An Optimal Distributed Bridge-Finding Algorithm

David Pritchard, Department of Combinatorics and Optimization
University of Waterloo, Ontario, Canada

Overview and Problem Statement

We model a computer network by a graph, where the vertices are computers and the edges are two-way communication links. It is a fundamental necessity that the graph be connected if the network is to operate as a whole. A *bridge* is an edge whose deletion causes the network to become disconnected. The bridges are the edges that are critical to network reliability. We give a simple, efficient distributed algorithm to determine the bridges of a network. Let the network be $\mathcal{G} = (V, E)$, and let $\mathsf{Diam}(\mathcal{G})$ be the largest distance between any pair of nodes in $\mathcal{G}$. Our protocol

- Assumes the network is synchronous.
- Assumes the existence of a distinguished *leader* node.
- Uses messages that are $O(\log V)$ bits long.
- Sends a total of $O(E)$ messages.
- Completes in $O(\mathsf{Diam}(\mathcal{G}))$ time.
- Has optimal message and time complexity (within a constant factor) on all graphs, under certain assumptions.

Background

Historically, algorithms for bridge-finding are linked to ones for determining the *cut points* — those nodes whose deletion disconnects the network — and *blocks* — maximal subgraphs having no cut vertices. The blocks partition $E(\mathcal{G})$, and from this partition, it is easy to determine the cut points and bridges. Note, we consider only small-message protocols since we could otherwise use a trivial Diam-time algorithm. This also aids our algorithm's practicality.

Table: A brief history of bridge-finding and block-finding algorithms.

When	Who	What	Model	Complexity
1972	Tarjan	Blocks	Sequential	$O(V+E)$ time
◇ Based off of properties of depth-first search				
1974	Tarjan	Bridges	Sequential	$O(V+E)$ time
◇ Extends older ideas to use any spanning tree, not just DFS				
1984	Tarjan-Vishkin	Blocks	Seq., Parallel	$O(V+E)$ time
◇ Works with any spanning tree. Blocks are connected components of an auxiliary graph $\mathcal{G}'$ with $V(\mathcal{G}') = E(\mathcal{G})$				
1989	Huang	Blocks	Distributed	$O(V)$ time, $O(E)$ msg
◇ Reformulation of Tarjan-Vishkin				
1990	Hohberg	Blocks	Distributed	$O(E + V\log V)$ msg
◇ Proved $\Omega(E + V\log V)$ msgs are necessary if no leader present				
1997	Thurimella	Blocks	Distributed	$\tilde{O}(\mathsf{Diam} + \sqrt{V})$ time
◇ Improves Huang/T-V. Uses subgraph $\mathcal{H} \subseteq \mathcal{G}$ and not auxiliary graph ◇ Calls fast MST algorithm to get connected components of $\mathcal{H}$				
2006	[this poster]	Bridges	Distributed	$O(\mathsf{Diam})$ time, $O(E)$ msg

Our Contribution

Thurimella's block-finding algorithm seems also to be the fastest known distributed bridge-finding algorithm. The main point of this work is that, if we only want to compute the bridges, then it is wasteful to compute the blocks. Specifically, Thurimella's algorithm takes $\Theta(\mathsf{Diam}+\sqrt{V}\log^* V)$ time, but ours takes only $O(\mathsf{Diam})$ time to find the bridges, and this seems to be optimal. Our algorithm uses the same key ideas as Tarjan's 1974 paper.

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Bridge-Finding Technique

Step 1: Find a spanning tree. Our algorithm requires a spanning tree $\mathcal{T}$ of the network. Any spanning tree will do, but a BFS tree leads to the best running time. Since each cycle of $\mathcal{G}$ containing edge $\{u, v\}$ corresponds to a simple u-v path in $G\backslash\{\{u, v\}\}$, we have

$$\text{An edge is a bridge of } \mathcal{G} \text{ if and only if it is contained in no simple cycle of } \mathcal{G}. \tag{1}$$

