ANALYSIS ON THE LEVI-CIVITA FIELD, A BRIEF OVERVIEW

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Abstract. Recent work on a non-Archimedean field extension of the real numbers will be presented. We review the algebraic and order structures of the Levi-Civita field which contains the real numbers as a subfield, but also has infinitely small and infinitely large numbers. Then we review the properties of power series over the Levi-Civita field, which are shown to behave as nicely as real power series. We show with examples that Calculus on the field is different from the real case and we present different approaches to generalize various results from the real numbers field to the new field such as the Intermediate value Theorem, the Inverse Function Theorem, optimization results, and integration. Using calculus on the Levi-Civita field and the existence of infinitely small numbers in the field, we can compute the derivatives of real functions representable on a computer fast and accurately up to very high orders (whenever they exist).

1. Introduction

In this paper, we present an overview of recent Analysis results obtained on the Levi-Civita field $\mathcal{R}$ [9, 10] and we present one application. We start with a review of some basic and useful terminology and refer the reader to [2, 18, 15, 4, 24, 19, 25, 20, 16, 21, 26, 22, 17, 23] for a more detailed study of the field.

Definition 1.1 (The set $\mathcal{R}$)**.** We define

$$\mathcal{R} = \{f : \mathbb{Q} \to \mathbb{R} \mid \{x | f(x) \neq 0\} \text{ is left-finite}\}.$$

So the elements of $\mathcal{R}$ are those real valued functions on $\mathbb{Q}$ that are non-zero only on a left-finite set, i.e. below every rational number q there are only finitely many points where the given functions do not vanish.

We denote elements of $\mathcal{R}$ by x, y, etc. and identify their values at $q \in \mathbb{Q}$ with brackets like $x[q]$. For the further discussion, it is convenient to introduce the following terminology:

Definition 1.2. (supp, λ, $\sim$, $\approx$, $=_r$) For $x \in \mathcal{R}$, we define

$\mathrm{supp}(x) = \{q \in \mathbb{Q} \mid x[q] \neq 0\}$ and call it the support of x

For $x \in \mathcal{R}$, we define $\lambda(x) = \min(\mathrm{supp}(x))$ for $x \neq 0$ (which exists because of left-finiteness) and $\lambda(0) = +\infty$.

Given $x, y \in \mathcal{R}$ and $r \in \mathbb{R}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$; and $x =_r y$ if $x[q] = y[q]$ for all $q \leq r$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The set $\mathcal{R}$ is endowed with formal power series multiplication and componentwise addition, which make it into a field [2] in which we can isomorphically embed $\mathbb{R}$ as a subfield via the map $\Pi : \mathbb{R} \to \mathcal{R}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases} . \tag{1.1}$$

Definition 1.3. (Order in $\mathcal{R}$) Let $x \neq y$ in $\mathcal{R}$ be given. Then we say $x > y$ if $(x-y)[\lambda(x-y)] > 0$; furthermore, we say $x < y$ if $y > x$.

With this definition of the order relation, $\mathcal{R}$ is an ordered field. Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$ in the natural way. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

Besides the usual order relations, some other notations are convenient.

Definition 1.4. ($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.5. (The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if and only if $q > 0$. Moreover, for all $x \in \mathcal{R}$, the elements of $\mathrm{supp}(x)$ can be arranged in ascending order, say $\mathrm{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as

$x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the topology induced by the absolute value [2].

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to [2, 18, 15, 4, 24, 19, 25, 20, 16, 21, 26, 22, 17, 23]. In particular, it is shown that $\mathcal{R}$ is complete with respect to the topology induced by the absolute value. In the wider context of valuation theory, it is interesting to note that the topology induced by the absolute value, the so-called strong topology, is the same as that introduced via the valuation λ.

It follows therefore that the field $\mathcal{R}$ is just a special case of the class of fields discussed in [14]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [13], and for an overview of the related valuation theory to the books by Krull [7], Schikhof [14] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [12].

Besides being the smallest totally ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and algebraically closed [2], the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [2]. Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer [18, 19], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

2. Computation of Derivatives of Real Functions

We start this section with the following definition.

Definition 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ is called a computer function if it can be obtained from intrinsic functions and the Heaviside function through a finite number of arithmetic operations and compositions.

