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ON STEFAN BANACH AND SOME OF HIS RESULTS

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Dedicated to Professor Themistocles M. Rassias

Submitted by P. Enflo

ABSTRACT. In the paper a short biography of Stefan Banach, a few stories about Banach and the Scottish Café and some results that nowadays are named by Banach's surname are presented.

1. INTRODUCTION

Banach Journal of Mathematical Analysis is named after one of the most outstanding mathematicians in the XXth century, Stefan Banach. Thus it is natural to recall in the first issue of the journal some information about Banach. A very short biography, some of his most eminent results and some stories will be presented in this article.

2. BANACH AND THE SCOTTISH CAFÉ

Stefan Banach was born in the Polish city Kraków on 30 March, 1892. Some sources give the date 20 March, however it was checked (in particular, in the parish sources by the author ([12]) and by Roman Kałuża ([26])) that the date 30 March is correct.

Banach's parents were not married. It is not much known about his mother, Katarzyna Banach, after whom he had a surname. It was just recently discovered (see [25]) that she was a maid or servant and Banach's father, Stefan Greczek, who

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was a soldier (probably assigned orderly to the officer under whom Katarzyna was a servant) could not marry Katarzyna because of some military rules. Banach was grew up with the owner of the laundry Franciszka Płowa and her niece Maria.

Banach attended school in Kraków and took there final exams. He was interested in mathematics and he was very good at it. However, he thought that in this area nothing much new can be discovered and he decided to study at the Technical University at Lvov (Politechnika Lwowska). In those times both Kraków and Lvov were in the territory governed by Austro-Hungary. Therefore Banach moved to Lvov. His studies there were interrupted by the First World War and Banach came back to Kraków.

As a mathematician, Banach was self-taught. He did not study mathematics, however he attended some lectures at the Jagiellonian University, especially delivered by Stanisław Zaremba. In 1916, a very important event took place. Hugo Steinhaus, an outstanding mathematician, then already well known, spent some time in Kraków. Once, during his evening walk at the Planty Park in the centre of Kraków, Steinhaus heard the words “Lebesgue integral”. In those times it was a very modern mathematical term, so Steinhaus, a little surprised, started to talk with two young men who were speaking about the Lebesgue measure. These two men were Banach and Otto Nikodym. Steinhaus told them about a problem he was currently working on, and a few days later Banach visited Steinhaus and presented him a correct solution.

Steinhaus realised that Banach had an incredible mathematical talent. Steinhaus was just moving to Lvov where he got a Chair. He offered Banach a position at the Technical University. Thus Banach started his academic career and teaching students, however he did not graduate.

Steinhaus used to say that the discovery of Banach was his greatest mathematical discovery. It must be noted here that many outstanding mathematical results are due to Steinhaus.

There is a curious story how Banach got his Ph.D. He was being forced to write a Ph.D. paper and take the examinations, as he very quickly obtained many important results, but he kept saying that he was not ready and perhaps he would invent something more interesting. At last the university authorities became nervous. Somebody wrote down Banach's remarks on some problems, and this was accepted as an excellent Ph.D. dissertation. But an exam was also required. One day Banach was accosted in the corridor and asked to go to a Dean's room, as “some people have come and they want to know some mathematical details, and you will certainly be able to answer their questions”. Banach willingly answered the questions, not realising that he was just being examined by a special commission that had come to Lvov for this purpose.

In 1922 Banach, aged 30, was appointed Professor at the Jan Kazimierz University at Lvov. After the First World War Poland got back its independence and Kraków and Lvov were again in Poland.

In fact, Banach was interested in nothing but mathematics. He wrote down only a small part of his results. He was speaking about mathematics, introducing new ideas, solving problems all the time. Andrzej Turowicz, who knew Banach very well, used to say that two mathematicians should have followed Banach all the time and written down everything he said. Then, probably the majority of what Banach did would have been saved. Nevertheless, his results are incredible. Some of them will be recalled in the sequel of this paper. For more details, see [11], [13].

An important role in mathematicians' life in Lvov was played by the Scottish Café (see [10]). It was a place of their meetings, where they were eating, drinking, speaking about mathematics, stating problems and solving them. They used to write solutions on marble tables in the café. However, after each such visit the tables were carefully cleaned by the staff. Probably some difficult proofs of important theorems disappeared in this way. Therefore after some time Banach's wife, Łucja, bought a special book (called later the Scottish Book) that was always kept by the waiters and given to a mathematician when ordered. The problems, solutions, and rewards were written down in the book. One reward became particularly famous. On 6 November 1936 Stanisław Mazur stated the following problem (in the Scottish Book the problem had the number 153).