It follows that every bridge is a tree edge. Let $desc(v)$ denote the descendants of v in $\mathcal{T}$, including v itself. Let $p(v)$ denote the *parent* of v in $\mathcal{T}$. From property (1) we can deduce that

$$\text{Edge } \{p(v), v\} \in \mathcal{T} \text{ is a bridge if and only if no other edge meets both } desc(v) \text{ and } V(\mathcal{G})\backslash desc(v). \tag{2}$$

Step 2: Compute subtree sizes and preorder. We would like to efficiently determine for each node v whether $\{p(v), v\}$ meets condition (2). We can accomplish this with a *pre-order*, which is an *order of discovery* of a DFS on $\mathcal{T}$. **Important: hereafter we refer to all nodes by their preorder label.** So for example $p(v) < v$ for all non-root v.

Step 3: Compute low **and** $high$ **values.** For a node v we define its *subtree-neighbourhood* $SN(v)$ to comprise of the subtree rooted at v, along with any further nodes of $\mathcal{G}$ reachable from the subtree by a single non-tree edge:

$$SN(v) := desc(v) \cup \{w \mid u \in desc(v), \{w, u\} \in E(\mathcal{G}\backslash\mathcal{T})\}.$$

For each node v the values $low(v)$ and $high(v)$ denote the minimum and maximum preorder label amongst its subtree-neighbourhood:

$$low(v) := \min SN(v) \qquad \text{and} \qquad high(v) := \max SN(v).$$

Then from property (2), and since in preorder $desc(v) = \{v, \dots, v + \#desc(v) - 1\}$, we have

$$\text{Edge } \{p(v), v\} \text{ is a bridge if and only if } (low(v) \geq v) \text{ and } (high(v) < v + \#desc(v)). \tag{3}$$

Our algorithm simply determines which v have property (3).

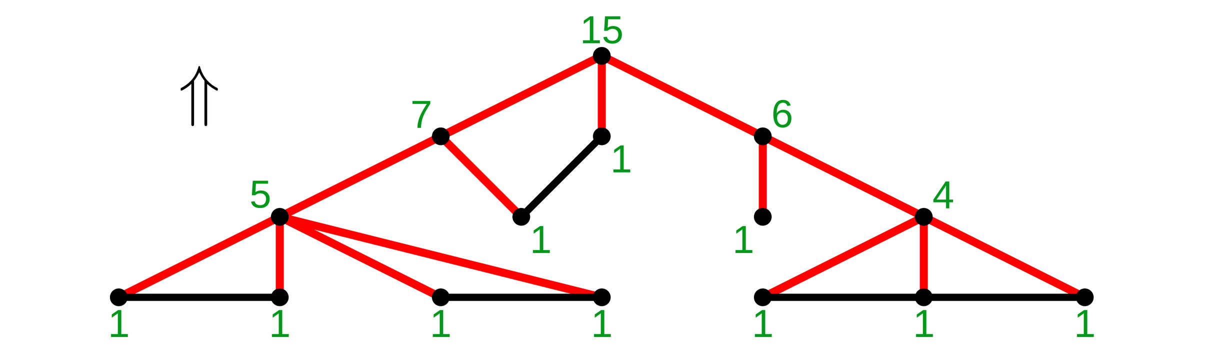
Implementation and Example

Step 1: We use a well-known greedy algorithm for distributively computing a BFS tree. That algorithm has time complexity $O(\mathsf{Diam})$ and uses $O(E)$ messages.

We repeatedly use two tree-based parallel communication techniques. In a *downcast*, messages are sent down each tree edge, starting at the root, and ending at the leaves. A *convergecast* is like a bottom-up acknowledgement for a downcast: node v waits for reports from each child, and then v reports to its parent.