Obviously, the class of computer functions describes all real functions that can be typed on a computer. Theorem 4.1 in Section 4 allows us to extend all intrinsic functions given by power series (and hence all real computer functions) to $\mathcal{R}$. The $\mathcal{R}$ numbers as well as the continuations of the intrinsic functions to $\mathcal{R}$ have all been implemented for use on a computer, using the code COSY INFINITY [6, 3]. Using Calculus on $\mathcal{R}$ and the fact that the field has infinitely small numbers represents a new method for computational differentiation [5] that avoids the well-known accuracy problems of numerical differentiation tools. It also avoids the often rather stringent limitations of formula manipulators that restrict the complexity of the function that can be differentiated, and the orders to which differentiation can be performed.

The general question of efficient differentiation is at the core of many parts of the work on perturbation and aberration theories relevant in Physics and Engineering; for an overview, see for example [5]. In this case, derivatives of highly complicated functions have to be computed to high orders. However, even when the derivative of the function is known to exist at the given point, numerical methods fail to give an accurate value of the derivative; the error increases with the order, and for orders greater than three, the errors often become too large for the results to be practically useful. On the other hand, while formula manipulators like Mathematica are successful in finding low-order derivatives of simple functions, they fail for high-order derivatives of very complicated functions. Consider, for example, the function

$$g(x) = \frac{\sin\left(x^3+2x+1\right) + \frac{3+\cos(\sin(\ln|1+x|))}{\exp\left(\tanh\left(\sinh\left(\cosh\left(\frac{\sin(\cos(\tan(\exp(x))))}{\cos(\sin(\exp(\tan(x+2))))}\right)\right)\right)\right)}}{2+\sin\left(\sinh\left(\cos\left(\tan^{-1}\left(\ln\left(\exp(x)+x^2+3\right)\right)\right)\right)\right)}. \tag{2.1}$$

Using the $\mathcal{R}$ calculus, we find $g^{(n)}(0)$ for $0 \le n \le 19$. These numbers are listed in table 1; we note that, for $0 \le n \le 19$, we list the CPU time needed to obtain all derivatives of g at 0 up to order n and not just $g^{(n)}(0)$. For comparison purposes, we give in table 2 the function value and the first six derivatives computed with Mathematica. Note that the respective values listed in tables 1 and 2 agree. However, Mathematica used much more CPU time to compute the first six derivatives, and it failed to find the seventh derivative as it ran out of memory. We also list in table 3 the first ten derivatives of g at 0 computed numerically using the numerical differentiation formulas

$$g^{(n)}(0) = (\Delta x)^{-n}\left(\sum_{j=0}^{n}(-1)^{n-j}\begin{pmatrix} n \\ j \end{pmatrix} g\left(j\Delta x\right)\right),\ \Delta x = 10^{-16/(n+1)},$$

for $1 \le n \le 10$, together with the corresponding relative errors obtained by comparing the numerical values with the respective exact values computed using $\mathcal{R}$ calculus.

TABLE 1. $g^{(n)}(0), 0 \leq n \leq 19$, computed with $\mathcal{R}$ calculus

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007115	1.820 msec
1	0.4601438089634254	2.070 msec
2	-5.266097568233224	3.180 msec
3	-52.82163351991485	4.830 msec
4	-108.4682847837855	7.700 msec
5	16451.44286410806	11.640 msec
6	541334.9970224757	18.050 msec
7	7948641.189364974	26.590 msec
8	-144969388.2104904	37.860 msec
9	-15395959663.01733	52.470 msec
10	-618406836695.3634	72.330 msec
11	-11790314615610.74	97.610 msec
12	403355397865406.1	128.760 msec
13	$0.5510652659782951 \times 10^{17}$	168.140 msec
14	$0.3272787402678642 \times 10^{19}$	217.510 msec
15	$0.1142716430145745 \times 10^{21}$	273.930 msec
16	$-0.6443788542310285 \times 10^{21}$	344.880 msec
17	$-0.5044562355111304 \times 10^{24}$	423.400 msec
18	$-0.5025105824599693 \times 10^{26}$	520.390 msec
19	$-0.3158910204361999 \times 10^{28}$	621.160 msec

TABLE 2. $g^{(n)}(0), 0 \leq n \leq 6$, computed with Mathematica

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007116	0.11 sec
1	0.4601438089634254	0.17 sec
2	-5.266097568233221	0.47 sec
3	-52.82163351991483	2.57 sec
4	-108.4682847837854	14.74 sec
5	16451.44286410805	77.50 sec
6	541334.9970224752	693.65 sec