Problem 2.1. Assume that a continuous function on the square $[0,1]^2$ and the number $\varepsilon > 0$ are given. Do there exist numbers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, $c_1, \ldots, c_n$, such that

$$\left| f(x,y) - \sum_{k=1}^{n} c_k f(a_k, y) f(x, b_k) \right| \leq \varepsilon$$

for any $x, y \in [0,1]$?

Now it is often said that the problem was about the existence of Schauder basis in arbitrary separable Banach space. However, that was not known at that time. It was only in 1955 that Alexandre Grothendieck showed ([19]) that the existence of such numbers is equivalent to so called "the approximation problem", i.e. the problem if *every compact linear operator T from a Banach space X into a Banach space Y is a limit in norm of operators of finite rank*. The problem was especially attractive as Mazur offered a prize: a live goose. The approximation problem (and, consequently, the original Mazur's problem) was solved only in 1972 by a Swedish mathematician Per Enflo (then 28 years old) ([17]) who shortly after giving a solution came to Warsaw and got from Mazur the prize.

The former Scottish Café in 2006 (now a bank)

There are plenty of stories about mathematicians in the Scottish Café. Here, let me recall two of them.

Once Mazur stated a problem and Herman Auerbach started thinking it over. After a while Mazur said that, to make a puzzle more interesting, he offered a bottle of wine as a reward. Then Auerbach said: "Ah, so I give up. Wine does not agree with me".

The second story is connected with Lebesgue's visit to Lvov in 1938. Lebesgue came to Lvov, delivered a lecture and after that was invited to the Scottish Café. A waiter gave Lebesgue the menu. Lebesgue, who didn't know Polish, studied the menu for a while, gave it back and said: "I eat only dishes which are well defined".

The period since the end of the First World War up to 1939 was the Golden age for Polish mathematics. In particular, in those times time Banach obtained a series of remarkable results. Banach can be regarded as the creator of functional analysis, which at that period could be named "the Polish branch of mathematics". In Lvov, together with Banach, there worked many outstanding mathematicians, for example Steinhaus, Mazur, Juliusz Schauder, Władysław Orlicz, Stan Ulam and Mark Kac (the last two moved to the USA before the war). Polish mathematicians worked actively also in other centers, especially in Warsaw; for example, Wacław Sierpiński, Karol Borsuk, Kazimierz Kuratowski and Stefan Mazurkiewicz should be mentioned.

Banach and Steinhaus initiated in 1928 a new journal, "Studia Mathematica" which published papers just on functional analysis. It was one of the first journals in history that was specialized in some particular areas of mathematics (the very first one was "Fundamenta Mathematicae").

In 1931 the fundamental monograph on functional analysis by Banach was published. It was "Operacje liniowe", in 1932 published in French "Théorie des opérations linéaires" as the first volume in the series "Mathematical Monographs". For many years it was the most basic book on functional analysis, up to the moment when the famous monograph [16] was published (see also [30]). It should be noted that only in 40s the term "functional analysis" was being used. In Banach times other names were of use, especially "the theory of linear operators".

In 1939 Lvov was captured by the Soviet Union, in 1941 Hitler's soldiers took Lvov for 4 years. Banach spent the Second World War in Lvov, living under extremely difficult conditions. After the war Lvov was taken by the Soviet Union again and Banach planned to go to Kraków where he would have taken a Chair at the Jagiellonian University. He died just a few days before the move. He is buried in Lychakov Cemetery (Cmentarz Łyczakowski) in Lvov. Now, in front of the building of the Mathematics Institute of the Jagiellonian University there is a monument of Banach.

The monument of Banach in Kraków

The Scottish Book survived the war. It was taken to Poland by Łucja Banach and later on translated to English by Steinhaus. Ulam let the problems from the book circulate in the United States. In 1981 the book was published in English in the version prepared by Dan Mauldin ([27]). This translation is remarkably valuable as besides the problems and solutions (if there are) it includes several comments and remarks about the continuation of the investigations inspired by the problems from the Scottish Book. It is a large and important piece of mathematics.

The international mathematical centre in Poland, created in 1972 is named after Banach. *The Banach Center* is a part of the Institute of Mathematics of the Polish Academy of Sciences and has its main office in Warsaw. Conferences took place in Warsaw, recently they have been organized also in Będlewo. There are *Banach Center Publications* that publishes proceedings of selected conferences and semesters held at the International Stefan Banach Mathematical Center. Up to now, 77 volumes were published.

3. Some results named by Banach's surname

Now we turn to some mathematical results of Banach. As mentioned above, only some of his discoveries were published; nevertheless, they present themselves an enormous collection. Here, we present only some of the most important results which are nowadays named after him; the results will be recalled and it will be said where they were published.

