Generalized dyadic spaces

by

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Abstract. We consider a generalization of dyadic spaces. We show that many dyadic theorems hold for this class of spaces. We study the clopen algebras of the 0-dimensional ones. These algebras are the union of countably many antichains. We show that there is a largest, purely combinatorial, subalgebra of $\mathscr{P}(\omega)$/fin. Examples, that the author has previously constructed, are shown to belong to this class of spaces.

1. Introduction. Dyadic spaces, i.e., Hausdorff continuous images of some power of 2, were defined by Alexandroff [1]. His amazing result that every compact metric space is the continuous image of the Cantor set was the genesis behind the notion. To quote Alexandroff [2]: "I became acquainted with the Cantor perfect set, which I immediately saw and still see to this day as one of the greatest wonders (and I mean a wonder, nothing less) discovered by the human mind."

The m-adic or polyadic spaces, i.e., Hausdorff continuous images of some power of the Alexandroff compactification of a discrete space, were defined by Mrówka [19] as a good generalization of the dyadic spaces. Many of the dyadic theorems held in this class of spaces and a chief benefit was that spaces of uncountable cellularity were now admitted into the study.

There have been further generalizations to important classes of spaces in their own right. We take the point of view that any generalization that admits the ordinal space ω_1+1 is decidedly non-dyadic.

The clopen algebras of our 0-dimensional „generalized dyadic" spaces, spaces that we call centered spaces, will manifest all possible combinatorial (i.e., disjointness) behaviour; while at the same time minimizing the order (i.e., inclusion) behaviour. We will show that many dyadic theorems remain true. A chief benefit of the generalization lies in the wealth of the examples. Loosely speaking, dyadic spaces are to centered collections as polyadic spaces are to disjoint collections as centered spaces are to all collections.

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2. Notation and definitions. Our set theory notation is standard. Cardinals are initial ordinals. The first infinite ordinal is ω and $\mathfrak{c}$ represents the continuum. $\mathcal{P}(X)$ represents the set of all subsets of X and $X \triangle Y$ is the symmetric difference $(X-Y) \cup (Y-X)$. If $\mathcal{S}$ is a collection of sets, then $\mathcal{S}^{\cap}$ is the set of all $\bigcap \mathcal{R}$ such that $\mathcal{R} \subset \mathcal{S}$. A collection of sets $\mathcal{S}$ is *centered* if $\bigcap \mathcal{F} \neq \emptyset$ for all finite subsets $\mathcal{F}$ of $\mathcal{S}$. If $\varphi\colon X \to Y$, $A \subset X$ and $B \subset Y$, then $\varphi[A] = \{\varphi(a)\colon a \in A\}$ and $\varphi^{-1}[B] = \{x \in X\colon \varphi(x) \in B\}$. If $R \subset X \cap Y$, then we can write $\varphi[R]$, $\varphi^{-1}[R]$ and even $\varphi^2[R] = \varphi[\varphi[R] \cap R]$ etc.

If B is a boolean algebra and G is a set of generators of B, then we write $B = [G]$. The space of all ultrafilters of B is denoted by $\operatorname{st} B$ and is given the Stone topology. An *antichain* of B is a subset of B in which no two elements are comparable. If X is compact and 0-dimensional, then we will write $B(X)$ for the boolean algebra of clopen sets of X.

N denotes the discrete space of natural numbers and $\boldsymbol{R}$ denotes the space of real numbers. A mapping is a continuous function. The Stone–Čech compactification of X is denoted by βX. The hyperspace of closed subsets of X, with the Vietoris topology, is denoted by $\operatorname{Exp} X$.

A collection of sets is n-linked if every n of the sets have a non-empty intersection. A space X has a σ-n-*linked* base, if X has an open base $\mathcal{B} = \bigcup_{i<\omega} \mathcal{B}_i$ where each $\mathcal{B}_i$ is n-linked. A space X has *compactness number at most* n if X possesses a closed subbase $\mathcal{S}$ such that every n-linked subcollection of $\mathcal{S}$ has a non-empty intersection. If m is the least such n, then we say that $\operatorname{cmpn}(X) = m$. If no such $n<\omega$ exists and X is compact then X *has infinite compactness number*. Spaces of cmpn 2 are called supercompact spaces, De Groot [16].

We use the following cardinal functions and refer the reader to Juhasz [17] for definitions. Weight, π-weight, character, π-character, cellularity, tightness and spread. A space of countable cellularity is said to be ccc. A space is Fréchet–Urysohn if whenever a point p is in the closure of a set K, then there is a sequence of points of K converging to p.

3. Definition of the spaces $\operatorname{Cen}(S)$. Let S be any infinite collection of sets. Put $\operatorname{Cen}(S) = \{T\colon T \text{ is a centered subcollection of } S\}$. The $\emptyset$ is included in $\operatorname{Cen}(S)$ and plays a crucial role. Put $\operatorname{Fin}(S) = \{F\colon F \text{ is a finite centered subcollection of } S\}$. For each $F \in \operatorname{Fin}(S)$ put $F^+ = \{T \in \operatorname{Cen}(S)\colon F \subset T\}$ and $F^- = \{T \in \operatorname{Cen}(S)\colon F \cap T = \emptyset\}$. If $s \in S$, we will write s^+ and s^- instead of $\{s\}^+$ and $\{s\}^-$. Use $\bigcup_{s\in S} \{s^+, s^-\}$ as a closed (also open) subbase for a topology on $\operatorname{Cen}(S)$. If we identify $\operatorname{Cen}(S)$ with $\{f\colon f \text{ is a characteristic function of a centered subcollection of } S\}$ then $\operatorname{Cen}(S)$ has the subspace topology inherited from the Tychonoff product 2^S. Being closed in 2^S, $\operatorname{Cen}(S)$ is a compact Hausdorff 0-dimensional space. $\operatorname{Cen}(S)$ has as a clopen base $\{F^+ \cap G^-\colon F \text{ and } G \text{ are finite subsets of } S\}$.