On the other hand, formula manipulators fail to find the derivatives of certain functions at given points even though the functions are differentiable at the respective points. For example, the functions

$$g_1(x) = |x|^{5/2} \cdot g(x) \text{ and } g_2(x) = \begin{cases} \frac{1-\exp(-x^2)}{x} \cdot g(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

TABLE 3. $g^{(n)}(0), 1 \leq n \leq 10$, computed numerically

Order n	$g^{(n)}(0)$	Relative Error
1	0.4601437841866840	54×10^{-9}
2	-5.266346392944456	47×10^{-6}
3	-52.83767867680922	30×10^{-5}
4	-87.27214664649106	0.20
5	19478.29555909866	0.18
6	633008.9156614641	0.17
7	-12378052.73279768	2.6
8	-1282816703.632099	7.8
9	83617811421.48561	6.4
10	91619495958355.24	149

where $g(x)$ is the function given in Equation (2.1), are both differentiable at 0; but the attempt to compute their derivatives using formula manipulators fails. This is not specific to g_1 and g_2, and is generally connected to the occurrence of non-differentiable parts that do not affect the differentiability of the end result, of which case g_1 is an example, as well as the occurrence of branch points in coding as in IF-ELSE structures, of which case g_2 is an example. Using the calculus on $\mathcal{R}$, we formulate a necessary and sufficient condition for the derivatives of a computer function to exist, and show how to find these derivatives whenever they exist (see [18, 19] for details on how that is done).

3. Calculus on $\mathcal{R}$

We define continuity and differentiability of a function at a point or on an interval in $\mathcal{R}$ like in $\mathbb{R}$ using the ϵ-δ arguments. The following examples show that, under similar conditions, we obtain results in $\mathcal{R}$ that are different from (and even opposite to) what we would expect in $\mathbb{R}$.

Example 3.1. Let $f_1 : [0,1] \to \mathcal{R}$ be given by

$$f_1(x) = \begin{cases} d^{-1} & \text{if } 0 \leq x < d \\ d^{-1/\lambda(x)} & \text{if } d \leq x \ll 1 \\ 1 & \text{if } x \sim 1 \end{cases}.$$

Then f_1 is continuous on $[0,1]$; but for $d \leq x \ll 1$, $f_1(x)$ grows without bound.

Example 3.2. Let $f_2 : [-1,1] \to \mathcal{R}$ be given by

$$f_2(x) = x - x[0].$$

Then f_2 is continuous on $[-1,1]$. However, f_2 assumes neither a maximum nor a minimum on $[-1,1]$. The set $f_2([-1,1])$ is bounded above by any positive real number and below by any negative real number; but it has neither a least upper bound nor a greatest lower bound.

Example 3.3. Let $f_3 : [0,1] \to \mathcal{R}$ be given by

$$f_3(x) = \begin{cases} 1 & \text{if } x \sim 1 \\ 0 & \text{if } x \ll 1 \end{cases}.$$

Then f_3 is continuous on $[0,1]$ and differentiable on $(0,1)$, with $f_3'(x) = 0$ for all $x \in (0,1)$. We have that $f_3(0) = 0$ and $f_3(1) = 1$; but $f_3(x) \neq 1/2$ for all $x \in [0,1]$. Moreover, f_3 is not constant on $[0,1]$ even though $f_3'(x) = 0$ for all $x \in (0,1)$.

Example 3.4. Let $f_4 : [-1,1] \to \mathcal{R}$ be given by

$$f_4(x) = \sum_{\nu=1}^{\infty} x_\nu d^{3q_\nu} \text{ when } x = x[0] + \sum_{\nu=1}^{\infty} x_\nu d^{q_\nu}.$$

Then $f_4'(x) = 0$ for all $x \in (-1,1)$. But f_4 is not constant on $[-1,1]$.

Example 3.5. Let $f_5 : (-1,1) \to \mathcal{R}$ be given by

$$f_5(x) = -f_4(x) + x^4.$$

Then $f_5' > 0$ on $(0,1)$; but f_5 is strictly decreasing on $\{x : 0 < x \ll 1\}$. Also f_5' is strictly increasing and $f_5'' \geq 0$ on $(-1,1)$; but f_5 is not convex on $(-1,1)$: $f_5(d) = -d^3 + d^4 < 0 = f_5(0) + f_5'(0)d$.