Before that, an important fact should be mentioned. It was checked what names appear most frequently in the titles of mathematical and physical papers published in the 20th century. It turned out that it was Banach's name that got the first place. The second place was obtained by Sophus Lie, the third by Bernhard Riemann.

Certainly, Banach deserves such a position mainly because of Banach spaces, nowadays one of the most important mathematical notions. A Banach space is *a normed complete vector space*. It was formally defined in the paper [5]; for some time mathematicians in Lvov called those spaces "a space of type B". The name "Banach space" was probably used for the first time by Maurice Fréchet, in 1928. Note that independently such spaces were introduced by Norbert Wiener, however Wiener thought that the spaces would not be of importance and gave up. A long time later Wiener wrote in his memoirs that the spaces quite justly should be named after Banach alone, as sometimes they were called "Banach-Wiener spaces". For more details, see [14] and [15].

There are some points which show why the introduction of Banach spaces was so important. For a variety of reasons function spaces are very useful in many investigations and applications. To a large extent, modern mathematics is concerned with the study of general structures. The essential thing is finding the right generalization. Insufficient generality can be too restrictive and a great deal of generality may result in a situation where little can be proved and applied. The space introduced by Banach, especially pointing out completeness, attests to his genius; he hit the traditional nail on the head.

Banach's great merit was that, in principle, it was thanks to him that the "geometric" way of looking at spaces was initiated. The elements of some general spaces might be functions or number sequences, but when fitted into the structure of a Banach space they were regarded as "points", as the elements of a "space". At times this resulted in remarkable simplifications.

Today, almost ninety years after its introduction, the notion of a Banach space remains fundamental in many areas of mathematics. The theory of Banach spaces is being developed to this day, and new, interesting, and occasionally surprising results are obtained be many researches. In particular, some really important results were obtained in the end of the 20th century by William Timothy Gowers. Some problems he solved waited for the solution since Banach's times. For his research, Gowers was awarded in 1998 with the Fields medal.

In the same paper [5] there is proved the famous Banach Fixed Point Theorem. It says

Theorem 3.1. *Let (X, d) be a complete metric space and a function $f : X \to X$ be a contracting operation, i.e. there exists a $\lambda \in (0, 1)$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for any $x, y \in X$. Then there exists a unique $p \in X$ such that $f(p) = p$.*

The theorem was the use of the method of successive approximations and a general version of the property that was known earlier in some concrete cases.

Now we turn to some fundamental theorems on functional analysis.

One of the most important of them is the Hahn-Banach Theorem.

Theorem 3.2. *Let X be a real normed vector space, $p : X \to \mathbb{R}$ a function such that $p(\alpha x + (1 - \alpha)y) \leq \alpha p(x) + (1 - \alpha)p(y)$ for each $x, y \in X$ and $\alpha \in [0, 1]$. Assume that $\varphi : Y \to \mathbb{R}$ is a linear functional defined on a vector subspace Y of X with the property $\varphi(x) \leq p(x)$ for all $x \in Y$. Then there exists a linear functional $\psi : X \to \mathbb{R}$ such that $\psi(x) = \varphi(x)$ for all $x \in Y$ and $\psi(x) \leq p(x)$ for all $x \in x$.*

The theorem has several versions (compare [29]).

It was published by Banach in [3]. Independently, it was (in a simper case) slightly earlier discovered by Hans Hahn ([20]) which was a generalization of his result from 1922. Banach did not know about Hahn's paper; neverthelesss, Banach's version was stronger as Hahn proved a theorem in the case where X is a Banach space. It should be noted that a simpler version of the theorem (in the case where X is the space of continuous real functions on a compact interval) was published much earlier by Eduard Helly ([21]; see [22]). Neither Banach nor Hahn knew about Helly's theorem.

The complex version of the theorem was proved later, in 1938 independently by G.A. Soukhomlinov and by H.F.Bohnenblust ([8]) and A.Sobczyk ([31]).

The Hahn-Banach Theorem is regarded by many authorities as one of three basic principles of functional analysis. The two others are the Banach-Steinhaus Theorem and the Banach Closed Graph Theorem.

The Banach-Steinhaus Theorem was proved in [7]. It says:

Theorem 3.3. *Let X be a Banach space, Y be a normed vector space. Consider the family $\mathcal{F}$ of all linear bounded functions from X to Y. If for any $x \in X$ the set $\{||T(x)|| : T \in \mathcal{F}\}$ is bounded, then the set $\{||T|| : T \in \mathcal{F}\}$ is bounded.*

Now recall the Banach Closed Graph Theorem.

Theorem 3.4. *Let X and Y be Banach spaces, and T be a linear operator from X to Y. Then T is bounded if and only if the graph of T is closed in $X \times Y$.*

This theorem is closely related to the very important Banach Open Mapping Principle.

