

# THE SYMPLECTIZATION OF SCIENCE

*Symplectic Geometry Lies at the Very  
Foundations of Physics and Mathematics*

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*Physics is geometry.* This dictum is one of the guiding principles of modern physics. It largely originated with Albert Einstein, whose most important contribution—via his General Theory of Relativity—was to view the phenomenon of gravity as a reflection of the curvature of the geometry of spacetime. Einstein’s vision is remarkable in its simplicity, has great conceptual power and is physically compelling. As well, it leads to a theory of gravity which is very accurate in its agreement with experiment and observation. A further triumph of the geometric point of view has been the development, over the past four decades, of the “gauge” or Yang-Mills field theories of fundamental physical processes. Now, not only is gravity a manifestation of geometry, so are electromagnetism and the nuclear forces. Work actively continues towards the ultimate “grand unification,” the marriage of *all* basic physical interactions with each other and geometry.

The geometry of general relativity and gauge field theories, known as *Riemannian geometry* after the great nineteenth century German mathematician Georg Friedrich Bernhard Riemann, is a curved generalization of the familiar (and ancient) geometry of Euclid. But there is another, less familiar and less intuitive, type of geometry which is even more deeply rooted in physics: *symplectic geometry*. This is the mathematics that underlies mechanics and hence is at the very foundation of classical physics. The behavior of systems and phenomena as diverse as spinning tops, magnetism, the propagation of water waves and even the gravitational field itself, can to a large extent be both described and understood in terms of this geometry.

Thus physics is indeed geometry—*symplectic* geometry. This is true not just at the formula, theoretical level, but at the practical, engineering level as well. Symplectic geometry is turning out to be an indispensable tool for comprehending the large scale behavior of complex physical systems like the superconducting supercollider, or the Galileo probe on its way to Jupiter. Understanding which may be lost in the study of the complicated differential equations governing the motion of such a body can be recovered by means of symplectic geometry. Such understanding is often crucial: it could have saved the late 1950s satellite Explorer I, which tumbled out of control when spun about a dynamically unstable axis.

Besides this key role in physics and engineering, symplectic geometry is found increasingly to play an important role in mathematics: not only are symplectic ideas prevalent throughout, there are indications that much of mathematics will eventually be “symplectized.” We may well be witnessing the advent of a symplectic revolution in fundamental science. Already, there is impressive evidence that symplectic geometry will come to be regarded as one of the most important and productive offshoots of, and links between, mathematics and physics in this century.

Box 1: *Some History*

Part of the reason symplectic geometry is relatively little known and was so long in developing is that it is a rather abstruse—and almost weird—sort of

geometry. To aid in its description, we shall compare and contrast it throughout with the more familiar Euclidean geometry (and its curved Riemannian generalization). The essence of both can be gleaned by studying the simplest possible case: the geometry of the plane  $\mathbf{R}^2$ .

As its Greek root *γεωμετρία* (“measure of land”) implies, geometry has its origins in the science of surveying. Thus it focuses on the measurement of the lengths of lines, and on measuring the angles between lines. All this information is encoded mathematically into the basic notion of Euclidean geometry: the *metric*  $g$ . This is an object which associates a number to every pair of vectors  $v = (v_1, v_2)$  and  $w = (w_1, w_2)$  in the plane according to the formula

$$g(v, w) = v_1 w_1 + v_2 w_2.$$

It is both *symmetric* (i.e.,  $g(v, w) = g(w, v)$ ) and *nondegenerate* (i.e.,  $g(v, w) = 0$  for all vectors  $w$  if and only if  $v = 0$ ). Using this device, one defines the length of a vector  $v$  to be  $\|v\| = \sqrt{g(v, v)}$ ; this gives the Pythagorean result

$$\|v\|^2 = (v_1)^2 + (v_2)^2.$$

Similarly, the angle  $\theta$  between two vectors  $v$  and  $w$  is determined via

$$\cos \theta = \frac{g(v, w)}{\|v\| \cdot \|w\|}.$$

In particular,  $v$  is at right angles to  $w$  if and only if  $g(v, w) = 0$ . The symmetry of  $g$  guarantees that the angle between  $v$  and  $w$  is the same as that between  $w$  and  $v$ , while nondegeneracy ensures that no nonzero vector can be perpendicular to every other vector. Thus various kinds of familiar geometric information can be recovered from the metric  $g$ .

In the same vein, the *standard symplectic form*  $\Omega$  on  $\mathbf{R}^2$  is also an object which associates to every two vectors  $v$  and  $w$  a number; in this case the formula is

$$\Omega(v, w) = v_1 w_2 - v_2 w_1.$$

Note both the ordering and the all important minus sign—because of these,  $\Omega$  is *antisymmetric*:  $\Omega(w, v) = -\Omega(v, w)$ . This means that, unlike the Euclidean metric  $g$ ,  $\Omega$  does not give rise to a notion of length or angle. Indeed, the “symplectic length”  $\|v\| = \sqrt{\Omega(v, v)}$  of a vector  $v$  always vanishes and, even worse, every vector is now perpendicular to itself!

But the symplectic form *does* do something sensible, and that is to make precise the concept of “oriented area.” Consider a parallelogram whose sides are formed by two vectors  $v$  and  $w$ ; then the area of this figure is given exactly by  $\|\Omega(v, w)\|$ . The sign of  $\Omega(v, w)$  is determined by comparing the orientation of the pair  $[v, w]$  with the standard orientation of the plane—the sign is positive if they agree, negative otherwise. Taking these vectors in the opposite order will then flip the orientation and reverse the sign; hence the antisymmetry. Like the

metric  $g$ , the symplectic form  $\Omega$  is nondegenerate, which in this context means that only the collapsed parallelogram (when  $v$  and  $w$  are parallel) has zero area.

So symplectic geometry is a purely “areal” type of geometry.<sup>1</sup> We shall see later how this seemingly obscure concentration on oriented area ties in with classical mechanics.