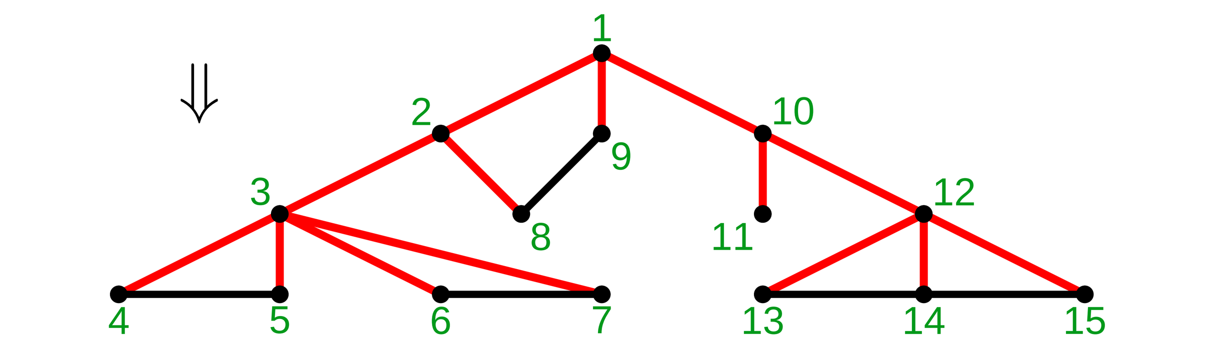
Step 2: Using a downcast, a request is sent that all nodes compute their subtree size $\#desc$. Then there corresponds a convergecast: each leaf node v immediately determines that $\#desc(v) = 1$ and reports this value to its parent; each non-leaf node v, upon learning the $\#desc$ values of its children $c_1, \dots, c_k$, computes $\#desc(v) = 1 + \sum_{i=1}^{k} \#desc(c_i)$, and reports this value to its parent. See Figure 1.

Figure 1: In this convergecast, each node v computes its subtree size $\#desc(v)$. Tree edges are red, with the root at the top.



Preordering is accomplished with a downcast. The root sets its preorder label to 1. Whenever a node v sets its preorder label to ℓ, it orders its children in $\mathcal{T}$ arbitrarily as $c_1, c_2, \dots$. Then v sends the message "Set your preorder label to ℓ_i" to each c_i, where v computes ℓ_i according to the formula $\ell_i = \ell + 1 + \sum_{j<i} \#desc(c_j)$. See Figure 2.

Figure 2: Using a downcast, each node computes its preorder label.

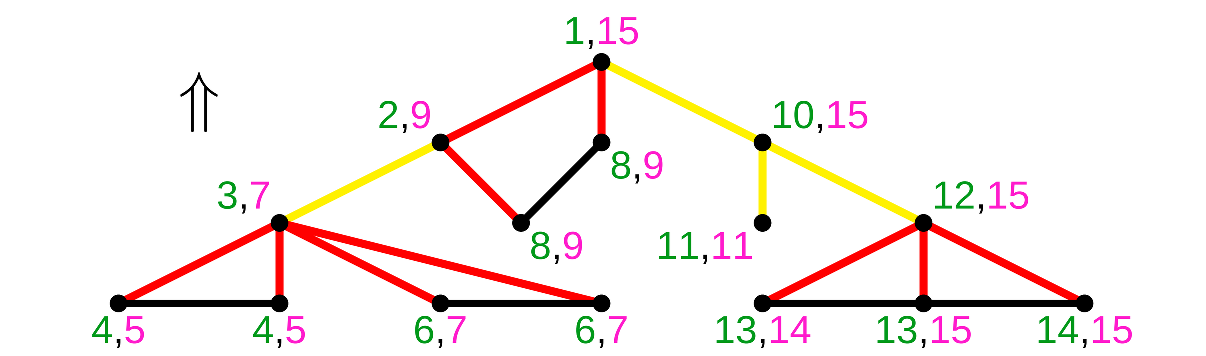


Step 3: Each node v initializes $low(v) \leftarrow v$ and $high(v) \leftarrow v$. Then, each node *announces* its preorder label to all of its neighbours except its parent and children. When node v hears such an announcement from u it sets $low(v) \leftarrow \min(low(v), u)$ and $high(v) \leftarrow \max(high(v), u)$. Using a convergecast, each node computes

$$low(v) \leftarrow \min\Big(\{low(v)\} \cup \{low(u) \mid u \text{ a child of } v\}\Big),$$
$$high(v) \leftarrow \max\Big(\{high(v)\} \cup \{high(u) \mid u \text{ a child of } v\}\Big).$$

See Figure 3. Finally, one additional message along each tree edge allows us to determine where property (3) holds.