Since $\{s^+\colon s \in S\}$ are distinct clopen sets, the weight of $\operatorname{Cen}(S)$ is exactly $|S|$. For each $n<\omega$, put $\operatorname{Fin}(S, n) = \{F \in \operatorname{Fin}(S)\colon |F| \leqslant n\}$. It is seen that each $\operatorname{Fin}(S, n)$ is a closed subspace of $\operatorname{Cen}(S)$, that each $\operatorname{Fin}(S, n+1) - \operatorname{Fin}(S, n)$ is discrete and that $\operatorname{Fin}(S)$ is dense in $\operatorname{Cen}(S)$.

If S is a centered collection, then $\operatorname{Cen}(S)$ is homeomorphic to 2^S and if S is a disjoint collection, then $\operatorname{Cen}(S)$ is homeomorphic to the Alexandroff compactification of the discrete space of size $|S|$. All other $\operatorname{Cen}(S)$ are between these two extremes.

If X is a compact 0-dimensional space, then we will write $\operatorname{Cen}(X)$ for $\operatorname{Cen}(B(X))$.

Remark. Many times, when we define a natural map between two spaces, we leave it to the reader to check continuity. There will be canonical subbases present and the reader need only check that preimages of their members are open or closed.

Remark. Another natural object of study is the subspace $\operatorname{Max}(S)$ of $\operatorname{Cen}(S)$ consisting of all maximal centered subcollections of S. Properties of $\operatorname{Max}(S)$ where S is a collection related to graphs or posets are investigated in Bell and Ginsburg [7]. In particular, the questions of when is $\operatorname{Max}(S)$ compact and which spaces are of this form are addressed.

4. Elementary properties of $\operatorname{Cen}(S)$. It is immediate that the spaces $\operatorname{Cen}(S)$ are determined by the disjointness properties of the collection S and not by the nature of the sets in S. Two collections S and T are said to be *combinatorially similar* if there exists a bijection $\varphi\colon S \to T$ such that for every finite $F \subset S$, $\bigcap F \neq \emptyset$ iff $\bigcap \varphi[F] \neq \emptyset$. In this case, we write $S \equiv T$. So, if $S \equiv T$, then $\operatorname{Cen}(S)$ is homeomorphic to $\operatorname{Cen}(T)$.

The intersection map proves useful. If $S \subset T$, then $\operatorname{Cen}(S)$ is a retract of $\operatorname{Cen}(T)$ by the mapping sending a centered subcollection T' of T to $T' \cap S$. Also, it is seen that each $\operatorname{Cen}(S)$ is a retract of its own hyperspace by the mapping sending a closed subset K of $\operatorname{Cen}(S)$ to $\bigcap K$. Let $\mathcal{S} = \bigcup_{s\in S} \{s^+, s^-\}$ be the canonical subbase for $\operatorname{Cen}(S)$. Each member K of $\mathcal{S}^{\cap}$ is of the form $\operatorname{Cen}(P)$ for some collection P. To see this, let $K = \bigcap_{s\in A} s^+ \cap \bigcap_{s\in B} s^-$ where A and B are subsets of S. Put $Q = S-(A \cup B)$. For each $q \in Q$, put $S_q = \{F\colon F \text{ is a finite subset of } Q,\ q \in F \text{ and } F \cup A \text{ is centered}\}$. Put $P = \{S_q\colon q \in Q\}$. It is seen that K is homeomorphic to $\operatorname{Cen}(P)$ by the mapping sending a centered subcollection T in K to $\{S_q\colon q \in T \cap Q\}$.

Let S_α, for each α in an index set A, be given. Form the Tychonoff product $\prod_{\alpha\in A} \operatorname{Cen}(S_\alpha)$. Put $S = \{p_\alpha^{-1}(s^+)\colon \alpha \in A \text{ and } s \in S_\alpha\}$ where p_α is the αth projection. Define a mapping φ from $\operatorname{Cen}(S)$ onto $\prod_{\alpha\in A} \operatorname{Cen}(S_\alpha)$ as follows: If T is a centered subcollection of S, define a function $\varphi(T)$ by $\varphi(T)(\alpha) = \{s \in S_\alpha\colon p_\alpha^{-1}(s^+) \in T\}$. It is seen that φ is a one-to-one map of $\operatorname{Cen}(S)$ onto $\prod_{\alpha\in A} \operatorname{Cen}(S_\alpha)$. Hence, $\operatorname{Cen}(S)$ is homeomorphic to $\prod_{\alpha\in A} \operatorname{Cen}(S_\alpha)$. Thus, the spaces $\operatorname{Cen}(S)$ are closed under arbitrary products.

The author had earlier [5] used graph spaces to solve a problem. These spaces were defined similarly to the Cen(S)'s. For a graph G, one considers the space $C(G)$ of all complete subgraphs of G using the v^+'s and v^-'s, where $v \in G$, as a subbase. It is seen that $\mathrm{Cen}(\{v^+ : v \in G\})$ is homeomorphic to $C(G)$ and thus the graph spaces are included among the Cen(S)'s. However, there are spaces Cen(S) that are not graph spaces, since all graph spaces are supercompact but not all Cen(S)'s are, see Section 6. Using analogous proofs to [5], one shows that Cen(S) is Fréchet–Urysohn iff all centered subcollections of S are countable and that Cen(S) is ccc iff Fin(S), partially ordered by $F \leq G$ iff $G \subset F$, is a ccc poset.

5. The clopen algebra of Cen(S)

THEOREM 5.1. *A compact 0-dimensional space X is homeomorphic to a* Cen(S) *iff $B(X)$ is generated (as a boolean algebra) by a subset G with the following property: For each pair of finite subsets F and H of G, $\bigcap F - \bigcup H \neq \emptyset$ iff $\bigcap F \neq \emptyset$ and $F \cap H = \emptyset$.*

Proof. The clopen algebra of Cen(S) is generated by $G = \{s^+ : s \in S\}$ and G has the property.