The symplectic and Euclidean geometries of the plane  $\mathbf{R}^2$  are already interesting and nontrivial. But physics and mathematics—as well as common experience—don’t all happen just on the flat plane, so it is necessary to generalize. There are two ways to make geometry more useful (and fascinating!): by adding dimensions and by allowing for more complicated (“warped”) spaces. Both generalizations are essential: indeed the physical universe—spacetime—in which we now reside is a curved 4-dimensional space. Each of these generalizations will now be discussed in turn, for both Euclidean and symplectic geometries.

For Euclidean geometry the generalization to higher dimensions is straightforward. There are lines, with angles between them, in everyday 3-dimensional space  $\mathbf{R}^3$ , as well as in the more abstract spaces  $\mathbf{R}^n$  of any number of dimensions  $n$ . Hence, in any of these spaces one can construct a metric which assigns lengths to vectors and angles to pairs of vectors in any directions.

The extension to higher dimensions is more complicated for symplectic geometry, since area is essentially a two-dimensional construct. In fact, one cannot consistently define “oriented area” for odd-dimensional spaces, such as  $\mathbf{R}^3$ , without introducing unwanted phenomena (“degeneracies”). On the other hand, it is possible to build a symplectic form on the four-dimensional space  $\mathbf{R}^4$  as follows. View  $\mathbf{R}^4$  as the “sum”  $\mathbf{R}^2 \oplus \mathbf{R}^2$  of two planes; then the oriented area of a parallelogram in this space is the sum of the oriented areas of its shadows on these planes. This approach also works for  $\mathbf{R}^6, \mathbf{R}^8$ , etc.; the upshot is that only *even*-dimensional spaces can carry symplectic structures. (We shall see a physical reason for this later.)

Let us now consider the other avenue of generalization necessary in geometry. The story is a familiar one in the Euclidean case: During the nineteenth century, geometers such as Bolyai, Gauss and Lobachevski wondered what geometry might be like if “parallel lines” didn’t stay parallel, but rather converged or diverged. Triangles, they found, could then have angles summing to more or to less than  $180^\circ$ , there could be six-sided “squares” (with all angles being right angles), and all the familiar mensuration formulas of Euclidean geometry were replaced by new and exotic ones.

Box 2: *Geometry on Manifolds*

Especially interesting are two features of these ‘non-Euclidean’ or “Riemannian” geometries: The first is that the geometry can vary considerably from point to point in the space. So triangles at one location might have angles which sum to  $200^\circ$ , while another triangle elsewhere might have angles summing to  $165^\circ$ . All this information can still be obtained from a metric  $g$ , but

this metric must be allowed to be different at each point. It is now a metric *function*. The second feature is that the spaces which underlie these non-Euclidean geometries are often structurally quite different from the spaces  $\mathbf{R}^n$ . The space could for example “curl up” on itself and close in a variety of ways. These “warped” or “twisted” spaces—called *manifolds*—are generalizations of surfaces such as the sphere. They arise naturally in a variety of contexts, both physical and mathematical, and can be quite complicated. Some examples are pictured in Boxes 2 and 5; further discussion of manifolds and their non-Euclidean geometries can be found in the Scientific American article “The Mathematics of Three-dimensional Manifolds,” by W.P. Thurston and J.R. Weeks (July, 1984).

Roughly speaking, then, Riemannian geometry is Euclidean geometry extended to curved spaces of arbitrary dimension. Élie Cartan captured the essence of Riemannian geometry when he observed: “A Riemannian manifold is really made up of an infinity of small pieces of Euclidean spaces.” On each of these infinitesimally small pieces (represented intuitively as flat “tangent spaces” to the manifold) the metric  $g$  takes a fixed value. Thus one may compute the length of a curve by decomposing it into a chain of “tangent vectors,” computing the length of each of these using the induced Euclidean metric on each tangent space (as illustrated previously in the two-dimensional case), and then summing.

Similar remarks apply to symplectic geometry. One generalizes the symplectic plane  $\mathbf{R}^2$  to an even-dimensional manifold and allows the geometry to vary from point to point. On each tangent space the symplectic form  $\Omega$  defines a notion of oriented area. Thus, for instance, one may compute the oriented area of a surface residing in a symplectic manifold by breaking the surface up into infinitesimal parallelograms and summing the oriented areas of these. As with Riemannian geometry, the “usual” rules for computing oriented areas will typically not remain valid in this more general setting. (See Box 2.)

At this juncture a crucial difference between symplectic and Riemannian geometries appears. One may think of constructing a Riemannian geometry piece-by-piece by “gluing together” the Euclidean geometries on each tangent space. This gluing can be done in an arbitrary way, so long as it is done smoothly. One may try to build a symplectic geometry in a similar fashion. But there is a constraint on the values of the symplectic form  $\Omega$  so obtained at neighboring points: these must be arranged so that the oriented surface area of every compact three-dimensional region is zero (“Jacobi’s identity”).<sup>2</sup> This condition is responsible for many of the most interesting—and occasionally frustrating—facets of symplectic geometry.

There is no such gluing condition in Riemannian geometry, with the consequence that *every* manifold (regardless of dimension) can be endowed with a Riemannian metric. But Jacobi’s identity cannot always be satisfied, so there are (even-dimensional) manifolds which do not carry symplectic forms (the six-dimensional sphere, for example).<sup>3</sup> Thus symplectic geometry is really rather special. In fact, it is not known exactly which manifolds are symplectic, and researchers have just begun to make progress in characterizing and classifying those that are. Along these lines we mention here an important recent result:

Mikhail Gromov’s discovery of an “exotic” symplectic structure on  $\mathbf{R}^4$ . This is yet a different symplectic geometry than the “standard” one described earlier. Although the existence of this exotic geometry was established theoretically several years ago, it was only in 1989 that an explicit expression for the symplectic form was found. Using computer graphics (Box 3), one can now get some insight into how this exotic structure behaves *vis-à-vis* the standard one. These and similar topics are the province of *symplectic topology*, currently one of the most active areas in the field. See the recent article *The Symplectic Camel* by Ian Stewart (Nature, September 1987) as well as the companion article by Claude Viterbo in this issue for more on this subject.