Figure 3: With another convergecast, each node v computes $low(v)$ and $high(v)$. The bridges are shown in yellow.



We further note: if all nodes begin simultaneously, we can derive a near time-optimal *local* algorithm with implicit termination by using Elkin's *neighbourhood cover* protocol.

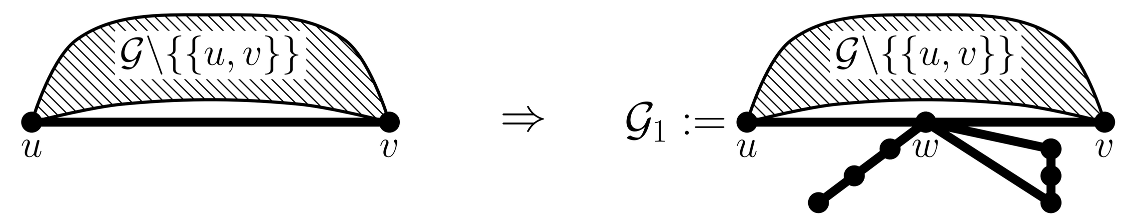
Complexity Analysis

Let h be the *height* of $\mathcal{T}$. Each downcast and convergecast takes h time and uses $V - 1$ messages. Since h is a BFS tree, $\frac{\mathsf{Diam}}{2} \leq h \leq \mathsf{Diam}$. The remaining operations — tree construction, announcement, and bridge identification — together take $O(\mathsf{Diam})$ time and $O(E)$ messages. Thus our algorithm's total complexity is $O(\mathsf{Diam})$ time and $O(E)$ messages. Each message has a single datum between 1 and V, and so can be encoded using $O(\log V)$ bits. Note, in an asynchronous environment, downcasts and convergecasts still take h time and $V - 1$ messages, but the complexity-dominating BFS tree construction step is more costly.

Universal Optimality

Our distributed algorithm is *deterministic, event-driven,* has a *single initiator,* and assumes neighbour identities are initially unknown (*no local knowledge*). Under these assumptions, *any* correct bridge-finding algorithm sends at least E messages and takes at least $\frac{\mathsf{Diam}}{2}$ time on *all* graphs.

Suppose that a bridge-finding protocol running on a graph $\mathcal{G}$ doesn't ever send a message on some edge, say $\{u, v\}$. Obtain graph $\mathcal{G}_1$ from $\mathcal{G}$ by subdividing $\{u, v\}$ with a new node w and attaching some cycles and bridges to w, as shown in the figure below. When we run the protocol on $\mathcal{G}_1$,



the same messages are sent; as no messages are sent on $\{u, w\}$ or $\{w, v\}$, and because the algorithm is event-driven, no messages reach the new parts of the graph. Hence the new edges cannot possibly be classified correctly. Thus in a correct protocol, every edge carries at least one message, and the E message lower bound follows.

The time lower bound has essentially the same proof. If a bridge-finding algorithm terminates in less than $\frac{\mathsf{Diam}(\mathcal{G})}{2}$ time on some graph $\mathcal{G}$, then some node receives no messages; we would attach new cycles and bridges to that node. So, no algorithm of the described form can beat ours by more than a constant factor on *any* graph. Other "optimal" algorithms are, in contrast, optimal only on *some* graphs.

Open Questions

Is bridge-finding strictly easier than finding blocks? Peleg and Rubinovich proved a $\tilde{\Omega}(\sqrt{n} + \mathsf{Diam})$ time lower bound for the minimum spanning tree problem. We may be able to adapt this proof to a lower bound on block-finding.

There are sequential algorithms for *strong components, triconnected components* and *planarity testing* in $O(V+E)$ time which are based on properties of DFS. In fact, if a spanning directed DFS tree is given, then our algorithm essentially computes the strong components. Do these problems admit $(o(n) + O(\mathsf{Diam}))$-time distributed solutions?