Assume that X is compact 0-dimensional and that $B(X) = [G]$ where G has the property. X is homeomorphic to $\mathrm{st}([G])$. Define a mapping φ from $\mathrm{st}([G])$ to Cen(G) by $\varphi(p) = p \cap G$. It is seen that φ is continuous. Also, φ is one-to-one since any ultrafilter of $[G]$ is determined by which generators are in it. If $T \in \mathrm{Cen}(G)$, then $T \cup \{\mathrm{st}([G]) - b : b \in G - T\}$ is centered, by the property, and generates an ultrafilter on $[G]$ which maps by φ onto T. Thus, φ is a homeomorphism. ■

We now prove an algebraic property about the clopen algebra of Cen(S). Namely, that it is the union of countably many antichains. This generalizes the fact that the clopen algebra of $2^\varkappa$, i.e., the free boolean algebra on $\varkappa$ generators, has this property. Galvin and Jonsson [13] have proven that the free lattice on $\varkappa$ generators has this property and Galvin points out that their proof easily handles the free boolean algebra case. However, in the absence of freeness, certain impediments present themselves.

We point out that we do this for all clopen sets and not for just a subbase, base or π-base.

If F and G are finite subsets of S and $\mathscr{F}$ and $\mathscr{G}$ are subsets of Cen(F) and Cen(G) respectively, then we say that $(F, \mathscr{F})$ is *equivalent to* $(G, \mathscr{G})$ if there exists a bijection $\varphi: F \to G$ such that

(a) $H \in \mathscr{F}$ iff $\varphi[H] \in \mathscr{G}$,

(b) $H \in \mathrm{Cen}(F)$ iff $\varphi[H] \in \mathrm{Cen}(G)$.

This is indeed an equivalence relation.

For each finite $F \subset S$ and for each $\mathscr{F} \subset \mathrm{Cen}(F)$ put $b(F, \mathscr{F}) = \bigcup_{H \in \mathscr{F}} H^+ \cap (F-H)^-$. We say that $b(F, \mathscr{F})$ is an irreducible representation for the clopen set b if $b = b(F, \mathscr{F})$ and $b \neq b(G, \mathscr{G})$ for any G with $|G| < |F|$. Every clopen set b has an irreducible representation $b(F, \mathscr{F})$ for some F and $\mathscr{F}$.

THEOREM 5.2. *For any collection S of sets, the clopen algebra of* Cen(S) *is the union of countably many antichains.*

Proof. Define an equivalence relation $\sim$ on the clopen algebra of Cen(S) as follows: $b \sim c$ iff there exist equivalent $(F, \mathscr{F})$ and $(G, \mathscr{G})$ such that $b(F, \mathscr{F})$ resp. $b(G, \mathscr{G})$ are irreducible representations of b resp. c. There are only countably many equivalence classes of $\sim$. We show that each class is an antichain.

Assume that $(F, \mathscr{F})$ and $(G, \mathscr{G})$ are equivalent via a bijection $\varphi: F \to G$, $b(F, \mathscr{F})$ and $b(G, \mathscr{G})$ are irreducible, and $b(F, \mathscr{F}) \subset b(G, \mathscr{G})$. We must show that $b(F, \mathscr{F}) = b(G, \mathscr{G})$. Put $R = F \cap G$. We will rule out the case that $|R| < |G|$. Then, we will have $F = G$. Take $H \in \mathscr{F}$. Then $H \in b(F, \mathscr{F})$ so $H \in b(G, \mathscr{G})$ so there exists $K \in \mathscr{G}$ such that $H \in K^+ \cap (G-K)^-$ so $K \subset H$ and $H \cap (G-K) = \emptyset$. Since $F = G$ we get $H = K$ so $H \in \mathscr{G}$. Since $\mathscr{F} \subset \mathscr{G}$ and since $|\mathscr{F}| = |\mathscr{G}|$ we get $\mathscr{F} = \mathscr{G}$. Hence $b(F, \mathscr{F}) = b(G, \mathscr{G})$.

Continuing, we assume that $|R| < |G|$. We now show that $b(G, \mathscr{G}) = b(R, \mathscr{G} \cap R)$ where $\mathscr{G} \cap R = \{K \cap R : K \in \mathscr{G}\}$ contradicting irreducibility of $b(G, \mathscr{G})$. Clearly, $b(G, \mathscr{G}) \subset b(R, \mathscr{G} \cap R)$, so it remains to prove that $b(R, \mathscr{G} \cap R) \subset b(G, \mathscr{G})$. If we can show that for each $K \in \mathscr{G}$, $K \cap R \in \mathscr{F}$, then we will be done. For, if $H \in P^+ \cap (R-P)^-$ where $P = K \cap R$ for some $K \in \mathscr{G}$, then $H \cap G \in (K \cap R)^+ \cap [F-(K \cap R)]^-$. Hence, $H \cap G \in M^+ \cap (G-M)^-$ for some $M \in \mathscr{G}$. Therefore, $H \in M^+ \cap (G-M)^-$ as well and we have proven that $b(R, \mathscr{G} \cap R) \subset b(G, \mathscr{G})$.

To this end, put $R_0 = R$ and $R_{n+1} = \varphi[R_n] \cap R$. The R_n's form a decreasing sequence of finite sets, hence there exists N such that for all $n \geq N$, $R_n = R_N$. It is seen that $\varphi \restriction R_N$ is a bijection from R_N onto R_N.