Box 3: *Symplectic Geometries on  $\mathbf{R}^4$*

The most important ramification of the Jacobi identity is that symplectic geometries are “flat.” This means that all symplectic manifolds of the same (even) dimension are *locally* indistinguishable; one cannot tell one from another if one looks at them with a magnifying glass. Only globally, when one observes the entire space, do differences between various symplectic manifolds begin to emerge. [It is partly for this reason that it is so difficult to get a handle on the exotic symplectic structure on  $\mathbf{R}^4$  – local measurements cannot differentiate it from the standard symplectic structure. In this regard, we point out that the twisting in the picture in Box 3 is a large scale phenomenon—it cannot be detected locally.] This best highlights how symplectic and Riemannian geometries differ, since the latter do not have this property; indeed, Riemannian geometries are typically curved. No portion of, say, a sphere—no matter how small—can be mapped onto the plane without distorting *shapes* (that is, lengths and angles). But it is possible to map part of the globe onto the plane in such a way that *relative sizes* (that is, areas) remain unaltered, a fact with which cartographers are well acquainted. We emphasize, however, that flatness does not rule out a complicated large scale structure: a circular cylinder is flat in both the symplectic and Riemannian senses, but is not planar.

Box 4: *Cartography*

Symplectic geometry is also quite “flexible,” at least in comparison with Riemannian geometry. To appreciate this, we dwell for a moment on the notion of *symmetry*. Consider again the round sphere. If we rigidly rotate it about any axis through its center, its Riemannian geometry remains the same. Thus rigid rotations are Riemannian symmetries, that is, transformations which leave lengths and angles invariant. But if instead we rotate the sphere differentially, with one point lagging behind a neighboring point, then angles and lengths are distorted and the geometry changes. We see that for a sphere there are a limited number of symmetries; and ellipsoid has fewer yet, and most Riemannian manifolds have no symmetries at all. By way of contrast, every symplectic manifold has a enormous number of symmetries (*i.e.*, transformations which

preserve oriented area). In fact, the set of all symmetries of a symplectic manifold is always infinite-dimensional!<sup>4</sup> Thus one can deform symplectic manifolds to a much greater extent than one can their Riemannian counterparts; this is because the former are all flat and hence much less rigid than the latter.

These observations provide some insight into the character of symplectic geometry. but symplectic geometry is not only of interest in itself; over the past two decades, there has been a burst of applications of symplectic techniques to other areas in mathematics. Possibly the most significant of these has been to the theory of group representations, culminating in what is known as “geometric quantization theory.” (We shall encounter this theory later in physics as well.) Various branches of analysis, number theory and lately even knot theory have also profited from symplectic ideas. One particularly interesting development relates to catastrophe theory, in which context symplectic geometry has been used to clear up several mysteries in laser optics. The book *Catastrophe Theory* by Vladimir I. Arnol’d (Springer-Verlag, 1986) contains a readable account of these results.

Potentially even more momentous, however, is the philosophy of “symplectization” advocated recently by Arnol’d. He cites mounting evidence which indicates that many “ordinary” mathematical ideas and constructions not only have analogues within the domain of symplectic geometry, but are in fact ultimately grounded there. (Certainly the corresponding assertion is true in classical physics.) This suggests that it may be possible to completely recast a great deal of mathematics in symplectic terms. Arnol’d has characterized this process of *symplectization* as “...one of the small number of operations of the highest level, which act[s] ... on all of mathematics at once.” It could conceivably lead to a revolution in mathematics comparable to the invention of complex numbers! In Stewart’s words:

*Mathematicians must have felt this way when they discovered that complex numbers were more than just one extra gimmick: virtually every idea of mathematics, from the geometry of curves to the analysis of partial differential equations, was ripe for complexification. Mathematics exploded overnight.*

Symplectic geometry may well light up the mathematical sky once again. But this is for the future; let us now delve into its role in the realm of physics.

Symplectic geometry was “invented” by Lagrange in 1808 during his seminal studies of celestial mechanics. It first appeared as an analytical technique whereby the equations of planetary motion could be written in a greatly simplified form. (See Alan Weinstein, *Lectures on Symplectic Manifolds* (American Mathematical Society, 1977), for nice mathematical discussions of both Lagrange’s work and symplectic geometry.) These techniques were substantially amplified and expanded by William Rowan Hamilton, who showed that Lagrange’s discoveries could be applied to mechanics as a whole. The resulting collection of ideas and calculational procedures is known as *Hamiltonian mechanics*. This theory was further expanded and refined by Jacobi, Liouville and



Poisson, among others, and now forms the structural basis for essentially all of classical physics.

The paradigm for Hamiltonian mechanics is particle dynamics—the study of the motion of an object subject to various forces, like an electron moving through the electromagnetic fields inside a cathode ray tube. To determine the trajectory which such an object will follow, it is not enough to know the object’s initial position; one must also know its initial velocity or, equivalently, its initial momentum. Only then is one able to “predict the future,” that is, divine the object’s location and movements at all future times.

This leads one to study particle dynamics on a space—called *phase space*—which consists both of all possible positions or “configurations”  $q$  and all possible velocities or “momenta”  $p$  of the particle. For example, the phase space of a particle moving in everyday three-dimensional space  $\mathbf{R}^3$  is  $\mathbf{R}^6$  whose points are labeled by six quantities  $(x, y, z, p_x, p_y, p_z)$  giving the three components of position and momentum. Similarly, the planar pendulum has a phase space which is a circular cylinder parametrized by  $(\theta, p_\theta)$ , where  $\theta$  is the angular position of the bob and  $p_\theta$  is its angular momentum. Other bodies, like relativistic particles with spin or coupled rigid bodies, have more complicated phase spaces.

