, translated by Y. Choudhary from the Russian original.

[Arhangel′skiĭ and Ponomarev(1968)] A. V. Arhangel′skiĭ and V. I. Ponomarev, "On dyadic bicompacta," *Soviet Mathematics, Doklady*, volume 9, (1968), pp. 1220–1224, translated by F. Cezus from the Russian original.

[Arhangel′skiĭ and Ponomarev(1984)] ———, *Fundamentals of General Topology. Problems and Exercises* (Kluwer, Holland, 1984), translated by V. Jain from the Russian original.

[Balogh(1989)] Z. Balogh, "On compact Hausdorff spaces of countable tightness," *Proceedings of the American Mathematical Society*, volume 105, no. 3, (1989), pp. 755–764.

[Birkhoff(1937)] G. Birkhoff, "Moore-Smith convergence in general topology," *The Annals of Mathematics*, volume 38, no. 1, (1937), pp. 39–56.

[Bourbaki(1966)] N. Bourbaki, *General Topology*, number III in The Elements of Mathematics (Addison Wesley, 1966).

[Cartan(1937)] H. Cartan, "Filtres et ultrafiltres," *Comptes Rendus de l'Académie des Sciences, Paris*, volume 205, (1937), pp. 777–779.

[Dudley(1964)] R. M. Dudley, "On sequential convergence," *Transactions of the American Mathematical Society*, volume 112, (1964), pp. 483–507.

[Dudley(1970)] ———, "Corrections to 'On sequential convergence'," *Transactions of the American Mathematical Society*, volume 148, (1970), pp. 623–624.

[Engelking(1989)] R. Engelking, *General Topology*, revised edition (Heldermann Verlag, 1989).

[Franklin(1965)] S. P. Franklin, "Spaces in which sequences suffice," *Fundamenta Mathematicae*, volume 57, (1965), pp. 107–115.

[Franklin(1966)] ———, "On unique sequential limits," *Nieuw Archief voor Wiskunde*, volume 14, (1966), pp. 12–14.

[Franklin(1967)] ———, "Spaces in which sequences suffice II," *Fundamenta Mathematicae*, volume 61, (1967), pp. 51–56.

[Franklin(1969)] ———, "On two questions of Moore and Mrówka," *Proceedings of the American Mathematical Society*, volume 21, no. 3, (1969), pp. 597–599.

[Franklin and Rajagopalan(1971)] S. P. Franklin and M. Rajagopalan, "Some examples in topology," *Transactions of the American Mathematical Society*, volume 155, no. 2, (1971), pp. 305–314.

[Fréchet(1906)] M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, volume 22, (1906), pp. 1–74.

[Hajnal and Juhász(1974)] A. Hajnal and I. Juhász, "On hereditarily α-Lindelöf and α-separable spaces II," *Fundamenta Mathematicae*, volume 81, (1974), pp. 147–158.

[Hodel(1984)] R. Hodel, "Cardinal functions I," in *Handbook of Set-Theoretic Topology* (edited by K. Kunen and J. E. Vaughan), pp. 1–62 (North-Holland, Amsterdam, 1984).

[Hu(1964)] S.-T. Hu, *Elements of General Topology* (Holden-Day, 1964).

[Juhász(1980)] I. Juhász, *Cardinal Functions in Topology—Ten Years Later* (Mathematisch Centrum, Amsterdam, 1980).

[Juhász(1984)] ———, "Cardinal functions II," in *Handbook of Set-Theoretic Topology* (edited by K. Kunen and J. E. Vaughan), pp. 63–110 (North-Holland, Amsterdam, 1984).

[Kelley(1955)] J. L. Kelley, *General Topology* (Van Nostrand, 1955).

[Kisyński(1960)] J. Kisyński, "Convergence du type L," *Colloquium Mathematicum*, volume 7, (1960), pp. 205–211.

[Levine(1976)] N. Levine, "On compactness and sequential compactness," *Proceedings of the American Mathematical Society*, volume 54, no. 1, (1976), pp. 401–402.

[Malyhin(1972)] V. I. Malyhin, "On tightness and Suslin number in $\exp X$ and in a product of spaces," *Soviet Mathematics, Doklady*, volume 31, no. 2, (1972), pp. 496–499, translated by Z. Skalsky from the Russian original.

[McCluskey and McMaster(1997)] A. McCluskey and B. McMaster, *Topology Course Lecture Notes* (Topology Atlas, `http://at.yorku.ca/topology/`, 1997).

[Moore and Mrówka(1964)] R. C. Moore and S. Mrówka, "Topologies determined by countable objects," *Notices of the American Mathematical Society*, volume 11, (1964), p. 554.

[Munkres(1975)] J. R. Munkres, *Topology: A First Course* (Prentice-Hall, 1975).

[Nagata(1985)] J. Nagata, *Modern General Topology*, 2nd edition (North-Holland, Amsterdam, 1985).

[Ostaszewski(1976)] A. J. Ostaszewski, "On countably compact, perfectly normal spaces," *Journal of the London Mathematical Society*, volume 14, (1976), pp. 505–516.

[Ponomarev(1960)] V. I. Ponomarev, "Axioms of countability and continuous mappings," *Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques*, volume 8, (1960), pp. 127–134.

[Shakhmatov(1992)] D. B. Shakhmatov, "Compact spaces and their generalisations," in *Recent Progress in General Topology* (edited by M. Hušek and J. van Mill), pp. 572–640 (North-Holland, Amsterdam, 1992).

[Steen and Seebach(1996)] L. A. Steen and J. A. Seebach, *Counterexamples in Topology*, 2nd edition (Dover, 1996).

[Tukey(1940)] J. W. Tukey, *Convergence and Uniformity in Topology*, number 2 in Annals of Mathematics Studies (Princeton, 1940).

[Vaughan(1984)] J. E. Vaughan, "Countably compact and sequentially compact spaces," in *Handbook of Set-Theoretic Topology* (edited by K. Kunen and J. E. Vaughan), pp. 569–602 (North-Holland, Amsterdam, 1984).

[Weiss(1978)] W. Weiss, "Countably compact spaces and Martin's Axiom," *Canadian Journal of Mathematics*, volume 30, (1978), pp. 243–249.

[Willard(1970)] S. Willard, *General Topology* (Addison-Wesley, 1970).