CLAIM. *If $K \in \mathscr{G}$, then $K \cap R \in \mathscr{F}$.*

Proof. Put $B_0 = K \cap R$ and $B_{n+1} = \varphi[B_n] \cap R$. Since $B_0 \subset K \in \mathscr{G}$ and $\mathscr{F}$ and $\mathscr{G}$ are equivalent via φ, we get that all the B_n's are centered. Assume that for some n, $B_{n+1} \in \mathscr{F}$. Well, $\varphi[B_n]$ is centered and $\varphi[B_n] \in B_{n+1}^+ \cap (F-B_{n+1})^-$ since $\varphi[B_n] \cap F \subset \varphi[B_n] \cap R = B_{n+1}$. Therefore, there exists $M \in \mathscr{G}$ such that $\varphi[B_n] \in M^+ \cap (G-M)^-$. Since $\varphi[B_n] \subset G$, $M \subset \varphi[B_n]$ and $\varphi[B_n] \cap (G-M) = \emptyset$, we get $\varphi[B_n] = M$. Thus, $\varphi[B_n] \in \mathscr{G}$ and so $B_n \in \mathscr{F}$. Hence, it suffices to find an n such that $B_n \in \mathscr{F}$.

Well, $B_N \subset R \cap \varphi[R] \cap \ldots \cap \varphi^N[R] = R_N$ and $B_N = \varphi^N[K \cap R_N]$. We now show that $\varphi^N[K \cap R_N] \in \mathscr{F}$.

Set $A_0 = \varphi^{-1}[K]$ and $A_{n+1} = \varphi^{-1}[A_n \cap R]$. $A_0 \in \mathscr{F}$ by equivalence of $\mathscr{F}$ and $\mathscr{G}$. Assume that $A_n \in \mathscr{F}$. We prove that $A_{n+1} \in \mathscr{F}$. Suffices to show that $A_n \cap R \in \mathscr{G}$. Since $A_n \in b(F, \mathscr{F})$, there exists $M \in \mathscr{G}$ such that $A_n \in M^+ \cap (G-M)^-$. Therefore, $M \subset A_n$ and $A_n \cap (G-M) = \emptyset$. Hence, $M \subset A_n \cap G = A_n \cap R \subset G \cap M = M$. So $A_n \cap R = M$, thus $A_n \cap R \in \mathscr{G}$. By induction on n, it follows that $A_n \in \mathscr{F}$ for all $n \geq 0$. Now

$$A_{N+1} \subset \varphi^{-1}[R] \cap \varphi^{-2}[R] \ldots \cap \varphi^{-(N+1)}[R] = \varphi^{-1}[R_N] = R_N$$

and

$$\varphi^{N+2}[A_{N+1}] = K \cap \varphi[R] \ldots \cap \varphi^{N+1}[R] = K \cap \varphi[R_N] = K \cap R_N.$$

Thus, $A_{N+1} = \varphi^{-(N+2)}[K \cap R_N]$. Since $\varphi \restriction R_N$ is a bijection of R_N onto R_N, there exists an $i>0$ such that $\varphi^{-i}[A_{N+1}] = \varphi^N[K \cap R_N]$. Hence, $\varphi^N[K \cap R_N] = A_{N+1+i} \in \mathscr{F}$. This completes the proof. ■

6. The largest combinatorial remainder of N. Which of the spaces $\mathrm{Cen}(S)$ are remainders of N, i.e., which are continuous images of $\beta N - N$? Equivalently, as the spaces $\mathrm{Cen}(S)$ are 0-dimensional, which have clopen algebras embeddable in $B(\beta N - N) = \mathscr{P}(\omega)/\mathrm{fin}$? Clearly, we must have $|S| \leqslant \mathfrak{c}$. Assuming CH, Parovicenko [20], this is sufficient. Otherwise, it is not sufficient. By adding ω_2 Cohen reals to any countable transitive set model of ZFC, the author [5] has constructed a Fréchet–Urysohn, ccc graph space $C(G)$ with $|G| \leqslant \mathfrak{c}$ which is not a remainder of N.

We now show that there is a largest such remainder of N. To do this, we use a result in [6]. Let H be the subalgebra of the power set algebra $\mathscr{P}(\omega^\omega)$ generated by the rectangles $\prod_{i<\omega} A_i$ where for each $i<\omega$, $A_i - A_{i+1}$ is finite. The result is that H is embeddable in $B(\beta N - N)$. For each $A \subset \omega$, put $R(A) = \prod_{i<\omega} (A - \{j: j<i\})$. Note the following fact: $\bigcap_{i<n} R(A_i) - \bigcup_{j<m} R(B_j) \neq \emptyset$ iff $\bigcap_{i<n} A_i$ is infinite and for every $j<m$, $\bigcap_{i<n} A_i \not\subset B_j$. Put $K = \{(I, F, \varphi)$: I is a finite subset of ω, $F \subset \{0, \ldots \max I\}$ and $\varphi \subset \mathscr{P}(\{0, \ldots, \max I\})\}$. Enumerate K as $\{k_i: i<\omega\}$. We will identify the algebra $\mathscr{P}(\omega^\omega)$ with $\mathscr{P}(K^\omega)$ for the purposes of realizing H. We redefine our function R above as follows: For each $A \subset K$ put $R(A) = \prod_{i<\omega} (A - \{k_j: j<i\})$. Then R satisfies the above fact as well.

THEOREM 6.1. *$\mathrm{Cen}(\beta N - N)$ is a remainder of N.*

Proof. We will embed the clopen algebra of $\mathrm{Cen}(\beta N - N)$ into H as realized in $\mathscr{P}(K^\omega)$. For each $A \subset \omega$ put $A^* = \{(I, F, \varphi)$: I is a finite subset of A, $F \subset \{0, \ldots, \max I\}$, $\varphi \subset \mathscr{P}(\{0, \ldots, \max I\})$ and $F \cap A \in \psi\}$. The clopen algebra of $\mathrm{Cen}(\beta N - N)$ is generated by $G = \{b^+: b \in B(\beta N - N)\}$. Using choice, let $h: B(\beta N - N) \to \mathscr{P}(N)$ such that for all b, $b = \mathrm{Cl}_{\beta N} h(b) \cap \beta N - N$. Then, $\bigcap_{i<n} b_i \neq \emptyset$ iff $\bigcap_{i<n} h(b_i)$ is infinite and, also, $a \neq b$ implies $h(a) \triangle h(b) \neq 0$.