Example 3.6. Let $f_6 : (-1,1) \to \mathcal{R}$ be given by

$$f_6(x) = -\left(f_4(x)\right)^2 + x^8.$$

Then f_6 is infinitely often differentiable on $(-1,1)$ with $f_6^{(j)}(0) = 0$ for $1 \leq j \leq 7$ and $f_6^{(8)}(0) = 8! > 0$. But f_6 has a relative maximum at 0.

The difficulties embodied in the examples above are not specific to $\mathcal{R}$, but are common to all non-Archimedean fields; and they result from the fact that the field $\mathcal{R}$ is disconnected in the topology induced by the order. This makes developing Analysis on the field more involved than in the real case; for example, the existence of nonconstant functions, the derivative of which is zero everywhere on an interval (as in example 3.4) makes integration much harder and the solutions of the simplest initial value problems (differential equations with initial conditions, like $y' = 0; y(0) = 0$) not unique [16]. To circumvent such difficulties, different approaches have been followed depending on the problem to be solved or the result to be extended from the real case to the new field. For example, by imposing stronger conditions on the function than in the real case, the function satisfies an intermediate value theorem and an inverse function theorem (see [20]); by using a stronger concept of continuity and differentiability than in the real case, the optimization results in one dimension and multi-dimensions have been generalized to $\mathcal{R}$ (see [25, 26]); by carefully defining a measure on the field, that measure proves to be a natural generalization of the Lebesgue measure and have similar properties and it leads naturally to an integration theory with similar properties to those of the Lebesgue integral of Real Analysis (see [21]); and by restricting solutions of initial value problems to functions given locally by power series (see Section 4 below for an overview of power series on $\mathcal{R}$), the uniqueness of the solutions will be assured under similar conditions as in the real case (see [16]).

4. Review of Power Series

Power series on the Levi-Civita field $\mathcal{R}$ have been studied in details in [15, 24, 22, 17, 23]. Prior to that, work on power series has been mostly restricted to power series with real coefficients. In [9, 10, 11, 8], they could be studied for infinitely small arguments only, while in [2], using the newly introduced weak topology, also finite arguments were possible. Moreover, power series over complete valued fields in general have been studied by Schikhof [14], Alling [1] and others in valuation theory, but always in the valuation topology.

In [24], we study the general case when the coefficients in the power series are Levi-Civita numbers, using the weak convergence of [2]. We derive convergence criteria for power series which allow us to define a radius of convergence η such that the power series converges weakly for all points whose distance from the center is smaller than η by a finite amount and it converges strongly for all points whose distance from the center is infinitely smaller than η.

In [22] it is shown that, within their radius of convergence, power series are infinitely often differentiable and the derivatives to any order are obtained by differentiating the power series term by term. Also, power series can be re-expanded around any point in their domain of convergence and the radius of convergence of the new series is equal to the difference between the radius of convergence of the original series and the distance between the original and new centers of the series. In [23], it is shown that, within their radius of convergence, power series satisfy the intermediate value theorem.

Of particular interest here is the following result which allows us to extend to $\mathcal{R}$ all real functions that are given locally by power series [2, 15, 19, 24].

Theorem 4.1 (Power Series with Purely Real Coefficients). *For each $n \in \mathbb{N}$, let $a_n \in \mathbb{R}$ and let $\sum_{n=0}^{\infty} a_n X^n$ be a power series with classical radius of convergence (in $\mathbb{R}$) equal to η. Let $x \in \mathcal{R}$, and let $A_n(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{R}$. Then, for $|x| < \eta$ and $|x| \not\approx \eta$, the sequence $(A_n(x))$ converges absolutely weakly. We define the limit to be the continuation of the power series to $\mathcal{R}$.*

Definition 4.2. Using Theorem 4.1, the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \ \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \text{ and } \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

converge absolutely weakly in $\mathcal{R}$ for any x, at most finite in absolute value. We define these series to be $\exp(x)$, $\cos(x)$, $\sin(x)$, $\cosh(x)$ and $\sinh(x)$, respectively.

A detailed study of the transcendental functions introduced in Definition 4.2 can be found in [15]. In particular, it is easily shown that addition theorems similar to the real ones hold for these functions.

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