Dissipation of energy in the locally isotropic turbulence†

By A. N. Kolmogorov

In my note (Kolmogorov 1941*a*) I defined the notion of local isotropy and introduced the quantities

$$\left.\begin{aligned} B_{dd}(r) &= \overline{[u_d(M')-u_d(M)]^2}, \\ B_{nn}(r) &= \overline{[u_n(M')-u_n(M)]^2}, \end{aligned}\right\} \tag{1}$$

where r denotes the distance between the points M and M', $u_d(M)$ and $u_d(M')$ are the velocity components in the direction $\overline{MM'}$ at the points M and M', and $u_n(M)$ and $u_n(M')$ are the velocity components at the points M and M' in some direction, perpendicular to MM'.

In the sequel we shall need the third moments

$$B_{ddd}(r) = \overline{[u_d(M')-u_d(M)]^3}. \tag{2}$$

For the locally isotropic turbulence in incompressible fluid we have the equation

$$4\bar{E}+\left(\frac{\mathrm{d}B_{ddd}}{\mathrm{d}r}+\frac{4}{r}B_{ddd}\right) = 6\nu\left(\frac{\mathrm{d}^2B_{dd}}{\mathrm{d}r^2}+\frac{4}{r}\frac{\mathrm{d}B_{dd}}{\mathrm{d}r}\right) \tag{3}$$

similar to the known equation of Kármán for the isotropic turbulence in the sense of Taylor. Herein $\bar{E}$ denotes the mean dissipation of energy in the unit of time per unit of mass. The equation (3) may be rewritten in the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{4}{r}\right)\left(6\nu\frac{\mathrm{d}B_{dd}}{\mathrm{d}r}-B_{ddd}\right) = 4\bar{E}, \tag{4}$$

and, in virtue of the condition $(\mathrm{d}/\mathrm{d}r)B_{dd}(0) = B_{ddd}(0) = 0$, yields

$$6\nu\,\mathrm{d}B_{dd}/\mathrm{d}r-B_{ddd} = \tfrac{4}{5}\bar{E}r. \tag{5}$$

For small r we have, as is known,

$$B_{dd} \sim \tfrac{1}{15}\bar{E}r^2/\nu, \tag{6}$$

i.e.

$$6\nu\,\mathrm{d}B_{dd}/\mathrm{d}r \sim \tfrac{4}{5}\bar{E}r.$$

Thus, the second term on the left-hand side of (5) is for small r infinitesimal in comparison with the first. For large r, on the contrary, the first term may be neglected in comparison with the second, i.e. we may assume that

$$B_{ddd} \sim -\tfrac{4}{5}\bar{E}r. \tag{7}$$

It is natural to assume that for large r the ratio

$$S = B_{ddd}:B_{dd}^{\frac{3}{2}}, \tag{8}$$

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i.e. the *skewness* of the distribution of probabilities for the difference $\Delta u_d = u_d(M') - u_d(M)$, remains constant. Under this assumption we have for large r

$$B_{dd} \sim C\bar{E}^{\frac{2}{3}} r^{\frac{2}{3}}, \tag{9}$$

where

$$C = (-4/5S)^{\frac{2}{3}}. \tag{10}$$

In Kolmogorov (1941*a*) the relation (9) was deduced from somewhat different considerations,† and the *local scale* of turbulence was introduced:

$$\lambda = (\nu^3/\bar{E})^{\frac{1}{4}} \tag{11}$$

and we established the assumption that

$$\left.\begin{aligned} B_{dd}(r) &= \surd(\nu\bar{E})\,\beta_{dd}(r/\lambda), \\ B_{nn}(r) &= \surd(\nu\bar{E})\,\beta_{nn}(r/\lambda), \end{aligned}\right\} \tag{12}$$

where β_{dd} and β_{nn} are universal functions, for which for small ρ

$$\beta_{dd}(\rho) \sim \tfrac{1}{15}\rho^2, \quad \beta_{nn}(\rho) \sim \tfrac{2}{15}\rho^2 \tag{13}$$

and for large ρ

$$\beta_{dd}(\rho) \sim C\rho^{\frac{2}{3}}, \quad \beta_{nn}(\rho) \sim \tfrac{4}{3}C\rho^{\frac{2}{3}}. \tag{14}$$