Now, define a function $\psi: G \to H$ by $\psi(b^+) = R(h(b)^*)$. To extend ψ to an embedding of the clopen algebra of $\mathrm{Cen}(\beta N - N)$ it suffices to prove the following:

CLAIM. *For each pair of finite subsets A and B of $B(\beta N - N)$,*

$$\bigcap_{a\in A} a^+ - \bigcup_{b\in B} b^+ \neq \emptyset \quad \textit{iff} \quad \bigcap_{a\in A} \psi(a^+) - \bigcup_{b\in B} \psi(b^+) \neq \emptyset .$$

Proof. $\bigcap_{a\in A} a^+ - \bigcup_{b\in B} b^+ \neq \emptyset$ iff P: $\bigcap_{a\in A} h(a)$ is infinite and $A \cap B = \emptyset$. $\bigcap_{a\in A} \psi(a^+) - \bigcup_{b\in B} \psi(b^+) \neq \emptyset$ iff Q: $\bigcap_{a\in A} h(a)^*$ is infinite and for each $b \in B$, $\bigcap_{a\in A} h(a)^* \not\subset h(b)^*$. This follows from the aforementioned fact. We show that P iff Q holds. Assume P. Since $\bigcap_{a\in A} h(a)$ is infinite it follows that $\bigcap_{a\in A} h(a)^*$ is infinite. Since $A \cap B = \emptyset$, choose for each $a \in A$ and $b \in B$, $p(a, b) \in h(a) \triangle h(b)$. Put $F = \{p(a, b)$: $a \in A$ and $b \in B\}$. Put $\varphi = \{F \cap h(a): a \in A\}$. Choose any $n \in \bigcap_{a\in A} h(a)$ such that $\max F < n$. Then, $(\{n\}, F, \varphi) \in \bigcap_{a\in A} h(a)^* - \bigcup_{b\in B} h(b)^*$. This is because $F \cap h(b)$ is not equal to any $F \cap h(a)$. Thus Q holds. Assume Q. Clearly $A \cap B = \emptyset$. If $\bigcap_{a\in A} h(a)$ was finite, then $\bigcap_{a\in A} h(a)^* = (\bigcap_{a\in A} h(a))^*$ would be finite. Hence $\bigcap_{a\in A} h(a)$ is infinite and P holds. ■

THEOREM 6.2. *If $\mathrm{Cen}(S)$ is a remainder of N, then $\mathrm{Cen}(S)$ is homeomorphic to a retract of $\mathrm{Cen}(\beta N - N)$.*

Proof. Embed the clopen algebra of $\mathrm{Cen}(S)$ into $B(\beta N - N)$. Thus, we consider $\{s^+: s \in S\}$ as a subset of $B(\beta N - N)$. By the second paragraph of Section 4, $\mathrm{Cen}(\{s^+: s \in S\})$ is a retract of $\mathrm{Cen}(\beta N - N)$. Since $S \equiv \{s^+: s \in S\}$, we get that $\mathrm{Cen}(S)$ is homeomorphic to a retract of $\mathrm{Cen}(\beta N - N)$. ■

This theorem essentially says that the clopen algebra of $\mathrm{Cen}(\beta N - N)$ is the largest purely combinatorial subalgebra of $B(\beta N - N)$.

In response to a question of van Douwen [8], the author [6] has constructed, for each $n \geqslant 2$, compact spaces X_n which are remainders of N, have σ-n-linked bases but are not separable. In Section 4, part A of [6], it is shown that these spaces satisfy the condition of Theorem 5.1. Hence, the spaces X_n are homeomorphic to spaces $\mathrm{Cen}(S_n)$.

In response to another question of van Douwen in the same paper, van Mill (unpublished result) has proven if X is a Hausdorff continuous image of a space of cmpn n and X has a σ-n-linked base, then X is separable. We mention that van Douwen had proven this if $\mathrm{cmpn}(X) = n$.

This allows us to prove:

COROLLARY 6.3. *$\mathrm{Cen}(\beta N - N)$ has infinite compactness number.*

Proof. By Theorem 6.2 the spaces $X_n = \mathrm{Cen}(S_n)$ are retracts of $\mathrm{Cen}(\beta N - N)$. If cmpn $(\mathrm{Cen}(\beta N - N)) = n$, then by van Mill's result, the space X_n would be separable but it isn't. ■

So, not all centered spaces are supercompact. It is an unsolved problem of whether all dyadic spaces are supercompact.

7. Centered spaces. We say that X is a *centered space* if X is a Hausdorff continuous image of some $\mathrm{Cen}(S)$. From Section 4, it follows that every product of centered spaces is again a centered space. Thus every polyadic space is a centered space.

The following is analogous to a dyadic theorem of Sanin [21].

THEOREM 7.1 (An image property). *If X is a centered space of π-weight $\varkappa$, then there exists a collection of sets T with $|T| \leqslant \varkappa$ such that $\mathrm{Cen}(T)$ maps onto X. Hence, the π-weight of a centered space equals the weight of the space.*

Proof. Let φ map $\mathrm{Cen}(S)$ onto X. Let $\{U_\alpha: \alpha<\varkappa\}$ be a π-base for X where $\varkappa$ is the π-weight of X. For each $\alpha<\varkappa$ choose finite subsets F_α and G_α of S such that $\emptyset \neq F^+_\alpha \cap G^-_\alpha \subset \varphi^{-1}[U_\alpha]$. Put $T = \bigcup_{\alpha<\varkappa} (F_\alpha \cup G_\alpha)$. The range of $\varphi|\mathrm{Cen}(T)$ is dense in X and thus X is an image of $\mathrm{Cen}(T)$. Since $\mathrm{Cen}(T)$ is compact and of weight $\leqslant \varkappa$, so is X. ■

COROLLARY 7.2. *Every centered compactification of a discrete space of size $\varkappa$ has weight $\varkappa$. In particular, every centered compactification of N is metrizable.*

So, βN is not centered.