Theorem 3.5. *Let X and Y be Banach spaces, and T be a linear operator from X onto Y (we assume that T is surjective). Then for any open subset U of X the set $T(U)$ is open in Y.*

Both theorems were published in [6].

Let us mention also another theorem on functional analysis, nowadays called frequently the Banach-Alaoglu Theorem. It says

Theorem 3.6. *Let X^* be the dual space of a Banach space X. Then the closed unit ball in X^* is compact in X^* with the weak–* topology.*

A proof of this theorem was given in 1940 by Leonidas Alaoglu ([1]) and in the case of separable normed vector spaces was published in 1929 by Banach ([4]).

Banach did not work only on functional analysis. For example, today his name is connected with famous Banach-Tarski Theorem on paradoxical decomposition of the ball. The theorem may be formulated in the following way.

Theorem 3.7. *If $B \subset \mathbb{R}^3$ is a three-dimensional ball, then there exist pairwise disjoint sets $A_1, \ldots, A_n$ and isometric transformations $I_1, \ldots, I_n$ such that $B = A_1 \cup \ldots \cup A_n$, and for some $k \in (1, n)$: $I_1(A_1), \ldots, I_k(A_k)$ are pairwise disjoint, $B = I_1(A_1) \cup \ldots \cup I_k(A_k)$, $I_{k+1}(A_{k+1}), \ldots, I_n(A_n)$ are pairwise disjoint, $B = I_{k+1}(A_{k+1}) \cup \ldots \cup I_n(A_n)$*

The theorem looks very strange, as it, in fact, says that we can double the volume! The point is that the ball is split into pieces that are non-measurable. The proof relays on the axiom of choice. It was published in [9]. For more information, see [18].

Not all the mathematical theorems and notions which are now frequently called by Banach's name were decribed above. Let us mention here, for instance, *Banach integral, Banach generalized limit* (introduced in [2]) and *Banach algebra*. Banach algebras were a kind of restructurization of Banach spaces (instead of a vector space there is taken a ring and in addition a multiplication of elements). Banach algebras were introduced in 1941 by a Russian mathematician Israil M. Gelfand. One should mention here also the *Banach-Mazur distance* (introduced in [6]), which is a suitable defined distance between two isomorphic Banach spaces.

The first volume of the Banach Journal of Mathematical Analysis is dedicated to Themistocles M. Rassias. It is nice to notice that some of the achievements of Th.M. Rassias have a particular connection with Banach and the mathematics from the Scottish Café.

One of the most important mathematicians of the Lvov group was Stan Ulam, who was very young in his Lvov days. In 1936 he moved to the USA where he later on became a very famous scientist. The reader is referred to the wonderful volume [32]. As was mentioned above, Ulam played a great role in circulating the mathematics from the Scottish Café after the Second World War.

With the names of Ulam and Rassias there is connected a mathematical term, now widely known as Ulam–Hyers–Rassias stability. Let X and Y be real Banach spaces. The stability of Ulam–Hyers–Rassias approximate isometries on restricted domains S (bounded or unbounded) for into mapping $f : S \to Y$ satisfying $||f(x) - f(y)|| - ||x - y|| \leq \varepsilon$ especially where Y is a Banach space.

In 1940 Stan Ulam stated the problem concerning the stability of homomorphisms: *Let G_1 be a group and let G_2 be a metric group with a metric d and let $\varepsilon > 0$ be given. Does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x, \in G_1$?* Roughly speaking: *When does a linear mapping near an "approximately linear" mapping exists*? T.M. Rassias gave a solution in [28], introducing some condition for mappings between Banach spaces. A particular case of Rassias's theorem was the result of Donald H. Hyers ([23]). Now it can be said that the study of Ulam–Hyers–Rassias stability in its present form was started by the paper of Th.M. Rassias [28]. For more information of such kind of stability and the basic papers on the subject, see [24].

Let us end with some more anecdotes.

There is a story to the effect that, upon publication, Banach's monograph "Theory of operations. Linear operations" was displayed in some Lvov bookshops on shelves labelled "Medical Books".

In some Polish "cities, including Kraków and Warsaw, there are streets named "Banach street". In Warsaw, now the Mathematical Institute of Warsaw University has its house at Banach street. The 1983 International Congress of Mathematicians took place in Warsaw. A few foreign mathematicians found out that there is a street in Warsaw called Banach Street, and this is the last stop on a certain trolley line. Curious about Banach Street, they got on the trolley, got off at the last stop, and were confronted by a sizable empty area. They arrived at the unanimous conclusion that what they were facing was not "Banach street" but rather a "Banach space".

The list of references is far from complete, as there are enormous numbers of papers connected with Banach. Here, except the original papers mentioned in the article, are given only some papers in English where the reader can find several additional information.

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