Box 5: *The Phase Flow for the Planar Pendulum*

The underlying idea is that once one knows a particle’s initial *state*  $(q, p)$  in phase space and also the forces that act on it, then one has enough information to chart the particle’s motion in time. All this can be neatly visualized (Box 5). To specify the (net) force is to assign a particular arrow (or “flow vector”) to every point in phase space. Then a particle starting out at some given state moves along a unique trajectory (“flow line”) as indicated by the arrows. The collection of all possible trajectories (the “flow”) fills in the phase space, with no two trajectories crossing. If one were to work, say, in just the space of all possible configurations ( $q$ ’s) rather than phase space ( $q$ ’s and  $p$ ’s), one would not have such an elegant and orderly description of particle motion. Thus phase space is the appropriate arena for dynamics.

The picture of particle motion we have been sketching here is both figuratively *and* literally analogous to the flow of a fluid. In fact dynamics, in a precise mathematical sense, is merely a fluid flow in phase space. This “phase flow,” however, has a very special property: it is *area-preserving*, that is the areas of two-dimensional sheets of fluid remain unchanged as they flow along. Since a fluid flow on a manifold may be viewed as a time-dependent transformation of that manifold, we conclude that dynamics consists of a time-dependent transformation of phase space which preserves areas. And where there are areas, there must be symplectic forms!

These observations constitute the fundamental links between mechanics and geometry, *viz.*, *the phase spaces of particle dynamics are symplectic manifolds, and dynamics corresponds to a time-dependent symplectic transformation.* These facts have had far-reaching consequences for physics. Indeed, it is the

presence of the symplectic structure on phase space which is largely responsible, in one way or another, for the tremendous success Hamiltonian mechanics has had in describing the physical world.

What precisely does the symplectic form do *physically*? Two interrelated things, primarily. First, it sorts out how the generalized positions  $q$  and momenta  $p$  fit together. For a complicated system such as the Galileo Jupiter probe, with literally hundreds of configuration variables  $q$  and momentum variables  $p$ , it is crucial that the right momentum be tied to the right position if the system's motion is to make sense. Now recall that, mathematically, the symplectic form locally serves to split a  $2n$ -dimensional space into a collection of  $n$  transverse planes. So in a sense it collects the  $2n$  independent directions  $(q_1, \dots, q_n, p_1, \dots, p_n)$  of phase space into pairs:  $(q_1, p_1; \dots; q_n, p_n)$ .<sup>5</sup> This explains, from a physical standpoint, why symplectic manifolds are even-dimensional: it is because each position is paired with a corresponding momentum. [As we will see, the intertwining of the  $p$ 's and  $q$ 's also has striking consequences in quantum mechanics.] Second, the symplectic form tells one exactly how to convert the forces which act on the system into an assignment of flow vectors on phase space, thereby enabling one to compute the phase flow and hence all allowable motions of the system. Succinctly, it converts *dynamic* data—about forces—into *kinematic* information—about the system's motion. For more on the “why” and “how” of symplectic geometry in physics, we refer the reader to the books *Foundations of Mechanics*, by Ralph Abraham and Jerrold E. Marsden (Benjamin-Cummings, second ed., 1978), and *Symplectic Techniques in Physics*, by Victor Guillemin and Shlomo Sternberg (Cambridge University Press, second ed., 1989).

Besides their primary duties, symplectic structures can be exploited for numerous other purposes in mechanics. They provide an efficient way to relate symmetries (translational, rotational, “internal” or “gauge,” etc.) of physical systems to quantities (energy-momentum, angular momentum, electric charge, etc.) which remain constant as the systems evolve in time. Such “conservation laws” are very helpful in the general analysis of how systems behave, especially in nonlinear dynamics, where it is often impossible to obtain exact quantitative results. Symplectic methods are also crucial for studying questions of stability (*e.g.*, could small oscillations of the Galileo probe's antenna grow uncontrollably in time?), and have greatly enhanced our ability to model and accurately predict the dynamical behavior of complex mechanical systems. One beautiful application of symplectic geometry (involving many of the above ideas) is to the age-old question of why a falling cat always lands on its feet. It turns out that the cat achieves this by behaving as if it were a particle moving in a certain Yang-Mills field !

Box 6: *The Yang-Mills Cat*

Thus the symplectic form is an essential ingredient both theoretically and practically in classical mechanics. And recently, in a formulation due to S.

Sternberg, it has assumed an even more transcendent role. In Sternberg's approach the forces themselves have been subsumed into the symplectic structure, which is now *all there is!* We have come full circle: Symplectic geometry is the mathematics of mechanics, and Hamiltonian mechanics is nothing but symplectic geometry on phase space. While symplectic geometry is most visible in the context of mechanics, its use extends far beyond that small branch of physics. Indeed, most classical systems, however complex, can be studied by means of Hamiltonian techniques. These include models of galaxy formation, electric circuits, and collective models of the nucleus. And if we generalize to infinite-dimensional phase spaces, then classical fields can be studied as well. In this way we have learned much about gravitational and other fields, about elasticity theory, about plasmas, and even about corrosion. Several close relatives of symplectic geometry are likewise important. One is *contact geometry*, which is an extension of symplectic geometry to *odd*-dimensional manifolds. It plays a role in optics which is similar to that of symplectic geometry in mechanics. [It is for this reason that mechanics and optics have had such a close and parallel development; they are mathematical first cousins.]

In our exposition thus far, we have been concerned exclusively with classical physics. One of the great lessons of the twentieth century, however, is that the fundamental description of most (if not all) physical systems must be *quantum mechanical*. The differences between the classical and quantum approaches to physics are pronounced as well as profound, but it would take us too far afield to discuss them in detail. We will therefore be content with making a few basic observations. (A nice account of some of the more interesting aspects of quantum theory can be found in Richard P. Feynman's book *QED* (Princeton University Press, 1985).)