The following generalizes a dyadic theorem of Engelking and Pełczyński [11].

COROLLARY 7.3. *If βX is centered, then X is pseudocompact.*

Proof. If X is non-pseudocompact, then by a Lemma in [11], there exists a mapping $\varphi: X \to R$ such that $\varphi[X]$ is a dense subspace of R. Therefore, βX maps onto βR. If βX is centered then so also is βR. But βR has countable π-weight and uncountable weight, contradicting Theorem 7.1. ■

The following, in the absence of ccc, is an analogue to a dyadic theorem of Sanin [21] on calibres. It is similar to the property $B(n)$ that Gerlits [14] proved all ζ-adic spaces satisfy, for all cardinals n.

THEOREM 7.4 (A clustering property). *For every uncountable regular cardinal $\varkappa$ and for every collection $\{V_\alpha: \alpha<\varkappa\}$ of non-empty open sets of a centered space X, there exists $A \subset \varkappa$ with $|A| = \varkappa$, points $p_\alpha \in V_\alpha$ for each $\alpha<\varkappa$, and a point $p \in X$ such that every neighbourhood of p contains all but finitely many p_α's.*

Proof. Since the clustering property is preserved under continuous images, it suffices to show that $\mathrm{Cen}(S)$ has the clustering property. Moreover, it suffices to prove it for V_α's coming from the canonical open base of $\mathrm{Cen}(S)$. So, let $\{F^+_\alpha \cap G^-_\alpha: \alpha<\varkappa\}$, where $\varkappa$ is uncountable and regular, be non-empty basic clopen sets of $\mathrm{Cen}(S)$. Note that some of the F_α's or G_α's may be the $\emptyset$. By Sanin's theorem, there exists $A \subset \varkappa$ with $|A| = \varkappa$ and an $F \in \mathrm{Fin}(S)$ such that for each $\alpha \neq \beta$ in A, $F_\alpha \cap F_\beta = F$. The points p_α's are the F_α's and the point p is F. ■

The Alexandroff double of the closed unit interval is not centered since it does not have the clustering property for a set of isolated points.

A space has *countable depth* if there does not exist a properly decreasing ω_1-sequence of open sets V_α such that $\alpha<\beta$ implies $\overline{V}_\beta \subset V_\alpha$. The following generalizes a polyadic theorem of Gerlits [15].

THEOREM 7.5 (An order property). *Every centered space X has countable depth. Moreover, X contains no uncountable chain (of any order type) of clopen sets.*

Proof. Let φ map $\mathrm{Cen}(S)$ onto X. If X had a properly decreasing ω_1-sequence of open sets contradicting countable depth, then the inverse images would be such a sequence in $\mathrm{Cen}(S)$. Since $\mathrm{Cen}(S)$ is 0-dimensional we could produce a properly decreasing ω_1-sequence of clopen sets in $\mathrm{Cen}(S)$. But this contradicts Theorem 5.2 which implies that there are no uncountable chains (of any order type) of clopen sets of $\mathrm{Cen}(S)$. ■

Since the „Two arrows" space of Alexandroff has an uncountable chain of clopen sets of order type the real, we see that this space is not centered. Also, ordinal

space ω_1+1 is not centered. Since ω_1+1 is a retract of $\mathrm{Exp}(\omega_1+1)$, $\mathrm{Exp}(\omega_1+1)$ also is not centered. We will use this fact in Theorem 7.11.

A space is *d-separable*, Arhangelskii [3] if it has a dense σ-discrete subspace. The following generalizes a dyadic theorem of Arhangelskii [3].

THEOREM 7.6 (A density property). *Every centered space X has a dense subspace which is the union of countably many compact spaces each of which is the union of finitely many discrete spaces.*

Proof. Let φ map $\mathrm{Cen}(S)$ onto X. Each $\mathrm{Fin}(S, n)$ is a compact space that is the union of $n+1$ discrete subspaces. It is a topological exercise to prove that any continuous image of such a space is also the union of at most $n+1$ discrete subspaces. Thus $\varphi[\mathrm{Fin}(S)] = \bigcup_n \varphi[\mathrm{Fin}(S, n)]$ is the required dense subspace. ■

The following generalizes a dyadic theorem of Engelking [10] and is his method of proof.

THEOREM 7.7 (A discrete property). *For every point p of character $\varkappa$ in a centered space X, X contains a discrete subspace A of size $\varkappa$ such that $A \cup \{p\}$ is homeomorphic to the Alexandroff compactification of A.*

Proof. Let φ map $\mathrm{Cen}(S)$ onto X. For each $p \in X$, say that p depends on s, where $s \in S$, if there exists centered subsets S_1 and S_2 of S such that $S_1-\{s\} = S_2-\{s\}$ and $\varphi(S_1) = p \neq \varphi(S_2)$. Assume that p has character $\varkappa$. Let $S_0 = \{s \in S: p \text{ depends on } s\}$. For each $s' \in S_0$, choose $T(s')$ and $R(s')$ centered subsets of S such that $\varphi(T(s')) = p \neq \varphi(R(s'))$ and $T(s')-\{s'\} = R(s')-\{s'\}$.