In the case of isotropic turbulence in the sense of Taylor the laws of locally isotropic turbulence must hold for distances, considerably less than the *integral scale* L of turbulence (as regards its rational definition cf. my note (Kolmogorov 1941*b*)). The correlation coefficients

$$\left.\begin{aligned} R_{dd}(r) &= (\overline{u_d(M')\,u_d(M)}) : b, \\ R_{nn}(r) &= (\overline{u_n(M')\,u_n(M)}) : b, \end{aligned}\right\} \tag{15}$$

where b is the mean value of the square of the velocity components, are here connected with $B_{dd}(r)$ and $B_{nn}(r)$ by the relations

$$\left.\begin{aligned} B_{dd} &= 2b(1-R_{dd}), \\ B_{nn} &= 2b(1-R_{nn}). \end{aligned}\right\} \tag{16}$$

In virtue of (16) and (12) we shall have for r, small in comparison with L,

$$\left.\begin{aligned} 1-R_{dd} &\sim (\surd(\nu\bar{E})/2b)\,\beta_{dd}(r/\lambda), \\ 1-R_{nn} &\sim (\surd(\nu\bar{E})/2b)\,\beta_{nn}(r/\lambda). \end{aligned}\right\} \tag{17}$$

If r is small in comparison with L, but large in comparison with λ, we have in virtue of (14) and (11)

$$1-R_{dd} \sim \tfrac{1}{2}C\bar{E}^{\frac{2}{3}} b^{-1} r^{\frac{2}{3}}, \tag{18$'$}$$

$$1-R_{nn} \sim \tfrac{2}{3}C\bar{E}^{\frac{2}{3}} b^{-1} r^{\frac{2}{3}}. \tag{18$''$}$$

The formulae (18) enable us to determine the constant C from the experimental data. The most accurate measurements of the correlation coefficients R_{dd} and R_{nn} were carried out by Dryden *et al.* (1937) in their paper. Rewriting (18″) in the form

$$1-R_{nn} \sim 2C(kr)^{\frac{2}{3}}, \quad k = \bar{E} : (3b)^{\frac{3}{2}}, \tag{19}$$

† A. Obukhov has found the relation (9) independently by computing the balance of the energy distribution of pulsations over the spectrum (cf. Obukhov 1941).

I calculated from the empirical formula (17) of Dryden *et al.*'s paper (using their notation, $b = \sqrt{\overline{u}^2}, \overline{E} = \frac{3}{2}U\,\mathrm{d}\sqrt{\overline{u}^2}/\mathrm{d}x$) the values of the coefficient k, corresponding to the turbulence at the distance of $40M$ from the grid with the width of mesh M equal to 1, $3\frac{1}{4}$ and 5 inches (2.4, 8.1 and 12.5 cm):

M/inches	1	$3\frac{1}{4}$	5
k/cm^{-1}	0.197	0.065	0.042.

With these values for k the graphs in fig. 5 of Dryden *et al.*, taking into account the wire-length correction, show for values of r, not too large in comparison with L, a good agreement with formula (19) for

$$C = \tfrac{3}{2}. \tag{20}$$

The local scale λ is, in the circumstances of experiments described in Dryden *et al.* (1937), so small that deviations from the relations (18) cannot be observed for small r.†

The curves in fig. 28 of Kolmogorov (1941*b*) cannot be directly used for the determination of C, since the wire-length correction is not introduced into them. They confirm, however, for r, small in comparison with L, the relation

$$(1-R_{nn}):(1-R_{dd}) = \tfrac{4}{3} \tag{21}$$

following from (18), with a satisfactory accuracy.

If we take into account the wire-length correction for the ratio (21), in Dryden *et al.* (1937, p. 29) is obtained the value 1.28, which is also sufficiently close to the theoretical value $\frac{4}{3}$, if we consider the limited accuracy of the experiment.

References

Dryden, H. L., Schubauer, G. B., Mock, W. C. Jr & Scramstad, H. K. 1937 *Rep NACA*, no. 581.
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† In connexion with this we may remark that the attempt of application to the observations of Dryden *et al.* (1937) of the theory of isotropic turbulence, neglecting the third moments, undertaken by Millionshtchikov (1939), must be considered as based on a misunderstanding. It is easily verified that under the circumstances of observations in Dryden *et al.*, in the equations connecting the second moments with the third ones (e.g. in equation (3)) the terms with the second moments are *considerably smaller*, than the terms with the third moments.

In fig. 3 of Millionshtchikov (1939) the comparison of the theoretical curve for R_{nn} (obtained under neglection of the third moments) with the experimental data of Dryden *et al.* is carried out in the right way, since (1) the used data refer to a definite spectral component of the pulsations, and not to the total pulsations, and (2) the experimental data do not correspond to the condition $8\nu t = 7.56$, by which the theoretical curve is determined. In his later papers Millionshtchikov himself gave, by means of rather fine considerations, using the third and fourth moments, an estimation of the error resulting from the neglection of the third moments.