The crux of the matter is that whereas classical physics is completely deterministic, quantum mechanics is inherently probabilistic. One consequence is that while classically an observer may (in principle) simultaneously measure physical quantities to any desired accuracy, an observer in a quantum mechanical world cannot: there are *unavoidable* limitations on the precision with which certain pairs of quantities may be simultaneously measured. This is the content of Heisenberg's famous "uncertainty principle." For instance, a measurement which seeks to pinpoint simultaneously the position  $q$  and momentum  $p$  of an electron must be uncertain in both, with the inaccuracies  $\Delta q$  and  $\Delta p$  satisfying the inequality

$$\Delta q \Delta p \geq h/4\pi,$$

where Planck's constant  $h = 6.6246 \times 10^{-34}$  Joules-seconds. Since this constant is very small, for most macroscopic systems such uncertainties are negligible. For microscopic systems, they are not.

Does symplectic geometry play a role in quantum physics comparable to that which it plays in the classical theory? It would not seem so. For one thing quantum mechanical states are not represented by points of the classical phase space. Rather, the quantum state space for a physical system is an

infinite-dimensional space called *Hilbert space*. This Hilbert space moreover is related to the classical configuration space (viz., the space of all positions  $q$ ) of the system and not to its phase space (em viz., the space of all momenta  $p$  as well as positions  $q$ ). Thus the basic symplectic object–phase space–of classical physics has “disappeared” in the quantum theory. However, like the Cheshire cat’s smile, remnants of symplectic geometry are still to be found in the quantum theory. One such remnant is apparent in the above formula for the uncertainty principle: the symplectic pairing of a position with its corresponding momentum. In fact, there is no uncertainty principle for quantities which are not symplectically intertwined. Other symplectic relics are the “Bohr-Sommerfeld-Maslov quantization rules,” which explain why certain physical parameters such as electric charge and elementary particle spin can only take on a discrete set of values.

But symplectic geometry is intimately involved in the transition from the classical description of a system to its quantum version, that is, *quantization*. To appreciate the significance of this, let us consider how the classical and quantum descriptions of a given physical system are related. *In principle*, as noted above, every physical system is quantum mechanical in nature. However, for most (sufficiently macroscopic) systems, the quantum description has a unique “classical limit”—a classical description which in a certain sense accurately approximates the quantum one. *In practice*, on the other hand, one almost always has a better *ab initio* understanding of the classical limit of a system than of its full-blown quantum formulation. [This is for two reasons: our everyday common-sense environment is classical, so that we are used to thinking in classical terms, and because the classical approximation is much simpler than the quantum description.] Thus physicists are more often confronted with the problem of constructing a quantum formulation of a system from a knowledge of its classical limit than they are with recovering the classical description from the quantum. In other words, it is occasionally necessary to “quantize” a classical system.

Unfortunately, this is not a straightforward proposition. One difficulty is that while the classical limit of a given quantum system is unique (if it exists), there are always many different quantum systems which have the *same* classical limit. Compounding matters is a “no-go” theorem, which asserts that it is *impossible* to find a quantization scheme which can be consistently applied to *every* classical system.

In spite of these obstacles, efforts to systematically quantize limited assortments of systems persist. One of the most successful such schemes is known as *geometric quantization theory*. Based on the work of Bertram Kostant at M.I.T. and Jean-Marie Souriau in Marseille, geometric quantization is a beautiful application of some of the most sophisticated ideas in symplectic geometry.

The idea behind geometric quantization is to use the symplectic geometry of the classical phase space to construct the quantum Hilbert space. The key step in this procedure is to *polarize* the phase space; that is, to invariantly separate the positions  $q$  from the momenta  $p$ . This distinction is used to cut the phase space down to the configuration space which, as we indicated earlier, is closely

related to the quantum Hilbert space. Once the phase space has been polarized, it is possible to generate a quantum theory of the system.

While geometric quantization is a powerful tool, it can be both difficult and subtle. For instance, there usually is some freedom in the choice of polarization, mirroring the fact that many quantum systems have identical classical limits. Thus the geometric quantization procedure may produce several inequivalent quantum theories, and it is usually necessary to resort to experiment to select the physically correct one. At the opposite extreme, there exist symplectic manifolds which cannot be polarized. It is not known if there are any genuine physical systems whose phase spaces have this property; in any event, it is not clear what to make of these “purely classical” symplectic geometries. One area in physics where geometric quantization may play a useful role is general relativity. The classical physics of the gravitational field is fairly well understood in terms of Einstein’s theory. Yet, of all the fundamental interactions in nature, gravity alone does not have a consistent quantum description. This a major puzzle of theoretical physics, and is why “quantum gravity” is an area of active research. One reason the gravitational field so strongly resists quantization is that its phase space is both infinite-dimensional and highly nonlinear. To gain some preliminary insight into quantum gravity, a favorite stratagem is to “freeze out” all but a finite number of these dimensions by demanding that the gravitational field be the same everywhere in space (while still allowing it to change in time). One thereby builds (relatively) tractable models of the Universe and its gravitational fields known as *homogeneous cosmologies*. These form handy “laboratories” for studying the quantization of gravity, and geometric quantization has proven most useful in these “experiments.”

One of the experiments the authors have run in this laboratory concerns the fascinating problem of gravitational *singularities*. These are unimaginably dense, gravitationally violent phases through which the Universe, according to general relativity, must pass at some point in its evolution. One such singularity—the “Big Bang”—has almost certainly occurred, at the moment of creation. Will there be another, final singularity—the “Big Crunch”? Observational data is inconclusive at this stage, but if gravitational attraction is strong enough to overcome the current expansion of the Universe, then it appears that the Universe will collapse, unrelentingly, to a fiery doom.

Box 7: *The Fate of the Universe*

This prediction is founded on *classical* general relativity. However, as gravity squeezes the Universe to submicroscopic dimensions, *quantum* effects should predominate, and there has been much speculation that such effects could slow or even halt the collapse. The authors have used geometric quantization to investigate this possibility within the framework of homogeneous cosmologies. Although the issue is far from settled, indications—unluckily!—are that quantum effects *cannot* prevent the final, catastrophic collapse of the Universe to a singularity.

In all of this discussion we have left unanswered one important question: what is the origin of the unusual name “symplectic”? It is derived from the Greek *συμπλεκτικός*, which is the antecedent of the Latin “complex.” Its mathematical usage is due to Hermann Weyl who, in an effort to avoid a certain semantic confusion, renamed the then obscure “line complex group” the “symplectic group.” But whatever its etymology, the adjective “symplectic” means “plaited together” or “woven.” This is wonderfully apt, for it is this intertwining—already evident in the above expression for the form—that most characterizes, and is in fact the essence of, both symplectic geometry and Hamiltonian mechanics. And it is the intricate plaiting together of mathematics and physics which gives symplectic geometry its power and its promise.

## FOOTNOTES

1. From the above discussion it may seem that the symplectic form is essentially the vector (or cross) product. This is a coincidence on  $\mathbf{R}^2$ ; in higher dimensions there is no relation between the two.
2. Technically, the Jacobi identity amounts to the 2-form  $\Omega$  being closed:  $d\Omega = 0$ . In this regard we observe that the exterior differential of forms is analogous to the Lie bracket of vector fields.
3.  $S^6$  cannot be symplectic for cohomological reasons. But manifolds may fail to be symplectic on other grounds. For instance, on  $S^4$  it is not even possible to define a nondegenerate antisymmetric bilinear form (closed or not), so the obstruction in this case is more “algebraic.”
4. The set of symmetries of a geometrical structure on a manifold forms a (Lie) group under composition. It is a well known fact that the *isometry* group of a Riemannian metric is always finite-dimensional. But in symplectic geometry the analogous object—the *symplectomorphism* group—is huge. (In fact, the Lie algebra of this group is isomorphic to the set of all closed 1-forms on the manifold in question.)
5. This is particularly evident in the local expression for the symplectic form, which is

$$\Omega = \sum_{i=1}^n dq_i \wedge dp_i,$$

where  $\wedge$  is the wedge product of forms.

Box 1: *Some History*

Menaechmus advised Alexander that “There is no royal road to Geometry.” But if there is no royal road to symplectic geometry, the historical path is surely replete with regal personages. For the roots of symplectic geometry can be traced back to the work of many of the most outstanding names in mathematics and physics of the nineteenth and early twentieth centuries: Lagrange, Poisson, Hamilton, Jacobi, Liouville, Hertz, Noether and Poincaré. Indeed, they made lasting contributions to the science of mechanics, from whence symplectic geometry springs and with which it is inextricably linked. As is often the case the mathematics owes its existence to physics, and this is especially true of symplectic geometry; physics, in turn, has been enriched by the techniques and insights afforded by the maturation of the mathematics it engendered.

Although symplectic geometry was really “there” from the beginning, the geometric nature of mechanics was obscured by an early emphasis on the analytical and computational aspects of the theory. This mindset was so overwhelming that Lagrange, in the preface to his monumental *Mécanique Analytique* [1788], could boast:

*The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure.*

It was not until more than a hundred years later, well after the groundbreaking mathematical work of Riemann in 1854, that one finds Darboux in 1889 and Hertz in 1899 employing geometrical ideas in mechanics, treating the motion of any system, no matter how complicated, as that of a “particle” moving in a certain higher-dimensional curved space. The death knell of the analytical era was finally rung by the great French mathematician Henri Poincaré in 1889, when he realized that purely quantitative techniques could not suffice to solve various problems in celestial mechanics, notably that of stability in the “ $n$ -body problem” (for instance, the motion of the planets subject to their mutual gravitational attractions).

This famous failure ushered in the “qualitative period” in mechanics, and geometry at long last got its due. The first theorem in symplectic geometry as such as Poincaré’s “last geometric theorem” in 1912 which, among other things, predicts the existence of periodic orbits in the nobody problem. Still, the geometrization of mechanics proceeded slowly, and by and large the symplectic aspects remained shadowy. Even as late as the 1940s, one finds physicists utilizing symplectic ideas rather cautiously and superficially. (A contemporary account of mechanics is given by Cornelius Lanczos in *The Variational Principles of Mechanics* (University of Toronto Press, 1949).)

Meanwhile, mathematicians were rediscovering symplectic geometry from entirely different angles, with the researches of Sophus Lie, Poincaré and Élie Cartan paving the way. But symplectic geometry, as a distinct mathematical



discipline, did not really appear until the 1940s with the (little-known) work of Hwa-Chung Lee in China. Within ten years, one finds French mathematicians such as Charles Ehresmann, André Lichnerowicz and Georges Reeb setting the stage for future developments and later applications to mechanics. By the mid 1960s, symplectic geometry was cast into the modern mathematical idiom, and the symplectic compact between geometry and mechanics was irreversibly sealed. A period of explosive growth, fueled primarily by emergent schools of American and Russian symplectic geometers, followed and continues unabated to this day.

Box 2: *Geometry on Manifolds*

The familiar mensuration formulas of Euclidean and symplectic geometry will change when the flat plane  $\mathbf{R}^2$  is replaced by a more general “warped” space, such as a sphere or a saddle. On both the (positively curved) sphere and the (negatively curved) saddle we draw a disk of radius  $r$  as illustrated below. Provided  $r$  is relatively small, the circumference  $C$  of the disk is given by

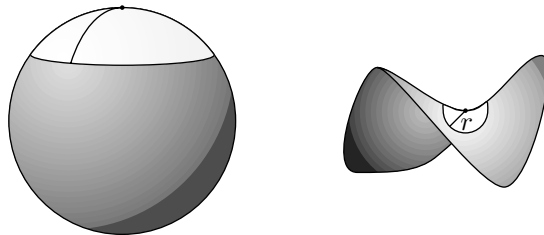
$$C = 2\pi \left( r - \frac{1}{6}\kappa r^3 + \dots \right) .$$

$\kappa$  is a constant which is  $+1$  for the sphere,  $0$  for the plane and  $-1$  for the saddle. Only when  $\kappa = 0$  do we recover the “standard” formula  $C = 2\pi r$ , showing that the Riemannian geometries of the sphere and saddle are genuinely non-Euclidean.

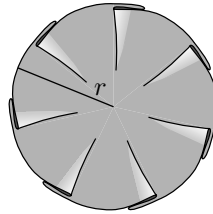
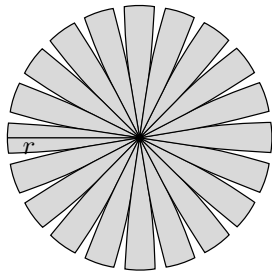
Similarly, the symplectic geometries of the sphere and saddle are distinct from that of the plane. For if we compute the area  $A$  of the disk, we find

$$A = \pi \left( r^2 - \frac{1}{2}\kappa r^4 + \dots \right) ,$$

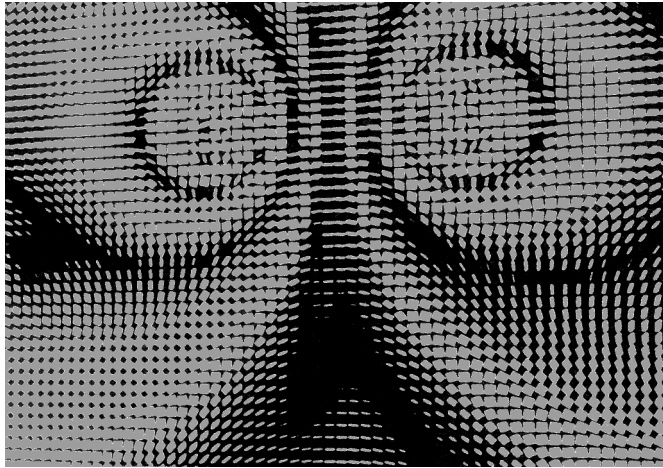
which deviates from the familiar result  $A = \pi r^2$  when  $\kappa \neq 0$ .



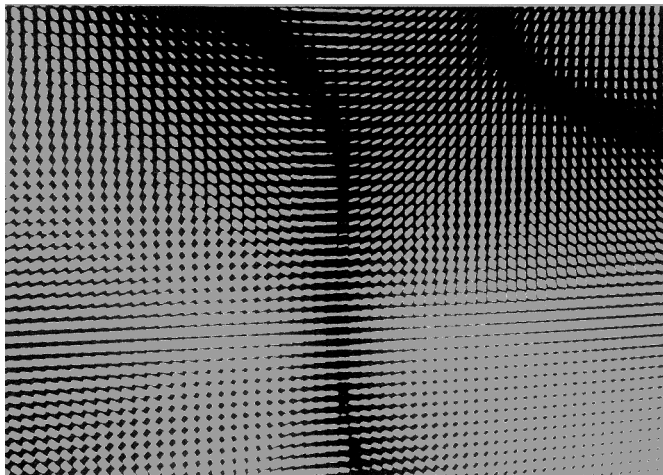
Note that on the sphere, the circumference and area of the disk are smaller than those of a disk of radius  $r$  in the plane. This can be most easily seen by cutting the disk out of the sphere and flattening it onto the plane; it would split apart as shown below left. If one does the same for the disk on the saddle, it would fold over on itself as shown below right, which explains why the values of  $C$  and  $A$  are greater on the saddle than the plane.



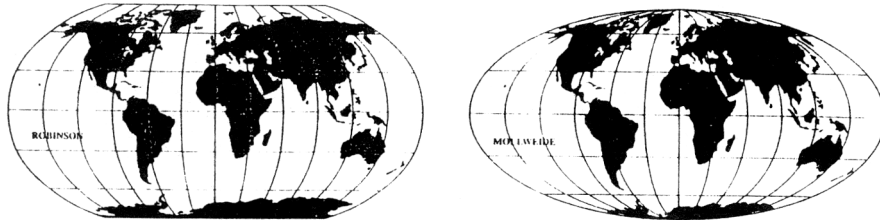
Box 3: *Symplectic Geometries on  $\mathbf{R}^4$*



To get a feeling for the nature of the exotic symplectic geometry on  $\mathbf{R}^4$ , it is useful to contrast it with the standard one. Since we cannot visualize 4-dimensional objects, we restrict attention to the unit 3-dimensional sphere centered at the origin of  $\mathbf{R}^4$ . When restricted to this sphere, both the exotic and standard symplectic forms define symplectic planes at each point. Some of these planes are plotted in the photographs here. Notice how twisted the pattern of planes for the exotic symplectic structure is (above) as compared to the standard one (below). (Computer graphics courtesy of Larry Bates and Charles Herr, Universities of Calgary and Alberta.)

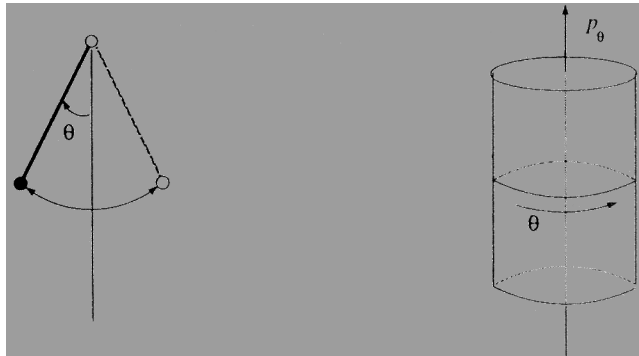


Box 4: *Cartography*

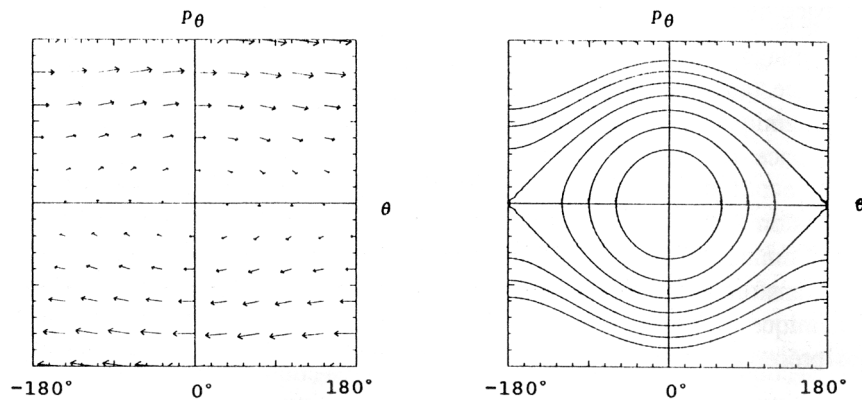


A map attempts to represent the surface of a sphere on a plane. Every map distorts shapes (Riemannian geometry) to some extent, even the currently accepted “best” map, the Robinson projection of 1963 (left). On the other hand, one can draw maps with *no* size distortion (symplectic geometry). One such map is the Mollweide elliptical equal-area projection of 1805 (right). This phenomenon reflects the “flatness” of symplectic manifolds.

Box 5: *The Phase Flow for the Planar Pendulum*



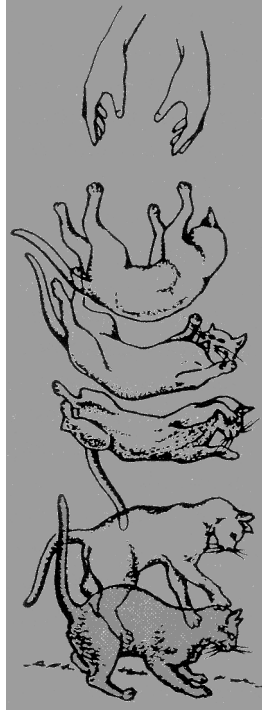
A planar pendulum is a mass at the end of a light rod which is free to swing in a given plane. All possible positions of the bob are parametrized by the angle  $\theta$ , with  $-180^\circ \leq \theta \leq 180^\circ$ . Its corresponding angular momentum  $p_\theta$  can take on any value. The phase space for the planar pendulum is therefore a circular cylinder, with  $\theta$  running counterclockwise around the cylinder and  $p_\theta$  running along its axis. Since  $180^\circ$  and  $-180^\circ$  represent the same position (“up”), these two configurations must be identified; hence the circle.)



The forces acting on the bob are gravity and the tension in the rod. The pattern of force vectors and resulting dynamical trajectories in the phase space are drawn above. For clarity, we have “unwrapped” the phase space into a rectangular strip; the two vertical lines corresponding to  $\theta = \pm 180^\circ$  must be identified with one another. The point in the center represents a stable equilibrium, with the pendulum motionless and hanging straight down. The ellipses surrounding this point correspond to rocking motions of the pendulum. These increase in

amplitude until the bob “just” reaches the top. There we have an unstable equilibrium (located at  $\theta = \pm 180^\circ$ ,  $p_\theta = 0$ ), with the pendulum stationary and pointing straight up. The remaining wavy trajectories represent motions wherein the bob swings entirely around the hinge going either clockwise (top) or counterclockwise (bottom).

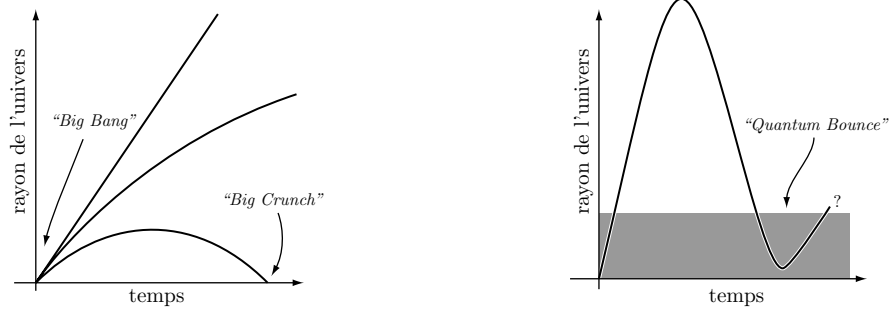
Box 6: *The Yang-Mills Cat*



The falling cat can be modeled mechanically by two cylinders connected with a ball-and-socket joint. In control theory terms, the problem is to maneuver this system, by rotating the cylinders both about their axes and relative to one another, to achieve a desired orientation in an optimal way. Symplectic geometry provides a succinct description of this problem and greatly facilitates its solution. In effect, the cat lands on its feet by solving the Yang-Mills equations! Contrary to an old wives' tale, the cat's tail is irrelevant; Manx cats land on their feet too. (This last assertion is rumored to have been experimentally verified.)



Box 7: *The Fate of the Universe*



Einstein's classical theory of gravity predicts that the Universe has evolved from an initial singularity—the Big Bang. But what will become of the Universe is not yet known; the standard scenarios (for homogeneous and isotropic models) are sketched above left. The Universe is currently expanding and its fate depends on how quickly gravitational attraction slows this expansion. This in turn depends crucially on the average mass density  $\rho$  of the Universe. The critical value of  $\rho$  is  $\rho_c \approx 10^{-30} \text{ gm/cm}^3$ . Models with  $\rho = \rho_c$  will just escape gravitational collapse (the middle curve). In models with  $\rho > \rho_c$  (such as the third curve), gravitational attraction overwhelms the expansion and the model collapses to a final singularity—the Big Crunch. Observationally  $\rho \approx 10^{-31}$ , but the issue is far from settled.

Cosmologists have wondered whether quantum effects, which are expected to be of primary importance when the Universe has been crushed to subatomic size (the shaded region in the figure on the right), might prevent the ultimate collapse. Conceivably the Universe could “bounce” into a new expansion phase.