Put $B = \{T \in \mathrm{Cen}(S): \text{there exists } R \in \varphi^{-1}(p) \text{ such that } T \cap S_0 = R \cap S_0\}$. Clearly $\varphi^{-1}(p) \subset B$. Conversely, take $T \in B$. Choose $R \in \varphi^{-1}(p)$ such that $T \cap S_0 = R \cap S_0$. For each finite subset F of $S-S_0$, $R-F \in \mathrm{Cen}(S)$ and $R-F \in \varphi^{-1}(p)$. Since $\varphi^{-1}(p)$ is closed, we get that $R \cap S_0 \in \varphi^{-1}(p)$. For each finite subset F of $T-S_0$, $(R \cap S_0) \cup F \in \mathrm{Cen}(S)$ and $(R \cap S_0) \cup F \in \varphi^{-1}(p)$. Since $\varphi^{-1}(p)$ is closed, we get that $T \in \varphi^{-1}(p)$. Hence, $\varphi^{-1}(p) = B$. Since B is the intersection of $|S_0|$ open sets and p has character $\varkappa$ we deduce that $|S_0| \geqslant \varkappa$. Put $R = \{R(s'): s' \in S_0\}$ and $T = \{T(s'): s' \in S_0\}$. Assume that there exists an infinite $S' \subset S_0$ such that for all $s' \in S'$, $R(s') = R_0$ for a fixed R_0. Put $T' = \{T(s'): s' \in S'\}$. Let $F^+ \cap G^-$, be an arbitrary neighbourhood of R_0. Choose $s' \in S'-(F \cup G)$. Since $R(s') = R_0 \in F^+ \cap G^-$, we get that $T(s') \in F^+ \cap G^-$. Hence, $R_0 \in \overline{T'}$, therefore $\varphi(R_0) = p$ which is a contradiction. Therefore, the association $s' \to R(s')$ has finite preimages and so $|R| = |S_0| \geqslant \varkappa$.

All cluster points of R are in $\varphi^{-1}(p)$. For, let Q be a cluster point of R. Let $F^+ \cap G^-$ be a neighbourhood of Q. Choose $s' \in S_0-(F \cup G)$ such that $R(s') \in F^+ \cap G^-$. Thus, $T(s') \in F^+ \cap G^-$. So $Q \in \overline{T}$, therefore $\varphi(Q) = p$. Now let $A = \varphi[R]$. ■

The following generalizes a dyadic theorem of Efimov [9].

COROLLARY 7.8. *Every non-isolated point in a centered space is the limit of a non-trivial convergent sequence.*

The following generalizes a dyadic theorem of Esenin-Vol'pin [12]. We use Marty's [18] method of proof.

THEOREM 7.9 (A character property). *The character of a centered space equals the weight of the space.*

Proof. Let φ map $\mathrm{Cen}(S)$ onto X and let $\varkappa$ be the character of X. We must show that the weight of X is $\varkappa$. A subset T of S is a support of a point x in $\mathrm{Cen}(S)$ if whenever $y \in \mathrm{Cen}(S)$ and $y \cap T = x \cap T$ then $\varphi(y) = \varphi(x)$.

CLAIM. *For each $x \in \mathrm{Cen}(S)$ there exists $S_x \subset S$ with $|S_x| \leqslant \varkappa$ such that S_x is a support of x.*

Proof. Let $\{V_\alpha: \alpha<\varkappa\}$ be a neighborhood base of $\varphi(x)$ in X. For each $\alpha<\varkappa$ choose finite subsets F_α and G_α of S such that $x \in F_\alpha^+ \cap G_\alpha^- \subset \varphi^{-1}[V_\alpha]$. Put $S_x := \bigcup \{F_\alpha \cup G_\alpha: \alpha<\varkappa\}$.

Now, choose $S_0 \subset S$ with $|S_0| \leqslant \varkappa$ such that S_0 is a support of $\emptyset$.

For each $x \in \mathrm{Fin}(S_0)$ choose S_x with $|S_x| \leqslant \varkappa$ such that S_x is a support of x. Put $S_1 = S_0 \cup \bigcup \{S_x: x \in \mathrm{Fin}(S_0)\}$.

For each $x \in \mathrm{Fin}(S_1)$ choose S_x with $|S_x| \leqslant \varkappa$ such that S_x is a support of x. Put $S_2 = S_1 \cup \bigcup \{S_x: x \in \mathrm{Fin}(S_1)\}$.

And so on ...

Now put $S_\infty = \bigcup_{i<\omega} S_i$. Note that $|S_\infty| \leqslant \varkappa$. We claim that $\varphi \restriction \mathrm{Cen}(S_\infty)$ maps $\mathrm{Cen}(S_\infty)$ onto X. Since $\mathrm{Cen}(S_\infty)$ has weight equal to $|S_\infty| \leqslant \varkappa$, we would get that the weight of X is at most $\varkappa$. To prove this claim, choose $x \in X$ and $T \in \mathrm{Cen}(S)$ such that $\varphi(T) = x$. For each $F \in \mathrm{Fin}(T \cap S_\infty)$, $F \cup (T-S_\infty) \in \mathrm{Cen}(S)$. T is in the closure of $\{F \cup (T-S_\infty): F \in \mathrm{Fin}(T \cap S_\infty)\}$. For each $F \in \mathrm{Fin}(T \cap S_\infty)$, there exists $i<\omega$ such that $F \in \mathrm{Fin}(S_i)$. But at the ith stage of our procedure we included a support of F in S_∞. Since $[F \cup (T-S_\infty)] \cap S_\infty = F \cap S_\infty$ we get that $\varphi(F \cup (T-S_\infty)) = \varphi(F)$. By continuity of φ, we get that $x = \varphi(T)$ is in the X closure of $\{\varphi(F): F \in \mathrm{Fin}(T \cap S_\infty)\}$, thus $x \in \varphi[\mathrm{Cen}(S_\infty)]$. ■

COROLLARY 7.10. *The spread of a centered space equals the weight of the space.*

Proof. Use Theorem 7.9 followed by Theorem 7.7. ■

EXAMPLE. Let $S = \{\{F\}: F$ is a finite subset of $R\} \cup \{\{F: F$ is a finite subset of R and $r \in F\}: r \in R\}$. It is seen that $\mathrm{Cen}(S)$ is a compactification of a discrete space of size $\mathfrak{c}$ whose non-isolated points are homeomorphic to $2^{\mathfrak{c}}$. So, there are non-trivial centered compactifications of uncountable discrete spaces.

EXAMPLE. With a little effort, it can be shown that $\mathrm{Cen}(\beta N)$ is separable. However, it is not polyadic. A reason is the following: For each $n \in N$, put $V_n = \{n\}^+$. Choose $p \in \beta N-N$ and consider the centered collection $T = \{U: U$ is clopen in βN and $p \in U\}$. Then $T \in \overline{\bigcup_{n\in N} V_n}$ but no sequence from $\bigcup_{n\in N} V_n$ converges to T. Whereas, Mrówka [19] has shown that in a polyadic space the closure of an open set is its sequential closure. Thus, separable centered spaces need not be polyadic.

EXAMPLE. Gerlits' theorem [15] for polyadic spaces that weight equals cellularity times tightness doesn't generalize to centered spaces. At least we have a consistent counterexample. The graph space $C(G)$ alluded to previously has countable cellularity and countable tightness but is of uncountable weight. Whether there is a „real" example we do not know.

The following is a generalization of a dyadic theorem of Sapiro [22].

THEOREM 7.11. $\mathrm{Exp}(2^{\omega_2})$ *is not centered.*

Proof. Assume not, i.e., let φ map $\mathrm{Cen}(S)$ onto $\mathrm{Exp}(2^{\omega_2})$. We will use the following result proved in [4]. Let $\mathscr{S} = \mathscr{S}^\cap$ be a closed subbase for a compact space X and let X map onto $\mathrm{Exp}(2^{\varkappa^+})$. If Y is a compact 0-dimensional space of weight at most $\varkappa$, then there exists an $S \in \mathscr{S}$ such that S maps onto $\mathrm{Exp}\, Y$.

Let $\mathscr{R} = \bigcup_{s\in S} \{s^+, s^-\}$ be the canonical closed subbase for $\mathrm{Cen}(S)$. Put $\mathscr{S} = \mathscr{R}^\cap$. Then $\mathscr{S} = \mathscr{S}^\cap$. Referring to the second paragraph of Section 4, we see that every member of $\mathscr{S}$ is of the form $\mathrm{Cen}(P)$ for some collection P. The above result implies that $\mathrm{Exp}\, Y$ is centered for every compact 0-dimensional space Y of weight ω_1. This is not true. $\mathrm{Exp}(\omega_1+1)$ is not centered. ■

We point out that Sirota [23] has proven that $\mathrm{Exp}(2^{\omega_1})$ is homeomorphic to 2^{ω_1}.

Some particularly interesting dyadic theorems that we have been unable to decide for centered spaces are: Is a closed G_δ of a centered space again centered? If a centered space has a point of π-character $\varkappa$, does it contain a copy of $2^\varkappa$?

Another intriguing question is: Is $\mathrm{Cen}(\beta N-N)$ homogeneous or not? We mention that one can show that every point of $\mathrm{Cen}(\beta N-N)$ has π-character $\mathfrak{c}$.

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On the transformers of the Zahorski classes of functions

by

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Abstract. Let $\mathcal{M}_i$ ($i = 1, \ldots, 5$) denote the Zahorski classes of functions defined on interval $I = (0, 1)$. By $H_{\mathcal{M}_i}$ we denote the set of all increasing homeomorphism h of interval I onto itself which leaves $\mathcal{M}_i$ invariant, i.e., such that for every $f \in \mathcal{M}_i$ $f \circ h \in \mathcal{M}_i$. In this paper it is shown that there are proper inclusion $H_{\mathcal{M}_i} \supset H_{\mathcal{M}_{i+1}}$, for $i = 1, 2, 3$. We give a characterization of the class $H_{\mathcal{M}_4}$ and the examples of homeomorphisms showing that classes $H_{\mathcal{M}_5} \setminus H_{\mathcal{M}_4}$ and $H_{\mathcal{M}_4} \setminus H_{\mathcal{M}_5}$ are nonempty. We also establish the inclusion $H_{\mathcal{M}_4} \supset H_{b\Delta}$ where $H_{b\Delta}$ is the class of homeomorphisms preserving bounded derivatives.

Let g be a homeomorphism of $(0, 1)$ onto itself, $g(0) = 0$, $g(1) = 1$. If $\mathcal{F}$ is an arbitrary family of real functions defined on $(0, 1)$ then g is said to be a transformer on $\mathcal{F}$ if $f \circ g \in \mathcal{F}$ holds for every $f \in \mathcal{F}$. Let $H_{\mathcal{F}}$ denote the class of transformers of $\mathcal{F}$.

M. Laczkovich and G. Petruska in [7] have given a necessary and sufficient condition for g to be a transformer on a class of derivatives.

In 1950, Z. Zahorski [9] considered a hierarchy of classes of functions, $\mathcal{M}_k$, $k = 0, \ldots, 5$. The largest class, $\mathcal{M}_0$, turned out to be Darboux and first class of functions: the smallest, $\mathcal{M}_5$, turned out to be class of approximately continuous functions. He showed how the classes of derivatives and bounded derivatives fit into the scheme.

In the present paper we study a hierarchy of classes $H_{\mathcal{M}_k}$, $k = 0, \ldots, 5$ (for the sake of simplicity we use the notation H_k, $k = 0, \ldots, 5$) and consider a number of related properties.

Throughout this paper the word "set" means a Lebesgue measurable subset of open interval $(0, 1)$, the word "homeomorphism" means an increasing homeomorphism of $(0, 1)$ onto itself. By a "function" we means a real function on $(0, 1)$. $A' = (0, 1) \setminus A$, $|A|$ being the Lebesgue measure of the set A and $d(A, a)$ being the density of the set A in the point a. As usual, $d^+(A, a)$ and $d^-(A, a)$ denote the righthand side and lefthand side densities and $\bar{d}(A, a)$ the upper density.

We begin with the definition of the Zahorski classes of sets.

Definition 1. Let E be a nonempty set of type F_σ. We say E *belongs to class*: